On kernel smoothing for extremal quantile regression

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Abstract

Nonparametric regression quantiles obtained by inverting a kernel estimator of the conditional distribution of the response are long established in statistics. Attention has been, however, restricted to ordinary quantiles staying away from the tails of the conditional distribution. The purpose of this paper is to extend their asymptotic theory far enough into the tails. We focus on extremal quantile regression estimators of a response variable given a vector of covariates in the general setting, whether the conditional extreme-value index is positive, negative, or zero. Specifically, we elucidate their limit distributions when they are located in the range of the data or near and even beyond the sample boundary, under technical conditions that link the speed of convergence of their (intermediate or extreme) order with the oscillations of the quantile function and a von-Mises property of the conditional distribution. A simulation experiment and an illustration on real data were proposed. The real data are the American electric data where the estimation of conditional extremes is found to be of genuine interest.

Key words : Regression, extreme quantile, extreme-value index, kernel smoothing, von-Mises condition, asymptotic normality

1 Introduction

Quantile regression plays a fundamental role in various statistical applications. It complements the classical regression on the conditional mean by offering a more useful tool for examining how a vector of regressors $X \in \mathbb{R}^p$ influences the entire distribution of a response variable $Y \in \mathbb{R}$. The nonparametric regression quantiles obtained by inverting a kernel estimator of the conditional distribution function are used widely in applied work and investigated extensively in theoretical statistics. See, for example [3, 25, 27, 28], among others. Attention has been, however, restricted to conditional quantiles having a fixed order $\alpha \in (0, 1)$. In the following, the order α has to be understood as the conditional probability to be larger than the conditional quantile. In result, the available large sample theory does not apply sufficiently far in the tails.

There are many important applications in ecology, climatology, demography, biostatistics, econometrics, finance, insurance, . . . where extending that conventional asymptotic theory further into the tails of the conditional distribution is an especially welcome development. This translates into considering the order $\alpha = \alpha_n \to 0$ or $\alpha_n \to 1$ as the sample size n goes to infinity. Motivating examples include the study of extreme rainfall as a function of the geographical location [14], the estimation of factors of high risk in finance [29], the assessment of the optimal cost of the delivery activity of postal services [7], the analysis of survival at extreme durations [22], the edge estimation in image reconstruction [23], the accurate description of the upper tail of the claim size distribution for reinsurers [2], the analysis of environmental time series with application to trend detection in ground-level ozone [26], the estimation of autoregressive models with asymmetric innovations [12], etc.

There have been several efforts to treat the asymptotics of extreme conditional quantile estimators in semi/parametric and other nonparametric regression models. For example, Chernozhukov [5] and Jurecková [21] considered the extreme quantiles in the linear regression model and derived their asymptotic distributions under various distributions of errors. Other parametric models are proposed in [10, 26], where some extreme-value based techniques are extended to the point-process view of high-level exceedances. A semi-parametric approach to modeling trends in sample extremes, based on local polynomial fitting of the Generalized extreme-value distribution, has been introduced in [9]. Hall and Tajvidi [19] suggested a nonparametric estimation of the temporal trend when fitting parametric models to extreme values. Another semi-parametric method has been developed in [1], where the regression is based on a Pareto-type conditional distribution of the response. Fully nonparametric estimators of extreme conditional quantiles have been discussed in [1, 4], where the former approach is based on the technique of local polynomial maximum likelihood estimation, while spline estimators are fitted in the latter by a maximum penalized likelihood method. Recently, [14, 15] proposed, respectively, a nearest-neighbor type of estimator and a moving-window based estimator for extreme quantiles of heavy-tailed conditional distributions, and they established their asymptotic properties.

In the context of kernel-smoothing, the asymptotic theory for quantile regression in the tails is relatively unexplored and still in full development. Daouia *et al.* [8] have extended the asymptotics further into the tails in the particular setting of a heavy-tailed conditional distribution, while [16] have analyzed the case $\alpha_n = 1/n$ in the particular situation where the response Y given $X = x$ is uniformly distributed. The purpose of this paper is to develop a unified asymptotic theory for the kernel-smoothed conditional extremes in the general setting where the conditional distribution can be short, light or heavy-tailed. We will focus on the $\alpha_n \to 0$ case, which corresponds to the class of large quantiles of the upper conditional tail. Similar considerations evidently apply to the case $\alpha_n \to 1$. Specifically, we first obtain the asymptotic normality of the extremal quantile regression under the 'intermediate' order condition $nh^p\alpha_n \to \infty$ where $h = h_n \to 0$ stands for the bandwidth involved in the kernel smoothing estimation. Next we extend the asymptotic normality far enough into the 'most extreme' order-β_n regression quantiles with $\beta_n/\alpha_n \to 0$, thus providing a conditional analog of modern extreme-value results [18]. We also analyze kernel-smoothed Pickands type estimators of the conditional extreme-value index as in the familiar nonregression case [11].

The paper is organized as follows. Section 2 contains the basic notations and assumptions. Section 3 states the main results. Section 4 presents some simulation evidence and practical guidelines. Section 5 provides a motivating example in production theory, and Section 6 collects the proofs.

2 The setting and assumptions

Let (X_i, Y_i), $i = 1, \ldots, n$, be independent copies of a random pair $(X, Y) \in \mathbb{R}^p \times \mathbb{R}$. The conditional survival function (csf) of Y given $X = x$ is denoted by $\bar{F}(y|x) = \mathbb{P}(Y > y|X = x)$ and the probability density function (pdf) of X is denoted by g. We address the problem of estimating extreme conditional quantiles

$$q(\alpha_n|x) = \bar{F}^{\leftarrow}(\alpha_n|x) = \inf\{t,\ \bar{F}(t|x) \leq \alpha_n\},$$

where $\alpha_n \to 0$ as n goes to infinity. In the following we denote by $y_F(x) = q(0|x) \in (-\infty, \infty]$ the endpoint of the conditional distribution of Y given $X = x$. The kernel estimator of $\bar{F}(y|x)$ is defined for all $(x,y) \in \mathbb{R}^p \times \mathbb{R}$ by

$$\hat{\bar{F}}_n(y|x) = \sum_{i=1}^n K_h(x - X_i)\mathbb{I}\{Y_i > y\} \Bigg/ \sum_{i=1}^n K_h(x - X_i), \tag{1}$$

where $\mathbb{I}\{.\}$ is the indicator function and $h = h_n$ is a nonrandom sequence such that $h \to 0$ as $n \to \infty$. We have also introduced $K_h(t) = K(t/h)/h^p$ where K is a pdf on $\mathbb{R}^p$. In this context, h is called the window-width. Similarly, the kernel estimators of conditional quantiles $q(\alpha|x)$ are defined via the generalized inverse of $\hat{\bar{F}}_n(.|x)$:

$$\hat{q}_n(\alpha|x) = \hat{\bar{F}}_n^{\leftarrow}(\alpha|x) = \inf\{t,\ \hat{\bar{F}}_n(t|x) \le \alpha\}, \tag{2}$$

for all $\alpha \in (0,1)$. Many papers are dedicated to the asymptotic properties of this type of estimator for fixed $\alpha \in (0,1)$: weak and strong consistency are proved respectively in [27] and [13], asymptotic normality being established in [28, 25, 3]. In Theorem 1 below, the asymptotic distribution of (2) is investigated when estimating extreme quantiles, *i.e* when $\alpha = \alpha_n$ goes to 0 as the sample size n goes to infinity. The asymptotic behavior of such estimators then depends on the nature of the conditional distribution tail. In this paper, we assume that the csf satisfies the following von-Mises condition, see for instance [18, equation (1.11.30)]:

(A.1) The function $\bar{F}(.|x)$ is twice differentiable and

$$\lim_{y \uparrow y_F(x)} \frac{\bar{F}(y|x)\bar{F}''(y|x)}{(\bar{F}')^2(y|x)} = \gamma(x) + 1,$$

where $\bar{F}'(.|x)$ and $\bar{F}''(.|x)$ are respectively the first and the second derivatives of $\bar{F}(.|x)$.

Here, $\gamma(.)$ is an unknown function of the covariate x referred to as the conditional extreme-value index. Let us consider, for all $z \in \mathbb{R}$, the classical K_z function defined for all $u \in \mathbb{R}$ by

$$K_z(u) = \int_1^u v^{z-1} dv.$$

The associated inverse function is denoted by K_z^{-1}. Then, **(A.1)** implies that there exists a positive auxiliary function $a(.|x)$ such that,

$$\lim_{y \uparrow y_F(x)} \frac{\bar{F}(y + t(x)a(y|x)|x)}{\bar{F}(y|x)} = \frac{1}{K^{-1}_{\gamma(x)}(t(x))}, \tag{3}$$

where $t(x) \in \mathbb{R}$ is such that $1 + t(x)\gamma(x) > 0$. Besides, (3) implies in turn that the conditional distribution of Y given $X = x$ is in the maximum domain of attraction (MDA) of the extreme-value distribution with shape parameter $\gamma(x)$, see [18, Theorem 1.1.8] for a proof. The case $\gamma(x) > 0$ corresponds to the Fréchet MDA and $\bar{F}(.|x)$ is heavy-tailed while the case $\gamma(x) = 0$ corresponds to the Gumbel MDA and $\bar{F}(.|x)$ is light-tailed. The case $\gamma(x) < 0$ represents most of the situations where $\bar{F}(.|x)$ is short-tailed, *i.e* $\bar{F}(.|x)$ has a finite endpoint $y_F(x)$, this is referred to as the Weibull MDA.

The convergence (3) is also equivalent to

$$b(t, \alpha|x) := \frac{q(t\alpha|x) - q(\alpha|x)}{a(q(\alpha|x)|x)} - K_{\gamma(x)}(1/t) \to 0 \tag{4}$$

for all $t > 0$ as $\alpha \to 0$, see [18, Theorem 1.1.6]. For all $(x, x') \in \mathbb{R}^p \times \mathbb{R}^p$, the Euclidean distance between x and x' is denoted by $d(x, x')$. The following Lipschitz condition is introduced:

(A.2) There exists $c_g > 0$ such that $\left|g(x) - g(x')\right| \le c_g d(x, x')$.

The last assumption is standard in the kernel estimation framework.

(A.3) K is a bounded pdf on $\mathbb{R}^p$, with support S included in the unit ball of $\mathbb{R}^p$.

3 Main results

Let $B(x,h)$ be the ball centered at x with radius h. The oscillations of the csf are controlled by

$$\Delta_\kappa(x,\alpha) := \sup_{(x',\beta)\in B(x,h)\times[\kappa\alpha,\alpha]} \left| \frac{\bar F(q(\beta|x)|x')}{\beta} - 1 \right|,$$

where $(\kappa,\alpha)\in(0,1)^2$. Under assumption **(A.1)**, $\bar F(.|x)$ is differentiable and the associated conditional density will be denoted in the sequel by $f(.|x)$. We first establish the asymptotic normality of $\hat q_n(\alpha_n|x)$.

Theorem 1. *Suppose* **(A.1)**, **(A.2)** *and* **(A.3)** *hold. Let $0<\tau_J<\cdots<\tau_2<\tau_1\le 1$ where J is a positive integer and $x\in\mathbb{R}^p$ such that $g(x)>0$. If $\alpha_n\to 0$ and there exists $\kappa\in(0,\tau_J)$ such that*

$$nh^p\alpha_n\to\infty,\ nh^p\alpha_n(h\vee\Delta_\kappa(x,\alpha_n))^2\to 0,$$

then, the random vector

$$\left\{ f(q(\alpha_n|x)|x)\sqrt{nh^p\alpha_n^{-1}}\,(\hat q_n(\tau_j\alpha_n|x)-q(\tau_j\alpha_n|x))\right\}_{j=1,\dots,J}$$

is asymptotically Gaussian, centered, with covariance matrix $\|K\|_2^2/g(x)\Sigma(x)$ where $\Sigma_{j,j'}(x) = (\tau_j\tau_{j'})^{-\gamma(x)}\tau_{j\wedge j'}^{-1}$ for $(j,j')\in\{1,\dots,J\}^2$.

Let us remark that, in the particular case where $J=1$, $\tau_1=1$ and $\alpha_n=\alpha$ is fixed in $(0,1)$, we find back the result of [3, Theorem 6.4]. Theorem 1 can be equivalently rewritten as

Corollary 1. *Under the assumptions of Theorem 1, the random vector*

$$\left\{ \sqrt{nh^p\alpha_n}\frac{q(\alpha_n|x)}{a(q(\alpha_n|x)|x)}\left(\frac{\hat q_n(\tau_j\alpha_n|x)}{q(\tau_j\alpha_n|x)}-1\right)\right\}_{j=1,\dots,J}$$

is asymptotically Gaussian, centered, with covariance matrix $\|K\|_2^2/g(x)\tilde\Sigma(x)$ where $\tilde\Sigma_{j,j'}(x) = (\tau_j\tau_{j'})^{-(\gamma(x)\wedge 0)}\tau_{j\wedge j'}^{-1}$ for $(j,j')\in\{1,\dots,J\}^2$.

Moreover [18, Theorem 1.2.5] and [18, page 33] show that

$$\lim_{y\uparrow y_F(x)} \frac{a(y|x)}{y} = \gamma(x)\vee 0. \tag{5}$$

Under the assumptions of Theorem 1, and from (5), it follows that $\hat q_n(\tau_j\alpha_n|x)/q(\tau_j\alpha_n|x)\xrightarrow{P}1$ when $n\to\infty$ which can be read as a weak consistency result for the considered estimator. Besides, if $\gamma(x)>0$, then collecting (5) and Corollary 1 shows that the random vector

$$\left\{ \sqrt{nh^p\alpha_n}\left(\frac{\hat q_n(\tau_j\alpha_n|x)}{q(\tau_j\alpha_n|x)}-1\right)\right\}_{j=1,\dots,J}$$

is asymptotically Gaussian, centered, with covariance matrix $\|K\|_2^2\gamma^2(x)/g(x)\tilde\Sigma(x)$ where the coefficients of the covariance matrix can be simplified $\tilde\Sigma_{j,j'}(x)=\tau_{j\wedge j'}^{-1}$ for $(j,j')\in\{1,\dots,J\}^2$. Our results thus build on and complement the analysis given by [8, Theorem 2] in the case $\gamma(x)>0$. As pointed out in [8], the condition $nh^p\alpha_n\to\infty$ implies $\alpha_n>\log^p(n)/n$ eventually. This condition provides a lower bound on the order of the extreme conditional quantiles for the asymptotic normality of kernel estimators to hold. We now propose a scheme to estimate extreme conditional quantiles without this restriction. Let $\alpha_n\to 0$ and $\beta_n/\alpha_n\to 0$ as $n\to\infty$. Suppose one has $\hat\gamma_n(x)$

and $\hat{a}_n(x)$ two estimators of $\gamma(x)$ and $a(q(\alpha_n|x)|x)$ respectively. Then, starting from the estimator $\hat{q}_n(\alpha_n|x)$ of $q(\alpha_n|x)$ defined in (2), it is possible to build an estimator $\tilde{q}_n(\beta_n|x)$ of $q(\beta_n|x)$ which is an extreme conditional quantile of higher order than $q(\alpha_n|x)$:

$$\tilde{q}_n(\beta_n|x) = \hat{q}_n(\alpha_n|x) + K_{\hat{\gamma}_n(x)}(\alpha_n/\beta_n)\hat{a}_n(x). \quad (6)$$

Let us consider, for all $z \in \mathbb{R}$, the function defined for all $u > 1$ by

$$K'_z(u) = \frac{\partial K_z(u)}{\partial z} = \int_1^u v^{z-1}\log(v)dv.$$

The following result provides a quantile regression analog of [18, Theorem 4.3.1].

Theorem 2. *Suppose* **(A.1)** *holds and let* $\alpha_n \to 0$, $\beta_n/\alpha_n \to 0$. *Let* $\hat{q}_n(\alpha_n|x)$ *be the kernel estimator of* $q(\alpha_n|x)$ *defined in (2). Let* $\hat{\gamma}_n(x)$ *and* $\hat{a}_n(x)$ *be two estimators of* $\gamma(x)$ *and* $a(q(\alpha_n|x)|x)$ *respectively such that*

$$\Lambda_n^{-1}\left(\hat{\gamma}_n(x) - \gamma(x), \frac{\hat{a}_n(x)}{a(q(\alpha_n|x)|x)} - 1, \frac{\hat{q}_n(\alpha_n|x) - q(\alpha_n|x)}{a(q(\alpha_n|x)|x)}\right)^t \xrightarrow{d} \zeta(x), \quad (7)$$

where $\zeta(x)$ *is a non degenerated* $\mathbb{R}^3$ *random vector,*

$$\Lambda_n \log(\alpha_n/\beta_n) \to 0 \text{ and } \Lambda_n^{-1}\frac{b(\beta_n/\alpha_n, \alpha_n|x)}{K'_{\gamma(x)}(\alpha_n/\beta_n)} \to 0$$

as $n \to \infty$. *Then,*

$$\Lambda_n^{-1}\left(\frac{\tilde{q}_n(\beta_n|x) - q(\beta_n|x)}{a(q(\alpha_n|x)|x)K'_{\gamma(x)}(\alpha_n/\beta_n)}\right) \xrightarrow{d} c(x)^t\zeta(x),$$

where $c(x)^t = \left(1, -(\gamma(x)\wedge 0), (\gamma(x)\wedge 0)^2\right)$.

As an illustration, for all $r \in (0,1)$, let us consider $\tau_j = r^{j-1}$, $j = 1, \dots, J$. The following estimators of $\gamma(x)$ and $a(q(\alpha_n|x)|x)$ are introduced

$$\begin{aligned}
\hat{\gamma}_n^{\mathrm{RP}}(x) &= \frac{1}{\log r}\sum_{j=1}^{J-2}\pi_j \log\left(\frac{\hat{q}_n(\tau_j\alpha_n|x) - \hat{q}_n(\tau_{j+1}\alpha_n|x)}{\hat{q}_n(\tau_{j+1}\alpha_n|x) - \hat{q}_n(\tau_{j+2}\alpha_n|x)}\right) \\
\hat{a}_n^{\mathrm{RP}}(x) &= \frac{1}{K_{\hat{\gamma}_n^{\mathrm{RP}}(x)}(r)}\sum_{j=1}^{J-2}\pi_j r^{\hat{\gamma}_n^{\mathrm{RP}}(x)j}(\hat{q}_n(\tau_j\alpha_n|x) - \hat{q}_n(\tau_{j+1}\alpha_n|x)),
\end{aligned}$$

where (π_j) is a sequence of weights summing to one. Let us highlight that $\hat{\gamma}_n^{\mathrm{RP}}(x)$ is an adaptation to the conditional case of the Refined Pickands estimator introduced in [11]. The joint asymptotic normality of $(\hat{\gamma}_n^{\mathrm{RP}}(x), \hat{a}_n^{\mathrm{RP}}(x), \hat{q}_n(\alpha_n|x))$ is established in the next theorem.

Theorem 3. *Suppose* **(A.1)**, **(A.2)** *and* **(A.3)** *hold. Let* $x \in \mathbb{R}^p$ *such that* $g(x) > 0$. *If* $\alpha_n \to 0$ *and there exists* $\kappa \in (0, \tau_J)$ *such that*

$$nh^p\alpha_n \to \infty,\ nh^p\alpha_n\left(h \vee \Delta_\kappa(x,\alpha_n) \vee \bigvee_{j=1}^{J} b(\tau_j, \alpha_n|x)\right)^2 \to 0,$$

as $n \to \infty$, *then the random vector*

$$\sqrt{nh^p\alpha_n}\left(\hat{\gamma}_n^{RP}(x) - \gamma(x), \frac{\hat{a}_n^{RP}(x)}{a(q(\alpha_n|x)|x)} - 1, \frac{\hat{q}_n(\alpha_n|x) - q(\alpha_n|x)}{a(q(\alpha_n|x)|x)}\right)^t$$

is asymptotically centered and Gaussian.

The asymptotic covariance matrix is denoted by $S(x)$. It can be explicitly calculated from (27) in the proof of Theorem 3, but the result would be two complicated to be reported here. As a consequence of the two above theorems, one obtains the asymptotic normality of the extreme conditional quantile estimator built on $\hat{\gamma}_n^{\mathrm{RP}}(x)$ and $\hat{a}_n^{\mathrm{RP}}(x)$:

$$\tilde{q}_n^{\mathrm{RP}}(\beta_n|x) := \hat{q}_n(\alpha_n|x) + K_{\hat{\gamma}_n^{\mathrm{RP}}(x)}(\alpha_n/\beta_n)\hat{a}_n^{\mathrm{RP}}(x).$$

Corollary 2. *Suppose* **(A.1)**, **(A.2)** *and* **(A.3)** *hold. Let* $x \in \mathbb{R}^p$ *such that* $g(x) > 0$. *If* $\alpha_n \to 0$, $\beta_n/\alpha_n \to 0$ *and there exists* $\kappa \in (0, \tau_J)$ *such that*

$$\frac{nh^p\alpha_n}{(\log(\alpha_n/\beta_n))^2} \to \infty, \ nh^p\alpha_n \left(h \vee \Delta_\kappa(x,\alpha_n) \vee \bigvee_{j=1}^{J} b(\tau_j,\alpha_n|x) \vee \frac{b(\beta_n/\alpha_n,\alpha_n|x)}{K'_{\gamma(x)}(\alpha_n/\beta_n)} \right)^2 \to 0,$$

as $n \to \infty$, *then*

$$\sqrt{nh^p\alpha_n}\left(\frac{\tilde{q}_n^{RP}(\beta_n|x) - q(\beta_n|x)}{a(q(\alpha_n|x)|x)K'_{\gamma(x)}(\alpha_n/\beta_n)} \right)$$

is asymptotically Gaussian, centered with variance $c(x)^t S(x) c(x)$.

Finally, two particular cases of $\hat{\gamma}_n^{\mathrm{RP}}(x)$ may be considered. First, constant weights $\pi_1 = \ldots \pi_{J-2} = 1/(J-2)$ yield

$$\hat{\gamma}_n^{\mathrm{RP},1}(x) = \frac{1}{(J-2)\log r} \log\left(\frac{\hat{q}_n(\tau_1\alpha_n|x) - \hat{q}_n(\tau_2\alpha_n|x)}{\hat{q}_n(\tau_{J-1}\alpha_n|x) - \hat{q}_n(\tau_J\alpha_n|x)} \right).$$

Clearly, when $J = 3$, this estimator reduces the kernel Pickands estimator introduced and studied in [8] in the situation where $\gamma(x) > 0$. Second, linear weights $\pi_j = 2j/((J-1)(J-2))$ for $j = 1, \ldots, J-2$ give rise to a new estimator

$$\hat{\gamma}_n^{\mathrm{RP},2}(x) = \frac{2}{(J-1)(J-2)\log r} \sum_{j=1}^{J-2} \log\left(\frac{\hat{q}_n(\tau_j\alpha_n|x) - \hat{q}_n(\tau_{j+1}\alpha_n|x)}{\hat{q}_n(\tau_{J-1}\alpha_n|x) - \hat{q}_n(\tau_J\alpha_n|x)} \right),$$

which can be read as the average of $J-1$ estimators $\hat{\gamma}_n^{\mathrm{RP},1}(x)$. These estimators are now compared on finite sample situations.

4 Some simulation evidence

Subsection 4.1 provides Monte Carlo evidence that the extreme quantile function estimator $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x)$ is efficient relative to the version $\tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$, whether $\gamma(x)$ is positive, negative or zero, and outperforms the estimator $\hat{q}_n(\beta_n|x)$ for heavy-tailed conditional distributions. Practical guidelines for selecting the bandwidth h and the order α_n are suggested in Subsection 4.2.

4.1 Monte Carlo experiments

To evaluate finite-sample performance of the conditional extreme-value index and extreme quantile estimators described above we have undertaken some simulation experiments following the model

$$Y_i = g(X_i) + \sigma(X_i)\,U_i, \quad i = 1, \ldots, n.$$

The local scale factor, $\sigma(x) = (1+x)/10$, is linearly increasing in x, while the local location parameter

$$g(x) = \sqrt{x(1-x)}\,\sin\left(\frac{2\pi(1+2^{-7/5})}{x+2^{-7/5}} \right)$$

has been introduced in [24, Section 17.5.1]. The design points X_i are generated following a standard uniform distribution. The U_i's are independent and their conditional distribution given $X_i = x$ is chosen to be standard Gaussian, Student $t_{k(x)}$, or Beta$(\nu(x), \nu(x))$, with

$$k(x) = [\nu(x)] + 1, \quad \nu(x) = \left\{ \left(\frac{1}{10} + \sin(\pi x) \right) \left(\frac{11}{10} - \frac{1}{2} \exp\{-64(x - 1/2)^2\} \right) \right\}^{-1},$$

and $[\nu(x)]$ being the integer part of $\nu(x)$. Let us recall that the Gaussian distribution belongs to the Gumbel MDA, *i.e* $\gamma(x) = 0$, the Student distribution $t_{k(x)}$ belongs to the Fréchet MDA with $\gamma(x) = 1/k(x) > 0$ and the Beta distribution belongs to the Weibull MDA with $\gamma(x) = -1/\nu(x) < 0$.

In all cases we have $q(\beta|x) = g(x) + \sigma(x)\bar{F}^{\leftarrow}_{U|X}(\beta|x)$, for $\beta \in (0, 1)$. All the experiments were performed over 400 simulations for $n = 200$, and the kernel function K was chosen to be the Triweight kernel

$$K(t) = \frac{35}{32}(1 - t^2)^3 \mathbb{I}\{-1 \le t \le 1\}.$$

Monte Carlo experiments were first devoted to accuracy of the two conditional extreme-value index estimators $\hat{\gamma}_n^{\mathrm{RP},1}(x)$ and $\hat{\gamma}_n^{\mathrm{RP},2}(x)$. The measures of efficiency for each simulation used were the mean squared error and the bias

$$\mathrm{MSE}\{\hat{\gamma}_n(.)\} = \frac{1}{L}\sum_{\ell=1}^{L} \{\hat{\gamma}_n(x_\ell) - \gamma(x_\ell)\}^2, \quad \mathrm{Bias}\{\hat{\gamma}_n(.)\} = \frac{1}{L}\sum_{\ell=1}^{L} \{\hat{\gamma}_n(x_\ell) - \gamma(x_\ell)\}$$

for $\hat{\gamma}_n(x) = \hat{\gamma}_n^{\mathrm{RP},1}(x)$, $\hat{\gamma}_n^{\mathrm{RP},2}(x)$, with the x_ℓ's being $L = 100$ points regularly distributed in $[0, 1]$. To guarantee a fair comparison among the two estimation methods, we used for each estimator the parameters (α_n, h) minimizing its mean squared error, with α_n ranging over $\mathcal{A} = \{0.1, 0.15, 0.2, \ldots, 0.95\}$ and the bandwidth h ranging over a grid $\mathcal{H}$ of 50 points regularly distributed between $h_{min} = \max_{1 \le i < n} |X_{(i+1)} - X_{(i)}|$ and $h_{max} = |X_{(n)} - X_{(1)}|/2$, where $X_{(1)} \le \cdots \le X_{(n)}$ are the ordered observations. The resulting values of MSE and bias are averaged on the 400 Monte Carlo replications and reported in Table 1 for $J \in \{3, 4, 5\}$ and $r \in \{1/J, (J - 1)/J\}$.

	$r = 1/J$				$r = (J-1)/J$			
	MSE		Bias		MSE		Bias	
	$\hat{\gamma}_n^{\mathrm{RP},1}(x)$	$\hat{\gamma}_n^{\mathrm{RP},2}(x)$	$\hat{\gamma}_n^{\mathrm{RP},1}(x)$	$\hat{\gamma}_n^{\mathrm{RP},2}(x)$	$\hat{\gamma}_n^{\mathrm{RP},1}(x)$	$\hat{\gamma}_n^{\mathrm{RP},2}(x)$	$\hat{\gamma}_n^{\mathrm{RP},1}(x)$	$\hat{\gamma}_n^{\mathrm{RP},2}(x)$
Gaussian								
J=3	0.2026	0.2026	-0.2415	-0.2415	0.7656	0.7656	-0.3213	-0.3213
J=4	0.1915	0.2018	-0.3270	-0.3501	0.6730	0.7960	-0.3455	-0.3747
J=5	NaN	NaN	NaN	NaN	0.7305	0.9128	-0.4104	-0.4107
Student								
J=3	0.2882	0.2882	-0.2964	-0.2964	1.1109	1.1109	-0.4497	-0.4497
J=4	0.3350	0.2837	-0.4167	-0.3480	0.9991	1.1997	-0.4384	-0.4597
J=5	NaN	NaN	NaN	NaN	1.1245	1.3331	-0.5715	-0.5872
Beta								
J=3	0.1157	0.1157	-0.0730	-0.0730	0.6737	0.6737	-0.2591	-0.2591
J=4	0.0510	0.0597	-0.0811	-0.0750	0.5861	0.6891	-0.2338	-0.2432
J=5	NaN	NaN	NaN	NaN	0.6431	0.8167	-0.2185	-0.2757

Table 1: *Performance of* $\hat{\gamma}_n^{RP,1}(x)$ *and* $\hat{\gamma}_n^{RP,2}(x)$ – *Results averaged on* 400 *simulations with* $n = 200$. *The results may not be available for* $r = 1/J$ *and* $J = 5$ *since the numerator* $\{\hat{q}_n(\tau_j\alpha_n|x) - \hat{q}_n(\tau_{j+1}\alpha_n|x)\}$ *and the denominator* $\{\hat{q}_n(\tau_{J-1}\alpha_n|x) - \hat{q}_n(\tau_J\alpha_n|x)\}$ *in the definitions of both estimators might be null when* n *is not large enough.*

It does appear that the results for $r = 1/J$ are superior to those for $r = (J - 1)/J$, uniformly in J. For these desirable results, it may be seen that the estimator $\hat{\gamma}_n^{\mathrm{RP},1}(x)$ performs better than $\hat{\gamma}_n^{\mathrm{RP},2}(x)$ in the Gaussian error model, whereas the latter is superior to the former in the Student error model. It may be also seen that there is no winner in the Beta error model in terms of both MSE and bias.

Turning to the performance of the extreme conditional quantile estimators, we consider as above the two measures of performance

$$\mathrm{MSE}\{q_n(\beta_n|\cdot)\} = \frac{1}{L}\sum_{\ell=1}^{L}\{q_n(\beta_n|x_\ell) - q(\beta_n|x_\ell)\}^2, \quad \mathrm{Bias}\{q_n(\beta_n|\cdot)\} = \frac{1}{L}\sum_{\ell=1}^{L}\{q_n(\beta_n|x_\ell) - q(\beta_n|x_\ell)\},$$

for $q_n(\beta_n|x) = \hat{q}_n(\beta_n|x)$, $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x)$, $\tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$. The averaged MSE and bias of these three estimators of $q(\beta_n|x)$, computed for $\beta_n \in \{0.05, 0.01, 0.005\}$, $J \in \{3,4\}$ and $r = 1/J$, over 400 Monte Carlo simulations are displayed in Table 2. Here also, we used for each estimator the smoothing parameters (α_n, h) minimizing its MSE over the grid of values $\mathcal{A}\times\mathcal{H}$ described above. When comparing the estimators $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x)$ and $\tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$ themselves with $\hat{q}_n(\beta_n|x)$, the results (both in terms of MSE and bias) indicate that $\tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$ is slightly less efficient than $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x)$ in all cases, and that the latter is appreciably more efficient than $\hat{q}_n(\beta_n|x)$ only in the Student error model. It may be also noticed that $\hat{q}_n(\beta_n|x)$ is more efficient but not by much (especially when $J = 3$) in the Gaussian and Beta error models.

$r = 1/J$, $\beta_n = 0.05$

	MSE			Bias		
	$\tilde{q}_n^{\mathrm{RP},1}(\beta_n\|x)$	$\tilde{q}_n^{\mathrm{RP},2}(\beta_n\|x)$	$\hat{q}_n(x)$	$\tilde{q}_n^{\mathrm{RP},1}(\beta_n\|x)$	$\tilde{q}_n^{\mathrm{RP},2}(\beta_n\|x)$	$\hat{q}_n(x)$
Gaussian						
J=3	.0110	.0110	.0108	.0001	.0001	.0063
J=4	.0591	.0796	.0108	.1136	.1131	.0063
Student						
J=3	.0307	.0307	.0771	-.0134	-.0134	.0871
J=4	.0532	.0743	.0771	.0792	.0792	.0871
Beta						
J=3	.0091	.0091	.0022	.0505	.0505	.0135
J=4	.0745	.1002	.0022	.1746	.1752	.0135

$r = 1/J$, $\beta_n = 0.01$

	MSE			Bias		
	$\tilde{q}_n^{\mathrm{RP},1}(\beta_n\|x)$	$\tilde{q}_n^{\mathrm{RP},2}(\beta_n\|x)$	$\hat{q}_n(x)$	$\tilde{q}_n^{\mathrm{RP},1}(\beta_n\|x)$	$\tilde{q}_n^{\mathrm{RP},2}(\beta_n\|x)$	$\hat{q}_n(x)$
Gaussian						
J=3	.0265	.0265	.0161	-.0776	-.0776	-.0360
J=4	.0693	.0926	.0161	.1092	.1225	-.0360
Student						
J=3	.1115	.1115	.6825	-.0895	-.0895	-.0959
J=4	.1304	.3992	.6825	.0018	.1089	-.0959
Beta						
J=3	.0143	.0143	.0034	.0523	.0523	.0212
J=4	.1038	.1265	.0034	.1964	.2064	.0212

$r = 1/J$, $\beta_n = 0.005$

	MSE			Bias		
	$\tilde{q}_n^{\mathrm{RP},1}(\beta_n\|x)$	$\tilde{q}_n^{\mathrm{RP},2}(\beta_n\|x)$	$\hat{q}_n(x)$	$\tilde{q}_n^{\mathrm{RP},1}(\beta_n\|x)$	$\tilde{q}_n^{\mathrm{RP},2}(\beta_n\|x)$	$\hat{q}_n(x)$
Gaussian						
J=3	.0354	.0354	.0203	-.0981	-.0981	-.0524
J=4	.0719	.0932	.0203	.0982	.1073	-.0524
Student						
J=3	.2919	.2919	.9782	-.1623	-.1623	-.2605
J=4	.4569	.9748	.9782	-.1920	.0280	-.2605
Beta						
J=3	.0155	.0155	.0038	.0536	.0536	.0239
J=4	.1130	.1337	.0038	.1871	.2111	.0239

Table 2: *Performance of* $\tilde{q}_n^{RP,1}(\beta_n|x)$, $\tilde{q}_n^{RP,2}(\beta_n|x)$ *and* $\hat{q}_n(\beta_n|x)$ *with* $\beta_n = 0.05$ *(top),* $\beta_n = 0.01$ *(middle) and* $\beta_n = 0.005$ *(bottom) – Results averaged on* 400 *simulations with* $n = 200$.

4.2 Illustration on one simulated sample

For selecting the bandwidth h involved in the estimates $\hat{\hat{F}}_n(\cdot|\cdot)$ from one sample, we employ here the cross-validation criterion introduced in [30] and described *e.g.* in [8], that is,

$$h_{cv} = \arg\min_{h\in\mathcal{H}} \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\mathbb{I}(Y_i \geq Y_j) - \hat{\hat{F}}_{n,-i}(Y_j|X_i)\right\}^2 ,$$

where $\hat{\hat{F}}_{n,-i}(\cdot|\cdot)$ is the estimator $\hat{\hat{F}}_n(\cdot|\cdot)$ computed from the sample $\{(X_j, Y_j), 1 \leq j \leq n, j \neq i\}$. Then, one way to decide what value of the sequence $\alpha_n = \alpha_n(x)$ should one use to compute the estimates $\hat{\gamma}_n^{\mathrm{RP}}(x)$ of the extreme-value index, is by making use of the automatic *'ad hoc'* data driving rule employed in [7]. The idea is to select first, for each x (in a chosen grid of values), a grid of values for α_n for estimating $\gamma(x)$. We choose $\alpha_n = k/n$, where k is an integer varying between 1 and n. We then evaluate the estimates $\hat{\gamma}_n^{\mathrm{RP}}(x)$ and select the k where the variation of the results is the smallest. This is achieved by computing the standard deviation of $\hat{\gamma}_n^{\mathrm{RP}}(x)$ over a *'window'* of $[n/10]$ successive values of k (this size corresponds to having a window large enough to cover around 10% of the possible values of α_n in the selected range of values for $\alpha_n(x)$). The value of k where this standard deviation is minimal defines the value of $\alpha_n(x) = k/n$. A difficulty when using such a technique is that $\hat{\gamma}_n^{\mathrm{RP}}(x)$ as a function of k may be so unstable that a reasonable value of k (which would correspond to the true value of $\gamma(x)$) may be hidden in the graph of the estimator. In result, $\hat{\gamma}_n^{\mathrm{RP}}(x)$ may exhibit considerable volatility as a function of x itself. Good results might require a large sample size of the order of several thousands, but sometimes some simple techniques of smoothing are revealing.

A typical realization in each simulated scenario is shown in Figure 1. We see the true extreme-value index function $\gamma(x)$ as the solid red curve and its estimator $\hat{\gamma}_n^{\mathrm{RP},1}(x)$ as the solid black curve. The latter is estimated by using $r = 1/J$ with the value $J = 3$ for which $\hat{\gamma}_n^{\mathrm{RP},1}(x)$ coincides with $\hat{\gamma}_n^{\mathrm{RP},2}(x)$. As it may be seen from the three panels, the resulting estimated curves look quite reasonable given the small sample size $n = 200$. By making use of these simple data driven methods for selecting the smoothing parameter h and then α_n, the difficult question of how to pick out (h, α_n) in an optimal way might thus become less urgent.

Using the same simulated sample in each scenario, the true quantile function $q(\beta_n|x)$ is compared on Figure 2 to the estimators $\hat{q}_n(\beta_n|x)$ and $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x) = \tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$ in the case $r = 1/J$, $J = 3$ and $\beta_n = 0.05$, 0.01. Here, the procedure for selecting values for h and α_n for estimating the extreme quantile function is the same as the one used in the estimation of the extreme-value index function. As may be seen from the figure, this *ad hoc* prescription results in satisfactory estimates in terms of both bias and stability, although they do not enjoy the desired optimality.

Our tentative conclusion is that the performance of the extreme quantile $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x)$ estimator is quite remarkable in comparison with its analog $\tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$, at least in terms of MSE. In comparison with the estimator $\hat{q}_n(\beta_n|x)$, the variability and the bias of both $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x)$ and $\tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$ are quite respectable and it seems that the heavier is the conditional tail, the better the estimators $\tilde{q}_n^{\mathrm{RP}}(\beta_n|x)$ are. The simulations also indicate that the choice of the tuning parameters J and r may be crucial for both estimators $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x)$ and $\tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$. The optimal choice of the parameter α_n is not addressed here and is still an open issue.

5 Data example

Data on 123 American electric utility companies were collected and the aim is to investigate the economic efficiency of these companies (see, *e.g.*, [17]). A possible way to measure this efficiency is by looking at the maximum level of produced goods which is attainable for a given level of inputs-usage. From a statistical point of view, this problem translates into studying the upper boundary of the set of possible inputs X and outputs Y, the so-called cost/econometric frontier in production theory. Hendricks and Koenker [20] stated: “In the econometric literature on the estimation of production technologies, there has been considerable interest in estimating so called frontier production models that correspond closely to models for extreme quantiles of a stochastic production surface”. The present paper may be viewed as the first ‘purely’ nonparametric work to actually investigate theoretically the idea of Hendricks and Koenker.

For our illustration purposes, we used the measurements on the variable $Y = \log(Q)$, with Q being the production output of a firm, and the variable $X = \log(C)$, with C being the total cost involved in the production. Figure 3 shows the $n = 123$ observations, together with estimated extreme conditional quantiles $\hat{q}_n(\beta_n|x)$ and $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x) = \tilde{q}_n^{\mathrm{RP},2}(\beta_n|x)$ at $r = 1/J$ with $J = 3$. Given the small sample size, it was enough to use $\beta_n = 0.05$ and $\beta_n = 1/n$ in describing the conditional distribution tails. The bandwidth $h_{cv} \simeq 0.8305$ was computed via the cross-validation criterion, and we maintained the automatic empirical data driving rule described above for selecting the order α_n. It appears that the extreme conditional quantile estimates are quite similar. Following their evolution, one may distinguish between two different behaviors of the upper tail of the production process. They indicate a short-tailed conditional distribution for companies working at (transformed) input-factors larger than, say, 1.5. In contrast, the tail distribution for the smallest companies with inputs $X_i < 1.5$ seems to be moderately heavy. This is reflected by the extreme-value index estimator $\hat{\gamma}_n^{\mathrm{RP},1}(x) = \hat{\gamma}_n^{\mathrm{RP},2}(x)$ graphed in Figure 4, which was found to vary significantly with the sample boundary. As a matter of fact, good extreme-value index estimates in this context of production data should vary between -1 and 0 with the following intuitive interpretation [7]:

i. When $\gamma(x) < -\{p+1\}^{-1}$, the conditional density rises up to infinity as it approaches to the optimal boundary, which would correspond to an ideal production activity.

ii. The situation hoped for by the practitioners is, at most, the case $\gamma(x) = -\{p+1\}^{-1}$ where the density is strictly positive at the frontier.

iii. The density of data often decays to zero smoothly as it approaches to the frontier, which corresponds to the case $\gamma(x) > -\{p+1\}^{-1}$.

It may be seen from Figure 4 that $\hat{\gamma}_n^{\mathrm{RP},1}(x) \geq -0.5$, and so the American electric utility data do not correspond to a heavily short-tailed production process. In other words, the theoretical economic assumption that producers should operate on the upper boundary of the joint support of (X, Y) rather than on its interior is clearly not fulfilled here, revealing a certain lack of production performance in this sector of activity. It may be also noticed that, due to the small sample size here, the estimator $\hat{\gamma}_n^{\mathrm{RP},1}(x)$ exhibits some volatility (for the smallest inputs) exceeding the upper limit bound 0, but it remains quite reasonable. The estimated graph of $\tilde{q}_n^{\mathrm{RP},1}(\beta_n|x)$, or similarly $\hat{q}_n(1/n|x)$ might be interpreted as the set of the most efficient firms. It is then clear that the firms achieve significantly lesser output than that predicted by the extremal quantile frontiers. This indicates a relative economic inefficiency especially in the population of the (sparse) smallest companies in terms of inputs.

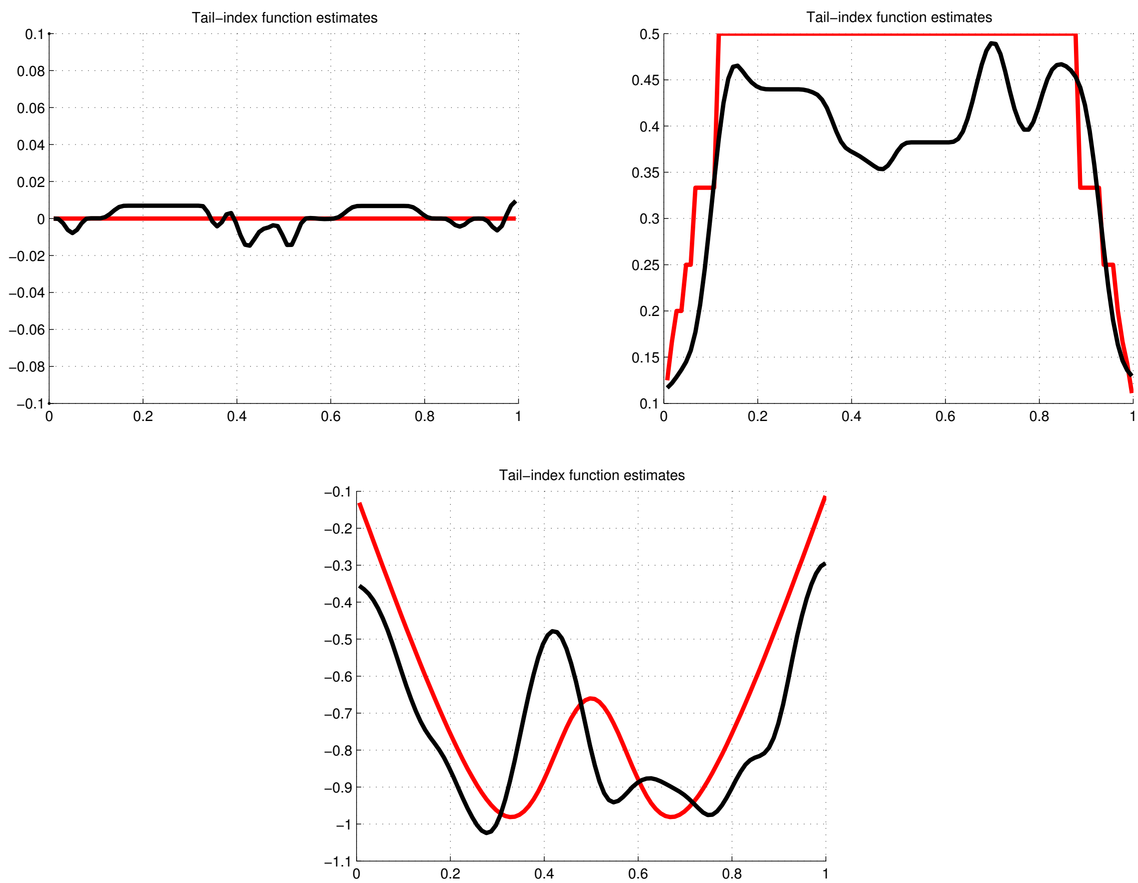


Figure 1: *The true extreme-value index* $\gamma(x)$ *in red and its estimator* $\hat{\gamma}_n^{RP,1}(x) = \hat{\gamma}_n^{RP,2}(x)$ *in black, with* $r = 1/J$ *and* $J = 3$*. From top to bottom,* $Y|X$ *is Gaussian, Student, Beta. Estimates are based on* $n = 200$ *simulated observations.*

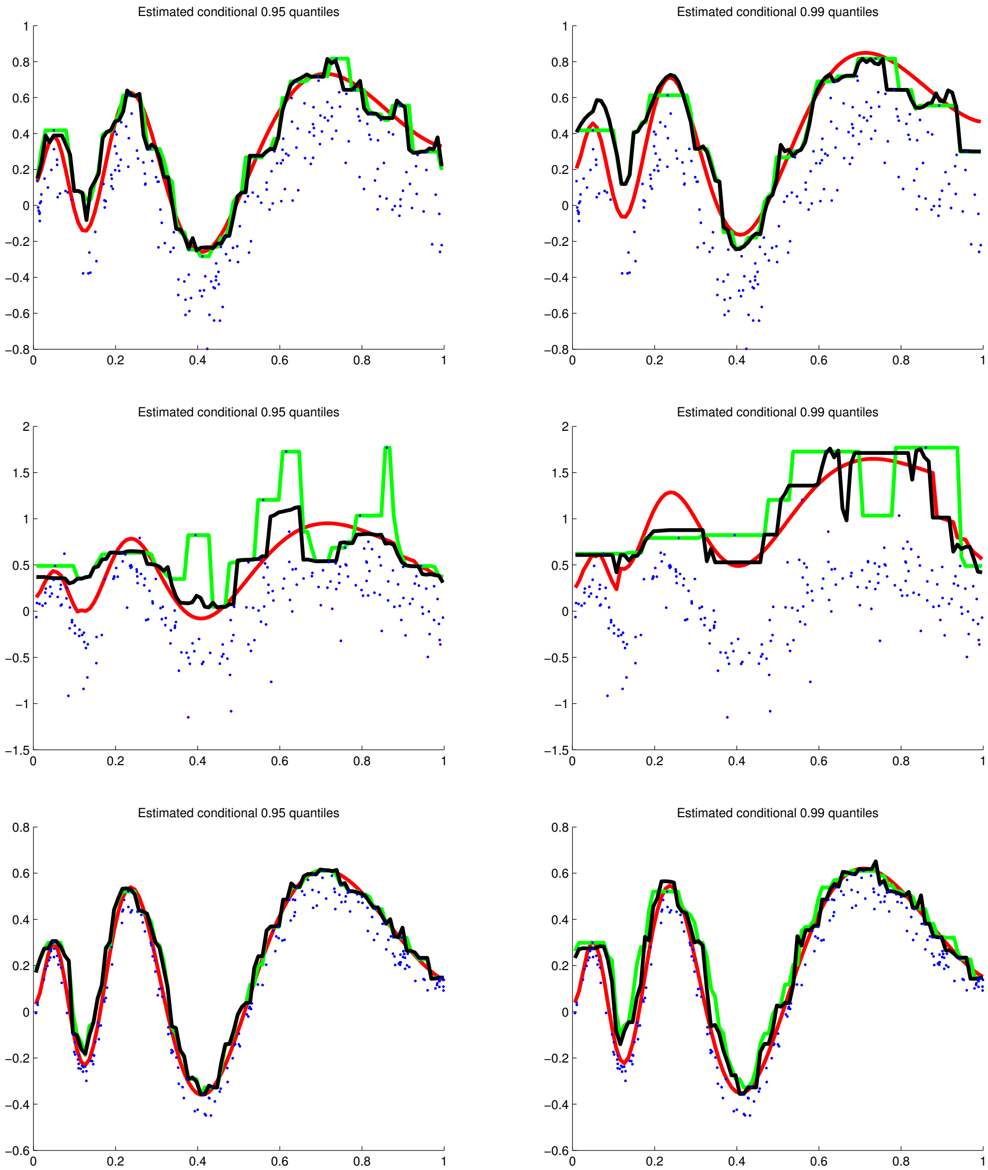
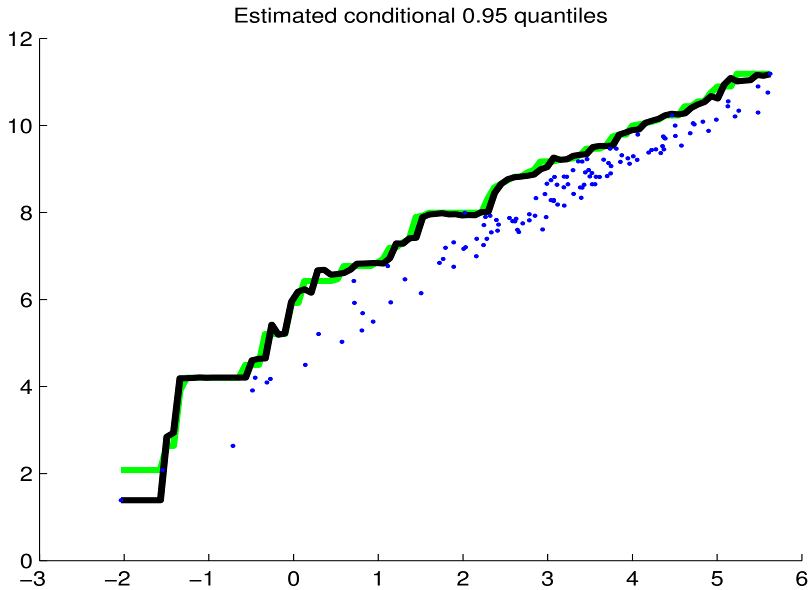


Figure 2: *The true quantile $q(\beta_n|x)$ in red, its estimators $\hat{q}_n(\beta_n|x)$ in green and $\tilde{q}_n^{RP,1}(\beta_n|x) = \tilde{q}_n^{RP,2}(\beta_n|x)$ in black, with $r = 1/J$ and $J = 3$. The observations (X_i, Y_i) are depicted as blue points. From left to right, $\beta_n = 0.05, 0.01$ with $n = 200$. From top to bottom, $Y|X$ is Gaussian, Student, Beta.*



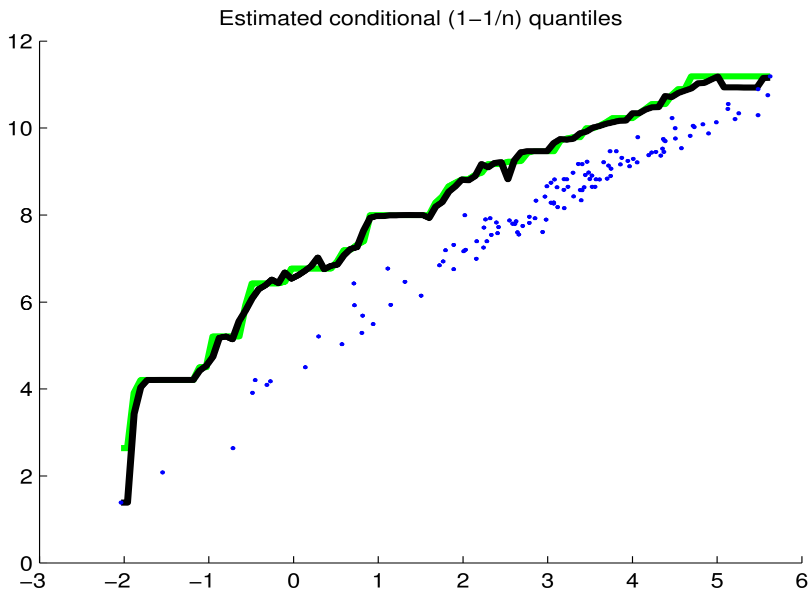


Figure 3: *Scatterplot of the American electric utility data. The solid green line is the* $\hat{q}_n(\beta_n|x)$ *estimator and the solid black line is the* $\tilde{q}_n^{RP,1}(\beta_n|x)$ *estimator with* $r = 1/J$ *and* $J = 3$*. Top:* $\beta_n = 0.05$*, bottom* $\beta_n = 1/n$*.*

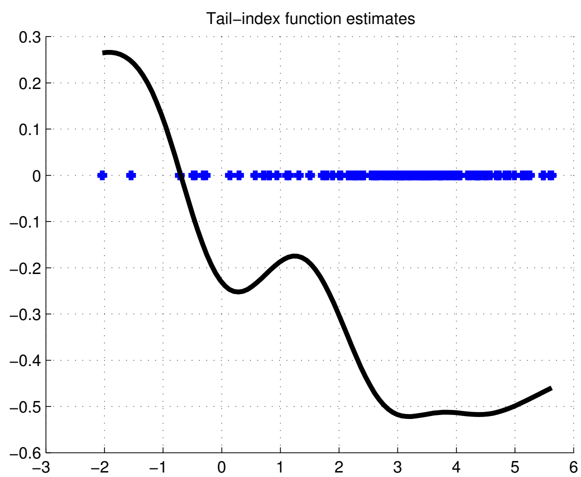


Figure 4: *Estimated extreme-value index. The inputs X_i are depicted as blue points, the estimator $\hat{\gamma}_n^{RP,1}(x) = \hat{\gamma}_n^{RP,2}(x)$ is drawn in black.*

6 Appendix: Proofs

6.1 Preliminary results

We begin with a homogeneous property of the quantile function.

Lemma 1. *Suppose* **(A.1)** *holds. If $\alpha_n \to 0$ as $n \to \infty$, then,*

$$\lim_{n\to\infty} \frac{q(\xi\alpha_n|x)}{q(\alpha_n|x)} = \xi^{-(\gamma(x)\vee 0)},$$

for all $\xi > 0$.

Proof. From (4), we have

$$\frac{q(\xi\alpha_n|x)}{q(\alpha_n|x)} = 1 + K_{\gamma(x)}(1/\xi)\frac{a(q(\alpha_n|x)|x)}{q(\alpha_n|x)}(1+o(1))$$

and the conclusion follows using (5). ■

The following lemma states that the convergence in (3) is uniform.

Lemma 2. *Under* **(A.1)**, *if $z_n(x) \uparrow y_F(x)$ as $n \to \infty$, then for all sequence of functions $t_n(x)$ such that $t_n(x) \to t_0(x)$ as $n \to \infty$ where $t_0(x)$ is such that there exists $\eta > 0$ for which $1+\gamma(x)t_0(x) \geq \eta$ then,*

$$\lim_{n\to\infty} \frac{\bar{F}(z_n(x) + t_n(x)a(z_n(x)|x)|x)}{\bar{F}(z_n(x)|x)} = \frac{1}{K^{-1}_{\gamma(x)}(t_0(x))}.$$

Proof. Since $t_n(x) \to t_0(x)$ as $n \to \infty$, for all $\varepsilon_1 > 0$ such that $|\gamma(x)|\varepsilon_1 < \eta$, there exists $N_1 \geq 0$ such that for all $n \geq N_1$, $t_0(x) - \varepsilon_1 \leq t_n(x) \leq t_0(x) + \varepsilon_1$. Since $a(z_n(x)|x) > 0$ and $\bar{F}(.|x)$ is a decreasing function, we have:

$$\begin{aligned} \frac{\bar{F}(z_n(x) + (t_0(x)+\varepsilon_1)a(z_n(x)|x)|x)}{\bar{F}(z_n(x)|x)} &\leq \frac{\bar{F}(z_n(x) + t_n(x)a(z_n(x)|x)|x)}{\bar{F}(z_n(x)|x)} \\ &\leq \frac{\bar{F}(z_n(x) + (t_0(x)-\varepsilon_1)a(z_n(x)|x)|x)}{\bar{F}(z_n(x)|x)}. \end{aligned}$$

Now, since $|\gamma(x)|\varepsilon_1 < \eta$, it is easy to check that $1 + \gamma(x)(t_0(x)+\varepsilon_1) \wedge 1 + \gamma(x)(t_0(x)-\varepsilon_1) > 0$. Hence, under **(A.1)**, for all $\varepsilon_2 > 0$, there exists $N_2 \geq 0$ such that for all $n \geq N_2$

$$\frac{1-\varepsilon_2}{K^{-1}_{\gamma(x)}(t_0(x)+\varepsilon_1)} \leq \frac{\bar{F}(z_n(x) + t_n(x)a(z_n(x)|x)|x)}{\bar{F}(z_n(x)|x)} \leq \frac{1+\varepsilon_2}{K^{-1}_{\gamma(x)}(t_0(x)-\varepsilon_1)}.$$

Since ε_1 and ε_2 can be chosen arbitrarily small, this concludes the proof. ■

Let us remark that the kernel estimator (1) can be rewritten as $\hat{\bar{F}}_n(y|x) = \hat{\psi}_n(y,x)/\hat{g}_n(x)$ where

$$\hat{\psi}_n(y,x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i)\mathbb{I}\{Y_i > y\}$$

is an estimator of $\psi(y,x) = \bar{F}(y|x)g(x)$ and $\hat{g}_n(x)$ is the classical kernel estimator of the pdf $g(x)$ defined by:

$$\hat{g}_n(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i).$$

Lemma 3 gives standard results on the kernel estimator (see [6], Proposition 2.1 and Proposition 2.2 for a proof) whereas Lemma 4 is dedicated to the asymptotic properties of $\hat{\psi}_n(y,x)$.

Lemma 3. *Suppose* **(A.2)**, **(A.3)** *hold. If* $nh^p \to \infty$, *then, for all* $x \in \mathbb{R}^p$ *such that* $g(x) > 0$,

(i) $\mathbb{E}(\hat{g}_n(x) - g(x)) = O(h)$,

(ii) $var(\hat{g}_n(x)) = \dfrac{g(x)\|K\|_2^2}{nh^p}(1 + o(1))$.

Therefore, under the assumptions of the above lemma, $\hat{g}_n(x)$ converges to $g(x)$ in probability. Let us introduce some further notations. In the following, $y_n(x)$ is a sequence such that $y_n(x) \uparrow y_F(x)$ and $y_{n,j}(x) = y_n(x) + K_{\gamma(x)}(1/\tau_j)a(y_n|x)(1+o(1))$ for all $j = 1, \ldots, K$. Recall that $0 < \tau_J < \cdots < \tau_2 < \tau_1 \le 1$. Moreover, the oscillations of the csf are controlled by

$$\omega_n(x) := \max_{j=1,\ldots,J} \sup_{x' \in B(x,h)} \left| \frac{\bar{F}(y_{n,j}(x)|x')}{\bar{F}(y_{n,j}(x)|x)} - 1 \right|.$$

Lemma 4. *Suppose* **(A.1)**, **(A.2)** *and* **(A.3)** *hold and let* $x \in \mathbb{R}^p$ *such that* $g(x) > 0$. *If* $\omega_n(x) \to 0$ *and* $nh^p \bar{F}(y_n(x)|x) \to \infty$ *then,*

(i) $\mathbb{E}(\hat{\psi}_n(y_{n,j}(x), x)) = \psi(y_{n,j}(x), x)\,(1 + O(\omega_n(x)) + O(h))$, *for* $j = 1, \ldots, J$.

(ii) *The random vector*

$$\left\{ \sqrt{nh^p \psi(y_n(x), x)} \left(\frac{\hat{\psi}_n(y_{n,j}(x), x) - \mathbb{E}(\hat{\psi}_n(y_{n,j}(x), x))}{\psi(y_{n,j}(x), x)} \right) \right\}_{j=1,\ldots,J}$$

is asymptotically Gaussian, centered, with covariance matrix $\|K\|_2^2 V$ *where* $V_{j,j'} = \tau_{j\wedge j'}^{-1}$ *for* $(j, j') \in \{1, \ldots, J\}^2$.

Proof. **(i)** Since the (X_i, Y_i), $i = 1, \ldots, n$ are identically distributed, we have

$$\mathbb{E}(\hat{\psi}_n(y_{n,j}(x), x)) = \int_{\mathbb{R}^p} K_h(x - t)\bar{F}(y_{n,j}(x)|t)g(t)dt = \int_S K(u)\bar{F}(y_{n,j}(x)|x - hu)g(x - hu)du,$$

under **(A.3)**. Let us now consider

$$|\mathbb{E}(\hat{\psi}_n(y_{n,j}(x), x)) - \psi(y_{n,j}(x), x)| \le \bar{F}(y_{n,j}(x)|x) \int_S K(u)|g(x - hu) - g(x)|du \quad (8)$$

$$+ \bar{F}(y_{n,j}(x)|x) \int_S K(u) \left| \frac{\bar{F}(y_{n,j}(x)|x - hu)}{\bar{F}(y_{n,j}(x)|x)} - 1 \right| g(x - hu)du. \quad (9)$$

Under **(A.2)**, and since $g(x) > 0$, we have

$$(8) \le \bar{F}(y_{n,j}(x)|x)c_g h \int_S d(u, 0)K(u)du = \psi(y_{n,j}(x), x)O(h), \quad (10)$$

and, in view of (10),

$$\begin{aligned}(9) &\le \bar{F}(y_{n,j}(x)|x)\omega_n(x) \int_S K(u)g(x - hu)du = \bar{F}(y_{n,j}(x)|x)\omega_n(x)g(x)(1 + o(1)) \\ &= \psi(y_{n,j}(x), x)\omega_n(x)(1 + o(1)). \quad (11)\end{aligned}$$

Combining (10) and (11) concludes the first part of the proof.
(ii) Let $\beta \neq 0$ in $\mathbb{R}^J$, $\Lambda_n(x) = (nh^p\psi(y_n(x), x))^{-1/2}$, and consider the random variable

$$\begin{aligned}
\Psi_n &= \sum_{j=1}^{J} \beta_j \left(\frac{\hat{\psi}_n(y_{n,j}(x), x) - \mathbb{E}(\hat{\psi}_n(y_{n,j}(x), x))}{\Lambda_n(x)\psi(y_{n,j}(x), x)} \right) \\
&= \sum_{i=1}^{n} \frac{1}{n\Lambda_n(x)} \left\{ \sum_{j=1}^{J} \frac{\beta_j K_h(x - X_i)\mathbb{I}\{Y_i \geq y_{n,j}(x)\}}{\psi(y_{n,j}(x), x)} - \mathbb{E}\left(\sum_{j=1}^{J} \frac{\beta_j K_h(x - X_i)\mathbb{I}\{Y_i \geq y_{n,j}(x)\}}{\psi(y_{n,j}(x), x)} \right) \right\} \\
&:= \sum_{i=1}^{n} Z_{i,n}.
\end{aligned}$$

Clearly, $\{Z_{i,n},\ i = 1, \ldots, n\}$ is a set of centered, independent and identically distributed random variables with variance

$$\mathrm{var}(Z_{i,n}) = \frac{1}{n^2h^{2p}\Lambda_n^2(x)} \mathrm{var}\left(\sum_{j=1}^{J} \beta_j K\left(\frac{x - X_i}{h}\right) \frac{\mathbb{I}\{Y_i \geq y_{n,j}(x)\}}{\psi(y_{n,j}(x), x)} \right) = \frac{1}{n^2h^p\Lambda_n^2(x)} \beta^t B\beta,$$

where B is the $J \times J$ covariance matrix with coefficients defined for $(j, j') \in \{1, \ldots, J\}^2$ by

$$\begin{aligned}
B_{j,j'} &= \frac{A_{j,j'}}{\psi(y_{n,j}(x), x)\psi(y_{n,j'}(x), x)}, \\
A_{j,j'} &= \frac{1}{h^p}\mathrm{cov}\left(K\left(\frac{x - X}{h}\right) \mathbb{I}\{Y \geq y_{n,j}(x)\},\ K\left(\frac{x - X}{h}\right) \mathbb{I}\{Y \geq y_{n,j'}(x)\} \right) \\
&= \|K\|_2^2 \mathbb{E}\left(\frac{1}{h^p} Q\left(\frac{x - X}{h}\right) \mathbb{I}\{Y \geq y_{n,j}(x) \vee y_{n,j'}(x)\} \right) \\
&- h^p\mathbb{E}(K_h(x - X)\mathbb{I}\{Y \geq y_{n,j}(x)\})\mathbb{E}(K_h(x - X)\mathbb{I}\{Y \geq y_{n,j'}(x)\}),
\end{aligned}$$

with $Q(.) := K^2(.)/\|K\|_2^2$ also satisfying assumption **(A.3)**. As a consequence, the three above expectations are of the same nature. Thus, remarking that, for n large enough, $y_{n,j}(x) \vee y_{n,j'}(x) = y_{n,j\vee j'}(x)$, part **(i)** of the proof implies

$$\begin{aligned}
A_{j,j'} &= \|K\|_2^2 \psi(y_{n,j\vee j'}(x), x) \left(1 + O(\omega_n(x)) + O(h)\right) \\
&- h^p \psi(y_{n,j}(x), x)\psi(y_{n,j'}(x), x) \left(1 + O(\omega_n(x)) + O(h)\right)
\end{aligned}$$

leading to

$$\begin{aligned}
B_{j,j'} &= \frac{\|K\|_2^2}{\psi(y_{n,j\wedge j'}(x), x)} \left(1 + O(\omega_n(x)) + O(h)\right) - h^p \left(1 + O(\omega_n(x)) + O(h)\right) \\
&= \frac{\|K\|_2^2}{\psi(y_{n,j\wedge j'}(x), x)} (1 + o(1)),
\end{aligned}$$

since $\psi(y_{n,j\wedge j'}(x), x) \to 0$ as $n \to \infty$. Now, from Lemma 2,

$$\lim_{n\to\infty} \frac{\psi(y_{n,j\wedge j'}(x), x)}{\psi(y_n(x), x)} = \frac{1}{K_{\gamma(x)}^{-1}(K_{\gamma(x)}(1/\tau_{j\wedge j'}))} = \tau_{j\wedge j'}$$

entailing

$$B_{j,j'} = \frac{\|K\|_2^2 V_{j,j'}}{\psi(y_n(x), x)} (1 + o(1)),$$

and therefore, $\mathrm{var}(Z_{i,n}) \sim \|K\|_2^2 \beta^t V\beta/n$, for all $i = 1, \dots, n$. As a preliminary conclusion, the variance of Ψ_n converges to $\|K\|_2^2 \beta^t V\beta$. Consequently, Lyapounov criteria for the asymptotic normality of sums of triangular arrays reduces to $\sum_{i=1}^n \mathbb{E}\,|Z_{i,n}|^3 = n\mathbb{E}\,|Z_{1,n}|^3 \to 0$. Remark that $Z_{1,n}$ is a bounded random variable:

$$|Z_{1,n}| \le \frac{2\|K\|_\infty \sum_{j=1}^J |\beta_j|}{n\Lambda_n(x) h^p \psi(y_{n,J}, x)} = \frac{2}{\tau_J}\|K\|_\infty \sum_{j=1}^J |\beta_j| \Lambda_n(x)(1 + o(1))$$

and thus,

$$\begin{aligned} n\mathbb{E}\,|Z_{1,n}|^3 &\le \frac{2}{\tau_J}\|K\|_\infty \sum_{j=1}^J n|\beta_j|\Lambda_n(x)\mathrm{var}(Z_{1,n})(1+o(1)) \\ &= \frac{2}{\tau_J}\|K\|_\infty \|K\|_2^2 \sum_{j=1}^J |\beta_j| \beta^t V\beta \Lambda_n(x)(1+o(1)) \to 0 \end{aligned}$$

as $n \to \infty$. As a conclusion, Ψ_n converges in distribution to a centered Gaussian random variable with variance $\|K\|_2^2 \beta^t V\beta$ for all $\beta \ne 0$ in $\mathbb{R}^p$. The result is proved. ∎

Let us now focus on the estimation of small tail probabilities $\bar{F}(y_n(x)|x)$ when $y_n(x) \uparrow y_F(x)$ as $n \to \infty$. The following result provides sufficient conditions for the asymptotic normality of $\hat{\bar{F}}_n(y_n(x)|x)$.

Proposition 1. *Suppose* **(A.1)**, **(A.2)** *and* **(A.3)** *hold and let* $x \in \mathbb{R}^p$ *such that* $g(x) > 0$. *If* $nh^p \bar{F}(y_n(x)|x) \to \infty$ *and* $nh^p \bar{F}(y_n(x)|x)(h \vee \omega_n(x))^2 \to 0$, *then, the random vector*

$$\left\{ \sqrt{nh^p \bar{F}(y_n(x)|x)} \left(\frac{\hat{\bar{F}}_n(y_{n,j}(x)|x)}{\bar{F}(y_{n,j}(x)|x)} - 1 \right) \right\}_{j=1,\dots,J}$$

is asymptotically Gaussian, centered, with covariance matrix $\|K\|_2^2/g(x)V$ *where* $V_{j,j'} = \tau_{j\wedge j'}^{-1}$ *for* $(j, j') \in \{1, \dots, J\}^2$.

Proof of Proposition 1. Keeping in mind the notations of Lemma 4, the following expansion holds

$$\Lambda_n^{-1}(x) \sum_{j=1}^J \beta_j \left(\frac{\hat{\bar{F}}_n(y_{n,j}(x)|x)}{\bar{F}(y_{n,j}(x)|x)} - 1 \right) = \frac{T_{1,n} + T_{2,n} - T_{3,n}}{\hat{g}_n(x)}, \tag{12}$$

where

$$\begin{aligned} T_{1,n} &= g(x)\Lambda_n^{-1}(x) \sum_{j=1}^J \beta_j \left(\frac{\hat{\psi}_n(y_{n,j}(x), x) - \mathbb{E}(\hat{\psi}_n(y_{n,j}(x), x))}{\psi(y_{n,j}(x), x)} \right) \\ T_{2,n} &= g(x)\Lambda_n^{-1}(x) \sum_{j=1}^J \beta_j \left(\frac{\mathbb{E}(\hat{\psi}_n(y_{n,j}(x), x)) - \psi(y_{n,j}(x), x)}{\psi(y_{n,j}(x), x)} \right) \\ T_{3,n} &= \left(\sum_{j=1}^J \beta_j \right) \Lambda_n^{-1}(x) \left(\hat{g}_n(x) - g(x) \right). \end{aligned}$$

Let us highlight that assumptions $nh^p \bar{F}(y_n(x)|x)\omega_n^2(x) \to 0$ and $nh^p \bar{F}(y_n(x)|x) \to \infty$ imply that $\omega_n(x) \to 0$ as $n \to \infty$. Thus, from Lemma 4(ii), the random term $T_{1,n}$ can be rewritten as

$$T_{1,n} = g(x)\|K\|_2 \sqrt{\beta^t V\beta}\,\xi_n, \tag{13}$$

where ξ_n converges to a standard Gaussian random variable. The nonrandom term $T_{2,n}$ is controlled with Lemma 4(i):

$$T_{2,n} = O(\Lambda_n^{-1}(x)(h + \Delta(y_n(x), x)) = O\left((nh^p \bar{F}(y_n(x)|x))^{1/2}(h \vee \omega_n(x))\right) = o(1). \quad (14)$$

Finally, $T_{3,n}$ is a classical term in kernel density estimation, which can be bounded by Lemma 3:

$$\begin{aligned} T_{3,n} &= O(h\Lambda_n^{-1}(x)) + O_P(\Lambda_n^{-1}(x)(nh^p)^{-1/2}) \\ &= O\left(nh^{p+2}\bar{F}(y_n(x)|x)\right)^{1/2} + O_P(\bar{F}(y_n(x)|x))^{1/2} = o_P(1). \quad (15) \end{aligned}$$

Collecting (12)–(15), it follows that

$$\hat{g}_n(x)\Lambda_n^{-1}(x)\sum_{j=1}^J \beta_j \left(\frac{\hat{\bar{F}}_n(y_{n,j}(x)|x)}{\bar{F}(y_{n,j}(x)|x)} - 1\right) = g(x)\|K\|_2\sqrt{\beta^t V \beta}\xi_n + o_P(1).$$

Finally, $\hat{g}_n(x) \xrightarrow{P} g(x)$ yields

$$\sqrt{nh^p\bar{F}(y_n(x)|x)}\sum_{j=1}^J \beta_j \left(\frac{\hat{\bar{F}}_n(y_{n,j}(x)|x)}{\bar{F}(y_{n,j}(x)|x)} - 1\right) = \|K\|_2\sqrt{\frac{\beta^t V\beta}{g(x)}}\xi_n + o_P(1)$$

and the result is proved. ■

The last lemma establishes that $K_{\hat{\gamma}_n(x)}(r_n)$ inherits from the convergence properties of $\hat{\gamma}_n(x)$.

Lemma 5. *Suppose* $\xi_n^{(\gamma)}(x) := \Lambda_n^{-1}(\hat{\gamma}_n(x) - \gamma(x)) = O_{\mathbb{P}}(1)$, *where* $\Lambda_n \to 0$. *Let* $r_n \geq 1$ *or* $r_n \leq 1$ *such that* $\Lambda_n \log(r_n) \to 0$. *Then,*

$$\Lambda_n^{-1}\left(\frac{K_{\hat{\gamma}_n(x)}(r_n) - K_{\gamma(x)}(r_n)}{K'_{\gamma(x)}(r_n)}\right) = \xi_n^{(\gamma)}(x)(1 + o_{\mathbb{P}}(1)).$$

Proof. Since $\hat{\gamma}_n(x) \xrightarrow{P} \gamma(x)$, a first order Taylor expansion yields

$$K_{\hat{\gamma}_n(x)}(r_n) = K_{\gamma(x)}(r_n) + \Lambda_n\xi_n^{(\gamma)}(x)K'_{\tilde{\gamma}_n(x)}(r_n),$$

where $\tilde{\gamma}_n(x) = \gamma(x) + \Theta_n\Lambda_n\xi_n^{(\gamma)}(x)$ with $\Theta_n \in (0, 1)$. As a consequence

$$\begin{aligned} \Lambda_n^{-1}\left(\frac{K_{\hat{\gamma}_n(x)}(r_n) - K_{\gamma(x)}(r_n)}{K'_{\gamma(x)}(r_n)}\right) &= \xi_n^{(\gamma)}(x)\frac{K'_{\tilde{\gamma}_n(x)}(r_n)}{K'_{\gamma(x)}(r_n)} \\ &= \xi_n^{(\gamma)}(x)\left(1 + \frac{\int_1^{r_n}(s^{\tilde{\gamma}_n(x)-\gamma(x)} - 1)s^{\gamma(x)-1}\log(s)ds}{\int_1^{r_n} s^{\gamma(x)-1}\log(s)ds}\right). \end{aligned}$$

Suppose for instance $r_n \geq 1$. The assumptions yield $(\log r_n)(\tilde{\gamma}_n(x) - \gamma(x)) \xrightarrow{P} 0$ and thus, for n large enough, with high probability,

$$\sup_{s\in[1,r_n]} |(s^{\tilde{\gamma}_n(x)-\gamma(x)} - 1)| \leq 2(\log r_n)|\tilde{\gamma}_n(x) - \gamma(x)| = o_{\mathbb{P}}(1).$$

As a conclusion,

$$\Lambda_n^{-1}\left(\frac{K_{\hat{\gamma}_n(x)}(r_n) - K_{\gamma(x)}(r_n)}{K'_{\gamma(x)}(r_n)}\right) = \xi^{(\gamma)}(x)(1 + o_{\mathbb{P}}(1))$$

and the result is proved. The case $r_n \leq 1$ is easily deduced since $K_{\gamma(x)}(1/r_n) = -K_{-\gamma(x)}(r_n)$ and $K'_{\gamma(x)}(1/r_n) = K'_{-\gamma(x)}(r_n)$. ■

6.2 Proofs of main results

Proof of Theorem 1. Let us introduce $v_n = (nh^p\alpha_n^{-1})^{1/2}$, $\sigma_n(x) = (v_n f(q(\alpha_n|x)|x))^{-1}$ and, for all $j = 1, \dots, J$,

$$\begin{aligned} W_{n,j}(x) &= v_n \left(\hat{\bar{F}}_n(q(\tau_j\alpha_n|x) + \sigma_n(x)z_j|x) - \bar{F}(q(\tau_j\alpha_n|x) + \sigma_n(x)z_j|x) \right) \\ a_{n,j}(x) &= v_n \left(\tau_j\alpha_n - \bar{F}(q(\alpha_{n,j}|x) + \sigma_n(x)z_j|x) \right) \end{aligned}$$

and $z_j \in \mathbb{R}$. We examine the asymptotic behavior of J-variate function defined by

$$\Phi_n(z_1, \dots, z_J) = \mathbb{P}\left(\bigcap_{j=1}^{J} \{\sigma_n^{-1}(x)(\hat{q}_n(\tau_j\alpha_n|x) - q(\tau_j\alpha_n|x)) \le z_j\} \right) = \mathbb{P}\left(\bigcap_{j=1}^{J} \{W_{n,j}(x) \le a_{n,j}(x)\} \right).$$

Let us first focus on the nonrandom term $a_{n,j}(x)$. For each $j \in \{1, \dots, J\}$ there exists $\theta_{n,j} \in (0,1)$ such that

$$a_{n,j}(x) = v_n\sigma_n(x)z_j f(q_{n,j}(x)|x) = z_j \frac{f(q_{n,j}(x)|x)}{f(q(\alpha_n|x)|x)},$$

where

$$\begin{aligned} q_{n,j}(x) &= q(\tau_j\alpha_n|x) + \theta_{n,j}\sigma_n(x)z_j \\ &= q(\tau_j\alpha_n|x) + \theta_{n,j}\frac{z_j}{\tau_j}(nh^p\alpha_n)^{-1/2}\frac{\tau_j\alpha_n}{f(q(\tau_j\alpha_n|x)|x)}\frac{f(q(\tau_j\alpha_n|x)|x)}{f(q(\alpha_n|x)|x)} \\ &= q(\tau_j\alpha_n|x) + \theta_{n,j}z_j\tau_j^{\gamma(x)}(nh^p\alpha_n)^{-1/2}\frac{\tau_j\alpha_n}{f(q(\tau_j\alpha_n|x)|x)}(1 + o(1)), \end{aligned}$$

since $y \mapsto f(q(y|x)|x)$ is regularly varying at 0 with index $\gamma(x) + 1$, see [18, Corollary 1.1.10, eq. 1.1.33]. Now, in view of [18, Theorem 1.2.6] and [18, Remark 1.2.7], a possible choice of the auxiliary function is

$$a(t|x) = \frac{\bar{F}(t|x)}{f(t|x)}(1 + o(1)), \tag{16}$$

leading to

$$q_{n,j}(x) = q(\tau_j\alpha_n|x) + \theta_{n,j}z_j\tau_j^{\gamma(x)}(nh^p\alpha_n)^{-1/2}a(q(\tau_j\alpha_n|x)|x)(1 + o(1)).$$

Applying Lemma 2 with $z_n(x) = q(\tau_j\alpha_n|x)$, $t_n(x) = \theta_{n,j}z_j\tau_j^{\gamma(x)}(nh^p\alpha_n)^{-1/2}(1+o(1))$ and $t_0(x) = 0$ yields

$$\frac{\bar{F}(q_{n,j}(x)|x)}{\tau_j\alpha_n} \to K_{\gamma(x)}^{-1}(0) = 1$$

as $n \to \infty$. Recalling that $y \mapsto f(q(y|x)|x)$ is regularly varying, we have

$$\frac{f(q_{n,j}(x)|x)}{f(q(\alpha_n|x)|x)} \to \tau_j^{\gamma(x)+1}$$

as $n \to \infty$ and therefore

$$a_{n,j}(x) = z_j\tau_j^{\gamma(x)+1}(1 + o(1)),\ j = 1, \dots, J. \tag{17}$$

Let us now turn to the random term $W_{n,j}(x)$. Let us define $z_{n,j}(x) = q(\tau_j\alpha_n|x) + \sigma_n(x)z_j$ for $j = 1, \dots, J$, $y_n(x) = q(\alpha_n|x)$, and consider the expansion

$$\frac{z_{n,j}(x) - y_n(x)}{a(y_n(x)|x)} = \frac{q(\tau_j\alpha_n|x) - q(\alpha_n|x)}{a(q(\alpha_n|x)|x)} + \frac{\sigma_n(x)z_j}{a(q(\alpha_n|x)|x)}.$$

From (4), we have

$$\lim_{n\to\infty} \frac{q(\tau_j\alpha_n|x) - q(\alpha_n|x)}{a(q(\alpha_n|x)|x)} = K_{\gamma(x)}(1/\tau_j),$$

and

$$\lim_{n\to\infty} \frac{\sigma_n(x) z_j}{a(q(\alpha_n|x)|x)} = 0,$$

leading to $z_{n,j}(x) = y_n(x) + K_{\gamma(x)}(1/\tau_j)a(y_n(x)|x)(1+o(1))$. Introducing $\beta_{n,j}(x) = \bar{F}(y_{n,j}(x)|x)$, the oscillation $\omega_n(x)$ can be rewritten as

$$\omega_n(x) = \max_{j=1,\dots,J} \sup_{x'\in B(x,h)} \left| \frac{\bar{F}(q(\beta_{n,j}(x)|x)|x')}{\beta_{n,j}(x)} - 1 \right|.$$

For all $\kappa \in (0, \tau_J)$ and $j = 1, \dots, J$, we eventually have $z_{n,j}(x) \in [y_n(x), z_n(x)]$ where $z_n(x) := y_n(x) + K_{\gamma(x)}(2/\kappa)a(y_n(x)|x)$ and thus $\beta_{n,j}(x) \in [\bar{F}(z_n(x)|x), \alpha_n]$ eventually. Now, Lemma 2 implies that $\bar{F}(z_n(x)|x)/\alpha_n \to \kappa/2$ as $n \to \infty$ and thus, for n large enough, $\beta_{n,j}(x) \in [\kappa\alpha_n, \alpha_n]$. Consequently, $\omega_n(x) \le \Delta_\kappa(\alpha_n, x)$. Applying Proposition 1 and Lemma 2 yields

$$W_{n,j}(x) = \frac{\bar{F}(z_{n,j}(x)|x)}{\alpha_n}\xi_{n,j} = \tau_j \xi_{n,j}(1+o(1))$$

where $\xi_n = (\xi_{n,1}, \dots, \xi_{n,j})^t$ converges to a centered Gaussian random vector with covariance matrix $\|K\|_2^2/g(x)V$. Taking into account of (17), the results follows. ■

Proof of Corollary 1. Let us remark that, from (16),

$$\begin{aligned}
&\left\{ f(q(\alpha_n|x)|x)\sqrt{nh^p\alpha_n^{-1}}\,(\hat{q}_n(\tau_j\alpha_n|x) - q(\tau_j\alpha_n|x)) \right\}_{j=1,\dots,J} \\
=&\ \left\{ q(\alpha_n|x)\frac{f(q(\alpha_n|x)|x)}{\alpha_n}\frac{q(\tau_j\alpha_n|x)}{q(\alpha_n|x)}\sqrt{nh^p\alpha_n}\left(\frac{\hat{q}_n(\tau_j\alpha_n|x)}{q(\tau_j\alpha_n|x)} - 1\right) \right\}_{j=1,\dots,J} \\
=&\ \left\{ \frac{q(\alpha_n|x)}{a(q(\alpha_n|x)|x)}\frac{q(\tau_j\alpha_n|x)}{q(\alpha_n|x)}\sqrt{nh^p\alpha_n}\left(\frac{\hat{q}_n(\tau_j\alpha_n|x)}{q(\tau_j\alpha_n|x)} - 1\right) \right\}_{j=1,\dots,J}(1+o(1)) \\
=&\ \left\{ \frac{q(\alpha_n|x)}{a(q(\alpha_n|x)|x)}\tau_j^{-(\gamma(x)\vee 0)}\sqrt{nh^p\alpha_n}\left(\frac{\hat{q}_n(\tau_j\alpha_n|x)}{q(\tau_j\alpha_n|x)} - 1\right) \right\}_{j=1,\dots,J}(1+o(1)),
\end{aligned}$$

in view of Lemma 1. The result follows from Theorem 1. ■

Proof of Theorem 2. By definition,

$$q_n(\beta_n|x) = q_n(\alpha_n|x) + (K_{\gamma(x)}(\alpha_n/\beta_n) + b(\beta_n/\alpha_n, \alpha_n))a(q(\alpha_n|x)|x)$$

and thus, the following expansion can be easily established:

$$\begin{aligned}
\Lambda_n^{-1}\left(\frac{\tilde{q}_n(\beta_n|x) - q(\beta_n|x)}{a(q(\alpha_n|x)|x)K'_{\gamma(x)}(\alpha_n/\beta_n)}\right) &= \Lambda_n^{-1}\left(\frac{\hat{q}_n(\alpha_n|x) - q(\alpha_n|x)}{a(q(\alpha_n|x)|x)K'_{\gamma(x)}(\alpha_n/\beta_n)}\right) \\
&+ \Lambda_n^{-1}\left(\frac{K_{\hat{\gamma}_n(x)}(\alpha_n/\beta_n) - K_{\gamma(x)}(\alpha_n/\beta_n)}{K'_{\gamma(x)}(\alpha_n/\beta_n)}\right)\frac{\hat{a}_n(x)}{a(q(\alpha_n|x)|x)} \\
&+ \Lambda_n^{-1}\frac{K_{\gamma(x)}(\alpha_n/\beta_n)}{K'_{\gamma(x)}(\alpha_n/\beta_n)}\left(\frac{\hat{a}_n(x)}{a(q(\alpha_n|x)|x)} - 1\right) \\
&+ \Lambda_n^{-1}\frac{b(\beta_n/\alpha_n, \alpha_n)}{K'_{\gamma(x)}(\alpha_n/\beta_n)} \\
&=: T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

Introducing

$$(\xi_n^{(\gamma)}(x), \xi_n^{(a)}(x), \xi_n^{(q)}(x)) := \Lambda_n^{-1} \left(\hat{\gamma}_n(x) - \gamma(x), \frac{\hat{a}_n(x)}{a(q(\alpha_n|x)|x)} - 1, \frac{\hat{q}_n(\alpha_n|x) - q(\alpha_n|x)}{a(q(\alpha_n|x)|x)} \right),$$

from and remarking that, when $u \to \infty$,

$$K_z'(u) = (1 + o(1)) \left| \begin{array}{ll} \frac{1}{z^2} & \text{if } z < 0, \\ \frac{\log^2(u)}{2} & \text{if } z = 0, \\ \frac{u^z \log(u)}{z} & \text{if } z > 0, \end{array} \right. \tag{18}$$

the first term can be rewritten as

$$T_{n,1} = \frac{\xi_n^{(q)}(x)}{K'_{\gamma(x)}(\alpha_n/\beta_n)} = (\gamma(x) \wedge 0)^2 \xi_n^{(q)}(x)(1 + o_{\mathbb{P}}(1)). \tag{19}$$

Second, $\Lambda_n \to 0$ and (7) entail $\hat{a}_n(x)/a(q(\alpha_n|x)|x) \xrightarrow{P} 1$ and thus

$$T_{n,2} = \Lambda_n^{-1} \left(\frac{K_{\hat{\gamma}_n(x)}(\alpha_n/\beta_n) - K_{\gamma(x)}(\alpha_n/\beta_n)}{K'_{\gamma(x)}(\alpha_n/\beta_n)} \right) (1 + o_{\mathbb{P}}(1)) = \xi_n^{(\gamma)}(x)(1 + o_{\mathbb{P}}(1)), \tag{20}$$

from Lemma 5. From (7), (18), and in view of

$$K_z(u) = (1 + o(1)) \left| \begin{array}{ll} -\frac{1}{z} & \text{if } z < 0, \\ \log(u) & \text{if } z = 0, \\ \frac{u^z}{z} & \text{if } z > 0. \end{array} \right.$$

the third term can be rewritten as

$$T_{n,3} = \xi_n^{(a)}(x) \frac{K_{\gamma(x)}(\alpha_n/\beta_n)}{K'_{\gamma(x)}(\alpha_n/\beta_n)} = -(\gamma(x) \wedge 0)\xi_n^{(a)}(x)(1 + o_{\mathbb{P}}(1)). \tag{21}$$

Finally, $T_{n,4} = o_{\mathbb{P}}(1)$ by assumption and the conclusion follows from (7), (19), (20) and (21). ■

Proof of Theorem 3. The proof consists in deriving asymptotic expansions for the three considered random variables. i) Let us first introduce

$$\gamma_{n,j}(x) = \frac{1}{\log r} \log \left(\frac{\hat{q}_n(\tau_j \alpha_n|x) - \hat{q}_n(\tau_{j+1}\alpha_n|x)}{\hat{q}_n(\tau_{j+1}\alpha_n|x) - \hat{q}_n(\tau_{j+2}\alpha_n|x)} \right) \tag{22}$$

such that $\hat{\gamma}_n^{\text{RP}}(x) = \sum_{j=1}^{J-2} \pi_j \gamma_{n,j}(x)$. From Theorem 1 and in view of (4), we have, for all $j = 1, \ldots, J$,

$$\hat{q}_n(\tau_j \alpha_n|x) = q(\alpha_n|x) + a(q(\alpha_n|x)|x)(K_{\gamma(x)}(1/\tau_j) + b(\tau_j, \alpha_n|x)) + \sigma_n(x)\xi_{j,n},$$

with $\sigma_n^{-1}(x) = f(q(\alpha_n|x)|x)\sqrt{nh^p\alpha_n^{-1}}$ and where the random vector $\xi_n = (\xi_{j,n})_{j=1,\ldots,J}$ is asymptotically Gaussian, centered, with covariance matrix $\|K\|_2^2/g(x)\Sigma(x)$. Introducing

$$\begin{aligned} \eta_n(x) &:= \max_{j=1,\ldots,J} |b(\tau_j, \alpha_n|x)| \\ \varepsilon_n &:= \sigma_n(x)/a(q(\alpha_n|x)|x) = (nh^p\alpha_n)^{-1/2}(1 + o(1)), \end{aligned}$$

see [18], it follows that

$$\begin{aligned}\frac{\hat{q}_n(\tau_j\alpha_n|x)-\hat{q}_n(\tau_{j+1}\alpha_n|x)}{a(q(\alpha_n|x)|x)} &= \varepsilon_n(\xi_{j,n}-\xi_{j+1,n}) \\ &+ K_{\gamma(x)}(1/\tau_j)-K_{\gamma(x)}(1/\tau_{j+1})+b(\tau_j,\alpha_n|x)-b(\tau_{j+1},\alpha_n|x) \\ &= \varepsilon_n(\xi_{j,n}-\xi_{j+1,n})+K_{\gamma(x)}(r)r^{-\gamma(x)j}+O(\eta_n(x)). \qquad (23)\end{aligned}$$

Replacing in (22), we obtain

$$(\log r)\gamma_{n,j}(x)=\log\left(\frac{\varepsilon_n(\xi_{j,n}-\xi_{j+1,n})+K_{\gamma(x)}(r)r^{-\gamma(x)j}+O(\eta_n(x))}{\varepsilon_n(\xi_{j+1,n}-\xi_{j+2,n})+K_{\gamma(x)}(r)r^{-\gamma(x)(j+1)}+O(\eta_n(x))}\right),$$

or equivalently,

$$\begin{aligned}(\log r)(\gamma_{n,j}(x)-\gamma(x)) &= \log\left(1+\frac{\varepsilon_n(\xi_{j,n}-\xi_{j+1,n})r^{\gamma(x)j}}{K_{\gamma(x)}(r)}+O(\eta_n(x))\right) \\ &- \log\left(1+\frac{\varepsilon_n(\xi_{j+1,n}-\xi_{j+2,n})r^{\gamma(x)(j+1)}}{K_{\gamma(x)}(r)}+O(\eta_n(x))\right)\end{aligned}$$

A first order Taylor expansion yields

$$(\log r)\varepsilon_n^{-1}(\gamma_{n,j}(x)-\gamma(x))=\frac{r^{\gamma(x)j}}{K_{\gamma(x)}(r)}\left(\xi_{j,n}-(1+r^{\gamma(x)})\xi_{j+1,n}+r^{\gamma(x)}\xi_{j+2,n}\right)+O(\varepsilon_n^{-1}\eta_n(x))+o_{\mathbb{P}}(1)$$

and thus, under the assumption $(nh^p\alpha_n)^{1/2}\eta_n(x)\to 0$ as $n\to\infty$,

$$\sqrt{nh^p\alpha_n}(\hat{\gamma}_n^{\mathrm{RP}}(x)-\gamma(x))=\frac{1}{(\log r)K_{\gamma(x)}(r)}\sum_{j=1}^{J-2}\pi_j r^{\gamma(x)j}\left(\xi_{j,n}-(1+r^{\gamma(x)})\xi_{j+1,n}+r^{\gamma(x)}\xi_{j+2,n}\right)+o_{\mathbb{P}}(1).$$

Defining for the sake of simplicity $\pi_{-1}=\pi_0=\pi_{J-1}=\pi_J=0$, $\beta_0^{(\gamma)}=\frac{1}{\log r}$, $\beta_1^{(\gamma)}=-\frac{1+r^{-\gamma(x)}}{\log(r)}$ and $\beta_2^{(\gamma)}=\frac{r^{-\gamma(x)}}{\log(r)}$, we end up with

$$\begin{aligned}\xi_n^{(\gamma)}(x) &:= \sqrt{nh^p\alpha_n}(\hat{\gamma}_n^{\mathrm{RP}}(x)-\gamma(x)) \\ &= \frac{1}{K_{\gamma(x)}(r)}\sum_{j=1}^{J} r^{\gamma(x)j}\left(\beta_0^{(\gamma)}\pi_j+\beta_1^{(\gamma)}\pi_{j-1}+\beta_2^{(\gamma)}\pi_{j-2}\right)\xi_{j,n}+o_{\mathbb{P}}(1). \qquad (24)\end{aligned}$$

ii) Second, let us now consider

$$a_{n,j}(x)=\frac{r^{\hat{\gamma}_n^{\mathrm{RP}}(x)j}(\hat{q}_n(\tau_j\alpha_n|x)-\hat{q}_n(\tau_{j+1}\alpha_n|x))}{K_{\hat{\gamma}_n^{\mathrm{RP}}(x)}(r)}$$

such that $\hat{a}_n(x)=\sum_{j=1}^{J-2}\pi_j a_{n,j}(x)$. From (23), it follows that, for all $j=1,\dots,J$,

$$\frac{a_{n,j}(x)}{a(q(\alpha_n|x)|x)}=\frac{r^{\hat{\gamma}_n^{\mathrm{RP}}(x)j}}{K_{\hat{\gamma}_n^{\mathrm{RP}}(x)}(r)}\left(\varepsilon_n(\xi_{j,n}-\xi_{j+1,n})+K_{\gamma(x)}(r)r^{-\gamma(x)j}+O(\eta_n(x))\right).$$

Remarking that $\hat{\gamma}_n^{\mathrm{RP}}(x)=\gamma(x)+(nh^p\alpha_n)^{-1/2}\xi_n^{(\gamma)}(x)$, Lemma 5 yields

$$\frac{a_{n,j}(x)}{a(q(\alpha_n|x)|x)}=\frac{1+\frac{r^{\gamma(x)j}}{K_{\gamma(x)}(r)}\varepsilon_n(\xi_{j,n}-\xi_{j+1,n})+O(\eta_n(x))}{1+\frac{K'_{\gamma(x)}(r)}{K_{\gamma(x)}(r)}(nh^p\alpha_n)^{-1/2}\xi_n^{(\gamma)}(x)(1+o_{\mathbb{P}}(1))}\exp\left(\xi_n^{(\gamma)}(x)j\log(r)(nh^p\alpha_n)^{-1/2}\right).$$

A first order Taylor expansion thus gives

$$\begin{aligned}\sqrt{nh^p\alpha_n}\left(\frac{a_{n,j}(x)}{a(q(\alpha_n|x)|x)}-1\right) &= \xi_n^{(\gamma)}(x)\left(j\log(r)-\frac{K'_{\gamma(x)}(r)}{K_{\gamma(x)}(r)}\right)+\frac{r^{\gamma(x)j}}{K_{\gamma(x)}(r)}(\xi_{j,n}-\xi_{j+1,n})\\ &+ O\left(\sqrt{nh^p\alpha_n}\eta_n(x)\right)+o_{\mathbb{P}}(1).\end{aligned}$$

Recalling that $\pi_{-1}=\pi_0=\pi_{J-1}=\pi_J=0$ and introducing

$$\begin{aligned}\mathbb{E}(\pi) &= \textstyle\sum_{j=1}^J j\pi_j, & \beta_1^{(a)} &= -r^{-\gamma(x)}-(r^{-\gamma(x)}+1)\left(\mathbb{E}(\pi)-\frac{K'_{\gamma(x)}(r)}{\log(r)K_{\gamma(x)}(r)}\right),\\ \beta_0^{(a)} &= 1+\mathbb{E}(\pi)-\frac{K'_{\gamma(x)}(r)}{\log(r)K_{\gamma(x)}(r)}, & \beta_2^{(a)} &= r^{-\gamma(x)}\left(\mathbb{E}(\pi)-\frac{K'_{\gamma(x)}(r)}{\log(r)K_{\gamma(x)}(r)}\right)\end{aligned}$$

it follows that

$$\begin{aligned}\xi_n^{(a)}(x) &:= \sqrt{nh^p\alpha_n}\left(\frac{a_n(x)}{a(q(\alpha_n|x)|x)}-1\right)\\ &= \left(\mathbb{E}(\pi)\log(r)-\frac{K'_{\gamma(x)}(r)}{K_{\gamma(x)}(r)}\right)\xi_n^{(\gamma)(x)}+\frac{1}{K_{\gamma(x)}(r)}\sum_{j=1}^J r^{\gamma(x)j}(\pi_j-r^{-\gamma(x)}\pi_{j-1})\xi_{j,n}+o_{\mathbb{P}}(1)\\ &= \frac{1}{K_{\gamma(x)}(r)}\sum_{j=1}^J r^{\gamma(x)j}\left(\beta_0^{(a)}\pi_j+\beta_1^{(a)}\pi_{j-1}+\beta_2^{(a)}\pi_{j-2}\right)\xi_{j,n}+o_{\mathbb{P}}(1) \qquad (25)\end{aligned}$$

in view of (24). iii) Third, Corollary 1 states that

$$\xi_{1,n}=\frac{\sqrt{nh^p\alpha_n}}{a(q(\alpha_n|x)|x)}\left(\hat{q}_n(\alpha_n|x)-q(\alpha_n|x)\right) \qquad (26)$$

is asymptotically Gaussian. Finally, collecting (24), (25) and (26),

$$(\xi_n^{(\gamma)}(x),\xi_n^{(a)}(x),\xi_{1,n})^t=\frac{1}{K_{\gamma(x)}(r)}A(x)\xi_n+o_{\mathbb{P}}(1),$$

where $A(x)$ is the $3\times J$ matrix defined by

$$\begin{aligned}A_{1,j}(x) &= r^{\gamma(x)j}\left(\beta_0^{(\gamma)}\pi_j+\beta_1^{(\gamma)}\pi_{j-1}+\beta_2^{(\gamma)}\pi_{j-2}\right)\\ A_{2,j}(x) &= r^{\gamma(x)j}\left(\beta_0^{(a)}\pi_j+\beta_1^{(a)}\pi_{j-1}+\beta_2^{(a)}\pi_{j-2}\right)\\ A_{3,j}(x) &= K_{\gamma(x)}(r)\mathbb{I}\{j=1\},\end{aligned}$$

for all $j=1,\dots,J$. It is thus clear that the random vector $(\xi_n^{(\gamma)}(x),\xi_n^{(a)}(x),\xi_{1,n})^t$ converges in distribution to a centered Gaussian random vector with covariance matrix

$$\frac{\|K\|_2^2}{g(x)K^2_{\gamma(x)}(r)}A(x)\Sigma(x)A^t(x)=:S(x). \qquad (27)$$

■

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