Large time behavior of differential equations with drifted periodic coefficients modeling Carbon storage in soil

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Abstract

This paper is concerned with the linear ODE in the form $y'(t) = \lambda\rho(t)y(t) + b(t)$, $\lambda < 0$ which represents a simplified storage model of the carbon in the soil. In the first part, we show that, for a periodic function $\rho(t)$, a linear drift in the coefficient $b(t)$ involves a linear drift for the solution of this ODE. In the second part, we extend the previous results to a classical heat non-homogeneous equation. The connection with an analytic semi-group associated to the ODE equation is considered in the third part. Numerical examples are given.

Keywords: Ordinary differential equations, Parabolic differential equations, analytic semi group, T-periodic function, linear drift, Cauchy sequence, series' estimates.
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1 Introduction

A lot of phenomena of evolution are described using ordinary differential equations ODE or systems in which the coefficients and/or the source terms are periodic. Let us mention some applications in physics (e.g. the harmonic oscillator , the resonance phenomena due to oscillatory source terms), electricity (let us mention the famous RLC circuit with an oscillatory generator), in biology (circadian cycle), in agricultural studies (due to seasonal effects).

The main question which is addressed here is when or under which conditions a slow perturbation of the coefficient in the ODE will induce a similar behavior on the solutions, in large time. More precisely, we are looking for conditions to ensure that a linear drift in the (periodic) coefficients of the ODE will lead to a linear deviation (and thus unbounded) of the solutions or, on the contrary, what kind of perturbation in the coefficients are compatible with a stable (bounded, periodic in large time) solutions.

Although these questions can be applied to large number of applications, our original motivation concerned with the effect of climate change on the seasonal variations in the organic carbon contained in the soil as claimed at the end of the conclusion of Martin et al. [13].

Since the readers may be not familiar with this domain. Let us recall some basis about the issues of the soil organic carbon (SOC) modelling.

The spreading is one of opportunities for organic materials of human origin (sludge of filtering treatment station and derivative products) and agricultural (manure). The organic matter spread contain significant amounts of organic carbon which, after application, a fraction is permanently stored in the soil. ; The remainder is returned to the atmosphere as CO2. The spreading can be accessed through the storage of organic carbon in the soil, helping to reduce CO2 emissions (a major greenhouse gas effect) compared for example to the incineration of organic matter that returns all carbon into the atmosphere. The optimization of spreading of organic materials is important in reducing emissions of greenhouse gas effect.

The dynamics of carbon in the soil, which determines the amount of organic carbon stored in soil and returned in the form of CO2, depends on soil type, agricultural practices, climate and quantities spread. The soil organic carbon (SOC) plays an important role in several environmental and land management issues. One of the most important issues is the role that SOC plays as part of the terrestrial carbon and might play as a regulation of the atmospheric CO2. Many factors are likely, in a near future, to modify the SOC content, including changes in agricultural practices [3,18,2] and global climate changes [9,5,8,10,11].

Understanding SOC as a function of soil characteristics, agricultural management and climatic conditions is therefore crucial and many models have been developed in this perspective. These models are used in a variety of ways and after for long term studies [6,15,16,17]. The behavior of the SOC system, over a long term and assuming that the environment of the system (inputs of organic carbon, climatic conditions) is stable, is reported to tend towards steady or periodic state.

Some of he SOC models have been formulated mathematically [14,4,1,12,13]. Here we consider the RothC model [7,13] which consists in splitting the soil carbon into four active compartments. Under a continuous form it can be written as

$$\frac{dC(t)}{dt} = \rho(t)AC(t) + B(t), \tag{1.1}$$

where C (t) is a vector with 4 components, each corresponding to a compartment storage of carbon in the soil: DPM (decomposable plant material), RPM (resistant plant material), BIO (microbial community) and HUM (humus). These components indicate the amount of carbon stored at the moment t.

In (1.1)

i) $\rho(t)$ is a positive function indicating the speed of mineralization of soil, which results in CO2 emissions.

ii) A is a matrix that can represent the percentage of clay in the soil and speeds of mineralization in each compartment of C in the soil.

iii) $B(t)$ indicates the amount of carbon brought in the soil (amount spread per unit time).

Let us recall that the initial goal of this study was to understand how long term evolution on climatic data imply variation in large time on the solution of ODE with periodic coefficient and/or source terms with a (linear) drift. The mathematical tools involved in this paper are rather classical and simple but there are, up to our knowledge, very few literature on the subject.

This article is organized as follows:

In section 2 we consider the asymptotic behavior oh the solution of the ODE associated to (1.1) and we derive the main property i.e. there exits a unique solution $y_\infty(t)$ of $y'(t) = \lambda\rho(t)\,y(t) + b(t)\,,\ 0 \leq t < +\infty$, such that $y_\infty(t+T) = y_\infty(t) + \gamma(t)$ where $\gamma(t)$ is a periodic function of period T. In section 3 we consider a classical heat non-homogeneous equation whose the rhs $f(x,t)$ satisfies the assumption derived from the 2nd section. So the properties concerning this equation can be deduced from the results of the ODE (theorem 3.2 and theorem 3.3). In the 3rd part (section 4) we connect the PDE and ODE equations with an analytic semi-group to extend the previous results given by the PDE and ODE equations (theorem 4.2). These results of sections 2 and 3 are illustrated by numerical tests in section 5.

2 Asymptotic behavior of the solution of the ODE

In this paper, we consider the linear differential equation

$$y'(t) = \lambda\rho(t)\,y(t) + b(t)\,,\ 0 \le t < +\infty, \tag{2.1}$$

where $\lambda < 0$, $\rho(t)$ and $b(t)$ are given functions satisfying the following conditions
(A_1) $\rho(t)$ is an T-periodic function, with $T > 0$ fixed.

(A_2) there exist the T-periodic function, $\beta(t)$, such that

$$b(t+T) = b(t) + \beta(t), \quad \forall t \in [0,\infty)\,. \tag{2.2}$$

The general solution of (2.1) has the form

$$y(t) = e^{a(t)}\left\{C_1 + \int_0^t b(s)e^{-a(s)}ds\right\}, \tag{2.3}$$

where C_1 is a constant and

$$a(t) \overset{def}{=} \lambda\int_0^t \rho(s)ds. \tag{2.4}$$

In this section, we prove that there exists a unique solution $y_\infty(t)$ of (2.1) satisfying

$$y_\infty(t+T) = y_\infty(t) + \gamma(t), \forall t \in [0,\infty), \tag{2.5}$$

where $\gamma(t)$ is an T-periodic function. Let first remark that

Lemma 2.1. *Let $b : R_+ \longrightarrow R$. The following properties are equivalent :*
(a) $\exists\beta(t)$ periodic with period T such that $b(t+T) = b(t) + \beta(t),\ \forall t \ge 0$
(b) $\exists\tilde{b}(t)$ periodic with period T such that $b(t) = \tilde{b}(t) + \frac{t}{T}\beta(t), \forall t \ge 0$

Proof. Let us first prove (a)$\Longrightarrow$(b). Choose $\tilde{b}(t) = b(t) - \frac{t}{T}\beta(t)$. Then $\tilde{b}(t)$ is periodic with period T since $\tilde{b}(t+T) = b(t+T) - \frac{t+T}{T}\beta(t+T) = b(t) + \beta(t) - (\frac{t}{T}+1)\beta(t) = b(t) - \frac{t}{T}\beta(t) = \tilde{b}(t)$.

Conversely (b)$\Longrightarrow$(a). Using $b(t) = \tilde{b}(t) + \frac{t}{T}\beta(t)$, we have $b(t+T) = \tilde{b}(t+T) + \frac{t+T}{T}\beta(t+T) = \tilde{b}(t) + (\frac{t}{T}+1)\beta(t) = \tilde{b}(t) + \frac{t}{T}\beta(t) + \beta(t) = b(t) + \beta(t)$. This concludes the proof. □

Let us now prove the main result of this section. First, we state the some lemmas

Lemma 2.2. *Let (A_1) hold. Then*

$$a(t+nT) = a(t) + a(nT) = a(t) + na(T), \forall t \ge 0, n \in \mathbb{N}. \tag{2.6}$$

Proof. From (2.4) we deduce that

$$a(t+nT) = \lambda\int_0^{nT} \rho(s)ds + \lambda\int_{nT}^{t+nT} \rho(s)ds = a(nT) + \lambda\int_{nT}^{t+nT} \rho(s)ds. \tag{2.7}$$

On the other hand, by the assumption (A_1), we have

$$\lambda\int_{nT}^{t+nT} \rho(s)ds = \lambda\int_0^t \rho(s+nT)ds = \lambda\int_0^t \rho(s)ds = a(t), \tag{2.8}$$

and

$$a(nT) = \lambda\sum_{k=0}^{n-1}\int_{kT}^{(k+1)T} \rho(s)ds = \lambda\sum_{k=0}^{n-1}\int_0^T \rho(s)ds = na(T). \tag{2.9}$$

Combining (2.7)-(2.9) we have (2.6). □

Lemma 2.3. *Let assumptions (A_1), (A_2) hold. For $n \in \mathbb{Z}_+$ and $t \in [0, \infty)$, we put*

$$y_n(t) = y(t + nT) = e^{a(t+nT)} \left(y(0) + \int_0^{t+nT} b(s)e^{-a(s)}ds \right). \tag{2.10}$$

Then,

$$y_n(t) = y_\infty(t) + \delta_n(t) + n\gamma(t), \tag{2.11}$$

where

$$y_\infty(t) = e^{a(t)} \left\{ \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s)e^{-a(s)}ds - \frac{e^{a(T)}}{\left(1 - e^{a(T)}\right)^2} \int_0^T \beta(s)e^{-a(s)}ds + \int_0^t b(s)e^{-a(s)}ds \right\}, \tag{2.12}$$

$$\delta_n(t) = e^{a(t)}e^{na(T)} \left\{ y(0) - \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s)e^{-a(s)}ds + \frac{e^{a(T)}}{\left(1 - e^{a(T)}\right)^2} \int_0^T \beta(s)e^{-a(s)}ds \right\}, \tag{2.13}$$

and

$$\gamma(t) = e^{a(t)} \left\{ \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s)e^{-a(s)}ds + \int_0^t \beta(s)e^{-a(s)}ds \right\}. \tag{2.14}$$

Proof. By the assumption (A_2), it follows from (2.10) and the lemma 2.2 that

$$y_n(t) = e^{a(t)} \left\{ y_n(0) + \int_0^t b(s)e^{-a(s)}ds + n \int_0^t \beta(s)e^{-a(s)}ds \right\}.$$

On the other hand, we have

$$\begin{aligned} y_n(0) = y(nT) &= e^{a(nT)} \left\{ y(0) + \int_0^{nT} b(s)\, e^{-a(s)}ds \right\} \\ &= e^{na(T)} \left\{ y(0) + \sum_{k=0}^{n-1} \int_0^T b(s + kT)\, e^{-a(s+kT)}ds \right\} \\ &= e^{na(T)} \left\{ y(0) + \int_0^T b(s)\, e^{-a(s)}ds \sum_{k=0}^{n-1} e^{-ka(T)} + \int_0^T \beta(s)\, e^{-a(s)}ds \sum_{k=0}^{n-1} ke^{-ka(T)} \right\}. \end{aligned}$$

By using the following equalities

$$\sum_{k=0}^{n-1} e^{-ka(T)} = \frac{1 - e^{-na(T)}}{1 - e^{-a(T)}},$$

and

$$\sum_{k=0}^{n-1} ke^{-ka(T)} = \frac{e^{-a(T)}}{\left(e^{-a(T)} - 1\right)^2} - \frac{e^{-(n+1)a(T)}}{\left(e^{-a(T)} - 1\right)^2} + n\frac{e^{-na(T)}}{e^{-a(T)} - 1},$$

thus we obtain

$$\begin{aligned} y_n(0) = {} & \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s)\, e^{-a(s)}ds - \frac{e^{a(T)}}{\left(1 - e^{a(T)}\right)^2} \int_0^T \beta(s)\, e^{-a(s)}ds \\ & + e^{na(T)} \left\{ y(0) - \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s)\, e^{-a(s)}ds + \frac{e^{a(T)}}{\left(1 - e^{a(T)}\right)^2} \int_0^T \beta(s)\, e^{-a(s)}ds \right\}. \\ & + n\frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s)\, e^{-a(s)}ds \end{aligned}$$

Combining previous equalities, we obtain (2.11). The proof of Lemma is complete. □

Now, we state the main theorem

Theorem 2.4. *Let* (A_1), (A_2) *hold. Then, there exists a unique solution* $y_\infty(t)$ *of (2.1) such that*

$$y_\infty(t+T) = y_\infty(t) + \gamma(t), \forall t \in [0,\infty), \tag{2.15}$$

where $\gamma(t)$ *is periodic function of period* T*, defined by*

$$\gamma(t) = e^{a(t)} \left\{ \frac{e^{a(T)}}{1-e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_0^t \beta(s) e^{-a(s)} ds \right\}. \tag{2.16}$$

Proof. For $n \in \mathbb{Z}_+$ and $t \geq 0$, let us define

$$u_n(t) \overset{def}{=} y_n(t) - n\gamma(t). \tag{2.17}$$

It follows from (2.11)-(2.14) and (2.17) that

$$\lim_{n\to+\infty} u_n(t) = y_\infty(t), \ \forall t \in [0,+\infty). \tag{2.18}$$

It is clear that $y_\infty(t)$ is a solution of equation (2.1) satisfies the value at $t=0$

$$y_\infty(0) = \frac{e^{a(T)}}{1-e^{a(T)}} \int_0^T b(s) e^{-a(s)} ds - \frac{e^{a(T)}}{\left(1-e^{a(T)}\right)^2} \int_0^T \beta(s) e^{-a(s)} ds \equiv L(T). \tag{2.19}$$

By (2.18), we have

$$\begin{aligned} y_\infty(t+T) &= \lim_{n\to+\infty} u_n(t+T) = \lim_{n\to+\infty} \{y_n(t+T) - n\gamma(t+T)\}. \\ &= \lim_{n\to+\infty} \{y_{n+1}(t) - (n+1)\gamma(t) + n[\gamma(t) - \gamma(t+T)] + \gamma(t)\} \end{aligned} \tag{2.20}$$

On the other hand, by the periodicity of $\beta(t)$, we get

$$\begin{aligned} \gamma(t+T) &= e^{a(t+T)} \left\{ \frac{e^{a(T)}}{1-e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_0^{t+T} \beta(s) e^{-a(s)} ds \right\} \\ &= e^{a(t)} \left\{ \frac{e^{a(T)}}{1-e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_T^{t+T} \beta(s) e^{a(T)-a(s)} ds \right\} \\ &= e^{a(t)} \left\{ \frac{e^{a(T)}}{1-e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_0^t \beta(s) e^{-a(s)} ds \right\} = \gamma(t). \end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21), we obtain

$$y_\infty(t+T) = \lim_{n\to+\infty} u_{n+1}(t) + \gamma(t) = y_\infty(t) + \gamma(t). \tag{2.22}$$

Uniqueness

Now, let $\widetilde{y}(t)$ be the solution of (2.1) corresponding to the initial value $\widetilde{y}(0) = A$ and

$$\widetilde{y}(t+T) = \widetilde{y}(t) + \widetilde{\gamma}(t), \tag{2.23}$$

where $\widetilde{\gamma}(t)$ is an T-periodic function. Then $y^*(t) = y_\infty(t) - \widetilde{y}(t)$ satisfy

$$\begin{cases} y'(t) = \lambda\rho(t) y(t), \quad 0 < t < +\infty, \\ y(0) = L(T) - A, \end{cases} \tag{2.24}$$

and

$$y^*\left(t+T\right)=y^*\left(t\right)+\gamma^*\left(t\right), \quad \gamma^*\left(t+T\right)=\gamma^*\left(t\right), \ \forall t\geq 0. \tag{2.25}$$

It follows from (2.24) that

$$y^*\left(t\right)=\left(L\left(T\right)-A\right)e^{a(t)}, \forall t\geq 0. \tag{2.26}$$

From (2.25) and (2.26) we deduce that

$$\gamma^*\left(t\right)=-\left(L\left(T\right)-A\right)\left(1-e^{a(T)}\right)e^{a(t)}, \forall t\geq 0. \tag{2.27}$$

Combining (2.25), (2.27) we get $A=L(T)$. By the uniqueness of Cauchy problem, the proof of theorem 2.8 is complete. $\square$

Remark If we consider the equation (2.1) where the assumptions (A_1) and (A_2) are replaced by

(A_1') $b\left(t\right)$ is an T-periodic function, with $T>0$ fixed

(A_2') there exist a T-periodic function, $\alpha(t)$, such that

$$\rho(t+T)=\rho(t)+\alpha(t), \quad \forall t\in[0,\infty). \tag{2.28}$$

In that case it is clear that there does not exist a solution which has the same property as the function $\rho(t)$, for instance if we consider the example with $b(t)=0$. Here the solution of (2.1) tends to 0 as $t\to+\infty$.

We end this section with an example. Consider the following Cauchy problem (2.1) with the choice

$$\lambda=-1, \ y_0=1, \ \rho(t)=\sin^2 t, \ b(t)=t \tag{2.29}$$

In fig.1 we have put the graphs of the functions $y_n(t)$ and $y_n(t+\pi)$ with $n=5$ and here we also note the drift property for the solution of (2.1) taking initial value $y_0=1$ at $t=0$.

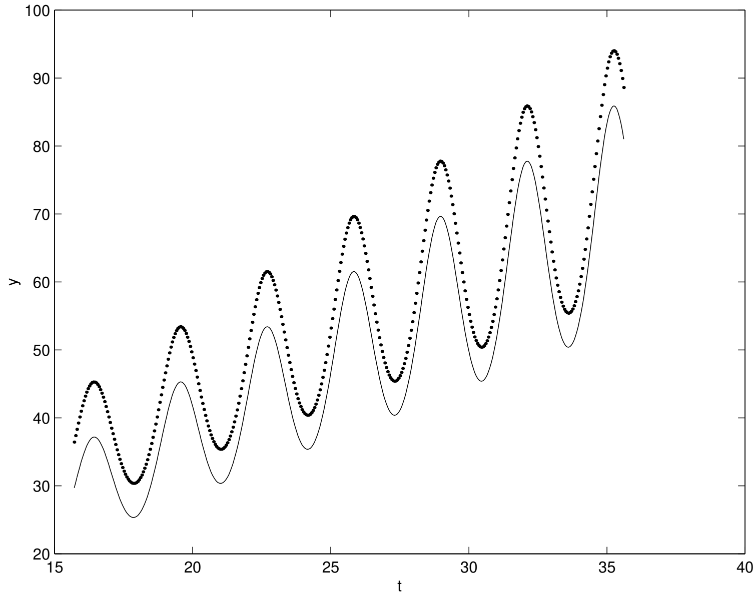

Fig1: Drift property in asymptotic behavior

3 Classical heat equation

Let $\Omega=(0,1)$ and $Q_T=\Omega\times(0,T)$, for $T>0$. In what follows we will denote

$$\langle u,v\rangle=\int_0^1 u(x)v(x)dx, \|v\|=\sqrt{\langle v,v\rangle}.$$

We also denote $u(x,t), u_t(x,t), u_x(x,t)$ by $u(t), u'(t), u_x(t)$ respectively. In this paper, first we consider the classical heat equation

$$u_t - \rho(t)u_{xx} = f(x,t),\ (x,t) \in Q_T \tag{3.1}$$
$$u(0,t) = u(1,t) = 0, \tag{3.2}$$
$$u(x,0) = u_0(x),\ . \tag{3.3}$$

In the next three theorems $\rho(t)$ will be taken equal to 1. We have the well known result for linear parabolic equation in the following theorem

Theorem 3.1 *Let $T > 0$ and assume that $u_0 \in L^2(\Omega), f \in L^2(Q_T)$. Then, the problem (3.1)-(3.3) has a unique solution u satisfying*

$$u \in L^2(0,T; H_0^1(\Omega)) \cap C(0,T; L^2(\Omega)),\ u_t \in L^2(0,T; H^{-1}(\Omega)).$$

Assumption A

Always with a rhs f belonging $L^2_{Loc}(Q_\infty), Q_\infty = \Omega \times \mathbb{R}_+$ we assume in addition that f satisfies the following hypothesis:
(A) there exists the T_1 -periodic function $\beta(x,t)$, such that

$$\begin{aligned} f(x,t+T_1) &= f(x,t) + \beta(x,t), \quad \forall (x,t) \in Q_\infty \\ \beta(x,t+T_1) &= \beta(x,t), \quad \forall (x,t) \in Q_\infty. \end{aligned} \tag{3.4}$$

It follows from (3.4) that the functions $f_j(t)$ satisfies a drift property i.e.

$$f_j(t+T_1) = f_j(t) + \beta_j(t) \tag{3.5}$$

where $\beta_j(t) = \int_0^1 \beta(x,t)w_j(x)dx$ is a T_1 periodic-function.

Theorem 3.2 *Let $T > 0$ and assume that f belonging $L^2_{Loc}(Q_\infty)$ and satisfies assumption (A). Then, the problem (3.1)-(3.2) has a unique solution $\tilde{u}$ satisfying*

$$\tilde{u} \in L^2(0,T; H_0^1(\Omega)) \cap C(0,T; L^2(\Omega)),\ \tilde{u}_t \in L^2(0,T; H^{-1}(\Omega)).$$

and which is T_1 periodic wrt the norm $L^2(\Omega)$.

Proof: If we associate the ODE with the drift-property (3.5) and theorem 3.1, we can then apply the previous results (Lemma 2.3 and theorem 2.4) in which we have proved that there exists a unique solution called $g_j^\infty(t)$ and which has also the same drift property $g_j^\infty(t+T_1) = g_j^\infty(t) + \gamma_j(t)$, $\gamma_j(t)$ being T_1 -periodic.

Consider now the function

$$\tilde{u}(x,t) = \sum_{j=1}^{\infty} g_j^\infty(t) w_j(x) \tag{3.6}$$

To prove that the function $\tilde{u}$ is a solution of the problem (3.1)-(3.3), then from theorem 3.1 it is sufficient to prove that the initial condition $\tilde{u}(x,0)$ is in $L^2(\Omega)$. We have

$$g_j^\infty(0) = \frac{\mathrm{e}^{-\lambda_j T_1}}{1-\mathrm{e}^{-\lambda_j T_1}} \int_0^{T_1} f_j(s)\mathrm{e}^{\lambda_j s} ds - \frac{\mathrm{e}^{-\lambda_j T_1}}{(1-\mathrm{e}^{-\lambda_j T_1})^2} \int_0^{T_1} \beta_j(s)\mathrm{e}^{\lambda_j s} ds. \tag{3.7}$$

By virtue of (3.7) and using the Cauchy-Schwarz inequality we get after some calculations

$$|g_j^\infty(0)|^2 \leq \frac{3}{\lambda_1(1-\mathrm{e}^{-\lambda_1 T_1})^4} \int_0^{2T_1} |f_j(s)|^2 ds,$$

since the definition of the function β_j in (3.5).

Hence

$$\|\tilde{u}(x,0\|^2 = \sum_{j=1}^{\infty} |g_j^{\infty}(0)|^2 \leq \frac{3}{\lambda_1(1-\mathrm{e}^{-\lambda_1 T_1})^4} \sum_{j=1}^{\infty} \int_0^{2T_1} |f_j(s)|^2 ds = C(T_1)\|f\|^2_{L^2(Q_{2T_1})}$$

which proves the first part of the theorem.

It remains to prove that the function $\tilde{u}(x,t)$ has the drift T_1 periodic property for the norm L^2. For that purpose consider $\|\tilde{u}(t+T_1) - \tilde{u}(t)\|^2 \equiv \Gamma(t)$. So he have from Lemma 2.3 and theorem 2.4

$$\Gamma(t) = \sum_{j=1}^{\infty} |g_j^{\infty}(t+T_1) - g_j^{\infty}(t)|^2 = \sum_{j=1}^{\infty} |\gamma_j(t)|^2$$

where

$$\gamma_j(t) = \frac{\mathrm{e}^{-\lambda_j t}}{1-\mathrm{e}^{-\lambda_j T_1}} \int_0^{T_1} \beta_j(s)\mathrm{e}^{-\lambda_j(T_1-s)} ds + \int_0^t \beta_j(s)\mathrm{e}^{-\lambda_j(t-s)} ds. \tag{3.8}$$

Using again the inequality of Cauchy-Schwarz we find from (3.8) that $\Gamma(t) \leq C(T_1)\|f\|^2_{L^2(Q_{2T_1})}$, $C(T_1)$ being a generic constant depending on T_1, which proves that $\Gamma(t)$ is well defined. The property T_1-periodic follows since the functions $\gamma_j(t)$ are themselves T_1-periodic.

Using Lemma 2.3 yet we give an asymptotic expansion for the solution of the problem (3.1)-(3.3). So we have

Theorem 3.3 *Let u be the solution of the initial and boundary value problem (3.1)-(3.3) then there exists a function $u_1 \in L^2_{Loc}(Q_\infty)$ such as*

$$\lim_{n\to\infty} \|u(t+nT_1) - \tilde{u}(t) - nu_1(t)\| = 0$$

$\tilde{u}$ denoting the solution given in theorem 3.2.

Proof: Let $u(x,t) = \sum_{j=1}^{\infty} g_j(t)w_j(x)$ the unique solution of the problem (3.1)-(3.3). From Lemma 2.3 the coefficient $g_j(t)$ have the following expansion

$$g_j(t+nT_1) = g_j^{\infty}(t) + \delta_{n,j}(t) + n\tilde{\gamma}_j(t) \tag{3.9}$$

where

$$\tilde{\gamma}_j(t) = \frac{\mathrm{e}^{-\lambda_j T_1}}{1-\mathrm{e}^{-\lambda_j T_1}} \int_0^{T_1} \beta_j(s)\mathrm{e}^{-\lambda_j(t-s)} ds + \int_0^t \beta_j(s)\mathrm{e}^{-\lambda_j(t-s)} ds$$

$$\begin{aligned} \delta_{n,j}(t) = -\mathrm{e}^{n\lambda_j T_1}[\mathrm{e}^{-jt}u_{0j} &- \frac{\mathrm{e}^{-\lambda_j T_1}}{1-\mathrm{e}^{-\lambda_j T_1}} \int_0^{T_1} f_j(s)\mathrm{e}^{-\lambda_j(t-s)} ds \\ &+ \frac{\mathrm{e}^{-\lambda_j T_1}}{(1-\mathrm{e}^{-\lambda_j T_1})^2} \int_0^{T_1} \beta_j(s)\mathrm{e}^{-\lambda_j(t-s)} ds] \end{aligned}$$

It is easily to prove as before that $\sum_{j=1}^{\infty} |\tilde{\gamma}_j(t)|^2 \leq C(T_1\|f\|^2_{L^2(Q_{2T_1})}$, . So the function $u_1(x,t) = \sum_{j=1}^{\infty} \tilde{\gamma}_j(t)w_j(x)$ is well defined. We have

$$\|u(t+nT_1) - \tilde{u}(t) - nu_1(t)\|^2 = \sum_{j=1}^{\infty} |\delta_{n,j}(t)|^2$$

Using the definition of $\delta_{n,j}(t)$ in (3.9) we get the following bound

$$|\delta_{n,j}(t)|^2 \leq 3\mathrm{e}^{-2\lambda_j t-2n\lambda_j T_1}\left[u_{0j}^2+\frac{6}{\lambda_1(1-\mathrm{e}^{-\lambda_1 T_1})^4}\int_0^{T_1} f_j^2(s)ds\right]$$

and the last inequality enables us to obtain

$$\sum_{j=1}^{\infty}|\delta_{n,j}(t)|^2 \leq C\mathrm{e}^{-2nT_1}\to 0 \quad \text{as} \quad n\to\infty, \quad C \quad \text{constant}$$

which proves our assertion.

Remark: The three previous theorems are still valid with $\rho(t)$ T_1-periodic.

4 Connection with the semi-group

1. **Preliminaries:** Let X be a Banach space with norm$\|\cdot\|$. We consider the linear evolution equation given by

$$x'(t) = -A(t)x(t) + f(t),\, t\in\mathbb{R}_+, \tag{4.1}$$

where $A(t)$ is a family of closed linear operators in X and $f(t)$ be an X valued function. Throughout this paper, we make the following assumptions

Assumption 1. For each initial value $x(0) = \zeta$ in X, there exists a unique mild solution x of equation (4.1) on $\mathbb{R}_+$, defined by

$$x(t) = U(t,0)\zeta + \int_0^t U(t,s)f(s)\,ds, \tag{4.2}$$

where $U(t,s),\quad 0\leq s\leq t$, is the evolution system associated with the family $\{A(t)\}\,,\, t\in\mathbb{R}_+$.

Assumption 2. The maps $t\longmapsto A(t)$ is η - periodic and, for each $t\in\mathbb{R}_+$, $A(t)$ is dissipative operator, that is for every $x\in D\left(A(t)\right)$, there exists a $x^*\in F(x)$ such that

$$Re\,\langle Ax, x^*\rangle \leq 0, \tag{4.3}$$

where

$$F(x) = \left\{x^*\in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}.$$

Assumption 3. There exists a X valued function $\beta(t)$, $t\geq 0$, which is η - periodic such that

$$f(t+\eta) = f(t) + \beta(t),\, \forall t\in\mathbb{R}_+. \tag{4.4}$$

Remark.

1. Assumption 1 follows from the following three conditions (Pazy [15])

   (i) The domain $D(A)$ of $\{A(t) : t\in\mathbb{R}_+\}$ is dense in X and independent of t.

   (ii) For each $t\in\mathbb{R}_+$, the resolvent $R\left(\lambda : A(t)\right)$ of $A(t)$ exists for all λ with $Re(\lambda)\leq 0$ and there exists a constant M such that

$$\|R\left(\lambda : A(t)\right)\| \leq \frac{M}{|\lambda|+1}. \tag{4.5}$$

(iii) There exist a constants L and $0 < \alpha \leq 1$ such that

$$\left\|(A(t) - A(s))\, A^{-1}(\tau)\right\| \leq L|t-s|^{\alpha}, \;\forall t, s, \tau \in \mathbb{R}_+. \tag{4.6}$$

2. The periodicity of $A(t)$ implies that $U(t+\eta, s+\eta) = U(t,s)$, for all $0 \leq s \leq t$.

3. From the assumptions 2, we deduce that $\|U(t,s)\| \leq 1$ for each $0 \leq s \leq t$.

2. **Results:** In this section, we prove that there exists a unique solution $x_\infty(t)$ of equation (4.1)satisfying

$$x_\infty(t+\eta) = x_\infty(t) + \gamma(t), \text{ for all } t \geq 0, \tag{4.7}$$

where $\gamma(t)$ is an η-periodic function. First, we need the following lemma

Lemma 4.1. Let $x(t)$ is a mild solution of equation (4.1) with the initial value $x(0) = \zeta$. For $n \in \mathbb{Z}_+$ and $t \in [0, +\infty)$, we put

$$x_n(t) = x(t+n\eta) = U(t+n\eta, 0)\zeta + \int_0^{t+n\eta} U(t+n\eta, s) f(s)\, ds. \tag{4.8}$$

Then we have

$$x_n(t) = x_\infty(t) + \delta_n(\eta) + n\gamma(t), \tag{4.9}$$

where

$$\begin{aligned} x_\infty(t) = U(t,0)\Big\{ &(I-U_0)^{-1} \int_0^\eta U(\eta,s) f(s)\, ds \\ &-(I-U_0)^{-2} \int_0^\eta U(\eta,s)\beta(s)\, ds \Big\} + \int_0^t U(t,s) f(s)\, ds \end{aligned} \tag{4.10}$$

$$\delta_n(t) = U(t,0)\, U_0^n \left[\zeta - (I-U_0)^{-1} \int_0^\eta U(\eta,s) f(s)\, ds + (I-U_0)^{-2} \int_0^\eta U(\eta,s)\beta(s)\, ds\right], \tag{4.11}$$

$$\gamma(t) = U(t,0)(I-U_0)^{-1} \int_0^\eta U(\eta,s)\beta(s)\, ds + \int_0^t U(t,s)\beta(s)\, ds, \tag{4.12}$$

with $U_0 = U(\eta, 0)$ such that $\|U_0\| < 1$.

Proof. By the assumption 3, the remark (ii), it follows from (4.8) that

$$\begin{aligned} x_n(t) &= U(t+n\eta,0)\zeta + \int_0^{n\eta} U(t+n\eta,s)f(s)ds + \int_{n\eta}^{t+n\eta} U(t+n\eta,s)f(s)ds \\ &= U(t+n\eta,n\eta)\left\{U(n\eta,0)\zeta + \int_0^{n\eta} U(n\eta,s)f(s)ds\right\} + \int_0^t U(t+n\eta,s+n\eta)f(s+n\eta)ds \\ &= U(t,0)x_n(0) + \int_0^t U(t,s)f(s)ds + n\int_0^t U(t,s)\beta(s)ds \end{aligned} \tag{4.13}$$

since we have $U(t+nT,s) = U(t+nT,nT)U(nT,s)$. On the other hand, we have

$$\begin{aligned} x_n(0) &= U(n\eta,0)\zeta + \int_0^{n\eta} U(n\eta,s)f(s)ds \\ &= U(n\eta,0)\zeta + \sum_{k=1}^{n} \int_{(k-1)\eta}^{k\eta} U(n\eta,s)f(s)ds \\ &= U(n\eta,0)\zeta + \sum_{k=1}^{n} \int_0^{\eta} U(n\eta,s+(k-1)\eta)f(s)ds \\ &\qquad + \sum_{k=1}^{n}(k-1)\int_0^{\eta} U(n\eta,s+(k-1)\eta)\beta(s)ds \end{aligned} \tag{4.14}$$

Using the following relations

$$U(n\eta,0) = U^n(\eta,0) = U_0^n, \tag{4.15}$$

and

$$U(n\eta,s+(k-1)\eta) = U_0^{n-k}U(\eta,s), \tag{4.16}$$

we deduce from (4.14) that

$$x_n(0) = U_0^n\zeta + \left(\sum_{k=1}^{n} U_0^{n-k}\right)\int_0^{\eta} U(\eta,s)f(s)ds + \left(\sum_{k=1}^{n}(k-1)U_0^{n-k}\right)\int_0^{\eta} U(\eta,s)\beta(s)ds. \tag{4.17}$$

Since

$$\sum_{k=1}^{n} U_0^{n-k} = (I-U_0)^{-1}(I-U_0^n), \tag{4.18}$$

and

$$\sum_{k=1}^{n}(k-1)U_0^{n-k} = -(I-U_0)^{-2} + (I-U_0)^{-2}U_0^n + n(I-U_0)^{-1}, \tag{4.19}$$

it follows from (4.15) that

$$\begin{aligned} x_n(0) &= (I-U_0)^{-1}\int_0^{\eta} U(\eta,s)f(s)ds - (I-U_0)^{-2}\int_0^{\eta} U(\eta,s)\beta(s)ds \\ &\quad + U_0^n\left\{\zeta - (I-U_0)^{-1}\int_0^{\eta} U(\eta,s)f(s)ds + (I-U_0)^{-2}\int_0^{\eta} U(\eta,s)\beta(s)ds\right\} \\ &\quad + n(I-U_0)^{-1}\int_0^{\eta} U(\eta,s)\beta(s)ds. \end{aligned} \tag{4.20}$$

Combining (4.13) and (4.20) we obtain (4.9). The proof of Lemma is complete. □

Theorem 4.2. Let the assumptions 1, 2, 3 hold. Then, there exists a unique solution $x_\infty(t)$ of (4.1) such that

$$x_\infty(t+\eta) = x_\infty(t) + \gamma(t),\ \ \forall t \geq 0, \tag{4.21}$$

where $\gamma(t)$ is an η-periodic function, defined by (4.12).

Proof. For $n \in \mathbb{Z}_+$ and $t \geq 0$, put

$$u_n(t) = x_n(t) - n\gamma(t). \tag{4.22}$$

By Remark 3, it follows (4.9)-(4.12) and (4.22) that

$$\lim_{n\to\infty} u_n(t) = x_\infty(t). \tag{4.23}$$

It is clear that x_∞ is a mild solution of equation (4.1), satisfies the initial value

$$x_\infty(0) = (I - U_0)^{-1}\int_0^\eta U(\eta, s)f(s)ds - (I - U_0)^{-2}\int_0^\eta U(\eta, s)\beta(s)ds. \tag{4.24}$$

From (4.23), we have

$$\begin{aligned} x_\infty(t+\eta) &= \lim_{n\to\infty} u_n(t+\eta) = \lim_{n\to\infty} \{x_n(t+\eta) - n\gamma(t+\eta)\} \\ &= \lim_{n\to\infty} \{x_{n+1}(t) - (n+1)\gamma(t) + n\left[\gamma(t) - \gamma(t+\eta)\right] + \gamma(t)\}. \end{aligned} \tag{4.25}$$

On the other hand, by the periodicity of $\beta(t)$, we get

$$\begin{aligned} \gamma(t+\eta) &= U(t+\eta, 0)(I - U_0)^{-1}\int_0^\eta U(\eta, s)\beta(s)ds + \int_0^{t+\eta} U(t+\eta, s)\beta(s)ds \\ &= U(t, 0)(I - U_0)^{-1}\int_0^\eta U(\eta, s)\beta(s)ds + \int_\eta^{t+\eta} U(t+\eta, s)\beta(s)ds \\ &= U(t, 0)(I - U_0)^{-1}\int_0^\eta U(\eta, s)\beta(s)ds + \int_0^t U(t, s)\beta(s)ds \equiv \gamma(t) \end{aligned} \tag{4.26}$$

Combining (4.25), (4.26) we obtain

$$x_\infty(t+\eta) = \lim_{n\to\infty} u_{n+1}(t) + \gamma(t) = x_\infty(t) + \gamma(t). \tag{4.27}$$

Now, let $\widetilde{x}(t)$ be the mild solution of the equation (4.1) corresponding to the initial value $\widetilde{x}(0)$ and

$$\widetilde{x}(t+\eta) = \widetilde{x}(t) + \widetilde{\gamma}(t), \tag{4.28}$$

where $\widetilde{\gamma}(t)$ is an η-periodic function. Then $\widehat{x}(t) = x_\infty(t) - \widetilde{x}(t)$ satisfy

$$\begin{cases} \widehat{x}'(t) = A(t)\widehat{x}(t),\ t \geq 0, \\ \widehat{x}(0) = x_\infty(0) - \widetilde{x}(0), \end{cases} \tag{4.29}$$

and

$$\widehat{x}(t+\eta) = \widehat{x}(t) + \widehat{\gamma}(t), \quad \widehat{\gamma}(t+\eta) = \widehat{\gamma}(t), \quad \forall t \geq 0. \tag{4.30}$$

It follows from (4.29) that

$$\widehat{x}(t) = U(t, 0)\left(x_\infty(0) - \widetilde{x}(0)\right). \tag{4.31}$$

From (4.30), (4.31) we deduce that $x_\infty(0) = \widetilde{x}(0)$. By the assumption 1, Theorem 4.2 is proved. □

5 Numerical results

In this section we consider the initial and boundary value problem (3.1)-(3.3) of section 3:

$$u_t - u_{xx} = f(x,t) \text{ in } (0,1)\times(0,\infty), \tag{5.1}$$

with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0 \tag{5.2}$$

and initial condition

$$u(x,0) = \widetilde{u}_0(x)). \tag{5.3}$$

The graphs of the approximated and exact solutions through two examples are very close each other.

For the first example the functions $\widetilde{u}_0$, and f are defined by

$$\widetilde{u}_0(x) = \frac{4\pi^3}{(\pi^4+4\pi^2)^2}\sin(\pi x) \tag{5.4}$$

$$f(x,t) = \sin(\pi x)t\sin(2\pi t), \tag{5.5}$$

the function $f(x,t)$ satisfying the assumption $f(x,t+1) = f(x,t)+\beta(x,t)$, $\beta(x,t)$ being 1-periodic in t for each $x \in [0,1]$. The exact solution of the problem (5.1) – (5.3) with $\widetilde{u}_0$ and f defined in (5.4) – (5.5) respectively, is the function u_{ex} given by

$$u_{ex}(x,t) = \sin(\pi x)\left[\cos(2\pi t)(a_1 t + b_1) + \sin(2\pi t)(a_2 t - b_2)\right], \tag{5.6}$$

with $\quad a_1 = \dfrac{-2\pi}{\pi^4+4\pi^2},\ b_1 = \dfrac{4\pi^3}{(\pi^4+4\pi^2)^2},\ a_2 = \dfrac{\pi^2}{pi^4+4\pi^2},\ \ b_2 = \dfrac{4\pi^3}{(pi^4+4\pi^2)^2}$.

To solve problem (5.1)-(5.3) numerically, we consider the differential system for the unknowns $v_j(t) \equiv u(x_j,t)$, with $x_j = j\Delta x$, $\Delta x = \frac{1}{p}$, $j = 0,1,...,p$:

$$\left\{\begin{array}{l} \dfrac{dv_1}{dt}(t) = -\dfrac{2}{\Delta x^2}v_1(t) + \dfrac{1}{\Delta x^2}v_2(t) + f_1(t) \\ \dfrac{dv_j}{dt}(t) = \dfrac{1}{\Delta x^2}v_{j-1}(t) - \dfrac{2}{\Delta x^2}v_j(t) + \dfrac{1}{\Delta x^2}v_{j+1}(t) + f_j(t),\ j = \overline{2,p-2} \\ \dfrac{dv_{p-1}}{dt}(t) = \dfrac{1}{\Delta x^2}v_{p-2}(t) - \dfrac{2}{\Delta x^2}v_{p-1}(t) + f_{p-1}(t) \\ v_j(0) = \widetilde{u}_0(x_j),\ f_j(t) = f(x_j,t), j = \overline{1,p-1}. \end{array}\right. \tag{5.7}$$

The system (5.7) is equivalent to:

$$\tfrac{d}{dt}X(t) = AX(t) + F(t), \tag{5.8}$$

$$\left\{\begin{array}{l} X(t) = (v_1(t),...,v_{p-1}(t))^T \\ F(t) = (f_1,...,f_{p-1})^T \end{array}\right. \tag{5.9}$$

the tridiagonal matrix A being defined by

$$A = \left\|\begin{array}{cccccc} -2\alpha & \alpha & & & & 0 \\ \alpha & -2\alpha & \alpha & & & \\ & \ddots & \ddots & \ddots & & \\ & & & \alpha & -2\alpha & \alpha \\ 0 & & & & \alpha & -2\alpha \end{array}\right\| \tag{5.10}$$

where $\alpha = \frac{1}{\Delta x^2}$

To solve the linear differential system (5.9), we use a spectral method with a time step $\Delta t = 0.05$ and a spacial step $\Delta x = 0.1$

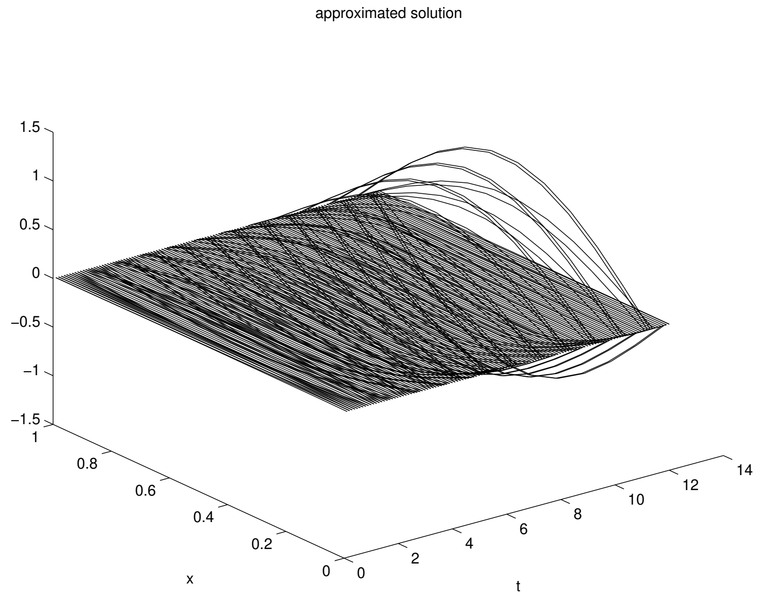

Figure 2: Numerical solution

In fig.2 we have drawn the approximated solution of the problem (5.1)-(5.3) while fig.3 represents his corresponding exact solution (5.6).

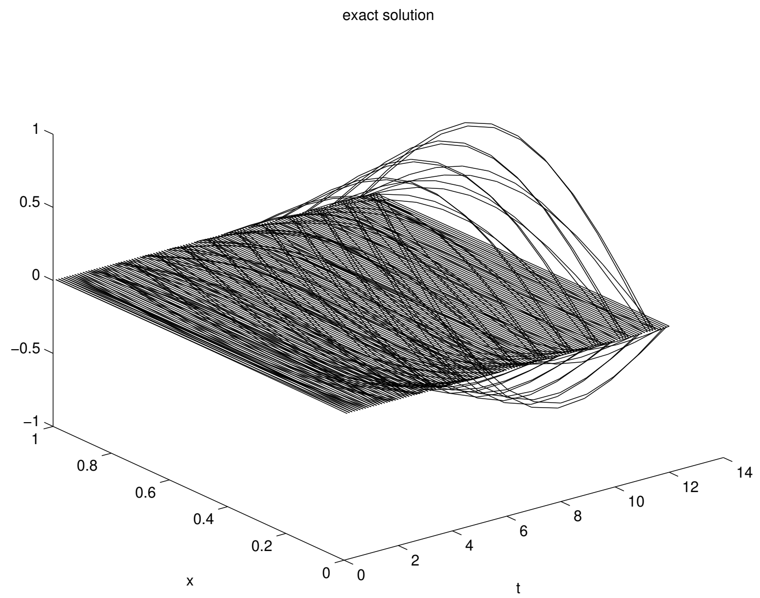

Figure 3: Exact solution

The fig.4 corresponds to the median approximated component $(x_j, t) \mapsto u(x_j, t)$. So we can see the drift property of this component generated by the the drift property of the second hand side $f(x, t)$

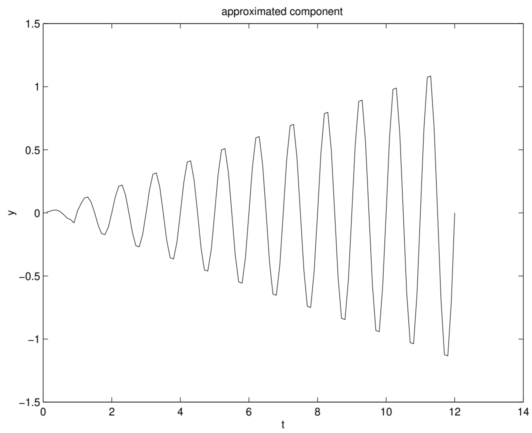

Figure 4: curve of $(x_j, t) \mapsto u(x_j, t)$

For one component x_j fixed we find again numerically the curve of the function $t \mapsto y_n(t)$ given in Lemma 2.3 (fig 1).

We have also considered another numerical example whose given data are

$$f(x,t) = t\sin(\pi x), \quad \widetilde{u}_0(x) = -\frac{\sin(\pi x)}{\pi^4} \tag{5.11}$$

The exact solution of (5.11) is $u_{ex}(x,t) = \sin(\pi x)\left(\dfrac{t}{\pi^2} - \dfrac{1}{\pi^4}\right)$. So with the same method as before, the corresponding surfaces and curve are drawn in fig.5, fig.6 and fig.7 (respectively approximated solution, exact solution and median component $(x_j, t) \mapsto u(x_j, t)$).

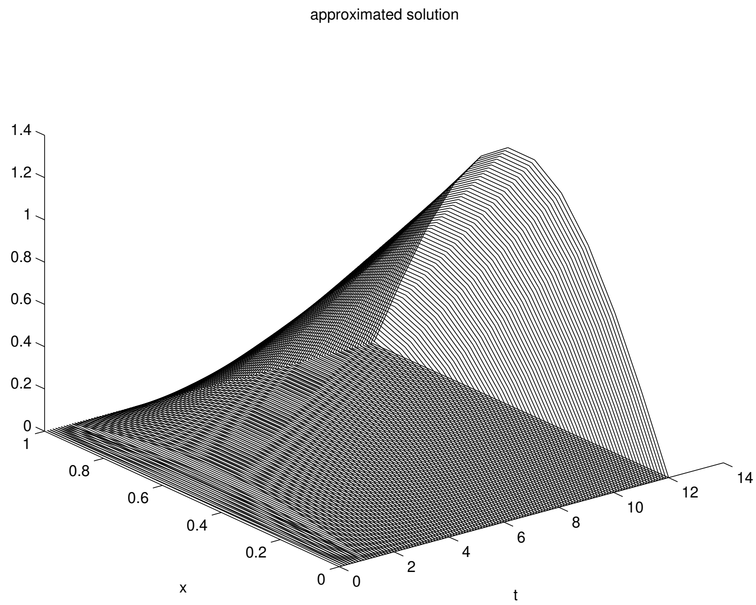

Figure 5: Numerical solution, case 2

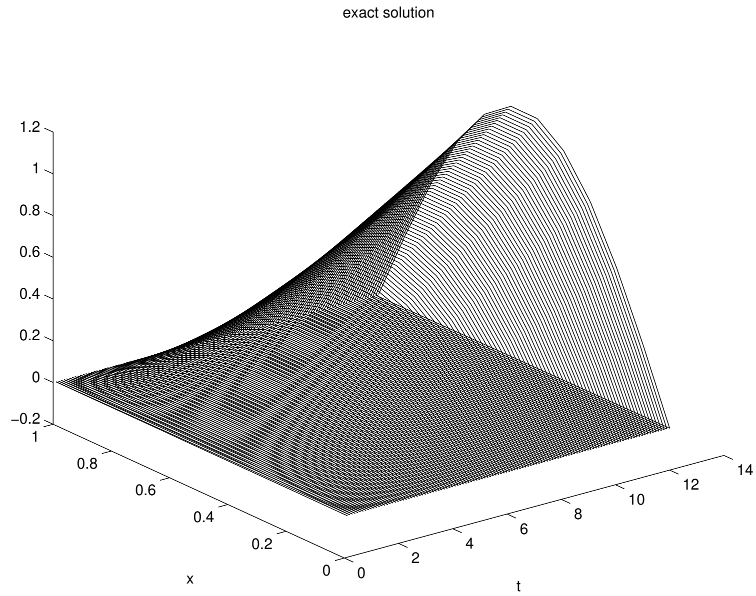

Figure 6: Exact solution, case 2

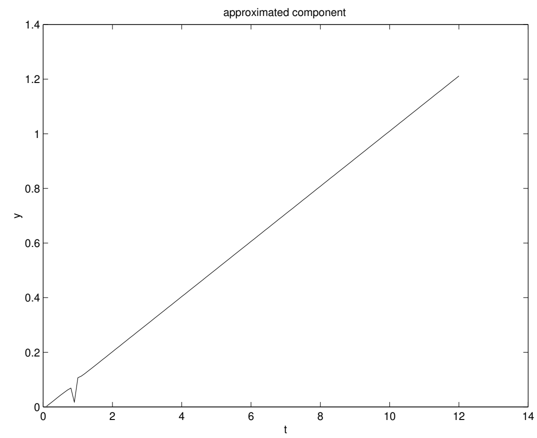

Figure 7: curve of $(x_j, t) \mapsto u(x_j, t)$

So through the results of the sections 2 and 3 we can see that a linear drift on the source term (the rhs) gives a linear drift on the solution. This property can be generalized via an analytic semi-group liking the ODE and PDE. On the other hand there exits a unit concerning the sections 2, 3 and 4 which dwells in the asymptotic behavior (lemma 2.3, Theorem 3.3 and theorem 4.1).

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