Cartan's Structural Equations for Degenerate Metric

Ovidiu Cristinel Stoica

CARTAN'S STRUCTURAL EQUATIONS FOR DEGENERATE METRIC

CRISTI STOICA

ABSTRACT. Cartan's structural equations show in a compact way the relation between a connection and its curvature, and reveals their geometric interpretation in terms of moving frames. On singular semi-Riemannian manifolds, because the metric is allowed to be degenerate, there are some obstructions in constructing the geometric objects normally associated to the metric. We can no longer construct local orthonormal frames and coframes, or define a metric connection and its curvature operator. But we will see that if the metric is radical stationary, we can construct objects similar to the connection and curvature forms of Cartan, to which they reduce if the metric is non-degenerate. We write analogs of Cartan's first and second structural equations. As a byproduct we will find a compact version of the Koszul formula.

CONTENTS

INTRODUCTION

In Riemannian and semi-Riemannian geometry, Cartan's first and second structural equations establish the relation between a local orthonormal frame, the connection, and its curvature. But in singular (semi-)Riemannian geometry, we cannot invert the metric to construct orthonormal coframes, there is no Levi-Civita connection, and no curvature operator. One important operation is the contraction between

Date: November 1, 2011.
Partially supported by Romanian Government grant PN II Idei 1187.

covariant indices, which requires the inverse of the metric tensor, absent in the degenerate case.

These problems were avoided in [5], where instead of the metric connection was used the Koszul form, and it was defined a Riemann curvature $R(\_,\_,\_,\_)$, which coincides to the usual Riemann curvature tensor if the metric is non-degenerate. As it was shown there, the covariant contraction at a point $p \in M$ can be defined only on the subspace of the cotangent space which consists on covectors $\omega \in T_p^*M$ which are of the form $\omega(V) = \langle U, V\rangle$, $U, V \in T_pM$. The contraction was shown to be well defined and has been extended to tensors of higher order. This contraction was used to define the Riemann curvature tensor $R(\_,\_,\_,\_)$.

In this article, I will show how to extend Cartan's formalism to singular semi-Riemannian geometry.

1. Brief review of singular semi-Riemannian manifolds

We recall some notions about singular semi-Riemannian manifolds, and some of the main results from [5], recapitulated in [6].

Definition 1.1. (see *e.g.* [1] for comparison) A *singular semi-Riemannian manifold* is a pair (M, g), where M is a differentiable manifold, and $g \in \Gamma(T^*M \odot_M T^*M)$ is a symmetric bilinear form on M, named *metric tensor* or *metric*. If the signature of g is allowed to vary from point to point, (M, g) is said to be with *variable signature*, otherwise it is said to be with *constant signature*. If g is non-degenerate, then (M, g) is named *semi-Riemannian manifold*, and if in addition it is positive definite, (M, g) is named *Riemannian manifold*.

The remaining of this section recalls very briefly the main notions and results on singular semi-Riemannian manifolds, as introduced in [5].

Definition 1.2. We denote by $T^\bullet M$ the subset of the cotangent bundle defined as

$$T^\bullet M = \bigcup_{p \in M} (T_pM)^\bullet \tag{1}$$

where $(T_pM)^\bullet \subseteq T_p^*M$ is the space of covectors at p which can be expressed as $\omega_p = Y_p^\flat$, *i.e.* $\omega_p(X_p) = \langle Y_p, X_p\rangle$ for some $Y_p \in T_pM$ and any $X_p \in T_pM$. $T^\bullet M$ is a vector bundle if and only if the signature of the metric is constant. We can define sections of $T^\bullet M$ in the general case, by

$$\mathcal{A}^\bullet(M) := \{\omega \in \Gamma(T^*M) | \omega_p \in (T_pM)^\bullet \text{ for any } p \in M\}. \tag{2}$$

Definition 1.3. On $T_p{}^\bullet M$ we can define a unique non-degenerate inner product $g_{\bullet p}$ by $g_{\bullet p}(\omega_p, \tau_p) := \langle X_p, Y_p \rangle$, where $X_p, Y_p \in T_p M$, $X_p^\flat = \omega_p$ and $Y_p^\flat = \tau_p$. We alternatively use the notations $\langle\langle \omega_p, \tau_p \rangle\rangle_\bullet = g_\bullet(\omega_p, \tau_p) = \omega_p(_\bullet)\tau_p(_\bullet)$, and call $\omega_p(_\bullet)\tau_p(_\bullet)$ the *covariant contraction* between ω_p and τ_p.

For a non-degenerate metric, we can define the covariant derivative of a vector field Y in the direction of a vector field X, where $X, Y \in \mathfrak{X}(M)$, by the *Koszul formula* (*cf. e.g.* [3], p. 61). For a metric which can be degenerate, the covariant derivative cannot be extracted from the Koszul formula, and we will have to work instead with the right hand side (named below *Koszul form*) of the Koszul formula, which is smooth.

Definition 1.4 (The Koszul form)**.** *The Koszul form* is defined as

$$\mathcal{K} : \mathfrak{X}(M)^3 \to \mathbb{R},$$

$$\begin{aligned} \mathcal{K}(X,Y,Z) \quad := \quad & \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ & -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\}. \end{aligned} \tag{3}$$

Theorem 1.5. The Koszul form of a singular semi-Riemannian manifold (M, g) has, for any $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathscr{F}(M)$, the following properties:

(1) It is additive and $\mathbb{R}$-linear in each of its arguments.
(2) It is $\mathscr{F}(M)$-linear in the first argument:
$\mathcal{K}(fX,Y,Z) = f\mathcal{K}(X,Y,Z)$.
(3) Satisfies the *Leibniz rule*:
$\mathcal{K}(X,fY,Z) = f\mathcal{K}(X,Y,Z) + X(f)\langle Y,Z\rangle$.
(4) It is $\mathscr{F}(M)$-linear in the third argument:
$\mathcal{K}(X,Y,fZ) = f\mathcal{K}(X,Y,Z)$.
(5) It is *metric*:
$\mathcal{K}(X,Y,Z) + \mathcal{K}(X,Z,Y) = X\langle Y,Z\rangle$.
(6) It is *symmetric* or *torsionless*:
$\mathcal{K}(X,Y,Z) - \mathcal{K}(Y,X,Z) = \langle [X,Y],Z\rangle$.
(7) Relation with the Lie derivative of g:
$\mathcal{K}(X,Y,Z) + \mathcal{K}(Z,Y,X) = (\mathcal{L}_Y g)(Z,X)$.
(8) $\mathcal{K}(X,Y,Z) + \mathcal{K}(Y,Z,X) = Y\langle Z,X\rangle + \langle [X,Y],Z\rangle$.

□

Definition 1.6 (see [2] Definition 3.1.3)**.** A singular semi-Riemannian manifold (M, g) is *radical-stationary* if it satisfies the condition

$$\mathcal{K}(X,Y,\_) \in \mathcal{A}^\bullet(M), \tag{4}$$

for any $X, Y \in \mathfrak{X}(M)$.

Definition 1.7. A *semi-regular semi-Riemannian manifold* is a radical-stationary singular semi-Riemannian manifold (M, g) which satisfies

$$\mathcal{K}(X, Y, {}_\bullet)\mathcal{K}(Z, T, {}_\bullet) \in \mathscr{F}(M). \tag{5}$$

for any $X, Y, Z, T \in \mathfrak{X}(M)$.

Remark 1.8. In [5] it was given a different definition for semi-regular semi-Riemannian manifolds. The Definition 1.7 was proved in [5] to be equivalent. Similarly, the definition of the Riemann curvature given in [5] is different than the one given below, but they are shown to be equivalent. I preferred here these definitions, because they simplify the task of finding the structure equations.

Definition 1.9. We define the *Riemann curvature tensor* as

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R}, \tag{6}$$

(7)
$$\begin{array}{rcl} R(X, Y, Z, T) & = & X\mathcal{K}(Y, Z, T) - Y\mathcal{K}(X, Z, T) - \mathcal{K}([X, Y], Z, T) \\ & & +\mathcal{K}(X, Z, {}_\bullet)\mathcal{K}(Y, T, {}_\bullet) - \mathcal{K}(Y, Z, {}_\bullet)\mathcal{K}(X, T, {}_\bullet) \end{array}$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$.

Remark 1.10. In [5] is shown that the Riemann curvature tensor has the same symmetries as in the non-degenerate case.

2. The first structural equation

Cartan's first structural equation shows how a moving coframe rotates when moving in one direction, due to the connection. In the following, we will derive the first structural equation for the case when the metric is allowed to be degenerate. Of course, in this case we will not have a notion of local orthonormal frame, and we will work instead with vectors and annihilator covectors. The following decomposition of the Koszul form will be needed to derive the first structural equation.

2.1. **The decomposition of the Koszul form.**

Lemma 2.1.

$$2\mathcal{K}(X, Y, Z) = (\mathrm{d}Y^\flat)(X, Z) + (\mathcal{L}_Y g)(X, Z). \tag{8}$$

Proof. From the formula for the exterior derivative we get:

$$\begin{array}{rcl} (\mathrm{d}Y^\flat)(X, Z) & = & X\left(Y^\flat(Z)\right) - Z\left(Y^\flat(X)\right) - Y^\flat([X, Z]) \\ & = & X\langle Y, Z\rangle - Z\langle X, Y\rangle + \langle Y, [Z, X]\rangle. \end{array}$$

The Lie derivative is

$$\begin{array}{rcl} (\mathcal{L}_Y g)(Z, X) & = & Yg(Z, X) - g([Y, Z], X) - g(Z, [Y, X]) \\ & = & Y\langle Z, X\rangle - \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle. \end{array}$$

The equation (8) follows then immediately. □

Corollary 2.2. To the properties of the Koszul form from Theorem 1.5 we can add the following:

$$(\mathrm{d}Y^\flat)(X,Z) = \mathcal{K}(X,Y,Z) - \mathcal{K}(Z,Y,X). \tag{9}$$

Proof. This is an immediate consequence of the Lemma 2.1. □

2.2. **The connection forms.** If $(E_a)_{a=1}^n$ is an orthonormal frame on a non-degenerate semi-Riemannian manifold, then its dual $(\omega^b)_{b=1}^n$ is also orthonormal. The *the connection forms* (*cf.* *e.g.* [4]) are the 1-forms ${\omega_a}^b$, $1 \leq a,b \leq n$ defined as

$${\omega_a}^b(X) := \omega^b(\nabla_X E_a). \tag{10}$$

It is important to be aware that the indices a, b label the connection one-forms ${\omega_a}^b$, and they don't represent the components of a form.

For general (possibly degenerate) metrics, there is no Levi-Civita connection $\nabla_X E_a$. Also, a frame $(E_a)_{a=1}^n$ cannot be orthonormal, only orthogonal, and its dual $(\omega^b)_{b=1}^n$ cannot be orthogonal, because the metric $\langle\langle\omega,\tau\rangle\rangle_\bullet = g_\bullet(\omega,\tau)$ is not defined for the entire T^*M, but only for $T^\bullet M$. Therefore, we need to find another way to define the connection one-forms.

Definition 2.3. Let $X, Y \in \mathfrak{X}(M)$ be two vector fields. Then, the *connection form* associated to X, Y is the one-form defined as

$$\omega_{XY}(Z) := \mathcal{K}(Z,X,Y). \tag{11}$$

In particular, we define ω_{ab} by

$$\omega_{ab}(X) := \omega_{E_aE_b}(X). \tag{12}$$

Remark 2.4. The fact that ω_{XY} is a one-form follows from the properties (1) and (2) of the Koszul form given in Theorem 1.5.

2.3. **The first structural equation.** Let (M, g) be a radical-stationary manifold (*cf.* def. 1.6).

Lemma 2.5 (The first structural equation)**.**

$$\mathrm{d}X^\flat = \omega_{X\bullet} \wedge {\bullet}^\flat \tag{13}$$

Proof. From the Lemma 2.1 and from the Definition 1.4 of the Koszul form, we have:

$$(\mathrm{d}X^\flat)(Y,Z) = \mathcal{K}(Y,X,Z) - \mathcal{K}(Z,X,Y). \tag{14}$$

By replacing the Koszul form with the connection one-form, we get:

$$(\mathrm{d}X^\flat)(Y,Z) = \omega_{XZ}(Y) - \omega_{XY}(Z). \tag{15}$$

By using the properties of the covariant contraction and the property of (M,g) of being radical-stationary, we can expand the Koszul form as

$$\mathcal{K}(X,Y,Z) = \mathcal{K}(X,Y,\bullet)\langle\bullet,Z\rangle = \mathcal{K}(X,Y,\bullet)\left(\bullet^{\flat}(Z)\right). \tag{16}$$

We can do the same for the connection one-form:

$$\begin{array}{rcl}\omega_{YZ}(X) & = & \omega_{Y\bullet}(X)\langle\bullet,Z\rangle = \omega_{Y\bullet}(X)\left(\bullet^{\flat}(Z)\right) \\ & = & \left(\omega_{Y\bullet}\otimes\bullet^{\flat}\right)(X,Z).\end{array} \tag{17}$$

The equation (15) becomes

$$\begin{array}{rcl}(\mathrm{d}X^{\flat})(Y,Z) & = & \left(\omega_{X\bullet}\otimes\bullet^{\flat}\right)(Y,Z) - \left(\omega_{X\bullet}\otimes\bullet^{\flat}\right)(Z,Y) \\ & = & \left(\omega_{X\bullet}\wedge\bullet^{\flat}\right)(Y,Z).\end{array} \tag{18}$$

□

The following corollary shows how we get the first structural equation as we know it.

Corollary 2.6. If the metric g is non-degenerate, $(E_a)_{a=1}^n$ is an orthonormal frame, and $(\omega^a)_{a=1}^n$ is its dual, then

$$\mathrm{d}\omega^a = -\omega_s{}^a \wedge \omega^s. \tag{19}$$

Proof. We have from Theorem 1.5 (5) that

$$\omega_{E_aE_b}(X) + \omega_{E_bE_a}(X) = X\langle E_a, E_b\rangle = X(\delta_{ab}) = 0, \tag{20}$$

and therefore

$$\omega_{E_aE_b} = -\omega_{E_bE_a}. \tag{21}$$

From equation (13) we obtain:

$$\mathrm{d}E_a^{\flat} = \omega_{E_aE_s}\wedge\omega^s. \tag{22}$$

Since $\omega_{E_aE_s} = -\omega_{E_sE_a}$ and $\omega^a = E_a^{\flat}$, the equation (19) follows. □

Remark 2.7. The version of the first structural equation obtained here has the advantage that it can be defined for general vector fields, which are not necessarily from an orthonormal local frame, or a local frame in general. It is well defined even if the metric becomes degenerate (but radical-stationary). Of course, at the points where the signature changes we should not expect to have continuity, but on the regions of constant signature the contraction is smooth. If the manifold (M,g) is semi-regular, the smoothness is ensured even at the points where the metric changes its signature.

3. The second structural equation

3.1. The curvature forms.

Definition 3.1. Let (M, g) be a radical-stationary singular semi-Riemannian manifold, and let $X, Y, Z, T \in \mathfrak{X}(M)$ be four vector fields. Then, the *curvature form* associated to X, Y is defined as

$$\Omega_{XY}(Z, T) := R(X, Y, Z, T). \tag{23}$$

In particular, if $(E_a)_{a=1}^n$ is a frame field, we define Ω_{ab} by

$$\Omega_{ab}(Z, T) := \Omega_{E_a E_b}(Z, T). \tag{24}$$

3.2. The second structural equation.

Lemma 3.2 (The second structural equation)**.** Let (M, g) be a radical-stationary singular semi-Riemannian manifold, and let $X, Y \in \mathfrak{X}(M)$ be two vector fields. Then

$$\Omega_{XY} = \mathrm{d}\omega_{XY} + \omega_{X\bullet} \wedge \omega_{Y\bullet}. \tag{25}$$

Proof. From the definition of the exterior derivative it follows that

$$\begin{array}{lll} \mathrm{d}\omega_{XY}(Z, T) & = & Z\left(\omega_{XY}(T)\right) - T\left(\omega_{XY}(Z)\right) - \omega_{XY}([T, Z]) \\ & = & Z\mathcal{K}(T, X, Y) - T\mathcal{K}(Z, X, Y) - \mathcal{K}([T, Z], X, Y). \end{array} \tag{26}$$

On the other hand,

$$\begin{array}{lll} \left(\omega_{X\bullet} \wedge \omega_{Y\bullet}\right)(Z, T) & = & \omega_{X\bullet}(Z)\omega_{Y\bullet}(T) - \omega_{X\bullet}(T)\omega_{Y\bullet}(Z) \\ & = & \mathcal{K}(Z, X, \bullet)\mathcal{K}(T, Y, \bullet) - \mathcal{K}(T, X, \bullet)\mathcal{K}(Z, Y, \bullet). \end{array} \tag{27}$$

From the equation (7), it follows that

$$\begin{array}{lll} R(X, Y, Z, T) & = & Z\mathcal{K}(T, X, Y) - T\mathcal{K}(Z, X, Y) - \mathcal{K}([Z, T], X, Y) \\ & & +\mathcal{K}(Z, X, \bullet)\mathcal{K}(T, Y, \bullet) - \mathcal{K}(T, X, \bullet)\mathcal{K}(Z, Y, \bullet), \end{array} \tag{28}$$

and from the identities (26), (27) and (28) the equation (25) follows. □

References

1. D. Kupeli, *Degenerate Manifolds*, Geom. Dedicata **23** (1987), no. 3, 259–290.
2. ______, *Singular semi-Riemannian geometry*, Kluwer Academic Publishers Group, 1996.
3. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure Appl. Math. (1983), no. 103, 468.
4. B. O'Neill and JM Stewart, *The Geometry of Kerr Black Holes*, Wellesley: A K Peters Ltd., 1995.
5. C. Stoica, *On Singular Semi-Riemannian Manifolds*, arXiv:math.DG/1105.0201 (2011).
6. ______, *Warped Products of Singular Semi-Riemannian Manifolds*, arXiv:math.DG/1105.3404 (2011).