Regular conditional distributions of max infinitely divisible processes

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Abstract

This paper is devoted to the *prediction problem* in extreme value theory. Our main result is an explicit expression of the *regular conditional distribution* of a max-stable (or max-infinitely divisible) process $\{\eta(t)\}_{t\in T}$ given observations $\{\eta(t_i)=y_i,\ 1\leq i\leq k\}$. Our starting point is the point process representation of max-infinitely divisible processes by Giné, Hahn and Vatan (1990). We carefully analyze the structure of the underlying point process, introduce the notions of *extremal function*, *sub-extremal function* and *hitting scenario* associated to the constraints and derive the associated distributions. This allows us to explicit the conditional distribution as a mixture over all hitting scenarios compatible with the conditioning constraints. This formula extends a recent related result by Wang and Stoev (2011) dealing with the case of spectrally discrete max-stable random fields. We believe this work offers new tools and perspective for prediction in extreme value theory together with numerous potential applications.

Key words: max-infinitely divisible process; max-stable process; regular conditional distribution; point process representation.
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1 Introduction

1.1 Motivations

Max-stable random fields turn out to be fundamental models for spatial extremes since they arise as the the limit of rescaled maxima. More precisely, consider the component-wise maxima

$$\eta_n(t) = \max_{1\leq i\leq n} X_i(t), \quad t\in T,$$

of independent realizations $\{X_i(t)\}_{t\in T}$, $i\geq 1$, of a random field $X=\{X(t)\}_{t\in T}$. If the random field $\eta_n=(\eta_n(t))_{t\in T}$ converges in distribution, as $n\to\infty$, under suitable affine normalization, then its limit $\eta=\{\eta(t)\}_{t\in T}$ is necessarily max-stable (see e.g. [7, 15]). Therefore, max-stable random fields play a central role in extreme value theory, just like Gaussian random fields do in the classical statistical theory based on the central limit Theorem.

Since the pioneer works by Fisher and Typett [10] and Gnedenko [12], the univariate theory of extremes is now well established with extensive studies on models, domains of attraction, parameter estimations, *etc.* (see e.g. de Haan and Fereira [7] and the references therein). The last decades have seen the quick development of multivariate and spatial extreme value theory: the emphasis is put on the characterization, modeling and estimation of the dependence structure of multivariate extremes. Among many others, the reader should refer to the excellent monographs [2, 7, 9, 15] and the reference therein.

In this framework, the *prediction problem* arise as an important and long-standing challenge in extreme value theory. Suppose that one already has a suitable max-stable model for the dependence structure of a random field $\eta=\{\eta(t)\}_{t\in T}$ and that the field is observed at some locations $t_1,\ldots,t_k\in T$. How can we take benefit from these observations and predict the random field η at other locations? We are naturally lead to consider the *conditional distribution* of $(\eta(t))_{t\in T}$ given the observations $\{\eta(t_i)=y_i,\ 1\leq i\leq k\}$. A formal definition of the notion of regular conditional distribution is deferred to the Appendix A.2.

In the classical Gaussian framework, i.e., if η is a Gaussian random field, it is well known that the corresponding conditional distribution remains Gaussian and simple formulas give the conditional mean and covariance structure. This theory is strongly linked with the theory of Hilbert spaces: the conditioned random field can be obtained as the L^2-projection of the initial random field η onto a suitable Gaussian subspace. In extreme value theory, the prediction problem turns out to be difficult. A first approach by Davis and Resncik [5, 6] is based on a L^1-metric between max-stable variables and on a kind of projection onto max-stable spaces. To some extent, this work mimics the corresponding L^2-theory for Gaussian spaces. However, unlike the Gaussian case, there is no clear relationship between the predictor obtained by projection onto the max-stable space generated by the variables $\{\eta(t_i),\ 1 \leq i \leq k\}$ and the conditional distributions of η with respect to these variables. A first major contribution to the conditional distribution problem is the work by Wang and Stoev [17]. The authors consider max-linear random fields, a special class of max-stable random fields with discrete spectral measure, and give an exact expression of the conditional distributions as well as efficient algorithms. The max-linear structure plays an essential role in their work and provides major simplifications since in this case η admits the simple representation

$$\eta(t) = \vee_{j=1}^{q} Z_j f_j(t), \quad t \in T,$$

where $f_1, \ldots, f_q$ are deterministic functions and $Z_1, \ldots, Z_q$ are independent and identically distributed (i.i.d.) random variables with unit Fréchet distribution. The authors determine the conditional distributions of $(Z_j)_{1\leq j\leq q}$ given observations $\{\eta(t_i) = y_i,\ 1 \leq i \leq k\}$. Their result relies on the important notion of *hitting scenario* defined as the subset of indices $j \in [\![1, q]\!]$ such that $\eta(t_i) = Z_j f(t_i)$ for some $i \in [\![1, k]\!]$, where, for $n \geq 1$, we note $[\![1, n]\!] = \{1, \ldots, n\}$. The conditional distribution of $(Z_j)_{1\leq j\leq q}$ is expressed as a mixture over all admissible hitting scenarios with minimal rank. The purpose of the present paper is to propose a general theoretical framework for conditional distributions in extreme value theory, covering not only the whole class of sample continuous max-stable random fields but also the class of sample continuous max-infinitely divisible (max-i.d.) random fields [1].

Our starting point is the general representation by Giné, Hahn and Vatan [11] of max-i.d. sample continuous random fields. It is possible to construct a Poisson random measure $\Phi = \sum_{i=1}^{N} \delta_{\phi_i}$ on the space of continuous functions on T such that

$$\eta(t) \overset{\mathcal{L}}{=} \vee_{i=1}^{N} \phi_i(t), \quad t \in T.$$

Here the random variable N is equal to the total mass of Φ that may be finite or infinite and $\overset{\mathcal{L}}{=}$ stands for equality of probability laws (see Theorem 1.1 below for a precise statement). We denote by $[\Phi] = \{\phi_i,\ 1 \leq i \leq N\}$ the set of atoms of Φ. Clearly, $\phi(t) \leq \eta(t)$ for all $t \in T$ and $\phi \in [\Phi]$. The observations $\{\eta(t_i) = y_i,\ 1 \leq i \leq k\}$ naturally lead to consider extremal points: an atom $\phi \in [\Phi]$ is called *extremal* if $\phi(t_i) = \eta(t_i)$ for some $i \in [\![1, k]\!]$, otherwise it is called *sub-extremal*. We show that under some mild condition, one can define a random partition $\Theta = (\theta_1, \ldots, \theta_\ell)$ of $\{t_1, \ldots, t_k\}$ and random functions $\phi_1^+, \ldots, \phi_\ell^+$ such that:

- $\{\phi_1^+, \ldots, \phi_\ell^+\}$ is exactly the set of extremal atoms of Φ;
- for all $j \in [\![1, \ell]\!]$ and $t \in \{t_1, \ldots, t_k\}$, $\phi_j^+(t) = \eta(t)$ if and only if $t \in \theta_j$.

Using the terminology of Wang and Stoev [17], we call hitting scenario the random partition Θ. It reflects the way how the extremal functions $\phi_1^+, \ldots, \phi_\ell^+$ hit the constraints $\phi_j^+(t_i) \leq \eta(t_i)$, $1 \leq i \leq k$. The main result of this paper is Theorem 3.2 where the conditional distribution of η given $\{\eta(t_i) = y_i,\ 1 \leq i \leq k\}$ is expressed as a mixture over all possible hitting scenarios.

The paper is structured as follows. In Section 2, the distribution of extremal and sub-extremal points is analyzed and a characterization of the hitting scenario distribution is given. In Section 3, we focus on conditional distributions: we compute the conditional distribution of the hitting scenario and extremal functions and then derive the conditional distribution of η. Section 4 is devoted to examples: we specify our results in the case of regular or max-stable models and then consider the max-stable model based on Gaussian random fields introduced by Kabluchko [13] (see also Kabluchko, Schlather and de Haan [14] in connection with Brown-Resnick processes) as well as the max-linear model considered by Wang and Stoev [17].

1.2 Framework

Let T be a compact metric space and $\mathbb{C} = \mathbb{C}(T, \mathbb{R})$ be the space of continuous functions on T endowed with the sup norm

$$\|f\| = \sup_{t \in T} |f(t)|, \quad f \in \mathbb{C}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A random process $\eta = \{\eta(t)\}_{t \in T}$ is said to be max-i.d. on $\mathbb{C}$ if η has a version with continuous sample path and if, for each $n \geq 1$, there exist $\{\eta_{ni},\ 1 \leq i \leq n\}$ independent and identically distributed (i.i.d.) sample continuous random fields on T such that

$$\eta \overset{\mathcal{L}}{=} \vee_{i=1}^n \eta_{ni},$$

where $\vee$ denotes pointwise maximum, i.e.,

$$(\vee_{i=1}^n \eta_{ni})(\omega, t) = \vee_{i=1}^n \eta_{ni}(\omega, t), \quad \omega \in \Omega,\ t \in T.$$

Giné, Hahn and Vatan (see [11] Theorem 2.4) give a representation of such processes in terms of Poisson random measure. For any function f on T and set $A \subset T$, we note $f(A) = \sup_{t \in A} f(t)$

Theorem 1.1. *(Giné, Hahn and Vatan [11])*
Let h be the vertex function of a sample continuous max-i.d. process η defined by

$$h(t) = \sup\{x \in \mathbb{R};\ \mathbb{P}(\eta(t) \geq x) = 1\} \in [-\infty, \infty), \quad t \in T,$$

and define $\mathbb{C}_h = \{f \in \mathbb{C};\ f \neq h, f \geq h\}$.
Under the condition that the vertex function h *is continuous, there exists a locally-finite Borel measure* μ *on* $\mathbb{C}_h$*, such that if* Φ *is a Poisson random measure* Φ *on* $\mathbb{C}_h$ *with intensity measure* μ*, then*

$$\{\eta(t)\}_{t\in T} \overset{\mathcal{L}}{=} \{\sup\{h(t), \phi(t); \phi \in [\Phi]\}\}_{t\in T}$$

where $[\Phi]$ *denotes the set of atoms of* Φ.
Furthermore, the following relations hold:

$$h(K) = \sup\{x \in \mathbb{R};\ \mathbb{P}(\eta(K) \geq x) = 1\}, \quad K \subset T \text{ closed}, \tag{1}$$

and

$$\mathbb{P}\left[\eta(K_i) < x_i,\ 1 \leq i \leq n\right] = \exp\left\{-\mu\left(\cup_{i=1}^n \{f \in \mathbb{C}_h;\ f(K_i) \geq x_i\}\right)\right\}, \tag{2}$$

where $n \in \mathbb{N}$, $K_i \subset T$ closed *and* $x_i > h(K_i)$.

Theorem 1.1 provides an almost complete description of max-i.d. continuous random processes, the only restriction being the continuity of the vertex function. Clearly, the distribution of η is completely characterized by the vertex function h and the so called *exponent measure* μ. Since the conditional distribution of η is easily deduced from that of $\eta - h$, we will assume throughout this paper that $h \equiv 0$ and the corresponding set $\mathbb{C}_0$ is the space of non negative and non null continuous functions on T.

We need some more notations from point process theory (see Daley and Vere-Jones [3, 4]). It will be convenient to introduce a measurable enumeration of the atoms of Φ (see [4] Lemma 9.1.XIII). The total mass of Φ is noted $N = \Phi(\mathbb{C}_0)$. If $\mu(\mathbb{C}_0) < \infty$, N has a Poisson distribution with mean $\mu(\mathbb{C}_0)$, otherwise $N = +\infty$ almost surely (a.s.). Using an ordered dissecting system of $\mathbb{C}_0$, one can construct $\mathbb{C}_0$-valued random variables $(\phi_i)_{i\geq 1}$ such that $\Phi = \sum_{i=1}^N \delta_{\phi_i}$.

Let $M_p(\mathbb{C}_0)$ be the space of point measures $M = \sum_{i\in I} \delta_{f_i}$ on $\mathbb{C}_0$ such that the set $\{f_i \in \mathbb{C}_0\colon \|f_i\| > \varepsilon\}$ is finite for all $\varepsilon > 0$. For $M \in M_p(\mathbb{C}_0)$, let $[M] = \{f_i,\ i \in I\}$ be the set of atoms of M. If M is non null, then for all $t \in T$, the set $\{f(t); f \in [M]\}$ is non empty and has at most one accumulation point equal to 0 so that the maximum $\max\{f(t); f \in [M]\}$ is reached. Furthermore by considering restrictions of the measure M to sets $\{f \in \mathbb{C}_0;\ \|f\| > \varepsilon\}$ and using the uniform convergence, it is easily shown that the application

$$\max(M)\colon \begin{cases} T \to [0, +\infty) \\ t \mapsto \max\{f(t); f \in [M]\} \end{cases}$$

is continuous where we use the convention that $\max(M) \equiv 0$ if $M = 0$. We endow $M_p(\mathbb{C}_0)$ with the σ-algebra $\mathcal{M}_p$ generated by the applications $M \mapsto M(A)$, for all Borel sets $A \subset \mathbb{C}_0$. Interestingly, it is possible to construct a distance d on $M_p(\mathbb{C}_0)$ so that

$M_p(\mathbb{C}_0)$ is a complete separable metric space, $\mathcal{M}_p$ coincides with the Borel σ-field and the application $M \mapsto \max(M)$ is continuous. This property is not needed for our purpose so we do not explicit the construction of the metric here (see Lemma 3.1 in [8] for more details). In Theorem 1.1 (with $h \equiv 0$), Equation (2) implies that the exponent measure μ assigns a finite mass to sets of the form $\{f \in \mathbb{C}_0;\ \|f\| > \varepsilon\}$. Consequently, we have $\Phi \in M_p(\mathbb{C}_0)$ a.s. and $\eta \overset{\mathcal{L}}{=} \max(\Phi)$.

2 Extremal points and related distributions

In the sequel, η denotes a sample continuous max-i.d. random process with vertex function $h \equiv 0$ and exponent measure μ on $\mathbb{C}_0$. On the same probability space, we suppose that a $M_p(\mathbb{C}_0)$-valued Poisson random measure $\Phi = \sum_{i=1}^N \delta_{\phi_i}$ with intensity measure μ is given and such that $\eta = \max(\Phi)$.

2.1 Definition and first properties

Let $K \subset T$ be a closed subset of T. We introduce here the notion of K-extremal points that will play a key role in this work. We use the following notations: if f_1, f_2 are two functions defined (at least) on K, we write

$$\begin{aligned} f_1 =_K f_2 &\quad \text{if and only if} \quad \forall t \in K,\ f_1(t) = f_2(t),\\ f_1 <_K f_2 &\quad \text{if and only if} \quad \forall t \in K,\ f_1(t) < f_2(t),\\ f_1 \not<_K f_2 &\quad \text{if and only if} \quad \exists t \in K,\ f_1(t) \geq f_2(t). \end{aligned}$$

Let $M \in M_p(\mathbb{C}_0)$. An atom $f \in [M]$ is called K-subextremal if $f <_K \max(M)$ and K-extremal otherwise. In words, a subextremal atom has no contribution to the maximum $\max(M)$ on K.

Definition 2.1. *Define the K-extremal random point measure Φ_K^+ and the K-subextremal random point measure Φ_K^- by*

$$\Phi_K^+ = \sum_{i=1}^N 1_{\{\phi_i \not<_K \eta\}} \delta_{\phi_i} \quad \textit{and} \quad \Phi_K^- = \sum_{i=1}^N 1_{\{\phi_i <_K \eta\}} \delta_{\phi_i}.$$

It should be noted that Φ_K^+ and Φ_K^- are well defined measurable random point measures. Technical details on measurability are deferred to Appendix A.3. Furthermore, it is straightforward from the definition that

$$\Phi = \Phi_K^+ + \Phi_K^-, \quad \max(\Phi_K^+) =_K \eta \quad \text{and} \quad \max(\Phi_K^-) <_K \eta.$$

Define the following measurable subsets of $M_p(\mathbb{C}_0)$ (see Appendix A.3):

$$\begin{aligned} C_K^+ &= \Big\{ M \in M_p(\mathbb{C}_0);\ \forall f \in [M],\ f \not<_K \max(M) \Big\} \quad (3) \\ C_K^-(g) &= \Big\{ M \in M_p(\mathbb{C}_0);\ \forall f \in [M],\ f <_K g \Big\} \quad (4) \end{aligned}$$

where g is any continuous function defined (at least) on K. Clearly, it always holds

$$\Phi_K^+ \in C_K^+ \quad \text{and} \quad \Phi_K^- \in C_K^-(\eta).$$

The following characterization of the K-extremal random point measure will be useful. If $M_1, M_2 \in M_p(\mathbb{C}_0)$ are such that $M_2 - M_1 \in M_p(\mathbb{C}_0)$, we call M_1 a sub-point measure of M_2.

Lemma 2.1. *The K-extremal point measure Φ_K^+ is the unique sub-point measure $\tilde{\Phi}$ of Φ such that*

$$\tilde{\Phi} \in C_K^+ \quad \text{and} \quad \Phi - \tilde{\Phi} \in C_K^-(\max(\tilde{\Phi})).$$

Proof of Lemma 2.1: First the condition $\Phi - \tilde{\Phi} \in C_K^-(\max(\tilde{\Phi}))$ implies

$$\max(\Phi - \tilde{\Phi}) <_K \max(\tilde{\Phi}) \quad \text{and} \quad \max(\tilde{\Phi}) =_K \max(\Phi).$$

Let $f \in [\Phi - \tilde{\Phi}]$. The condition $\Phi - \tilde{\Phi} \in C_K^-(\max(\tilde{\Phi}))$ implies $f <_K \max(\tilde{\Phi})$. Since $\tilde{\Phi}$ is a sub-point measure of Φ, $\max(\tilde{\Phi}) \leq \max(\Phi)$ so that $f <_K \max(\Phi)$ and f is Ksub-extremal in Φ.

Conversely for $f \in [\tilde{\Phi}]$, the condition $\tilde{\Phi} \in C_K^+$ implies the existence of $t_0 \in K$ such that $f(t_0) = \max(\tilde{\Phi})(t_0)$. Hence $f(t_0) = \max(\Phi)(t_0)$ and f is K-extremal in Φ. □

2.2 Distribution of (Φ_K^+, Φ_K^-)

The following Theorem characterizes the joint distribution (Φ_K^+, Φ_K^-) given that Φ_K^+ is finite.

Theorem 2.1. *For all measurable $A, B \in\subset M_p(\mathbb{C}_0)$,*

$$\mathbb{P}\big[(\Phi_K^+, \Phi_K^-) \in A \times B,\ \Phi_K^+(\mathbb{C}_0) = 0\big] = \exp[-\mu(\mathbb{C}_0)]\, \delta_0(A)\, \delta_0(B),$$

and for $k \geq 1$,

$$\begin{aligned}&\mathbb{P}\big[(\Phi_K^+,\Phi_K^-)\in A\times B,\ \Phi_K^+(\mathbb{C}_0)=k\big]\\ =\ &\frac{1}{k!}\int_{\mathbb{C}_0^k} 1_{\{\sum_{i=1}^k \delta_{f_i}\in A\cap C_K^+\}}\mathbb{P}[\Phi\in B\cap C_K^-(\vee_{i=1}^k f_i)]\,\mu^{\otimes k}(df_1,\dots,df_k),\end{aligned}$$

Proof of Theorem 2.1:
First note that $\Phi_K^+(\mathbb{C}_0)=0$ if and only if $\Phi=0$. This occurs with probability $\exp[-\mu(\mathbb{C}_0)]$ and in this case $\Phi_K^+=\Phi_K^-=0$. The first claim follows.
Next, let $k\geq 1$. According to Lemma 2.1, $\Phi_K^+(\mathbb{C}_0)=k$ if and only if there exists a k-uplet $(\phi_1,\dots,\phi_k)\in[\Phi]^k$ such that

$$\sum_{i=1}^k \delta_{\phi_i}\in C_K^+\quad\text{and}\quad \Phi-\sum_{i=1}^k\delta_{\phi_i}\in C_K^-(\vee_{i=1}^k\phi_i).$$

When this holds, the k-uplet $(\phi_1,\dots,\phi_k)$ is unique up to a permutation of the coordinates and we have

$$\sum_{i=1}^k \delta_{\phi_i}=\Phi_K^+\quad\text{and}\quad \Phi-\sum_{i=1}^k\delta_{\phi_i}=\Phi_K^-.$$

Hence the sum

$$\int_{\mathbb{C}_0^k} 1_{\{\sum_{i=1}^k\delta_{\phi_i}\in A\cap C_K^+,\ \Phi-\sum_{i=1}^k\delta_{\phi_i}\in C_K^-(\vee_{i=1}^k\phi_i)\}}\,\Phi(d\phi_1)\,(\Phi-\delta_{\phi_1})(d\phi_2)\cdots(\Phi-\sum_{j=1}^{k-1}\delta_{\phi_j})(d\phi_k)$$

is equal to $k!1_{\{(\Phi_K^+,\Phi_K^-)\in A\times B\}}$ if $\Phi_K^+(\mathbb{C}_0)=k$ and 0 otherwise. Using this and Slyvniak's formula (see Appendix A.1), we get

$$\begin{aligned}&\mathbb{P}\big[(\Phi_K^+,\Phi_K^-)\in A\times B,\ \Phi_K^+(\mathbb{C}_0)=k\big]\\ =\ &\frac{1}{k!}\mathbb{E}\Big[\int_{\mathbb{C}_0^k} 1_{\{\sum_{i=1}^k\delta_{\phi_i}\in A\cap C_K^+,\ \Phi-\sum_{i=1}^k\delta_{\phi_i}\in B\cap C_K^-(\vee_{i=1}^k\phi_i)\}}\,\Phi(d\phi_1)\cdots(\Phi-\sum_{j=1}^{k-1}\delta_{\phi_j})(d\phi_k)\Big]\\ =\ &\frac{1}{k!}\int_{\mathbb{C}_0^k} 1_{\{\sum_{i=1}^k \delta_{f_i}\in A\cap C_K^+\}}\mathbb{P}[\Phi\in B\cap C_K^-(\vee_{i=1}^k f_i)]\,\mu^{\otimes k}(df_1,\dots,df_k).\end{aligned}$$

This proves Theorem 2.1. □

We can interpret Theorem 2.1 in terms of marginal and conditional distributions. It is convenient to introduce the tail functional $\bar{\mu}_K$ defined by

$$\bar{\mu}_K(g)=\mu(\{f\in\mathbb{C}_0;\ f\not<_K g\})$$

for any continuous function g defined (at least) on K.

COROLLARY 2.1.

1. *The cardinal of* Φ_K^+ *has distribution given by* $\mathbb{P}[\Phi_K^+(\mathbb{C}_0)=0]=\exp[-\mu(\mathbb{C}_0)]$ *and for* $k\geq 1$
$$\mathbb{P}[\Phi_K^+(\mathbb{C}_0)=k]=\frac{1}{k!}\int_{\mathbb{C}_0^k}\exp[-\bar{\mu}_K(\vee_{i=1}^k f_i)]\,1_{\{\sum_{i=1}^k\delta_{f_i}\in C_K^+\}}\,\mu^{\otimes k}(df_1,\dots,df_k).$$

2. *Let* $k\geq 1$*. Given that* $\Phi_K^+(\mathbb{C}_0)=k$*, the conditional distribution of* Φ_K^+ *is given by*
$$\begin{aligned}&\mathbb{P}[\Phi_K^+\in A\mid\Phi_K^+(\mathbb{C}_0)=k] \qquad\qquad (5)\\ =\;&\frac{1}{k!\mathbb{P}[\Phi_K^+(\mathbb{C}_0)=k]}\int_{\mathbb{C}_0^k}\exp[-\bar{\mu}_K(\vee_{i=1}^k f_i)]\,1_{\{\sum_{i=1}^k\delta_{f_i}\in A\cap C_K^+\}}\,\mu^{\otimes k}(df_1,\dots,df_k)\end{aligned}$$
for all measurable $A\subset M_p(\mathbb{C}_0)$*.*

3. *Given that* $\Phi_K^+=\sum_{i=1}^k f_i$*, the conditional distribution of* Φ_K^- *is equal to the distribution of a Poisson random measure with intensity* $1_{\{f<_K\vee_{i=1}^k f_i\}}\mu(df)$*.*

REMARK 2.1. The second point of Corollary 2.1 can be reformulated in terms of Janossy measures (see e.g. Daley and Vere-Jones [3] section 5.3). The Janossy measure of order k of the K-extremal point measure Φ_K^+ is given by

$$J_k(df_1,\dots,df_k)=\exp[-\bar{\mu}_K(\vee_{i=1}^k f_i)]\,1_{\{\sum_{i=1}^k\delta_{f_i}\in C_K^+\}}\mu^{\otimes k}(df_1,\dots,df_k).$$

Proof of Corollary 2.1:

1. The first point is a direct consequence of Theorem 2.1 with $A=B=M_p(\mathbb{C}_0)$ since
$$\mathbb{P}[\Phi\in C_K^-(\vee_{i=1}^k f_i)]=\exp[-\bar{\mu}_K(\vee_{i=1}^k f_i)]$$
is the probability that no point of Φ lies in the set $\{f\in\mathbb{C}_0; f\not<_K\vee_{i=1}^k f_i\}$.

2. The second point follows from Theorem 2.1 with $B=M_p(\mathbb{C}_0)$ and from the definition of conditional probability.

3. For the third point, consider a measurable subset $A\subset\{N\in M_p(\mathbb{C}_0);\ N(\mathbb{C}_0)=k\}$. Since
$$\mathbb{P}\big[(\Phi_K^+,\Phi_K^-)\in A\times B\big]=\int_{\mathbb{C}_0^k}\frac{\mathbb{P}[\Phi\in B\cap C_K^-(\vee_{i=1}^k f_i)]}{\exp[-\bar{\mu}_K(\vee_{i=1}^k f_i)]}1_{\{\sum_{i=1}^k\delta_{f_i}\in A\}}J_k(df_1,\dots,df_k),$$

the conditional probability $\Phi_K^- \in B$ given $\Phi_K^+ = \sum_{i=1}^k \delta_{f_i}$ equals

$$\frac{\mathbb{P}[\Phi \in B \cap C_K^-(\vee_{i=1}^k f_i)]}{\exp[-\bar{\mu}_K(\vee_{i=1}^k f_i)]}.$$

We recognize the distribution of the Poisson random measure Φ conditioned to lie in $C_K^-(\vee_{i=1}^k f_i)$, i.e., to have no atom in $\{f \in \mathbb{C}_0; f \not<_K \vee_{i=1}^k f_i\}$. It is equal in law to a Poisson random measure with intensity $1_{\{f <_K \vee_{i=1}^k f_i\}}\mu(df)$.

□

Theorem 2.1 fully characterizes the joint distribution of (Φ_K^+, Φ_K^-) provided that $\Phi_K^+(\mathbb{C}_0)$ is almost surely finite. We now focus on this last condition.

PROPOSITION 2.1.
The K-extremal point measure Φ_K^+ is a.s. finite if and only if one of the following condition holds:

i) $\mu(\mathbb{C}_0) < +\infty$*;*

ii) $\mu(\mathbb{C}_0) = +\infty$ and $\inf_{t \in K} \eta(t) > 0$ a.s.

It should be noted that any simple max-stable random field (with unit Frécher margins) satisfies condition ii) above. See for example Corollary 3.4 in [11].

Proof of Proposition 2.1: In the case $\mu(\mathbb{C}_0) < +\infty$, Φ and *a fortiori* Φ_K^+ are a.s. finite. Suppose now $\mu(\mathbb{C}_0) = +\infty$, so that Φ is a.s. infinite. If $\inf_{t \in K} \eta(t) = 0$, then there is $t_0 \in K$ such that $\eta(t_0) = 0$ (recall η is continuous and K compact). This implies that $\phi(t_0) = 0$ for all $\phi \in [\Phi]$ and hence $\Phi_K^+ = \Phi$ is infinite. If $\inf_{t \in K} \eta(t) = \varepsilon > 0$, then the support of Φ_K^+ is included in the set $\{f \in \mathbb{C}_0;\ f(K) \geq \varepsilon\}$. From the definition of $M_p(\mathbb{C}_0)$, this set contains only a finite number of atoms of Φ so that Φ_K^+ must be finite. □

2.3 Extremal functions

Let $t \in T$. We will see that under a natural condition, there is a.s. a unique $\{t\}$-extremal point in Φ. This extremal point will be referred to as the *t-extremal function* and noted ϕ_t^+. We denote by μ_t the measure on $(0, +\infty)$ defined by

$$\mu_t(A) = \mu(\{f \in \mathbb{C}_0; f(t) \in A\}), \quad A \subset (0, +\infty) \text{ Borel set.}$$

The associated tail function is noted $\bar{\mu}_t(x) = \mu_t((x, +\infty))$, $x \geq 0$. Note that μ_t is the exponent measure associated with the max-i.d. random variable $\eta(t)$.

Proposition 2.2. *For $t \in T$, the following statements are equivalent:*

i) $\Phi^+_{\{t\}}(\mathbb{C}_0) = 1$ *almost surely;*

ii) $\bar{\mu}_t(0) = +\infty$ *and* $\bar{\mu}_t$ *is continuous on* $(0, +\infty)$*;*

iii) the distribution of $\eta(t)$ *has no atom.*

If these conditions are met, we define the t-extremal function ϕ^+_t by the relation $\Phi^+_{\{t\}} = \delta_{\phi^+_t}$ a.s.. For all measurable $B \subset \mathbb{C}_0$ we have

$$\mathbb{P}(\phi^+_t \in B) = \int_B \exp[-\bar{\mu}_t(x)]\mu_t(dx).$$

A very important class of processes satisfying the conditions of Proposition 2.2 is the class of max-stable processes (see section 4.2 below).

Proof of Proposition 2.2:

- According to equation (2),

$$\mathbb{P}[\eta(t) < x] = \exp[-\bar{\mu}_t(x)], \quad \text{if } x > 0,$$

and $\mathbb{P}[\eta(t) < x] = 0$ if $x \leq 0$. Hence, for all $x > 0$,

$$\mathbb{P}[\eta(t) = x] = \exp[-\bar{\mu}_t(x)] - \exp[-\bar{\mu}_t(x^-)],$$

and $\mathbb{P}[\eta(t) = 0] = \exp[-\bar{\mu}_t(0)]$. The equivalence between ii) and iii) follows.
The equivalence between i) and ii) is a consequence of Corollary 2.1 point 1 from which we get

$$\mathbb{P}[\Phi^+_{\{t\}}(\mathbb{C}_0) = 1] = \int_{[0,+\infty)} \exp[-\bar{\mu}_t(y)]\mu_t(dy).$$

It remains to prove that this probability is equal to 1 if and only if ii) is satisfied. To this aim, we compute

$$\mathbb{P}[\Phi^+_{\{t\}}(\mathbb{C}_0) = 1] = \int_{[0,+\infty)^2} e^{-x} 1_{\{\bar{\mu}_t(y) \leq x\}}\, dx \mu_t(dy) = \int_{[0,+\infty)} e^{-x} \mu_t(A_x)\, dx, \tag{6}$$

where $A_x = \{y \geq 0; \bar{\mu}_t(y) \leq x\}$. Since $\bar{\mu}_t$ is càd-làg, non-decreasing and tends to 0 at ∞, $A_x = [\inf A_x, +\infty) \neq \varnothing$ for all $x > 0$. Furthermore using equation (6) and the fact that $\mu_t(A_x) \leq x$, we get that $\mathbb{P}[\Phi^+_{\{t\}}(\mathbb{C}_0) = 1]$ if and only if $\mu_t(A_x) = x$ for all $x > 0$. This is easily seen to be equivalent to condition ii) and completes the equivalence between i) and ii).

- The second statement is a direct application of Corollary 2.1. If i) holds, the distribution of $\Phi^+_{\{t\}}$ is equal to its conditional distribution given that $\Phi^+_{\{t\}}(\mathbb{C}_0) = 1$ and is given by Equation (5) with $K = \{t\}$, $k = 1$. The distribution of ϕ_t^+ follows by setting $A = \{\delta_f \in M_p(\mathbb{C}_0),\ f \in B\}$ in equation (5). Note that in this case, $A \subset C^+_{\{t\}}$.

□

2.4 Hitting scenarios

Proposition 2.2 gives the distribution of Φ_K^+ when $K = \{t\}$ is reduced to a single point. Going a step further, we consider the case when K is finite. In the sequel, we suppose that the following assumption is satisfied:

$$(\mathbf{A})\quad K = \{t_1, \dots, t_k\} \text{ is finite and, for all } t \in K,\ \bar{\mu}_t \text{ is continuous and } \bar{\mu}_t(0) = +\infty.$$

Roughly speaking, this ensures that the maximum $\eta(t) = \max(\Phi)(t)$ is uniquely reached for all $t \in K$. This will provide combinatorial simplifications. More precisely, under Assumption $(\mathbf{A})$, the event $\Omega_K = \bigcap_{t\in K}\{\Phi_{\{t\}}(\mathbb{C}_0) = 1\}$ is of probability 1 and the extremal functions $\phi^+_{t_1}, \dots, \phi^+_{t_k}$ are well defined. In the next Definition, we introduce the notion of hitting scenario that reflects the way how these extremal functions hit the maximum η on K.

Let $\mathcal{P}_K$ be the set of partitions of K. It is convenient to think about K as an ordered set, say $t_1 < \cdots < t_k$. Then each partition $\tau \in \mathcal{P}_K$ can be written uniquely in the standardized form $\tau = (\tau_1, \dots, \tau_\ell)$ where $\ell = \ell(\tau)$ is the length of the partition, $\tau_1 \subset K$ is the component of t_1, $\tau_2 \subset K$ is the component containing $\min(K \setminus \tau_1)$ and so on. With this convention, the components $\tau_1, \dots, \tau_\ell$ of the partition are labeled such that $\min \tau_1 < \cdots < \min \tau_\ell$.

Definition 2.2. *Suppose that Assumption* $(\mathbf{A})$ *is met. Define* $\sim$ *the (random) equivalence relation on* $K = \{t_1, \dots, t_k\}$ *by*

$$t \sim t' \quad \textit{if and only if} \quad \phi_t^+ = \phi_{t'}^+.$$

We call hitting scenario the partition $\Theta = (\theta_1, \dots, \theta_{\ell(\Theta)})$ *of* K *into equivalence classes. For* $j \in [\![1, \ell(\Theta)]\!]$ *denote* ϕ_j^+ *the extremal function associated to the component* θ_j*, i.e., such that* $\phi_j^+ = \phi_t^+$ *for all* $t \in \theta_j$.

Clearly a point $\phi \in [\Phi]$ is K-extremal if and only if it is t-extremal for some $t \in K$, so that $[\Phi_K^+] = \{\phi_t^+,\ t \in K\}$. Furthermore, the random measure Φ_K^+ is almost surely simple, i.e. all atoms have a simple multiplicity, otherwise the condition $\Phi_{\{t\}}(\mathbb{C}_0) = 1$ a.s. would not be satisfied for some $t \in K$. These considerations entail that

$$\Phi_K^+ = \sum_{j=1}^{\ell(\Theta)} \delta_{\phi_j}^+.$$

In particular, the length $\ell(\Theta)$ of the hitting scenario is equal to $\Phi_K^+(\mathbb{C}_0)$. Furthermore the extremal functions satisfy

$$\forall j \in [\![1,\ell]\!],\ \forall t \in \theta_j, \quad \phi_j^+(t) > \vee_{j'\neq j}\phi_{j'}^+(t). \tag{7}$$

The distribution of the hitting scenario and extremal functions is given by the following Proposition. The proof relies on Theorem 2.1.

Proposition 2.3. *Suppose Assumption* (**A**) *is met.*
Then, for all partition $\tau = (\tau_1, \dots, \tau_\ell) \in \mathcal{P}_K$, and all Borel sets $A \subset \mathbb{C}_0^\ell$, $B \subset M_p(\mathbb{C}_0)$, we have

$$\begin{aligned}
&\mathbb{P}[\Theta = \tau,\ (\phi_1^+, \dots, \phi_\ell^+) \in A,\ \Phi_K^- \in B] \\
= \ &\int_A 1_{\{\forall j \in [\![1,\ell]\!],\ f_j >_{\tau_j} \vee_{j'\neq j} f_{j'}\}} \mathbb{P}[\Phi \in B \cap C_K^-(\vee_{j=1}^l f_j)]\, \mu^{\otimes \ell}(df_1, \dots, df_\ell).
\end{aligned} \tag{8}$$

Proof of Proposition 2.3: First note that the inequalities (7) characterize the hitting scenario. Let $\tau = (\tau_1, \dots, \tau_\ell) \in \mathcal{P}_K$ and define the sets

$$\tilde{C}_\tau = \Big\{ (f_1, \dots, f_\ell) \in \mathbb{C}_0^\ell;\ \forall j \in [\![1,\ell]\!],\ f_j >_{\tau_j} \vee_{j'\neq j} f_{j'} \Big\}.$$

and

$$C_\tau = \Big\{ \sum_{j=1}^{\ell} \delta_{f_j} \in M_p(\mathbb{C}_0);\ (f_1, \dots, f_\ell) \in \tilde{C}_\tau \Big\}.$$

Note that $C_\tau \subset C_K^+$ and that $\Theta = \tau$ if and only if $\Phi_K^+ \in C_\tau$.
Furthermore, $\Theta = \tau$ and $(\phi_1^+, \dots, \phi_\ell^+) \in A$ if and only if $\Phi_K^+ \in A_\tau$ with

$$A_\tau = \Big\{ \sum_{j=1}^{\ell} \delta_{f_j} \in M_p(\mathbb{C}_0);\ (f_1, \dots, f_\ell) \in C_\tau \cap A \Big\}.$$

Hence the following events are equal

$$\{\Theta = \tau,\ (\phi_1^+, \dots, \phi_\ell^+) \in A,\ \Phi_K^- \in B\} = \{\Phi_K^+ \in A_\tau,\ \Phi_K^- \in B,\ \Phi_K^+(\mathbb{C}_0) = \ell\}$$

and Theorem 2.1 implies

$$\begin{aligned} & \mathbb{P}[\Theta=\tau,\ (\phi_1^+,\dots,\phi_\ell^+)\in A,\ \Phi_K^-\in B] \\ = \ & \frac{1}{\ell!}\int_{\mathbb{C}_0^\ell} 1_{\{\sum_{j=1}^{\ell}\delta_{f_j}\in A_\tau\}}\mathbb{P}[\Phi\in B\cap C_K^-(\vee_{j=1}^{l}f_j)]\,\mu^{\otimes\ell}(df_1,\dots,df_\ell). \end{aligned} \tag{9}$$

Finally, $\sum_{j=1}^{\ell}\delta_{f_j}\in A_\tau$ if and only if there exists a permutation σ of $[\![1,\ell]\!]$ such that $(f_{\sigma(1)},\dots,f_{\sigma(1)}\})\in A\cap\tilde{C}_\tau$. Such a permutation is unique and this proves the equivalence of Equations (8) and (9). □

Proposition 2.3 has the following interpretation in terms of marginal and conditional distributions.

Corollary 2.2. *Suppose Assumption* (**A**) *holds true.*

1. *The distribution of the hitting scenario* Θ *satisfies*

$$\mathbb{P}(\Theta=\tau)=\int_{\mathbb{C}_0^\ell} 1_{\{\forall j\in[\![1,\ell]\!],\ f_j>_{\tau_j}\vee_{j'\neq j}f_{j'}\}}\exp[-\bar{\mu}_K(\vee_{j=1}^{l}f_j)]\,\mu^{\otimes\ell}(df_1,\dots,df_\ell)$$

   for all $\tau=(\tau_1,\dots,\tau_\ell)\in\mathcal{P}_K$.

2. *Conditionally on* $\Theta=\tau$, *the distribution of* $(\phi_1^+,\dots,\phi_\ell^+)$ *satisfies*

$$\begin{aligned} & \mathbb{P}[(\phi_1^+,\dots,\phi_\ell^+)\in A\mid\Theta=\tau] \\ = \ & \frac{1}{\mathbb{P}(\Theta=\tau)}\int_A 1_{\{\forall j\in[\![1,\ell]\!],\ f_j>_{\tau_j}\vee_{j'\neq j}f_{j'}\}}\exp[-\bar{\mu}_K(\vee_{i=1}^{k}f_i)]\,\mu^{\otimes\ell}(df_1,\dots,df_\ell). \end{aligned}$$

   for all measurable $A\subset\mathbb{C}_0^\ell$

3. *Conditionally on* $\Theta=\tau$ *and* $(\phi_1^+,\dots,\phi_\ell^+)=(f_1,\dots,f_\ell)$, *the regular conditional distribution of* Φ_K^- *is equal to the distribution of a Poisson random measure with intensity* $1_{\{f<_K\vee_{i=1}^{k}f_i\}}\mu(df)$.

Proof of Corollary 2.2: The proof is very similar to the proof of Corollary 2.1 and the details are omitted. □

3 Regular conditional distribution of max-id processes

We now focus on conditional distributions. We will need some notations.

If $\mathbf{s} = (s_1, \dots, s_k) \in T^l$ and $f \in \mathbb{C}_0$, we note $f(\mathbf{s}) = (f(s_1), \dots, f(s_l))$. Let $\mu_{\mathbf{s}}$ be the exponent measure of the max-i.d. random vector $\eta(\mathbf{s})$, i.e., the measure on $[0,+\infty)^l \setminus \{0\}$ defined by

$$\mu_{\mathbf{s}}(A) = \mu(\{f \in \mathbb{C}_0;\ f(\mathbf{s}) \in A\}), \quad A \subset [0,+\infty)^l \setminus \{0\} \text{ Borel set.}$$

Define the corresponding tail function

$$\bar{\mu}_{\mathbf{s}}(\mathbf{x}) = \mu(\{f \in \mathbb{C}_0;\ f(\mathbf{s}) \not< \mathbf{x} \text{ and } f(\mathbf{s}) \neq 0\}), \quad \mathbf{x} \in [0,+\infty)^l.$$

Let $\{P_{\mathbf{s}}(\mathbf{x}, df);\ \mathbf{x} \in [0,+\infty)^l \setminus \{0\}\}$ be a regular version of the conditional measure $\mu(df)$ given $f(\mathbf{s}) = \mathbf{x}$. Then for all measurable function

$$F : [0,+\infty)^l \times \mathbb{C}_0 \to\ [0,+\infty)$$

vanishing on $\{0\} \times \mathbb{C}_0$, we have

$$\int_{\mathbb{C}_0} F(f(\mathbf{s}), f)\mu(df) = \int_{[0,+\infty)^l \setminus \{0\}} \int_{\mathbb{C}_0} F(\mathbf{x}, f) P_{\mathbf{s}}(\mathbf{x}, df) \mu_{\mathbf{s}}(d\mathbf{x}).$$

Let $\mathbf{t} = (t_1, \dots, t_k)$ and $\mathbf{y} = (y_1, \dots, y_k) \in [0,+\infty)^k$. We note $K = \{t_1, \dots, t_k\}$. For all non empty $L \subset K$, we define $\tilde{L} \subset [\![1, k]\!]$ such that $i \in \tilde{L}$ if and only if $t_i \in L$ and we set $\mathbf{t}_L = (t_i)_{i \in \tilde{L}}$ and $\mathbf{y}_L = (y_i)_{i \in \tilde{L}}$.

Theorem 3.1. *Suppose assumption* $(\mathbf{A})$ *is satisfied.*

- *For* $\tau \in \mathcal{P}_K$, *define the measure* $\nu_{\mathbf{t}}^{\tau}$ *on* $[0,+\infty)^k$ *by*

  $$\nu_{\mathbf{t}}^{\tau}(C) = \mathbb{P}(\eta(\mathbf{t}) \in C;\ \Theta = \tau), \quad C \subset [0,+\infty)^k \text{ Borel set.}$$

  Then,

  $$\nu_{\mathbf{t}}^{\tau}(d\mathbf{y}) \quad = \quad \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})] \otimes_{j=1}^{\ell} \left\{ P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\})\ \mu_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j}) \right\}$$

  and the distribution $\nu_{\mathbf{t}}$ *of* $\eta(\mathbf{t})$ *is equal to* $\nu_{\mathbf{t}} = \sum_{\tau \in \mathcal{P}_K} \nu_{\mathbf{t}}^{\tau}$.

- *The regular conditional distribution of* Θ *w.r.t.* $\eta(\mathbf{t}) = \mathbf{y}$ *is* $\nu_{\mathbf{t}}(d\mathbf{y})$*-a.e. equal to*

  $$\pi_{\mathbf{t}}(\mathbf{y}, \tau) = \frac{d\nu_{\mathbf{t}}^{\tau}}{d\nu_{\mathbf{t}}}(\mathbf{y}), \quad \tau \in \mathcal{P}_K, \tag{10}$$

  where $\frac{d\nu_{\mathbf{t}}^{\tau}}{d\nu_{\mathbf{t}}}$ *denotes the Radon-Nykodym derivative of* $\nu_{\mathbf{t}}^{\tau}$ *w.r.t.* $\nu_{\mathbf{t}}$.

- *The regular conditional distribution of* $(\phi_1^+,\ldots,\phi_\ell^+)$ *w.r.t.* $\eta(\mathbf{t})=\mathbf{y}$ *and* $\Theta=\tau$ *is* $\nu_{\mathbf{t}}(d\mathbf{y})\pi_{\mathbf{t}}(\mathbf{y},d\tau)$*-a.e. equal to*

$$Q_{\mathbf{t}}^{\tau}(\mathbf{y},df_1\cdots df_\ell)=\otimes_{j=1}^{\ell}\Big\{\frac{1_{\{f_j(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\}}P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},df_j)}{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},\{f(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\})}\Big\}. \tag{11}$$

- *The regular conditional probability of* Φ_K^- *w.r.t.* $\eta(\mathbf{t})=\mathbf{y}$*,* $\Theta=\tau$ *and* $(\phi_1^+,\ldots,\phi_\ell^+)=(f_1,\ldots,f_\ell)$ *is a.e. equal to the distribution of a Poisson Random measure with intensity with intensity* $1_{\{f(\mathbf{t})<\mathbf{y}\}}\mu(df)$*.*

- *The joint regular conditional distribution of* $((\phi_j^+)_{1\le j\le\ell(\Theta)},\Phi_K^-)$ *w.r.t.* $\eta(\mathbf{t})=\mathbf{y}$ *is* $\nu_{\mathbf{t}}(d\mathbf{y})$*-a.e. equal to*

$$Q_{\mathbf{t}}(\mathbf{y},A\times B)=\Big\{\sum_{\tau\in\mathcal{P}_K}\pi_{\mathbf{t}}(\mathbf{y},\tau)Q_{\mathbf{t}}^{\tau}(\mathbf{y},A)\Big\}\frac{\mathbb{P}[\Phi\in B\cap C_K^-(\mathbf{y})]}{\mathbb{P}[\Phi\in C_K^-(\mathbf{y})]} \tag{12}$$

  for all Borel sets $A\subset\cup_{\ell=1}^{k}\mathbb{C}_0^{\ell}$ *and* $B\subset M_p(\mathbb{C}_0)$*.*

Under some extra regularity assumptions, one can even get an explicit density function for $\nu_{\mathbf{t}}$ (see the section 4.1 on regular models below).

Proof of Theorem 3.1: Note that $\eta(\mathbf{t})$ can be expressed in terms of the hitting scenario and the extremal function as follows. For $\tau\in\mathcal{P}_K$, define the mapping $\Gamma_\tau:\mathbb{C}_0^{\ell}\to[0,+\infty)^k$ by

$$\Gamma_\tau(f_1,\ldots,f_\ell)=(y_1,\ldots,y_k)\quad\text{with } y_i=f_j(t_i)\text{ if } t_i\in\tau_j.$$

Definition (2.2) entails that for all $t\in\theta_j$, $\eta(t)=\phi_j^+(t)$. This can be rewritten as $\eta(\mathbf{t})=\Gamma_\Theta(\phi_1^+,\ldots,\phi_\ell^+)$. Using this, the probability

$$P(\tau,A,B,C)=\mathbb{P}[\Theta=\tau,\ (\phi_1^+,\ldots,\phi_\ell^+)\in A,\ \Phi_K^-\in B,\ \eta(\mathbf{t})\in C]$$

can be computed thanks to Proposition 2.3:

$$\begin{aligned}&P(\tau,A,B,C)\\ =\ &\mathbb{P}[\Theta=\tau,\ (\phi_1^+,\ldots,\phi_\ell^+)\in A\cap\Gamma_\tau^{-1}(C),\ \Phi_K^-\in B,\ \eta(\mathbf{t})\in C]\\ =\ &\int_{A\cap\Gamma_\tau^{-1}(C)}1_{\{\forall j\in[\![1,\ell]\!],\ f_j>_{\tau_j}\vee_{j'\neq j}f_{j'}\}}\mathbb{P}[\Phi\in B\cap C_K^-(\vee_{j=1}^{\ell}f_j)]\,\mu^{\otimes\ell}(df_1,\ldots,df_\ell).\end{aligned}$$

Now for each $j\in[\![1,\ell]\!]$, we condition the measure $\mu(df_j)$ with respect to $f_j(\mathbf{t}_{\tau_j})$ so that

$$\begin{aligned}&P(\tau,A,B,C) \qquad (13)\\ =\ &\int_C\int_A 1_{\{\forall j\in[\![1,\ell]\!],\ f_j>_{\tau_j}\vee_{j'\neq j}f_{j'}\}}\mathbb{P}[\Phi\in B\cap C_K^-(\vee_{j=1}^{\ell}f_j)]\ \otimes_{j=1}^{\ell}\Big\{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},df_j)\,\mu_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j})\Big\}\\ =\ &\int_C\int_A 1_{\{\forall j\in[\![1,\ell]\!],\ f_j(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\}}\mathbb{P}[\Phi\in B\cap C_K^-(\mathbf{y})]\ \otimes_{j=1}^{\ell}\Big\{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},df_j)\,\mu_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j})\Big\}.\end{aligned}$$

In the last equality, we use the fact that $f_j(\mathbf{t}_{\tau_j}) = \mathbf{y}_{\tau_j}$ a.s. under $P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)$, whence

$$1_{\{\forall j \in [\![1,\ell]\!],\ f_j >_{\tau_j} \vee_{j' \neq j} f_{j'}\}} = 1_{\{\forall j \in [\![1,\ell]\!],\ f_j <_{\tau_j^c} \vee_{j' \neq j} f_{j'}\}} = 1_{\{\forall j \in [\![1,\ell]\!],\ f_j(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\}}.$$

- Set $A = \mathbb{C}_0^\ell$ and $B = M_p(\mathbb{C}_0)$ in Equation (13). This yields

  $$\begin{aligned} &\mathbb{P}[\Theta = \tau,\ \eta(\mathbf{t}) \in C] \\ = &\int_C \int_{\mathbb{C}_0^\ell} 1_{\{\forall j \in [\![1,\ell]\!],\ f_j(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\}} \mathbb{P}[\Phi \in B \cap C_K^-(\mathbf{y})] \otimes_{j=1}^\ell \Big\{ P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)\, \mu_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j}) \Big\}. \end{aligned}$$

  Using the fact that $\mathbb{P}[\Phi \in C_K^-(\mathbf{y})] = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]$ and performing integration with respect to $\otimes_{j=1}^\ell P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)$, we obtain Equation (10).

- Combining Equations (10), (10) and (11) from Theorem 3.1 and Equation (13) yields

  $$\begin{aligned} P(\tau, A, B, C) &= \int_C \int_A \frac{\mathbb{P}[\Phi \in B \cap C_K^-(\mathbf{y})]}{\mathbb{P}[\Phi \in C_K^-(\mathbf{y})]}\, Q_{\mathbf{t}}^\tau(\mathbf{y}, df_1 \cdots df_j) \nu_{\mathbf{t}}^\tau(d\mathbf{y}) \\ &= \int_C \int_A \frac{\mathbb{P}[\Phi \in B \cap C_K^-(\mathbf{y})]}{\mathbb{P}[\Phi \in C_K^-(\mathbf{y})]}\, Q_{\mathbf{t}}^\tau(\mathbf{y}, df_1 \cdots df_j) \pi_{\mathbf{t}}(\mathbf{y}, \tau) \nu_{\mathbf{t}}(d\mathbf{y}). \end{aligned}$$

  In this equation, we recognize the different conditional probabilities (see Appendix A.2).

- This implies that if $A \subset \cup_{\ell=1}^k \mathbb{C}_0^\ell$,

  $$\begin{aligned} &\mathbb{P}[(\phi_j^+)_{1 \le j \le \ell(\Theta)} \in A,\ \Phi_K^- \in B,\ \eta(\mathbf{t}) \in C] \\ = &\sum_{\tau \in \mathcal{P}_K} P(\tau, A, B, C) \\ = &\int_C \Big\{ \sum_{\tau \in \mathcal{P}_K} \pi_{\mathbf{t}}(\mathbf{y}, \tau) Q_{\mathbf{t}}^\tau(\mathbf{y}, A) \Big\} \frac{\mathbb{P}[\Phi \in B \cap C_K^-(\mathbf{y})]}{\mathbb{P}[\Phi \in C_K^-(\mathbf{y})]}\, \nu_{\mathbf{t}}(d\mathbf{y}) \\ = &\int_C Q_{\mathbf{t}}(\mathbf{y}, A \times B)\, \nu_{\mathbf{t}}(d\mathbf{y}). \end{aligned}$$

  We recognize the equation defining the regular conditional joint probability of $((\phi_j^+)_{1 \le j \le \ell(\Theta)}, \Phi_K^-)$ with respect to $\eta(\mathbf{t})$.

$\square$

We can now state and prove our main Theorem, giving the conditional distribution of η with respect to $\eta(\mathbf{t})$.
For any closed $L \subset T$ and continuous function $g : L \to [0, +\infty]$, we set $D(g) = \{f \in \mathbb{C}_0; f <_L g\}$.

Theorem 3.2. *The regular conditional probability* $R_{\mathbf{t}}\{\mathbf{y}, D(g)\}$ *of* η *with respect to* $\eta(\mathbf{t}) = \mathbf{y}$ *is* $\nu_{\mathbf{t}}(d\mathbf{y})$*-a.e. equal to:*

$$R_{\mathbf{t}}(\mathbf{y}, D(g)) \;=\; \Big\{ \sum_{\tau\in\mathcal{P}_K} \pi_{\mathbf{t}}(\mathbf{y},\tau) Q_{\mathbf{t}}^{\tau}(D(g)^{\ell(\tau)}) \Big\} \Big\{ \frac{\exp[-\mu(\{f \not<_L g\} \cup \{f \not<_K \mathbf{y}\})]}{\exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]} \Big\}$$

for all continuous $g : L \to [0, +\infty]$*,* $L \subset T$ *closed.*

Proof of Theorem 3.2: Remark that

$$\{\eta \in D(g)\} = \{(\Phi_K^+, \Phi_K^-) \in C_L^-(g) \times C_L^-(g)\}$$

where $C_L^-(g)$ is given by Equation (4). Using this, Theorem 3.1 implies that for all measurable $C \subset [0, +\infty)^k$,

$$\begin{aligned} \mathbb{P}(\eta \in D(g),\ \eta(\mathbf{t}) \in C) &= \mathbb{P}\Big[\Phi_K^+ \in C_T^-(g),\ \Phi_K^- \in C_T^-(g),\ \eta(\mathbf{t}) \in C\Big] \\ &= \int_C Q_{\mathbf{t}}(\mathbf{y}, (\cup_{\ell=1}^k D(g)^{\ell}) \times C_T^-(g))\, \nu_{\mathbf{t}}(d\mathbf{y}). \end{aligned}$$

This shows that the conditional probability $R_{\mathbf{t}}(\mathbf{y}, D(g))$ is $\nu_{\mathbf{t}}(d\mathbf{y})$-a.e. equal to

$$\begin{aligned} R_{\mathbf{t}}(\mathbf{y}, D(g)) &= Q_{\mathbf{t}}(\mathbf{y}, (\cup_{\ell=1}^k D(g)^{\ell}) \times C_T^-(g)) \\ &= \Big\{ \sum_{\tau\in\mathcal{P}_K} \pi_{\mathbf{t}}(\mathbf{y},\tau) Q_{\mathbf{t}}^{\tau}\{\mathbf{y}, D(g)^{\ell}(\tau)\} \Big\} \frac{\mathbb{P}[\Phi \in C_T^-(g) \cap C_K^-(\mathbf{y})]}{\mathbb{P}[\Phi \in C_K^-(\mathbf{y})]} \end{aligned}$$

with

$$Q_{\mathbf{t}}^{\tau}(\mathbf{y}, D(g)^{\ell}) = \prod_{j=1}^{\ell} \frac{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c} \text{ and } f <_L g\})}{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\})}$$

and

$$\frac{\mathbb{P}[\Phi \in C_T^-(g) \cap C_K^-(\mathbf{y})]}{\mathbb{P}[\Phi \in C_K^-(\mathbf{y})]} = \frac{\exp[-\mu(\{f \not<_L g \text{ or } f \not<_K \mathbf{y}\})]}{\exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]}.$$

□

Remark 3.1. Let us mention that Theorems 3.1 and 3.2 suggest a three step procedure for sampling from the conditional distribution $R_{\mathbf{t}}(\mathbf{y}, df)$ of η with respect to $\eta(\mathbf{t}) = \mathbf{y}$:

1. Draw a random partition τ with distribution $\pi_{\mathbf{t}}(\mathbf{y}, \cdot)$.
2. Given $\tau = \{\tau_1, \dots, \tau_\ell\}$, draw ℓ independent functions $\psi_1, \dots, \psi_\ell$, with ψ_j following the distribution $P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df)$ conditioned on $f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}$.

3. Independently of the above two steps, draw $\sum_{i\in I}\delta_{\phi_i}$ a Poisson random measure on $\mathbb{C}_0$ with intensity $1_{\{f(\mathbf{t})<\mathbf{y}\}}\mu(df)$. It can be obtained from a Poisson random measure with intensity $\mu(df)$ by removing those points not satisfying the constraint $f(\mathbf{t})<\mathbf{y}$.

Then, the random field

$$\tilde{\eta}(t) = \max\{\psi_1(t),\ldots,\psi_\ell(t)\} \vee \max\{\phi_i(t),\ i\in I\}, \quad t\in T,$$

has the required conditional distribution. In this procedure, we believe the major difficulty is the first step, i.e., sampling the hitting scenario τ from the distribution $\pi_{\mathbf{t}}(\mathbf{y},\cdot)$. Indeed the state space $\mathcal{P}_K$ will often be very large and the definition (10) abstract, even when we do have more explicit formulas as we will see in the next section. Another issue in step 3 is that the Poisson random measure $\sum_{i\in I}\delta_{\phi_i}$ may involve an infinite number of points. A forthcoming paper will address these issues and the computational aspects of conditional sampling.

4 Examples

This section is devoted to examples where we apply our general results to specific models to get more tractable expressions.

4.1 Regular models

The exponent measure μ is said to be *regular* (with respect to the Lebesgue measure) if for any $l\geq 1$ and $\mathbf{s}\in T^l$ with mutually distinct components, the measure $\mu_{\mathbf{s}}(d\mathbf{y})$ is absolutely continuous with respect to the Lebesgue measure $d\mathbf{y}$ on $[0,+\infty)^l$. We denote by $h_{\mathbf{s}}$ the corresponding Radon-Nykodym derivative, i.e., $\mu_{\mathbf{s}}(d\mathbf{y})=h_{\mathbf{s}}(\mathbf{y})d\mathbf{y}$. Using this it is possible to get more explicit expressions in Theorem 3.1.

Proposition 4.1. *Assume the assumptions of Theorem 3.1 are met and that μ is a regular exponent measure. Then,*

- *the distribution $\nu_{\mathbf{t}}$ of $\eta(\mathbf{t})$ is absolutely continuous with respect to the Lebesgue measure and*

$$\frac{d\nu_{\mathbf{t}}}{d\mathbf{y}}(y) = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})] \sum_{\tau\in\mathcal{P}_K} \prod_{j=1}^{\ell(\tau)} \int_{\{\mathbf{z}_j<\mathbf{y}_{\tau_j^c}\}} h_{(\mathbf{t}_{\tau_j},\mathbf{t}_{\tau_j^c})}(\mathbf{y}_{\tau_j},\mathbf{z}_j)\,d\mathbf{z}_j \quad d\mathbf{y}\text{-}a.e.; \tag{14}$$

- *the conditional distribution of the hitting scenario satisfies, for* $\tau \in \mathcal{P}_K$

$$\pi_{\mathbf{t}}(\mathbf{y},\tau) = \frac{\prod_{j=1}^{\ell(\tau)} \int_{\{\mathbf{z}_j < \mathbf{y}_{\tau_j^c}\}} h_{(\mathbf{t}_{\tau_j},\mathbf{t}_{\tau_j^c})}(\mathbf{y}_{\tau_j},\mathbf{z}_j)d\mathbf{z}_j}{\sum_{\tau'\in\mathcal{P}_K} \prod_{j=1}^{\ell(\tau')} \int_{\{\mathbf{z}_j < \mathbf{y}_{\tau_j'^c}\}} h_{(\mathbf{t}_{\tau_j'},\mathbf{t}_{\tau_j'^c})}(\mathbf{y}_{\tau_j'},\mathbf{z}_j)d\mathbf{z}_j}, \quad \nu_{\mathbf{t}}(d\mathbf{y})\text{-}a.e.; \tag{15}$$

- *the finite dimensional distributions of the transition kernel* $Q_{\mathbf{t}}^{\tau}(\mathbf{y}, df_1 \dots df_\ell)$ *are characterized as follows: for* $\mathbf{s} \in T^l$ *with components mutually distinct and distinct of those of* $\mathbf{t}$*,*

$$\begin{aligned} & Q_{\mathbf{t}}^{\tau}(\mathbf{y},(f_1(\mathbf{s}),\dots f_\ell(\mathbf{s}))\in d\mathbf{u}_1\cdots d\mathbf{u}_\ell) \\ = & \Big(\prod_{j=1}^{\ell}\frac{1}{C_j}\int_{\{\mathbf{z}_j<\mathbf{y}_{\tau_j^c}\}} h_{(\mathbf{t}_{\tau_j},\mathbf{t}_{\tau_j^c},\mathbf{s})}(\mathbf{y}_{\tau_j},\mathbf{z}_j,\mathbf{u}_j)d\mathbf{z}_j\Big)d\mathbf{u}_1\cdots d\mathbf{u}_\ell \quad \nu_{\mathbf{t}}(d\mathbf{y})\text{-}a.e. \end{aligned} \tag{16}$$

with $C_j = \int_{\{\mathbf{z}_j<\mathbf{y}_{\tau_j^c},\ \mathbf{u}_j\in(0,+\infty)^l\}} h_{(\mathbf{t}_{\tau_j},\mathbf{t}_{\tau_j^c},\mathbf{s})}(\mathbf{y}_{\tau_j},\mathbf{z}_j,\mathbf{u}_j)d\mathbf{z}_jd\mathbf{u}_j$, $1\le j\le \ell$.

- *the finite dimensional distributions of the transition kernel* $R_{\mathbf{t}}(\mathbf{y}, df)$ *giving the conditional distribution of* η *with respect to* $\eta(\mathbf{t})$ *are characterized as follows: for* $\mathbf{s}\in T^l$ *with components mutually distinct and distinct of those of* $\mathbf{t}$ *and* $\mathbf{z}\in(0,+\infty)^l$*,*

$$\begin{aligned} & R_{\mathbf{t}}(\mathbf{y},\{f(\mathbf{s})\le\mathbf{z}\}) \\ = & \frac{\sum_{\tau\in\mathcal{P}_K}\prod_{j=1}^{\ell}\int_{\{\mathbf{u}<\mathbf{y}_{\tau_j^c},\ \mathbf{v}<z\}} h_{(\mathbf{t}_{\tau_j},\mathbf{t}_{\tau_j^c},\mathbf{s})}(\mathbf{y}_{\tau_j},\mathbf{u},\mathbf{v})\,d\mathbf{u}d\mathbf{v}}{\sum_{\tau\in\mathcal{P}_K}\prod_{j=1}^{\ell}\int_{\{\mathbf{u}<\mathbf{y}_{\tau_j^c}\}} h_{(\mathbf{t}_{\tau_j},\mathbf{t}_{\tau_j^c})}(\mathbf{y}_{\tau_j},\mathbf{u})\,d\mathbf{u}} \frac{\exp(-\bar{\mu}_{(\mathbf{t},\mathbf{s})}(\mathbf{y},\mathbf{z}))}{\exp(-\bar{\mu}_{\mathbf{t}}(\mathbf{y}))} \quad \nu_{\mathbf{t}}(d\mathbf{y})\text{-}a.e. \end{aligned} \tag{17}$$

Proof of Proposition 4.1:
The Proposition is a straightforward reformulation of Theorems 3.1 and 3.2 based on the notion of regular exponent measure and on the following simple expression of the finite dimensional marginals of the conditional distribution $P_{\mathbf{s}}(\mathbf{y}, df)$. If all the components of $\mathbf{s}_1 \in T^{l_1}$ and $\mathbf{s}_2 \in T^{l_2}$ are mutually distinct, then

$$P_{\mathbf{s}_1}(\mathbf{y}_1, f(\mathbf{s}_2)\in d\mathbf{y}_2) = \frac{h_{(\mathbf{s}_1,\mathbf{s}_2)}(\mathbf{y}_1,\mathbf{y}_2)}{h_{\mathbf{s}_1}(\mathbf{y}_1)}d\mathbf{y}_2, \quad \mu_{\mathbf{s}_1}(d\mathbf{y}_1)\text{-a.e..} \tag{18}$$

In particular,

$$h_{\mathbf{s}_1}(\mathbf{y}_1)P_{\mathbf{s}_1}(\mathbf{y}_1,\{f(\mathbf{s}_2)<\mathbf{y}_2\}) = \int_{\{\mathbf{z}<\mathbf{y}_2\}} h_{(\mathbf{s}_1,\mathbf{s}_2)}(\mathbf{y}_1,\mathbf{z})d\mathbf{z} \quad \mu_{\mathbf{s}_1}(d\mathbf{y}_1)\text{-a.e..}$$

For example, Equations (18) and (10) together imply, for any $\tau\in\mathcal{P}_K$,

$$\frac{d\nu_{\mathbf{t}}}{d\mathbf{y}}(y) = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]\prod_{j=1}^{\ell(\tau)} P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},\{f(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\})h_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j}) \quad d\mathbf{y}\text{-a.e.,}$$

and also

$$\frac{d\nu^{\tau}_{\mathbf{t}}}{d\mathbf{y}}(y) = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})] \sum_{\tau\in\mathcal{P}_K} \prod_{j=1}^{\ell(\tau)} P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\}) h_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j}) \quad d\mathbf{y}\text{-a.e.},$$

Using this, Equation (10) entails, for $\tau \in \mathcal{P}_K$ and $\nu_{\mathbf{t}}(d\mathbf{y})$-a.e.

$$\begin{aligned} \pi_{\mathbf{t}}(\mathbf{y},\tau) &= \frac{d\nu^{\tau}_{\mathbf{t}}}{d\nu\mathbf{t}}(y) = \frac{\frac{d\nu^{\tau}_{\mathbf{t}}}{d\mathbf{y}}(y)}{\frac{d\nu_{\mathbf{t}}}{d\mathbf{y}}(y)} \\ &= \frac{\prod_{j=1}^{\ell(\tau)} P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\}) h_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j})}{\sum_{\tau'\in\mathcal{P}_K} \prod_{j=1}^{\ell(\tau')} P_{\mathbf{t}_{\tau'_j}}(\mathbf{y}_{\tau'_j}, \{f(\mathbf{t}_{\tau_j'^c}) < \mathbf{y}_{\tau_j'^c}\}) h_{\mathbf{t}_{\tau'_j}}(d\mathbf{y}_{\tau'_j})}. \end{aligned}$$

The remaining details are omitted for the sake of brevity. ☐

Remark 4.1. The notion of regular model has been introduced in order to provide simple expressions for the density of $\nu_{\mathbf{t}}$ with respect to the Lebesgue measure and for the distribution $\pi_{\mathbf{t}}(\mathbf{y},\cdot)$. This raises the natural question to determine the class of regular models. An answer is given by the following observation. For all $\mathbf{s} \in T^l$ with pairwise distinct components, the following statements are equivalent.

i) $\mu_{\mathbf{s}}$ is absolutely continuous with respect to the Lebesgue measure and $\bar{\mu}_{\mathbf{s}}(0) = +\infty$;

ii) $\nu_{\mathbf{s}}$ is absolutely continuous with respect to the Lebesgue measure.

The proof of this equivalence is based on Equation (10) and the details are omitted. As a consequence, if $\mu_t((0,+\infty)) = +\infty$ for all $t \in T$, then the model is regular if and only if the distribution $\nu_{\mathbf{s}}$ is absolutely continuous with respect to the Lebesgue measure for all $\mathbf{s} \in T^l$ with pairwise distinct components.

4.2 Max-stable models

We put the emphasis here on max-stable random fields. For convenience and without loss of generality, we focus on simple max-stable random fields η, i.e., with standard unit Fréchet margins

$$\mathbb{P}(\eta(t) \le x) = \exp[-x^{-1}]1_{\{x>0\}}, \quad x \in \mathbb{R},\ t \in T.$$

Any general max-stable random field in $\mathbb{C}(T)$ can be related to such a simple max-stable random field by simple transformation of the margins (see e.g. Corollary 3.6 in [11]).

Any simple max-stable random field η is max-i.d. with vertex function $h \equiv 0$. The corresponding exponent measure μ on $\mathbb{C}_0$ is homogeneous of order -1. More precisely, the mapping $f \mapsto (f/\|f\|, \|f\|)$ allows us to identify $\mathbb{C}_0$ with $\mathbb{S}_{\mathbb{C}_0} \times (0,+\infty)$ where $\mathbb{S}_{\mathbb{C}_0} = \{f \in \mathbb{C}_0;\ \|f\| = 1\}$. According to Proposition 3.2 in [11], η is simple max-stable if and only if μ is of the form $d\mu = d\sigma \times r^{-2}dr$ for some finite Borel measure σ on $\mathbb{S}_{\mathbb{C}_0}$ such that

$$\int_{\mathbb{S}_{\mathbb{C}_0}} f(t)\,\sigma(df) = 1, \quad t \in T.$$

The measure σ is unique and referred to as the *spectral measure*.

It is worth noting that we can associate a simple max-stable random field η to any exponent measure of the form $d\mu = d\tilde{\sigma} \times r^{-2}dr$ where $\tilde{\sigma}$ is a Borel measure on $\mathbb{C}_0$ such that

$$\int_{\mathbb{C}_0} \|f\|\,\tilde{\sigma}(df) < \infty \quad \text{and} \quad \int_{\mathbb{S}_{\mathbb{C}_0}} f(t)\,\tilde{\sigma}(df) = 1, \quad t \in T. \tag{19}$$

The spectral measure σ is related to $\tilde{\sigma}$ by the relation

$$\sigma(A) = \int_{\mathbb{C}_0} \|f\| 1_{\{f/\|f\| \in A\}}\,\tilde{\sigma}(df), \quad A \subset \mathbb{S}_{\mathbb{C}_0} \text{ Borel set.}$$

We apply Proposition 2.2 in this max-stable framework. We easily check that in this case, for all $t \in T$, $\mu_t(dy) = y^{-2}1_{\{y>0\}}dy$ and $\bar{\mu}_t(y) = y^{-1}$, $y > 0$. Taking into account the product form of μ, standard computation entails the following Proposition.

Proposition 4.2. *Let η be a simple max-stable random field with exponent measure $d\mu = d\tilde{\sigma} \times r^{-2}dr$ satisfying* (19)*. Then, for any $t \in T$ the extremal function ϕ_t^+ associated to the point t is a.s. uniquely defined and its distribution satisfies*

$$\mathbb{P}(\phi_t^+ \in B) = \int_{\mathbb{C}_0 \times (0,+\infty)} 1_{\{yf/f(t) \in B\}} f(t)\tilde{\sigma}(df) \exp[-y^{-1}] y^{-2}dy, \quad B \subset \mathbb{C}_0 \text{ Borel set.}$$

Furthermore, the regular conditional distribution of ϕ_t^+ given $\phi_t^+(t) = y$ satisfies $\mu_t(dy)$-a.s.

$$P_t(y, B) = \int_{\mathbb{C}_0} 1_{\{\frac{y}{f(t)} f \in B\}} f(t)\tilde{\sigma}(df), \quad B \subset \mathbb{C}_0 \text{ Borel set.}$$

Details of the proof are omitted here for the sake of brevity. It should be noted that, for $\mathbf{t} \in T^l$ and $\mathbf{y} \in (0,+\infty)^l$, the conditional measures $P_{\mathbf{t}}(\mathbf{y}, df)$ can be obtained from $P_{t_1}(y_1, df)$ by successively conditioning with respect to $f(t_2) = y_2, f(t_3) = y_3, \ldots, f(t_k) = y_k$.

Proposition 4.2 together with Theorem 3.2 is enough to provide an explicit expression of the conditional distribution of η with respect to $\eta(t) = y$, i.e., when the number of conditioning point is equal to one. To deal with more conditioning points, we have to deal with the conditional distribution of the hitting scenario. To push the computations forward, we will need to be more specific about the form of the measure $\tilde{\sigma}$. We will mostly consider two cases:

- max-stable models based on Gaussian random fields as in Kabluchko [13] or Kabluchko, Schlather and de Hahn [14]. These random fields are connected to extremes of Gaussian processes and also to Brown-Resnick processes. Here $\tilde{\sigma}$ is the distribution of
$$Y(t) = \exp[W(t) - \sigma^2(t)/2], \quad t \in T,$$
for some sample continuous centered Gaussian random field $W = \{W(t)\}_{t\in T}$ with covariance function $\gamma(s,t) = \mathbb{E}[W(s)W(t)]$ and variance $\sigma^2(t) = \gamma(t,t) = \mathbb{E}[W(t)^2]$. Note in [13, 14], the authors consider max-stable random fields with Gumbel marginals rather than unit Fréchet.

- max-linear processes, as considered by Wang and Stoev [17]. Here $\tilde{\sigma}$ is a discrete measure: $\tilde{\sigma} = \sum_{i=1}^q p_i \delta_{f_i}$ for some $q \geq 1$, $(p_i)_{1\leq i\leq q} \in (0,+\infty)^q$ and $(f_i)_{1\leq i\leq q} \in \mathbb{C}_0^q$ such that $\sum_{i=1}^q p_i f_i(t) = 1$, $t \in T$.

4.3 Max-stable models based on Gaussian random fields

Let W be a sample continuous centered Gaussian process with covariance function $\gamma(s,t) = \mathbb{E}[W(s)W(t)]$ and variance $\sigma^2(t) = \gamma(t,t) = \mathbb{E}[W(t)^2]$. Denote by $\tilde{\sigma}(df)$ the distribution of the log-normal process

$$Y(t) = \exp[W(t) - \sigma^2(t)/2], \quad t \in T.$$

Note for all $t \in T$, $\mathbb{E}[Y(t) = 1]$ which is equivalent to the condition $\int_{\mathbb{C}_0} f(t)\,\tilde{\sigma}(df) = 1$.

Let $t \in T$ and $y > 0$. According to Proposition 4.2, the conditional distribution $P_t(y, df)$ of the extremal function ϕ_t^+ given $\eta(t) = y$ satisfies

$$\begin{aligned} P_t(y,B) &= \mathbb{E}[Y(t)1_{\{yY(\cdot)/Y(t)\in B\}}] \\ &= \mathbb{E}[e^{W(t)-\sigma^2(t)/2}1_{\{ye^{W(\cdot)-W(t)-(\sigma^2(\cdot)-\sigma^2(t))/2}\in B\}}] \end{aligned}$$

Standard results on weighted Gaussian processes imply that $P_t(y, df)$ is the distribution of a log-normal random field:

$$P_t(y,B) = \mathbb{P}[ye^{G_t} \in B]$$

where $G_t = \{G_t(s)\}_{s\in T}$ is a Gaussian random field with mean and covariance

$$\mathbb{E}[G_t(s)] = \gamma(s,t) - \frac{1}{2}\{\gamma(t,t)+\gamma(s,s)\}, \quad s \in T,$$
$$\mathrm{Cov}(G_t(s_1), G_t(s_2)) = \gamma(t,t) + \gamma(s_1,s_2) - \gamma(s_1,t) - \gamma(s_2,t), \quad s_1, s_2 \in T.$$

More generally, for $\mathbf{t} \in T^l$ and $\mathbf{y} \in (0,+\infty)^l$, the distribution $P_t(y, df)$ defines a log-normal process: it can be obtained from the log-normal distribution $P_{t_1}(y_1, df)$ by further conditioning with respect to $f(t_2) = y_2, \ldots, f(t_l) = y_l$.

To push the computations forward, we will assume for the sake of simplicity that the Gaussian process W is non-degenerate, in the sense that for all $l \geq 1$ and $\mathbf{s} \in T^l$ with pairwise distinct coordinates, the Gaussian vector $W(\mathbf{s}) = (W(s_1), \ldots, W(s_l))$ has a positive definite covariance matrix $\Gamma_{\mathbf{s}}$. In this case, the distribution of $W(\mathbf{s})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^l$ and has density

$$g_{\mathbf{s}}(\mathbf{z}) = (2\pi)^{-l/2}\det(\Gamma_{\mathbf{s}})^{-1/2}\exp\Big[-\frac{1}{2}\mathbf{z}'\Gamma_{\mathbf{s}}^{-1}\mathbf{z}\Big], \quad \mathbf{z} \in \mathbb{R}^l.$$

We consider the simple max-stable process with exponent measure $d\mu = d\tilde{\sigma} \times r^{-2}dr$, i.e.,

$$\mu(A) = \int_0^\infty \mathbb{P}(rY \in A)\, r^{-2}dr, \quad A \subset \mathbb{C}_0 \text{ Borel set.}$$

We prove below that this max-stable model is regular in the sense of section 4.1. For all $\mathbf{s} \in T^l$ with pairwise distinct coordinates and all Borel set $B \subset [0,+\infty)^l$, we have

$$\mu_{\mathbf{s}}(B) = \int_0^\infty \mathbb{P}(rY(\mathbf{s}) \in B)\, r^{-2}dr = \int_0^\infty \int_{\mathbb{R}^l} 1_{\{re^{(\mathbf{z}-\sigma_{\mathbf{s}}^2/2)} \in B\}}\, g_{\mathbf{s}}(\mathbf{z})d\mathbf{z}\, r^{-2}dr$$

where $\sigma_{\mathbf{s}}^2 = \{\sigma^2(s_i)\}_{1\leq i\leq l}$. The change of variable $\mathbf{y} = re^{(\mathbf{z}-\sigma_{\mathbf{s}}^2/2)}$ yields

$$\mu_{\mathbf{s}}(A) = \int_0^\infty \int_B g_{\mathbf{s}}(\ln\mathbf{y} - \ln r + \sigma_{\mathbf{s}}^2/2)\prod_{i=1}^l y_i^{-1}\, d\mathbf{y}\, r^{-2}dr = \int_B h_{\mathbf{s}}(\mathbf{y})\, d\mathbf{y}$$

where the Radon-Nikodym derivative of $\mu_{\mathbf{s}}$ with respect to the Lebesgue measure equals

$$h_{\mathbf{s}}(\mathbf{y}) = \prod_{i=1}^l y_i^{-1}\int_0^\infty g_{\mathbf{s}}(\ln\mathbf{y} - \ln r + \sigma_{\mathbf{s}}^2/2)\, r^{-2}dr.$$

Let $1_{\mathbf{s}} = (1)_{1\leq i\leq l}$. Standard computations for Gaussian integrals reveal that

$$h_{\mathbf{s}}(\mathbf{y}) = C_{\mathbf{s}}\exp\Big[-\frac{1}{2}\ln\mathbf{y}'Q_{\mathbf{s}}\ln\mathbf{y} + L_{\mathbf{s}}\ln\mathbf{y}\Big]\prod_{i=1}^l y_i^{-1}$$

with

$$
\begin{aligned}
Q_{\mathbf{s}} &= \Gamma_{\mathbf{s}}^{-1} - \frac{\Gamma_{\mathbf{s}}^{-1}1_{\mathbf{s}}1_{\mathbf{s}}'\Gamma_{\mathbf{s}}^{-1}}{1_{\mathbf{s}}'\Gamma_{\mathbf{s}}^{-1}1_{\mathbf{s}}},\\
L_{\mathbf{s}} &= \frac{1}{2}\Big(\frac{1_{\mathbf{s}}'\Gamma_{\mathbf{s}}^{-1}\sigma_{\mathbf{s}}^2 - 2}{1_{\mathbf{s}}'\Gamma_{\mathbf{s}}^{-1}1_{\mathbf{s}}}1_{\mathbf{s}}'\Gamma_{\mathbf{s}}^{-1} - {\sigma^{2}}'_{\mathbf{s}}\Gamma_{\mathbf{s}}^{-1}\Big),\\
C_{\mathbf{s}} &= (2\pi)^{(1-l)/2}\det(\Gamma_{\mathbf{s}})^{-1/2}(1_{\mathbf{s}}'\Gamma_{\mathbf{s}}^{-1}1_{\mathbf{s}})^{-1/2}\exp\Big[\frac{1}{2}\frac{(1_{\mathbf{s}}'\Gamma_{\mathbf{s}}^{-1}\sigma_{\mathbf{s}}^2 - 1)^2}{1_{\mathbf{s}}'\Gamma_{\mathbf{s}}^{-1}1_{\mathbf{s}}} - \frac{1}{2}{\sigma^{2}}'_{\mathbf{s}}\Gamma_{\mathbf{s}}^{-1}\sigma_{\mathbf{s}}^2\Big].
\end{aligned}
$$

Please recall that μ and hence $\mu_{\mathbf{s}}$ are measures with infinite total mass. The function $h_{\mathbf{s}}$ looks like a log-normal density but the matrix $Q_{\mathbf{s}}$ is not definite positive since $1_{\mathbf{s}}'Q_{\mathbf{s}}1_{\mathbf{s}} = 0$.

Proposition 4.1 can be used to determine the different distributions and a forthcoming paper will be devoted to the computational aspect of conditional sampling for max-stable models based on Gaussian random fields.

4.4 Max-linear models

Following Wang and Stoev [17], we consider max-linear processes, i.e., spectrally discrete max stable processes. Here $\tilde{\sigma} = \sum_{i=1}^q p_i\delta_{f_i}$ is a discrete measure, with $q \geq 1$, $p_1, \ldots, p_q > 0$ and $f_1, \ldots, f_q \in \mathbb{C}_0$ such that $\sum_{i=1}^q p_i f_i(t) = 1$, $t \in T$. For $\mathbf{s} \in T^l$ and $A \subset [0, +\infty)^l$ Borel set,

$$\mu_{\mathbf{s}}(A) = \sum_{i=1}^q p_i \int_0^\infty 1_A(rf_i(\mathbf{s}))r^{-2}dr. \tag{20}$$

Note $\mu_{\mathbf{s}}(\{0\}) = +\infty$ if $f_i(\mathbf{s}) = 0$ for some $i \in [\![1, q]\!]$. The support of $\mu_{\mathbf{s}}$ is equal to $V_{\mathbf{s}} = \cup_{i\in I_{\mathbf{s}}} V_{\mathbf{s}}^i$ where

$$V_{\mathbf{s}}^i = \{rf_i(\mathbf{s}),\ r \geq 0\} \subset [0, +\infty)^l, \quad I_{\mathbf{s}} = \{i \in [\![1, q]\!],\ f_i(\mathbf{s}) \neq 0\}$$

For $i \in I_{\mathbf{s}}$, denote by $\lambda_{V_{\mathbf{s}}^i}$ the Lebesgue measure on the half-line $V_{\mathbf{s}}^i$ endowed with the restriction of the canonical euclidean product on $\mathbb{R}^l$ and the associated euclidean norm $\|\cdot\|_2$. Please note that

$$\lambda_{V_{\mathbf{s}}^i}(A) = \|f_i(\mathbf{s})\|_2 \int_0^\infty 1_A(rf_i(\mathbf{s}))dr, \quad A \subset [0, +\infty)^l \text{ Borel set}$$

so that the segment $\{rf_i(\mathbf{s});\ r \in [r_1, r_2]\}$ has measure $(r_2 - r_1)\|f_i(\mathbf{s})\|_2$ for all $r_2 > r_1$. Equation (20) entails that the restriction of $\mu_{\mathbf{s}}$ on $[0, +\infty)^l \setminus \{0\}$ is equal to

$$\mu_{\mathbf{s}}(d\mathbf{y}) = \sum_{i\in I_{\mathbf{s}}} p_i\|f_i(\mathbf{s})\|_2\|\mathbf{y}\|_2^{-2}\lambda_{V_{\mathbf{s}}^i}(d\mathbf{y}).$$

According to Theorem 3.1 Equation (10), for $\tau \in \mathcal{P}_K$,

$$\begin{aligned} &\nu^\tau_{\mathbf{t}}(d\mathbf{y}) && (21)\\ = \;& \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]\bigotimes_{j=1}^{\ell}\Big\{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},\{f(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\})\sum_{i_j\in I_{\mathbf{t}_{\tau_j}}} p_{i_j}\|f_{i_j}(\mathbf{t}_{\tau_j})\|_2\|\mathbf{y}_{\tau_j}\|_2^{-2}\lambda_{V^{i_j}_{\mathbf{t}_{\tau_j}}}(d\mathbf{y}_{\tau_j})\Big\}. \end{aligned}$$

Denote by $I^\tau_{\mathbf{t}}$ the set of multi-indices $\mathbf{i} = (i_1, \dots, i_\ell)$ such that $i_j \in I_{\mathbf{t}_{\tau_j}}$ for all $j = 1, \dots, \ell$. For $\mathbf{i} \in I^\tau_{\mathbf{t}}$, define

$$\omega_{\tau,\mathbf{i}}(by) = \prod_{j=1}^{\ell} P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},\{f(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\})\; p_{i_j}\|f_{i_j}(\mathbf{t}_{\tau_j})\|_2\|\mathbf{y}_{\tau_j}\|_2^{-2}$$

and $V^{\tau,\mathbf{i}}_{\mathbf{t}}$ the ℓ-dimensional cone

$$V^{\tau,\mathbf{i}}_{\mathbf{t}} = \{\mathbf{y} \in [0,+\infty)^k;\ \forall j \in [\![1,\ell]\!],\ \mathbf{y}_{\tau_j} \in V^{i_j}_{\mathbf{t}_{\tau_j}}\}.$$

Let $\lambda_{V^{\tau,\mathbf{i}}_{\mathbf{t}}}$ be the Lebesgue measure on $V^{\tau,\mathbf{i}}_{\mathbf{t}}$ and note that $V^{\tau,\mathbf{i}}_{\mathbf{t}}(d\mathbf{y}) = \otimes_{j=1}^{\ell}\lambda_{V^{i_j}_{\mathbf{t}_{\tau_j}}}(d\mathbf{y}_{\tau_j})$. Using this, Equation (21) becomes

$$\nu^\tau_{\mathbf{t}}(d\mathbf{y}) = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]\sum_{\mathbf{i}\in I^\tau_{\mathbf{t}}}\omega_{\tau,\mathbf{i}}(\mathbf{y})\lambda_{V^{\tau,\mathbf{i}}_{\mathbf{t}}}(d\mathbf{y}). \tag{22}$$

For $\mathbf{y} \in (0,+\infty)^l$, define $I^\tau_{\mathbf{t}}(\mathbf{y}) = \{\mathbf{i} \in I^\tau_{\mathbf{t}};\ \omega_{\tau,\mathbf{i}} \neq 0\}$. The support $S_{\mathbf{t}}$ of $\nu_{\mathbf{t}}$ is equal to the sets of $\mathbf{y}$'s such that $I^\tau_{\mathbf{t}}(\mathbf{y}) \neq \varnothing$ for some $\tau \in \mathcal{P}_K$. Given $\mathbf{y} \in S_{\mathbf{t}}$, we define the set of admissible hitting scenarios

$$\mathcal{P}_K(y) = \{\pi \in \mathcal{P}_K;\ I^\tau_{\mathbf{t}}(\mathbf{y}) \neq \varnothing\}.$$

Let $\hat{\ell}(\mathbf{y}) = \min\{\ell(\tau);\ \tau \in \mathcal{P}_K(\mathbf{y})$ be the minimal length of an admissible hitting scenario and

$$\widehat{\mathcal{P}}_K(y) = \{\pi \in \mathcal{P}_K(\mathbf{y});\ \ell(\tau) = \hat{\ell}(\mathbf{y})\}$$

the set of admissible hitting scenarios of minimal length. The hitting scenario conditional distribution $\pi_{\mathbf{t}}(\mathbf{y},\tau) = \frac{d\nu^\tau_{\mathbf{t}}}{d\nu_{\mathbf{t}}}(\mathbf{y})$ is given for $\mathbf{y} \in S_{\mathbf{t}}$ by

$$\pi_{\mathbf{t}}(\mathbf{y},\tau) = \frac{\omega_\tau(\mathbf{y})1_{\{\tau\in\widehat{\mathcal{P}}_K(y)\}}}{Z_{\mathbf{t}}(\mathbf{y})}, \tag{23}$$

with $\omega_\tau(\mathbf{y}) = \sum_{\mathbf{i}\in I^\tau_{\mathbf{t}}}\omega_{\tau,\mathbf{i}}(\mathbf{y})$ and $Z_{\mathbf{t}}(\mathbf{y}) = \sum_{\tau\in\widehat{\mathcal{P}}_K(y)}\omega_\tau(\mathbf{y})$ the normalization constant. Clearly, the distribution $\pi_{\mathbf{t}}(\mathbf{y},\cdot)$ is supported by $\widehat{\mathcal{P}}_K(y)$. In order to prove Equation (23), it is enough to show that

$$\nu^\tau_{\mathbf{t}}(A) = \int_A \pi_{\mathbf{t}}(\mathbf{y},\tau)\nu_{\mathbf{t}}(d\mathbf{y}), \quad A \subset [0,+\infty)^k \text{ Borel set.} \tag{24}$$

Clearly, Equation (22) entails

$$\int_A \pi_{\mathbf{t}}(\mathbf{y},\tau)\nu_{\mathbf{t}}(d\mathbf{y}) = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]\int_A \frac{\omega_\tau(\mathbf{y})1_{\{\tau\in\widehat{\mathcal{P}}_K(y)\}}}{Z_{\mathbf{t}}(\mathbf{y})}\sum_{\tau'\in\mathcal{P}_K}\sum_{\mathbf{i}\in I_{\mathbf{t}}^{\tau'}}\omega_{\tau',\mathbf{i}}(\mathbf{y})\lambda_{V_{\mathbf{t}}^{\tau',\mathbf{i}}}(d\mathbf{y})$$

and

$$\nu_{\mathbf{t}}^{\tau}(A) = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]\int_A\sum_{\mathbf{i}\in I_{\mathbf{t}}^{\tau}}\omega_{\tau,\mathbf{i}}(\mathbf{y})\lambda_{V_{\mathbf{t}}^{\tau,\mathbf{i}}}(d\mathbf{y}).$$

Using this, Equation (24) follows from the following observations:

- if $\tau'\notin\widehat{\mathcal{P}}_K(y)$, then $1_{\{\tau\in\widehat{\mathcal{P}}_K(y)\}}=0$ $\lambda_{V_{\mathbf{t}}^{\tau',\mathbf{i}}}(d\mathbf{y})$-a.e, because the set
$$\{y;\tau\in\widehat{\mathcal{P}}_K(y)\}\subset\cup_{\mathbf{i}\in I_{\mathbf{t}}^{\tau}}V_{\mathbf{t}}^{\tau,\mathbf{i}}$$
has dimension $\hat{\ell}(\mathbf{y})$ and the measure $\lambda_{V_{\mathbf{t}}^{\tau',\mathbf{i}}}$ is a Lebesgue measure in dimension $\ell(\tau')>\hat{\ell}(y)$;
- if $\tau_1',\tau_2'\in\widehat{\mathcal{P}}_K(y)$, the measures $\lambda_{V_{\mathbf{t}}^{\tau_1',\mathbf{i}}}(d\mathbf{y})$ and $\lambda_{V_{\mathbf{t}}^{\tau_2',\mathbf{i}}}(d\mathbf{y})$ are either equal (if $V_{\mathbf{t}}^{\tau_1',\mathbf{i}}=V_{\mathbf{t}}^{\tau_2',\mathbf{i}}$) or mutually singular (if $V_{\mathbf{t}}^{\tau_1',\mathbf{i}}\neq V_{\mathbf{t}}^{\tau_2',\mathbf{i}}$).

According to Proposition 4.2, the conditional probability measure $P_t(y,df)$ is given for $t\in T$ and $y>0$ by

$$P_t(y,df)=\sum_{i=1}^q p_if_i(t)\delta_{\frac{y}{f_i(t)}f_i}(df).$$

For $\mathbf{y}>0$ in the support $V_{\mathbf{s}}$ of $\mu_{\mathbf{s}}$, $P_{\mathbf{s}}(\mathbf{y},df)$ is obtained by conditioning $P_{s_1}(y_1,df)$ with respect to $f(s_2)=y_2,\dots,f(s_l)=y_l$. We obtain

$$P_{\mathbf{s}}(\mathbf{y},df)=\frac{\sum_{i=1}^q p_if_i(s_1)1_{\{\mathbf{y}\in V_{\mathbf{s}}^i\}}\delta_{\frac{y_1}{f_i(s_1)}f_i}(df)}{\sum_{i=1}^q p_if_i(s_1)1_{\{\mathbf{y}\in V_{\mathbf{s}}^i\}}}.$$

The above computations can be used to determine the conditional distributions $Q_{\mathbf{t}}(\mathbf{y},\cdot)$ and $R_{\mathbf{t}}(\mathbf{y},\cdot)$ given by Theorems 3.1 and 3.2. Comparison with the results of Wang and Stoev [17] is not straightforward because the authors use the alternative representation

$$(\eta(t))_{t\in T}\overset{\mathcal{L}}{=}(\vee_{i=1}^q Z_ip_if_i(t))_{t\in T}$$

where $Z_1,\dots,Z_q$ are i.i.d. random variables with unit Fréchet distribution, i.e.

$$\mathbb{P}[Z_i\le z]=\exp(-1/z),\quad z>0.$$

They determine the conditional distribution of $\mathbf{Z}=(Z_1,\dots,Z_q)$ with respect to $\eta(\mathbf{t})=\mathbf{y}$. A similar notion of hitting scenario with minimal rank is a key tool in their expression.

A Auxiliary results

A.1 Slyvniak's formula

Palm Theory deals with conditional distribution for point processes. We recall here one of the most famous formula of Palm theory, known as Slyvniak's Theorem. This will be the main tool in our computations. For a general reference on Poisson point processes, Palm theory and their applications, the reader is invited to refer to the monograph [16] by Stoyan, Kendall and Mecke.

Let $M_p(\mathbb{C}_0)$ be the set of locally-finite point measures N on $\mathbb{C}_0$ endowed with the σ-algebra generated by the family of mappings $\{N \mapsto N(A),\ A \subset \mathbb{C}_0 \text{ Borel set}\}$.

Theorem A.1 (Slyvniak's Formula).
Let Φ be a Poisson point process on $\mathbb{C}_0$ with intensity measure μ. For all measurable function $F : \mathbb{C}_0^k \times M_p(\mathbb{C}_0) \to [0, +\infty)$,

$$\begin{aligned} & \mathbb{E}\Big[\int_{\mathbb{C}_0^k} F\Big(\phi_1, \dots, \phi_k, \Phi - \sum_{i=1}^{k} \delta_{\phi_i}\Big)\, \Phi(d\phi_1)\, (\Phi - \delta_{\phi_1})(d\phi_2) \cdots (\Phi - \sum_{j=1}^{k-1} \delta_{\phi_j})(d\phi_k)\Big] \\ = & \int_{\mathbb{C}_0^k} \mathbb{E}[F(f_1, \dots, f_k, \Phi)]\, \mu^{\otimes k}(df_1, \dots, df_k). \end{aligned}$$

A.2 Regular conditional distribution

We recall here briefly the notion of regular conditional probability (see e.g. Proposition A1.5.III in Daley and Vere-Jones [3]). Let $(\mathcal{Y}, \mathcal{G})$ be a complete separable metric space with its associated σ-algebra of Borel sets, $(\mathcal{X}, \mathcal{F})$ an arbitrary measurable space, and π a probability measure on the product space $(\mathcal{X} \times \mathcal{Y}, \mathcal{F} \otimes \mathcal{G})$. Let $\pi_{\mathcal{X}}$ denote the $\mathcal{X}$-marginal of π, i.e. $\pi_{\mathcal{X}}(A) = \pi(A \times \mathcal{Y})$ for all $A \in \mathcal{F}$. Then there exists a family of kernels $K(x, B)$ such that

- $K(x, \cdot)$ is a probability measure on $(\mathcal{Y}, \mathcal{G})$ for all fixed $x \in \mathcal{X}$;
- $K(\cdot, B)$ is an $\mathcal{F}$-measurable function on $\mathcal{X}$ for each fixed $B \in \mathcal{G}$;
- $\pi(A \times B) = \int_A K(x, B) \pi_{\mathcal{X}}(dx)$ for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$.

These tree properties define the notion of regular conditional probability. We also speak of disintegration of π. Existence is ensured by the assumption that $\mathcal{Y}$ is a complete and separable metric space. Furthermore, for all $\mathcal{F} \otimes \mathcal{G}$-measurable non-negative function f on $X \times Y$, it holds

$$\int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\pi(dx,dy) = \int_{\mathcal{X}}\int_{\mathcal{Y}} f(x,y)Q(x,dy)\pi_{\mathcal{X}}(dx).$$

A.3 Measurability properties

Measurability of Φ_K^+ and Φ_K^-:
We show that the random point measures Φ_K^+ and Φ_K^- from Definition 2.1 are measurable from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(M_p(\mathbb{C}_0), \mathcal{M}_p)$. To see this, it is enough to prove that that the events $\{\phi_i <_K \eta\} \in \mathcal{F}$ and $\{\phi_i \not<_K \eta\}$ are $\mathcal{F}$-measurable. Let K_0 be a dense countable subset of K and note that $\phi <_K \eta$ if and only if there is some rational $\varepsilon > 0$ so that $\phi(t) < \eta(t) - \varepsilon$ for all $t \in K_0$. Hence, for all $n \in \mathbb{N} \cup \{+\infty\}$,

$$\begin{aligned} \{\phi_i <_K \eta\ ;\ N = n\} &= \bigcup_{\varepsilon>0}\bigcap_{t\in K_0}\{N = n\ ;\ \phi_i(t) < \eta(t) - \varepsilon\} \\ &= \bigcup_{\varepsilon>0}\bigcap_{t\in K_0}\bigcup_{j\le n}\{N = n\ ;\ \phi_i(t) < \phi_j(t) - \varepsilon\} \end{aligned} \tag{25}$$

and $\{\phi_i <_K \eta\} = \bigcup_{n=0}^{\infty}\{\phi_i <_K \eta\ ;\ N = n\} \in \mathcal{F}$.

Measurability of C_K^+ and $C_K^-(g)$:
Let g be a continuous function defined at least on K and consider the Borel set

$$A = \{f \in \mathbb{C}_0;\ f \not<_K g\} \subset \mathbb{C}_0.$$

The set $C_K^-(g)$ defined by Equation (4) is equal to

$$C_K^-(g) = \{M \in M_p(\mathbb{C}_0);\ M(A) = 0\}$$

and is $\mathcal{M}_p$-measurable.
In order to prove the measurability of C_K^+ defined by Equation (3), we introduce a measurable enumeration of the atoms of a point measure M (see Lemma 9.1.XIII in Daley and Vere-Jones [4]). Given an ordered dissecting system of $\mathbb{C}_0$, one can construct measurable applications $\kappa : M_p(\mathbb{C}_0) \to \mathbb{N} \cup \{\infty\}$ and $\psi_i : M_p(\mathbb{C}_0) \to \mathbb{C}_0$, $i \geq 1$, such that

$$M = \sum_{i=1}^{\kappa(M)} \delta_{\psi_i(M)}, \quad M \in M_p(\mathbb{C}_0).$$

A point measure M does not lie in C_K^+ if and only if it has a K-subsextremal atom:

$$M_p(\mathbb{C}_0) \setminus C_K^+ = \bigcup_{k=0}^{+\infty}\bigcup_{i=1}^{k}\{\kappa(M) = k;\ \psi_i(M) <_K \max(M)\}.$$

Similar computations as in Equation (25) entail

$$\{\kappa(M)=k;\ \psi_i(M)<_K \max(M)\} = \bigcup_{\varepsilon>0}\bigcap_{t\in K_0}\bigcup_{j\leq k}\{\kappa=k\ ;\ \psi_i(t)<\psi_j(t)-\varepsilon\},$$

whence C_K^+ is $\mathcal{M}_p$-measurable.

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