hal-00630204, version 1 - 7 Oct 2011

Very high-order finite volume method for one-dimensional convection diffusion problems

STÉPHANE CLAIN
Universidade do Minho
Dep. Matemática e Aplicações
Azurém, 4800-058 Guimarães
PORTUGAL
clain@math.uminho.pt

STEVEN DIOT
Université Paul Sabatier
Inst. Mathématiques de Toulouse
118 Narbonne, F-31062 Toulouse
FRANCE
steven.diot@math.univ-toulouse.fr

RAPHAEL LOUBÈRE
Université Paul Sabatier
Inst. Mathématiques de Toulouse
118 Narbonne, F-31062 Toulouse
FRANCE
raphael.loubere@math.univ-toulouse.fr

GASPAR MACHADO
Universidade do Minho
Dep. Matemática e Aplicações
Azurém, 4800-058 Guimarães
PORTUGAL
gjm@math.uminho.pt

RUI RALHA
Universidade do Minho
Dep. Matemática e Aplicações
Gualtar, 4710-057 Braga
PORTUGAL
r_ralha@math.uminho.pt

RUI M. S. PEREIRA
Universidade do Minho
Dep. Matemática e Aplicações
Azurém, 4800-058 Guimarães
PORTUGAL
rmp@math.uminho.pt

Abstract: We propose a new finite volume method to provide very high-order accuracy for the convection diffusion problem. The main tool is a polynomial reconstruction based on the mean-value to provide the best order. Numerical examples are proposed to show the method efficiency.

Key–Words: Finite Volume, very high-order, convection-diffusion, polynomial reconstruction.

1 Introduction

Efficient numerical schemes to solve convection-diffusion problems are a constant challenge due to the wide range of problems which concern the coupling of the two major physical phenomena. Finite differences and finite elements are very popular to produce numerical approximations ([1, 2, 3]) and a lot of academic or commercial codes are based on such techniques.

In the last decade, the finite volume method appears to be an interesting alternative for several reasons, the simplicity (one information per cell), the built-in conservative property, the capacity to handle unstructured and non conformal meshes. Important developments have been realized ([4, 5, 6]) in this way but a serious drawback of the finite volume method is the large amount of numerical viscosity and the weak convergence rate (at most second-order convergence). The paper is devoted to a new class of finite volume schemes for convection-diffusion problem able to reach the sixth-order accuracy. We present the method for the simple one-dimensional case and restrict to the steady-state case for the sake of consistency. The main tool is an accurate local polynomial reconstruction based on the mean values approximations in place of the pointwise values approximations.

The rest of the paper is as follows. The second section recalls the classical finite volume scheme for convection diffusion problem (namely the Patankar method). Then, we introduce the polynomial reconstruction and the high-order schemes in section three. The last section is devoted to some numerical tests to show the scheme capacity to provide sixth-order accuracy.

2 Classical finite volume schemes

For the sake of simplicity, we consider the one-dimensional convection-diffusion problem in domain $\Omega :=]0,1[$ with Dirichlet boundary conditions:

$$-(au')' + (vu)' = f \quad \text{on } \Omega$$
$$u(0) = u_{\text{lf}}, \quad u(1) = u_{\text{rg}},$$

where we assume that a and v are regular functions on $\overline{\Omega}$ with $a(x) \geq \alpha > 0$ for all $x \in \Omega$.

We denote by $\mathcal{T}_h$ a mesh of Ω constituted of cells $K_i := [x_{i-1/2}, x_{i+1/2}]$, $i := 1, \ldots, I$, of centroid c_i, where $x_{1/2} := 0$ and $x_{i+1/2} := x_{i-1/2} + h_i$, and set $h := \min_{i=1}^{I} h_i$ as the characteristic length of the mesh.

In the finite volume context, u_i denotes an approximation of the mean value over cell K_i, that is

hal-00630204, version 1 - 7 Oct 2011

$u_i \approx \frac{1}{h_i}\int_{K_i} u\,\mathrm{d}x$. We recall the classical scheme for the interior cells K_i, $i := 2,\dots,I-1$:

$$-\left[\mathcal{F}^\ell_\mathsf{d}(u_i,u_{i+1};x_{i+\frac12}) - \mathcal{F}^\ell_\mathsf{d}(u_{i-1},u_i;x_{i-\frac12})\right] + \left[\mathcal{F}^\ell_\mathsf{c}(u_i,u_{i+1};x_{i+\frac12}) - \mathcal{F}^\ell_\mathsf{c}(u_{i-1},u_i;x_{i-\frac12})\right] = f_i,$$

where the diffusion and the convection fluxes write

$$\mathcal{F}^\ell_\mathsf{d}(u_i,u_{i+1};x_{i+\frac12}) := a(x_{i+\frac12})\frac{2(u_{i+1}-u_i)}{(h_i+h_{i+1})},$$
$$\mathcal{F}^\ell_\mathsf{c}(u_i,u_{i+1};x_{i+\frac12}) := [v(x_{i+\frac12})]^+u_i + [v(x_{i+\frac12})]^-u_{i+1},$$

with $[\alpha]^+ := \frac{\alpha+|\alpha|}{2}$ and $[\alpha]^- := \frac{\alpha-|\alpha|}{2}$, while f_i denotes an approximation of the mean value of f over cell K_i, that is $f_i \approx \frac{1}{h_i}\int_{K_i} f\,\mathrm{d}x$ (for the first cell and the last cell the expressions change in the classical way). We obtain the following linear system

$$A^\ell U = F + F_\mathcal{D}$$

with $U := (u_1,\dots,u_I)^t \in \mathbb{R}^I$ the unknown values, $F := (f_1,\dots,f_I)^t \in \mathbb{R}^I$ the source term and

$$F_\mathcal{D} := \left(\left\{\frac{2a(0)}{h_1} + [v(0)]^+\right\}u_{\mathrm{lf}}, 0,\dots,0, \left\{\frac{2a(1)}{h_I} + [v(1)]^-\right\}u_{\mathrm{rg}}\right)^t \in \mathbb{R}^I$$

the vector collecting the contributions deriving from the Dirichlet boundary conditions. Remark that the finite volume scheme is not equivalent to the finite element scheme or finite difference one for non-uniform meshes.

3 Very high-order schemes

Very high-order schemes are based on local polynomial reconstruction. For each cell K_i, $i := 1,\dots,I$, we denote by $\widetilde{u}_i(x) \in \mathbb{P}_\mathsf{d}$ the polynomial reconstruction on K_i, where d is the polynomial degree. The reconstruction we detail in the sequel is a linear operator $\mathcal{P}$, that from $U := (u_1,\dots,u_I)^t$ and Dirichlet conditions u_{lf}, u_{rg} provides $\widetilde{u}_i(x) \in \mathbb{P}_\mathsf{d}$.

Based on the reconstruction, we then define the new numerical fluxes. For the interior cells we have

$$\mathcal{F}^\mathsf{d}_\mathsf{d}(x_{i+\frac12}) := a(x_{i+\frac12})\frac{\widetilde{u}'_i(x_{i+\frac12}) + \widetilde{u}'_{i+1}(x_{i+\frac12})}{2},$$
$$\mathcal{F}^\mathsf{d}_\mathsf{c}(x_{i+\frac12}) := [v(x_{i+\frac12})]^+\widetilde{u}_i(x_{i+\frac12}) + [v(x_{i+\frac12})]^-\widetilde{u}_{i+1}(x_{i+\frac12}),$$

while for the first cell and the last cell we consider, for the diffusive flux the expressions

$$\mathcal{F}^\mathsf{d}_\mathsf{d}(0) := a(0)\widetilde{u}'_1(0),$$
$$\mathcal{F}^\mathsf{d}_\mathsf{d}(1) := a(1)\widetilde{u}'_I(1).$$

At last, we define the affine operator $U \to G^\mathsf{d}(U;u_{\mathrm{lf}},u_{\mathrm{rg}})$ from $\mathbb{R}^I$ into $\mathbb{R}^I$ with

$$G^\mathsf{d}_i(U;u_{\mathrm{lf}},u_{\mathrm{rg}}) := -\left[\mathcal{F}^\mathsf{d}_\mathsf{d}(u_i,u_{i+1};x_{i+\frac12}) - \mathcal{F}^\mathsf{d}_\mathsf{d}(u_{i-1},u_i;x_{i-\frac12})\right] + \left[\mathcal{F}^\mathsf{d}_\mathsf{c}(u_i,u_{i+1};x_{i+\frac12}) - \mathcal{F}^\mathsf{d}_\mathsf{c}(u_{i-1},u_i;x_{i-\frac12})\right] - f_i$$

To provide the numerical solution, one has to find the solution U of the linear problem

$$G^\mathsf{d}(U;u_{\mathrm{lf}},u_{\mathrm{rg}}) = 0_I := (0,\dots,0)^t. \qquad (1)$$

We now detail the polynomial reconstruction which is the crucial point of the numerical technique. We shall consider two different situations whether we take the boundary cells (K_1 and K_I) into account or the interior cells (K_i, $i := 2,\dots,I-1$).

The general case concerns the inner cells. To this end, for each cell K_i we associate a index subset $\nu(i)$ such that for a given polynomial degree d, we choose the nearest $\mathsf{d}+2$ cells to K_i. We then consider the polynomial function of the form

$$\widetilde{u}_i(x) := u_i + \sum_{\alpha=1}^{\mathsf{d}} \mathcal{R}^i_\alpha\left[(x-c_i)^\alpha - \frac{1}{h_i}\int_{K_i}(x-c_i)^\alpha\mathrm{d}x\right]$$

and we seek the coefficients $\mathcal{R}^i_\alpha$, $\alpha := 1,\dots,\mathsf{d}$, which minimize the functional

$$E(\mathcal{R}^i_\alpha) := \sum_{j\in\nu(i)}\left[\frac{1}{h_j}\int_{K_j}\widetilde{u}_i\,\mathrm{d}x - u_j\right]^2.$$

The important point is that we minimize the functional with respect to the mean values and not identify (as usually done in finite volume context) the mean values with pointwise values at the centroid.

For the first cell, we slightly modify the minimizing functional. We associate the index subset $\nu(1) := \{2,\dots,\mathsf{d}-1\}$ and augment the functional with the Dirichlet condition

$$E(\mathcal{R}^1_\alpha) := (\widetilde{u}_1(0) - u_{\mathrm{lf}})^2 + \sum_{j\in\nu(1)}\left[\frac{1}{h_j}\int_{K_j}\widetilde{u}_1\mathrm{d}x - u_j\right]^2.$$

The reconstruction for the last cell is analogous to the first cell.

A simple way to solve the affine problem (1) consists in evaluating the associated matrix A^{d} and the right-hand side F^{d} and then to solve the linear problem $A^{\mathsf{d}}U = F^{\mathsf{d}}$ using standard methods such as the LU method. To this end, we first compute the right-hand side setting $F^{\mathsf{d}} := G^{\mathsf{d}}(0_I; u_{\mathrm{lf}}, u_{\mathrm{rg}})$, then we evaluate each column A^{d}_j of the matrix A^{d} with

$$A^{\mathsf{d}}_j := G^{\mathsf{d}}(\mathsf{e}^j; u_{\mathrm{lf}}, u_{\mathrm{rg}}) - F^{\mathsf{d}},$$

where e^j is the canonical vector with the $j-$component equal to 1.

It should be stressed that we construct the associated full matrix A^{d} for the sake of simplicity. In matter fact, since we only focus on the numerical scheme itself, that construction provides a simple method to test the convergence rate.

4 Numerical tests

To compare the numerical approximation with the exact solution, we introduce three kind of errors, namely the consistency error, the L^∞ error of the solution and the L^∞ error of the derivative (we intentionally use the L^∞-norm since we deal with regular functions). More precisely, consistency error is given by

$$E_{\mathcal{C}}(\overline{U}) := \max_{i=1}^{I} \left| G^{\mathsf{d}}_i(\overline{U}; u_{\mathrm{lf}}, u_{\mathrm{rg}}) \right|, \qquad (2)$$

where

$$\overline{U} := (\overline{u}_1, \dots, \overline{u}_I), \quad \overline{u}_i := \frac{1}{h_i} \int_{K_i} u \, \mathrm{d}x,$$

are the exact mean values of the solution of the continuous problem. The L^∞ error between the mean values is given by

$$E_0 := \max_{i=1}^{I} |u_i - \overline{u}_i|,$$

while we define the L^∞ error for the derivative with

$$E_1 := \max_{i=1}^{I} \Big[\Big| \widetilde{u}'_i(x_{i-\frac{1}{2}}) - u'(x_{i-\frac{1}{2}}) \Big|, \Big| \widetilde{u}'_i(x_{i+\frac{1}{2}}) - u'(x_{i+\frac{1}{2}}) \Big| \Big].$$

We consider a uniform mesh of I cells and test three types of polynomial reconstructions, namely the $\mathbb{P}_1$, $\mathbb{P}_3$, and $\mathbb{P}_5$.

The first case concerns the exponential function with constant diffusion and velocity, *i.e.*, $a := 1$ and $v := 1$ leading to a Peclet number of Pe=1. We state that the solution is $\exp(x)$ and deduce that $f = 0$, $u_{\mathrm{lf}} = 1$, and $u_{\mathrm{rg}} = \mathrm{e}$.

Table 1: Example 1 — errors and rates of convergence for the exponential function using a uniform mesh with Pe=1.

	I	$E_{\mathcal{C}}$ err	$E_{\mathcal{C}}$ ord	E_0 err	E_0 ord	E_1 err	E_1 ord
low	10	7.8e-02	—	8.3e-03	—	7.4e-02	—
	20	4.2e-02	0.9	4.6e-03	0.8	3.8e-02	1
	40	2.2e-02	0.9	2.5e-03	0.9	1.9e-02	1
	80	1.1e-02	1	1.3e-03	1	9.8e-03	1
$\mathbb{P}_1$	10	1.8e-01	—	1.4e-01	—	3.2e-01	—
	20	9.1e-02	1	4.9e-02	1.5	1.5e-01	1.1
	40	4.6e-02	1	1.7e-02	1.5	7.3e-02	1.1
	80	2.3e-02	1	5.8e-03	1.5	3.5e-02	1
$\mathbb{P}_3$	10	9.7e-04	—	6.0e-05	—	4.0e-04	—
	20	1.3e-04	2.9	4.7e-06	3.7	5.5e-05	2.9
	40	1.7e-05	2.9	3.3e-07	3.9	7.2e-06	2.9
	80	2.2e-06	3	2.2e-08	3.9	9.3e-07	3
$\mathbb{P}_5$	10	4.4e-06	—	1.0e-07	—	1.4e-06	—
	20	1.6e-07	4.8	2.3e-09	5.5	5.2e-08	4.8
	40	5.3e-09	4.9	4.1e-11	5.8	1.7e-09	4.9
	80	1.7e-10	5	6.8e-13	5.9	5.6e-11	4.9

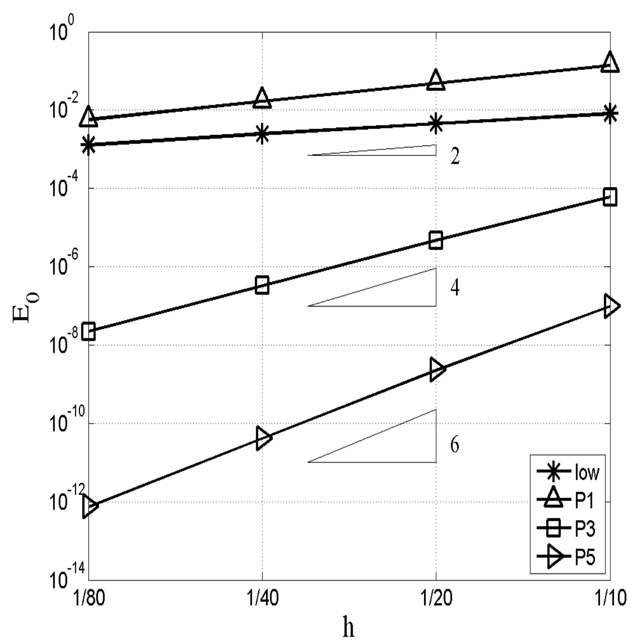

Figure 1: Example 1 — convergence curves of E_0 for the exponential function using a uniform mesh with Pe=1.

hal-00630204, version 1 - 7 Oct 2011

hal-00630204, version 1 - 7 Oct 2011

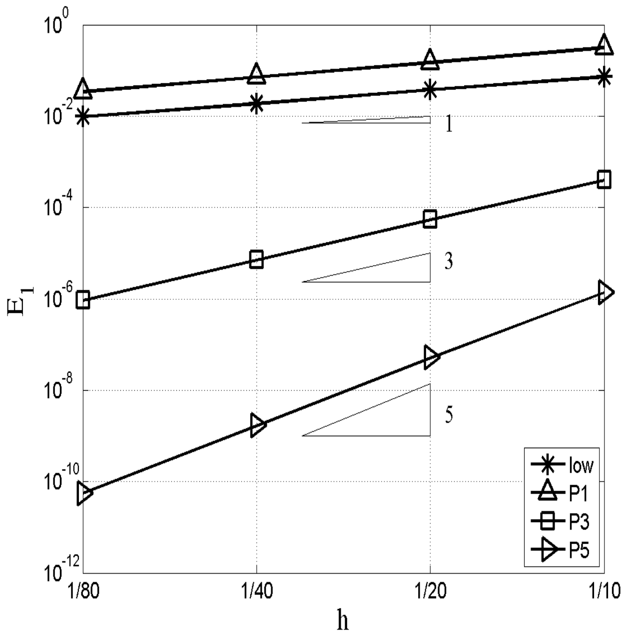

Figure 2: Example 1 — convergence curves of E_1 for the exponential function using a uniform mesh with Pe=1.

Table 1 provides the consistency error and the numerical errors both for the function and its derivative for the L^∞ norm while we plot in Figures 1 and 2 the convergence curves for the solution and its derivative. We obtain an effective $\mathsf{d}+1$-order of convergence rate for the functions and d-order convergence rate for the solution derivative when employing the reconstruction with polynomial functions in $\mathbb{P}_\mathsf{d}$. Consistency error also enjoys the convergence property since the scheme provides the optimal d-order rate (remember that the consistency error (2) contains derivatives).

The second case concerns the sine function with constant diffusion and velocity $a := 1$ and $v := 100$ such that Pe=100. The goal is to handle problems with a large Peclet number providing a convection-dominated problem. We state that the solution is $\sin(\frac{\pi x}{2})$ and deduce that $f(x) = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi x}{2}\right) + 50\pi \cos\left(\frac{\pi x}{2}\right)$, $u_{\text{lf}} = 0$, and $u_{\text{rg}} = 1$.

For a large Peclet number, the numerical schemes manage to provide the optimal order convergence, with the exception of the $\mathbb{P}_1$ case, as can be seen in Table 2 and Figures 3 and 4. For the $\mathbb{P}_1$ polynomial reconstruction, large error are derived from the conflicting outflow boundary condition since the fluid flow out at $x = 1$. A pure convection problem does

Table 2: Example 2 — errors and rates of convergence for the sine function using a uniform mesh with Pe=100.

	I	$E_\mathcal{C}$ err	ord	E_0 err	ord	E_1 err	ord
low	10	7.8e+00	—	6.5e-02	—	1.3e+00	—
	20	3.9e+00	1	2.8e-02	1.2	1.1e+00	0.2
	40	2.0e+00	1	1.1e-02	1.4	8.7e-01	0.4
	80	9.8e-01	1	5.2e-03	1.1	6.0e-01	0.5
$\mathbb{P}_1$	10	2.9e-01	—	3.9e-03	—	2.0e-01	—
	20	2.9e-02	3.3	5.2e-04	2.9	1.1e-01	0.9
	40	1.4e-02	1.1	1.3e-04	2	5.6e-02	0.9
	80	1.4e-02	0	1.8e-04	0.5	3.0e-02	0.9
$\mathbb{P}_3$	10	5.1e-03	—	5.3e-05	—	1.0e-03	—
	20	2.9e-04	4.1	2.9e-06	4.2	1.3e-04	2.9
	40	2.1e-05	3.8	1.6e-07	4.2	1.8e-05	2.9
	80	3.1e-06	2.7	1.1e-08	3.9	2.4e-06	2.9
$\mathbb{P}_5$	10	9.4e-05	—	4.2e-07	—	1.2e-05	—
	20	1.5e-06	6	6.3e-09	6.1	4.4e-07	4.8
	40	1.8e-08	6.3	9.3e-11	6.1	1.5e-08	4.9
	80	6.2e-10	4.9	1.3e-12	6.2	4.7e-10	5

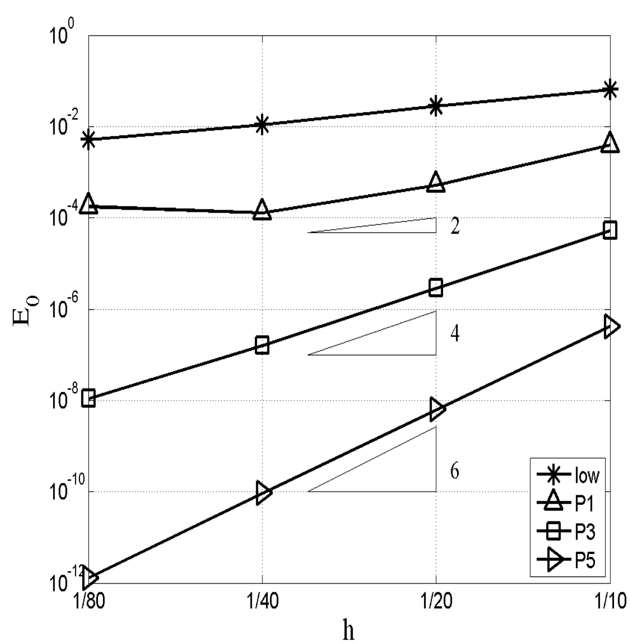

Figure 3: Example 2 — convergence curves of E_0 for the sine function using a uniform mesh with Pe=100.

hal-00630204, version 1 - 7 Oct 2011

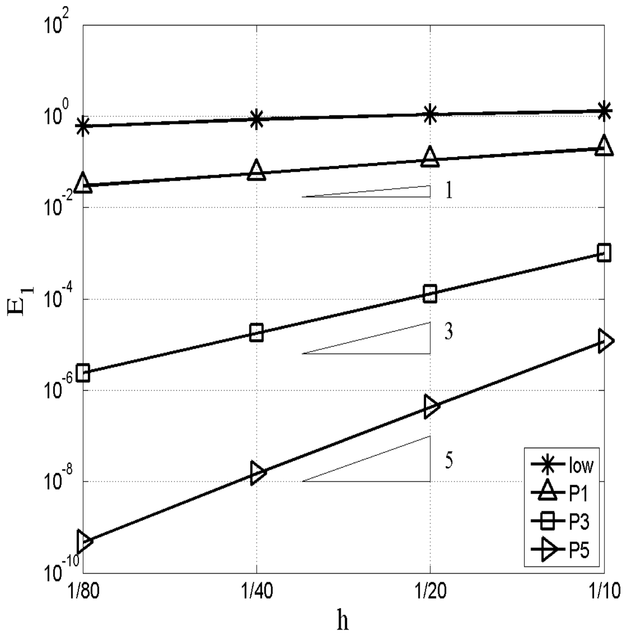

Figure 4: Example 2 — convergence curves of E_1 for the sine function using a uniform mesh with Pe=100.

not require the Dirichlet condition and we observe an ill-conditioning matrix with large Peclet number. To solve this phenomena, we intend to introduce another class of polynomial reconstruction adapted to pure convection problem as it done in ([7]).

5 Conclusions and further work

We have presented a new finite volume method for one-dimensional convection-diffusion problem which provides very high-order accuracy. Numerical simulations have been carried out to show the capacity of the method to provide a sixth-order scheme. Extensions for two-dimensional domains and the consideration of both Dirichlet and Neumann boundary conditions are under investigation. Another difficulty concerns the solution stability when dealing with rough data. A strategy based on the MOOD method ([7, 8]) is also under investigation.

Acknowledgements: This research was financed by FEDER Funds through Programa Operacional Factores de Competitividade — COMPETE and by Portuguese Funds through FCT — Fundação para a Ciência e a Tecnologia, within the Project PEst-C/MAT/UI0013/2011.

References:

[1] Z. Chen, Finite element methods and their applications Springer-Verlag, 2005.

[2] C. Grossmann, H.-G. Roos, M. Stynes, Numerical treatment of partial differential equations, Springer-Verlag, 2007.

[3] A. Tveito, R. Winther, Introduction to partial differential equations: a computational approach Springer-Verlag, 1998.

[4] L. Piar, F. Babik, R. Herbin, J.-C. Latché, A second order cell centered scheme for convection-diffusion equations on unstructured non-conforming grids, preprint available at http://hal.archives-ouvertes.fr/hal-00556911/fr/.

[5] R. Eymard, T. Gallouët, R. Herbin, The finite volume method, Handbook for Numerical Analysis, Ph. Ciarlet J.L. Lions eds., North Holland, 2000, pp. 715–1022.

[6] C. Zhongdi, X. Lifeng, A Streamline Diffusion Method for a Singularly Perturbed Conservative Convection-diffusion Problem, Proceedings of the 3rd WSEAS International Conference on Computer Engineering and Applications (CEA'09), 2009, pp. 211–215.

[7] S. Clain, S. Diot, R. Loubère, A high-order polynomial finite volume method for hyperbolic system of conservation laws with Multidimensional Optimal Order Detection (MOOD), Journal of Computational Physics, 230 (2011) 4028-4050.

[8] S. Clain, S. Diot, R. Loubère, Multi-dimensional Optimal Order Detection (MOOD) — A very high-order Finite Volume Scheme for conservation laws on unstructured meshes, Proceeding for the sixth Finite Volume and Complex Application, Springer Verlag Editor, Vol. 1, 2011, pp. 263–271.