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SCHWARZSCHILD'S SINGULARITY IS SEMI-REGULARIZABLE

CRISTI STOICA

Abstract. We show that the singularity of the Schwarzschild spacetime can be made semi-regular, by a proper choice of coordinates. A semi-regular singularity doesn't destroy the topology, and allows the field equations to be rewritten in a form which avoids the infinities, as it was shown elsewhere [3, 4]. In the new coordinates, the Schwarzschild solution extends beyond the singularity. This suggests a possibility that after the evaporation of a spacelike singularity, the topology and the information are restored.

Contents

Introduction

The Schwarzschild black hole solution, expressed in the Schwarzschild coordinates, has the following metric tensor:

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2, \tag{1}$$

where

$$\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 \tag{2}$$

is the metric of the unit sphere S^2, m the mass of the body, and the units were chosen so that $c = 1$ and $G = 1$ (see *e.g.* [1], p. 149).

The first two terms in the right hand side of equation (1) don't depend on the coordinates θ and ϕ, and $\mathrm{d}\sigma^2$ is independent on the coordinates r and

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t. Therefore we can view this solution as a warped product between a two-dimensional semi-Riemannian space and the sphere S^2 with the canonical metric (2). Hence, as long as we change only the coordinates r and t, we can ignore the term $r^2 \mathrm{d}\sigma^2$ in calculations, and reintroduce it at the end.

The singularity at $r = 2m$, which makes the coefficient $\left(1 - \dfrac{2m}{r}\right)^{-1}$ become infinite, is only apparent, as shown by the Eddington-Finkelstein coordinates ([1], p. 150).

As $r \searrow 0$, the coefficient $\left(1 - \dfrac{2m}{r}\right)^{-1}$ tends to 0, and the coefficient $-\left(1 - \dfrac{2m}{r}\right)$ tends to $+\infty$. This is a genuine singularity, as we can see from the fact that the scalar $R_{abcd}R^{abcd}$ tends to ∞. This seems to suggest that the Schwarzschild metric cannot be made smooth at $r = 0$. In fact, as we will see, we can find coordinate systems in which the metric, although degenerate, doesn't have infinite coefficients and is analytic, even at the genuine singularity given by $r = 0$. Moreover, we will see that we can find an analytic extension of the Schwarzschild spacetime, which is *semi-regular*.

In [3, 4, 2] it was developed the singular semi-Riemannian geometry for metrics which are allowed to change their signature, in particular to be degenerate. Such metrics g_{ab} are smooth, but g^{ab} tends to ∞ when the metric becomes degenerate. The notion of Levi-Civita connection cannot be defined, and the curvature cannot be defined canonically. But in the special case of semi-regular metrics we can construct a canonical Riemann curvature tensor R_{abcd}, which is smooth, although $R^a{}_{bcd}$ is not canonically defined and is singular. It admits canonical Ricci and scalar curvatures, which may be discontinuous or infinite at the points where the metric changes its signature. The usual tensorial and differential operations, normally obstructed by the degeneracy of the metric, can be replaced by equivalent operations which work fine, if the metric is semi-regular.

In this paper we will show that the Schwarzschild solution can be extended analytically to such a well behaved semi-regular solution.

1. Semi-regular extension of the Schwarzschild spacetime

Theorem 1.1. The Schwarzschild metric admits an analytic extension in which the singularity at $r = 0$ is semi-regular.

Proof. To show that the metric is semi-regular, it is enough to show that there is a coordinate system in which

$$g^{st}\Gamma_{abs}\Gamma_{cdt} \tag{3}$$

are all smooth [3], where Γ_{abc} are Christoffel's symbols of the first kind. If we find a coordinate system in which the metric is smooth, although degenerate, then Christoffel's symbols of the first kind are also smooth. But the inverse metric g^{st} is not smooth for $r = 0$, so we have to show then that the full expression (3) is smooth. It is enough to make the coordinate change in a neighborhood of the singularity – in the region $r < 2m$. On that region, the coordinate r is timelike, and t is spacelike. We choose the coordinates τ and ξ, so that

$$\begin{cases} r &= \tau^2 \\ t &= \xi\tau^4 \end{cases} \tag{4}$$

Then, we have

$$\frac{\partial r}{\partial \tau} = 2\tau,\ \frac{\partial r}{\partial \xi} = 0,\ \frac{\partial t}{\partial \tau} = 4\xi\tau^3,\ \frac{\partial t}{\partial \xi} = \tau^4. \tag{5}$$

Recall that the metric coefficients in the Schwarzschild coordinates are

$$g_{tt} = \frac{2m - r}{r},\ g_{rr} = \frac{r}{r - 2m},\ g_{tr} = g_{rt} = 0. \tag{6}$$

Let's calculate the metric coefficients in the new coordinates.

$$\begin{aligned} g_{\tau\tau} &= \left(\frac{\partial r}{\partial \tau}\right)^2 \frac{r}{r - 2m} + \left(\frac{\partial t}{\partial \tau}\right)^2 \frac{2m - r}{r} \\ &= 4\tau^2 \frac{\tau^2}{\tau^2 - 2m} + 16\xi^2\tau^6 \frac{2m - \tau^2}{\tau^2} \end{aligned}$$

Hence

$$g_{\tau\tau} = 4\frac{\tau^4}{\tau^2 - 2m} + 32m\xi^2\tau^4 - 16\xi^2\tau^6. \tag{7}$$

$$\begin{aligned} g_{\tau\xi} &= \frac{\partial r}{\partial \tau}\frac{\partial r}{\partial \xi}\frac{r}{r - 2m} + \frac{\partial t}{\partial \tau}\frac{\partial t}{\partial \xi}\frac{2m - r}{r} \\ &= 0 + 4\xi\tau^7 \frac{2m - \tau^2}{\tau^2} \end{aligned}$$

Hence

$$g_{\tau\xi} = 8m\xi\tau^5 - 4\xi\tau^7. \tag{8}$$

$$\begin{aligned} g_{\xi\xi} &= \left(\frac{\partial r}{\partial \xi}\right)^2 \frac{r}{r - 2m} + \left(\frac{\partial t}{\partial \xi}\right)^2 \frac{2m - r}{r} \\ &= 0 + \tau^8 \frac{2m - \tau^2}{\tau^2} \end{aligned}$$

Hence

$$g_{\xi\xi} = 2m\tau^6 - \tau^8. \tag{9}$$

We can see that the singularity of the new metric resides in it being degenerate, and none of its coefficient become infinite at $r = 0$.

Let's calculate now the determinant of the metric in the new basis:

$$\begin{aligned}\det g &= g_{\tau\tau}g_{\xi\xi} - g_{\tau\xi}^2 \\ &= \left(4\tau^2\frac{\tau^2}{\tau^2-2m} + 16\xi^2\tau^6\frac{2m-\tau^2}{\tau^2}\right)\left(\tau^8\frac{2m-\tau^2}{\tau^2}\right) \\ &\quad - \left(4\xi\tau^7\frac{2m-\tau^2}{\tau^2}\right)^2 \\ &= -4\tau^{10} + 16\xi^2\tau^{14}\left(\frac{2m-\tau^2}{\tau^2}\right)^2 - 16\xi^2\tau^{14}\left(\frac{2m-\tau^2}{\tau^2}\right)^2\end{aligned}$$

Therefore

$$\det g = -4\tau^{10}. \tag{10}$$

The inverse of the metric has the coefficients given by $g^{\tau\tau} = g_{\xi\xi}/\det g$, $g^{\xi\xi} = g_{\tau\tau}/\det g$, and $g^{\tau\xi} = g^{\xi\tau} = -g_{\tau\xi}/\det g$. It follows from (7), (8), and (9) that

$$g^{\tau\tau} = -\frac{m}{2}\tau^{-4} + \frac{1}{4}\tau^{-2} \tag{11}$$

$$g^{\tau\xi} = -2m\xi\tau^{-5} + \xi\tau^{-3} \tag{12}$$

$$g^{\xi\xi} = \frac{\tau^{-6}}{2m-\tau^2} - 8m\xi^2\tau^{-6} + 4\xi^2\tau^{-4}. \tag{13}$$

Christoffel's symbols of the first kind are given by

$$\Gamma_{abc} = \frac{1}{2}\left(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}\right), \tag{14}$$

so we have to calculate the partial derivatives of the coefficients of the metric. From (5), (7), (8), and (9) we have:

$$\begin{aligned}\partial_\tau g_{\tau\tau} &= \partial_\tau\left(4\frac{\tau^4}{\tau^2-2m} + 32m\xi^2\tau^4 - 16\xi^2\tau^6\right) \\ &= 4\frac{4\tau^3(\tau^2-2m) - 2\tau^4\tau}{(\tau^2-2m)^2} + 128m\xi^2\tau^3 - 96\xi^2\tau^5\end{aligned}$$

$$\partial_\tau g_{\tau\tau} = 8\frac{\tau^5 - 4m\tau^3}{(\tau^2-2m)^2} + 128m\xi^2\tau^3 - 96\xi^2\tau^5 \tag{15}$$

Similarly,

$$\partial_\tau g_{\tau\xi} = 40m\xi\tau^4 - 28\xi\tau^6, \tag{16}$$

$$\partial_\tau g_{\xi\xi} = 12m\tau^5 - 8\tau^7, \tag{17}$$

$$\partial_\xi g_{\tau\tau} = 64m\xi\tau^4 - 32\xi\tau^6, \tag{18}$$

$$\partial_\xi g_{\tau\xi} = 8m\tau^5 - 4\tau^7, \tag{19}$$

and

$$\partial_\xi g_{\xi\xi} = 0. \tag{20}$$

To check that the expression (3) is smooth, it is enough now to check that it doesn't contain negative powers of τ. The least power of τ in the partial derivatives of the metric is 3, as we can see by inspecting equations (15), (16), (17), (18), (19), and (20). The least power of τ in the inverse metric is -6, as it follows from the equations (11), (12), and (13). Hence, the least power of τ in the expression (3) is at least $-6+3+3=0$, guaranteeing the smoothness.

To go back to the four-dimensional spacetime, we take the warped product between the metric above and the sphere S^2, with the warping function τ^2. The warping function satisfies

$$\mathrm{d}(\tau^2) = \frac{\partial \tau^2}{\partial \tau}\mathrm{d}\tau + \frac{\partial \tau^2}{\partial \xi}\mathrm{d}\xi = 2\tau \mathrm{d}\tau, \tag{21}$$

which is smooth and cancels at the singular points $r=0$. Hence, according to the central theorem of semi-regular warped products from [4], the warped product between the two-dimensional extension (τ,ξ) and the sphere S^2, with warping function τ^2, is semi-regular. This is the needed extension of the Schwarzschild solution. □

It is useful to extract from the proof the expression of the metric:

Corollary 1.2. The metric

$$\begin{aligned} \mathrm{d}s^2 \;=\; & \left(4\frac{\tau^4}{\tau^2-2m} + 32m\xi^2\tau^4 - 16\xi^2\tau^6\right)\mathrm{d}\tau^2 \\ & +2\left(8m\xi\tau^5 - 4\xi\tau^7\right)\mathrm{d}\tau\mathrm{d}\xi + \left(2m\tau^6 - \tau^8\right)\mathrm{d}\xi^2 + \tau^4\mathrm{d}\sigma^2, \end{aligned} \tag{22}$$

is an analytic extension of the Schwarzschild metric, and is semi-regular at the singularity $r=0$.

□

Remark 1.3. Apparently, it is impossible to extend the Schwarzschild metric so that it becomes smooth at the singularity, instead of becoming infinite. We could do this because the coordinate change we used are themselves singular, and the singularity in the coordinate change coincides with the singularity of the Schwarzschild metric. This can be viewed as analogous to the Eddington-Finkelstein coordinate change, which removed the apparent singularity on the event horizon. In both cases the metric is made smooth at points where it was thought to be infinite, the only difference is that in our case, where $r=0$, the metric becomes degenerate.

Remark 1.4. This is not the unique way to extend the Schwarzschild metric so that it is smooth at the singularity. Not all smooth extensions ensure the semi-regularity of the metric. It may seem strange to have non-unique analytic extension, but this happens again because the coordinate changes from the Schwarzschild coordinates to coordinates which ensure the smoothness of the metric are singular.

Remark 1.5. The Riemann curvature tensor R_{abcd} is smooth, because the metric is semi-regular. How can it be smooth, when we know that the Riemann curvature of the Schwarzschild metric tends to infinity when it approaches the singularity $r = 0$? The answer is that the coefficients of R_{abcd} depend on the coordinate system, and our coordinates are themselves singular with respect to the usual coordinates used with the Schwarzschild black hole solution. But this should not be a big surprise, because for the Schwarzschild solution the Ricci tensor is 0, hence the scalar curvature is 0 too, and the Einstein's equation is simply $T_{ab} = 0$. On the other hand, the Kretschmann invariant $R_{abcd}R^{abcd}$ still becomes infinite at $r = 0$, of course. But only because R^{abcd} becomes infinite.

2. Penrose-Carter coordinates for the semi-regular solution

To move to Penrose-Carter coordinates, we apply the same steps as one usually applies for the Schwarzschild black hole ([1], p. 150-156). More precisely, the lightlike coordinates for the Penrose-Carter diagram are

$$\left\{\begin{array}{lcl} v'' & = & \arctan\left((2m)^{-1/2}\exp\left(\dfrac{v}{4m}\right)\right) \\ w'' & = & \arctan\left(-(2m)^{-1/2}\exp\left(-\dfrac{w}{4m}\right)\right) \end{array}\right. \tag{23}$$

where v, w are the Eddington-Finkelstein lightlike coordinates

$$\left\{\begin{array}{lcl} v & = & t + r + 2m\ln(r - 2m) \\ w & = & t - r - 2m\ln(r - 2m). \end{array}\right. \tag{24}$$

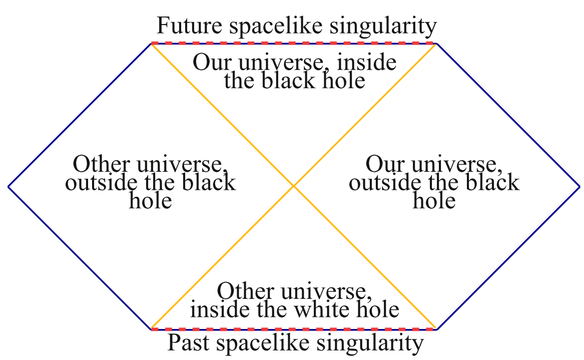

Figure 1. The maximally extended Schwarzschild solution, in Penrose-Carter coordiantes.

Usually, in the Penrose-Carter diagram of the Schwarzschild black hole, is considered that the maximal analytic extension is given by the conditions $v'' + w'' \in (-\pi, \pi)$ and $v'', w'' \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ (see Fig. 1). This is because we have to stop at the singularity $r = 0$, because of the infinite values we get prevent the analytic continuation.

In our case, we use the substitution (4):

$$\left\{\begin{array}{lcl} v & = & \xi\tau^4 + \tau^2 + 2m\ln(\tau^2 - 2m) \\ w & = & \xi\tau^4 - \tau^2 - 2m\ln(\tau^2 - 2m). \end{array}\right. \tag{25}$$

Our coordinates allow us to go beyond the singularity. As we can see from equation (22), our solution extends to negative τ as well. From (25) we see that it is symmetric with respect to the hypersurface $\tau = 0$. This leads to the Penrose-Carter diagram from Fig. 2.

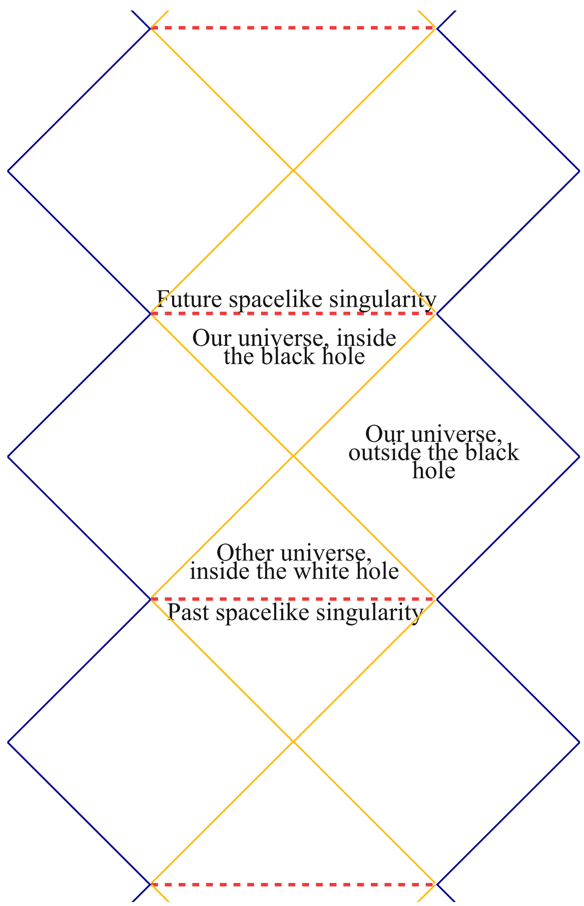

FIGURE 2. The maximally extended semi-regular Schwarzschild solution, in Penrose-Carter coordiantes.

3. The significance of the semi-regular solution

The main consequence of the extensibility of the Schwarzschild solution to a semi-regular solution beyond the singularity is that the information is not lost there. This can apply as well to the case of an evaporating black hole (see Fig. 3).

Because of the no-hair theorem, the Schwarzschild solution is representative for non-rotating and electrically neutral black holes. If the black hole evaporates, the singularity becomes visible to the distant observers. If the singularity is semi-regular, it doesn't destroy the topology of spacetime. Moreover, there are operators equivalent to the covariant derivative and other differential operators (as shown in [3]), which allow the rewriting of the field equations for the semi-regular case, without running into infinities. This ensures that the field equations can go beyond the singularity.

In the case of a black hole which is not primordial and evaporates completely in a finite time, all of the light rays traversing the singularity reach the past and future infinities. This means that the presence of a spacelike evaporating black hole is compatible with the global hyperbolicity, as in the diagram 3.

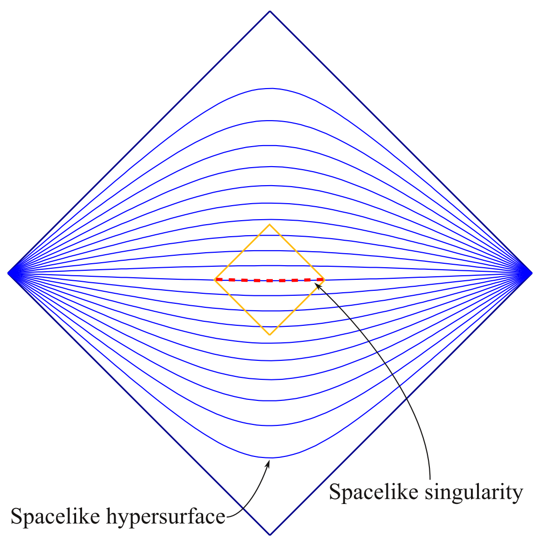

FIGURE 3. The Penrose-Carter diagram for a non-rotating and electrically neutral black hole formed at a finite time, which evaporates after a finite time.

The singularity is accompanied by violent forces, and it is very destructive. But, as we can see from the semi-regular formulation, there is no reason to consider that it destroys the information or the structure of spacetime.

References

1. S. Hawking and G. Ellis, *The Large Scale Structure of Space Time*, Cambridge University Press, 1995.
2. C. Stoica, *Cartan's Structural Equations for Degenerate Metric*, arXiv:math.DG /1111.0646 (2011).
3. ______, *On Singular Semi-Riemannian Manifolds*, arXiv:math.DG /1105.0201 (2011).
4. ______, *Warped Products of Singular Semi-Riemannian Manifolds*, arXiv:math.DG /1105.3404 (2011).