A shape theorem for an epidemic model in dimension $d \geq 3$

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Abstract

We prove a shape theorem for the set of infected individuals in a spatial epidemic model with 3 states (susceptible-infected-recovered) on $\mathbb{Z}^d, d \geq 3$, when there is no extinction of the infection. For this, we derive percolation estimates (using dynamic renormalization techniques) for a locally dependent random graph in correspondence with the epidemic model.

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1 Introduction

Mollison has introduced in [13], [14] a stochastic spatial general epidemic model on $\mathbb{Z}^d$, describing the evolution of individuals submitted to infection by contact contamination of infected neighbors. More precisely, each site of

$\mathbb{Z}^d$ can be healthy, infected, or immune. At time 0, there is an infected individual at the origin, and all other sites are occupied by healthy individuals. An infected individual emits germs according to a Poisson process, it stays infected for a random time, then it recovers and becomes immune to further infection. A germ emitted from $x \in \mathbb{Z}^d$ goes to one of the neighbors $y \in \mathbb{Z}^d$ of x chosen at random. If the individual at y is healthy then it becomes infected and begins to emit germs; if this individual is infected or immune, nothing happens. The germ emission processes and the durations of infections of different individuals are mutually independent.

Since its introduction, this epidemic model has given rise to many studies, and to other models that are variations of this "SIR" (Susceptible-Infected-Recovered) structure. A first direction is whether the different states aymptotically survive or not, according to the values of the involved parameters (e.g. the infection and recovery rates). A second direction is the obtention of a shape theorem for the asymptotic behavior of infected individuals, when there is no extinction of the infection (throughout this paper, extinction is understood as extinction of the infection).

Kelly in [8] proved that for $d = 1$, extinction is almost sure for the spatial general epidemic model. Kuulasmaa in [10] has studied the threshold behavior of this model in dimension $d \geq 2$. He proved that the process has a critical infection rate below which extinction is almost certain, and above which there is survival. His work (as well as the following ones on this model) is based on the analysis of a directed oriented percolation model, that he calls a locally dependent random graph, in correspondence with the epidemic model. See also the related paper [11].

Cox & Durrett have derived in [5] a shape theorem for the set of infected individuals in the spatial general epidemic model on $\mathbb{Z}^2$, when there is no extinction, and the contamination rule is nearest neighbor. This result was extended to a finite range contamination rule by Zhang in [15]. The proofs in [5], [15] are based on the correspondence with the locally dependent random graph, and they refer to [4] (which deals with first passage percolation). They rely on the introduction of circuits to delimit and control open paths, hence cannot be used above dimension $d = 2$.

Chabot proved in [3] the shape theorem for the infected individuals of the spatial general epidemic model in dimension $d \geq 3$, with the restriction to deterministic durations of infection: in that case the oriented percolation model is comparable to a non-oriented Bernoulli percolation model (as noticed in

[10], the case with constant durations of infection is the only one where the edges are independent). He also exploited the papers [1] by Antal & Pisztora, and [7] by Grimmett & Marstrand on dynamic renormalization to deal with dimension $d \geq 3$. He introduced some random neighborhoods for points in $\mathbb{Z}^d$ and with these instead of circuits he was able to extend the proof of [5].

In the present work, we prove the shape theorem for infected individuals in the spatial general epidemic model in dimension $d \geq 3$, when the durations of infection are random variables. There, the comparison with non oriented percolation done in [3] is not longer valid. Our approach requires to adapt techniques of [7], and to derive sub-exponential estimates to play the role of the exponential estimates of [1]. It is then possible to follow the skeleton of [3].

In Section 2 we define the spatial general epidemic model, the locally dependent random graph, and we state our main result, Theorem 2.1. In Section 3 we derive the necessary percolation estimates on the locally dependent random graph for Theorem 2.1. We prove the latter in Section 4, thanks to an analysis of the passage times for the epidemic.

2 The model: definition and result

Let $d \geq 3$. The epidemic model on $\mathbb{Z}^d$ is represented by a Markov process $(\eta_t)_{t\geq 0}$ of state space $\Omega = \{0, i, 1\}^{\mathbb{Z}^d}$. The value $\eta_t(x) \in \{0, i, 1\}$ is the state of individual x at time t: state 1 if the individual is healthy (but not immune), state i if it is infected, or state 0 if it is immune. To describe how the epidemic propagates, we introduce a locally dependent oriented bond percolation model on $\mathbb{Z}^d$.

For $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d, y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$, $\|x-y\|_1 = \sum_{i=1}^d |x_i - y_i|$ denotes the l^1 norm of $x - y$, and we write $x \sim y$ if x, y are neighbors, that is $\|x - y\|_1 = 1$. Let $(T_x, e(x, y) : x, y \in \mathbb{Z}^d, x \sim y)$ be independent random variables on a probability space, whose probability is denoted by P_λ for a parameter $\lambda > 0$, such that

1) the T_x's are positive with a common distribution satisfying $P_\lambda(T_x = 0) < 1$;

2) the $e(x, y)$'s are exponentially distributed with parameter λ.

We define

$$X(x,y) = \begin{cases} 1 & \text{if } e(x,y) < T_x; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The oriented bond (x, y) is said *open with passage time* $e(x, y)$ (abbreviated

λ-*open*, or *open* when the parameter is fixed) if $X(x,y)=1$ and *closed* (with infinite passage time) if $X(x,y)=0$.

For a given infected individual x, T_x denotes the amount of time x stays infected; during this time of infection, x emits germs according to a Poisson process of parameter $2d\lambda$; when T_x is over, x recovers and its state becomes 0 forever. An emitted germ from x reaches one of the $2d$ neighbors of x uniformly. If this neighbor is in state 1, it immediately changes to state i, and begins to emit germs according to the same rule; if this neighbor is in state 0 or i, nothing happens.

We denote by C_o^o the set of sites $x \in \mathbb{Z}^d$ that will ever become infected if, at time 0, the origin $o=(0,\dots,0)$ is the only infected site while all other sites are healthy, that is

$$C_o^o = \{x \in \mathbb{Z}^d : \exists\, t \geq 0, \eta_t(x) = i | \eta_0(o) = i, \forall\, z \neq o, \eta_0(z) = 1\}. \tag{2}$$

It was proven in [5, (1.2)] (see also [13, p. 322], [10, Lemma 3.1]) that C_o^o is the set of sites that can be reached from the origin following *an open path*, that is a path of open oriented bonds.

More generally, for each $x \in \mathbb{Z}^d$ we define the *ingoing and outgoing clusters to and from* x to be

$$C_x^i = \{y \in \mathbb{Z}^d : y \to x\}, \qquad C_x^o = \{y \in \mathbb{Z}^d : x \to y\}, \tag{3}$$

and the corresponding critical values to be

$$\lambda_c^i = \inf\{\lambda : P(|C_x^i| = +\infty) > 0\}, \qquad \lambda_c^o = \inf\{\lambda : P(|C_x^o| = +\infty) > 0\}, \tag{4}$$

where "$x \to y$" means that there exists (at least) an open path $\Gamma_{x,y} = (x_0 = x, x_1, \dots, x_n = y)$ from x to y, and $|A|$ denotes the cardinality of a set A. Although we are using the symbol o for both the origin and the outgoing cluster we believe no confusion will arise because in the former case it appears as a subindex while in the latter it does so as a superindex.

We will prove in Section 3 the following proposition.

Proposition 2.1

$$\lambda_c^i = \lambda_c^o.$$

This common value will be denoted by $\lambda_c = \lambda_c(\mathbb{Z}^d)$.

We can now state our main result:

Theorem 2.1 *Assume $\lambda > \lambda_c$. Define, for $t \geq 0$,*

$$\begin{aligned} \xi_t &= \{x \in \mathbb{Z}^d : x \textit{ is immune at time } t\} &&= \{x \in \mathbb{Z}^d : \eta_t(x) = 0\}; \\ \zeta_t &= \{x \in \mathbb{Z}^d : x \textit{ is infected at time } t\} &&= \{x \in \mathbb{Z}^d : \eta_t(x) = i\}. \end{aligned}$$

Then there exists a convex subset $D \subset \mathbb{Z}^d$ such that, for all $\varepsilon > 0$ we have

$$\Big((1-\varepsilon)tD \cap C_o^o\Big) \subset \Big(\xi_t \cup \zeta_t\Big) \subset \Big((1+\varepsilon)tD \cap C_o^o\Big) \textit{ a.s. for } t \textit{ large enough;} \quad (5)$$

and if $E(T_z^d) < \infty$ we also have

$$\zeta_t \subset \Big((1+\varepsilon)tD \setminus (1-\varepsilon)tD\Big) \textit{ a.s. for } t \textit{ large enough.} \quad (6)$$

In other words, the epidemic's progression follows linearly the boundary of a convex set.

To derive this theorem we follow some of the fundamental steps of [5], but since in dimensions three or higher, circuits are not useful as in dimension 2, this is not a straightforward adaptation. On the percolation model, we first construct for each site $z \in \mathbb{Z}^d$ a neighborhood $\mathcal{V}(z)$ in such a way that two neighborhoods are always connected by open paths. For $x, y \in \mathbb{Z}^d$, we show that the time $\tau(x, y)$ for the epidemic to go from x to y is 'comparable' (in a sense to be precised later on) to the time $\widehat{\tau}(x, y)$ it takes to go from $\mathcal{V}(x)$ to $\mathcal{V}(y)$. Then we approximate the passage times between different sites by a subadditive process, we derive through Kingman's subbaditive theorem a radial limit $\varphi(x)$ (for all x), which is asymptotically the linear growth speed of the epidemic in direction x. We establish that the global propagation speed is at most linear in z for $\widehat{\tau}(o, z)$. Finally we prove an asymptotic shape theorem for $\widehat{\tau}(o, \cdot)$, from which we deduce Theorem 2.1.

3 Percolation estimates

In this section we collect some results concerning the locally dependent random graph, that is the oriented percolation model given by the random variables $(X(x, y), x, y \in \mathbb{Z}^d)$ introduced in Section 2. Note that although these r.v.'s are not independent, the random vectors $\{X(x, x + \mathrm{e}_1), \ldots, X(x, x + \mathrm{e}_d), X(x, x - \mathrm{e}_1), \ldots, X(x, x - \mathrm{e}_d) : x \in \mathbb{Z}^d\}$ (where $(\mathrm{e}_1, \ldots, \mathrm{e}_d)$ denotes the canonical basis of $\mathbb{Z}^d$) are i.i.d. This small dependence forces us to explain why some results known for independent percolation remain valid in this context.

Remark 3.1 *The function* $X(x,y)$ *is increasing in the independent random variables* T_x *and* $-e(x,y)$*. It then follows as in [5, Lemma (2.1)] that the r.v.* $(X(x,y) : x,y \in \mathbb{Z}^d, y \sim x)$ *satisfy the following property: If* U *and* V *are bounded increasing functions of the random variables* $(X(x,y) : x,y \in \mathbb{Z}^d, y \sim x)$*, then* $E(UV) \geq E(U)E(V)$*.*

For $n \in \mathbb{N} \setminus \{0\}$, let $B_n = [-n,n]^d$, let ∂B_n denote the boundary vertex points of B_n, and, for $x \in \mathbb{R}^d$, $B(x,n) = x + B_n$. For $A, R \subset \mathbb{Z}^d$, "$A \to R$" means that there exists an open path $\Gamma_{x,y}$ from some $x \in A$ to some $y \in R$.

Theorem 3.1 *Suppose* $\lambda < \lambda_c^o$*, then there exists* $\beta_o > 0$ *such that for all* $n > 0$*,*

$$P_\lambda(o \to \partial B_n) \leq \exp(-\beta_o n).$$

This is a special case of [2, Theorem (3.1)], whose proof can be adapted to obtain:

Theorem 3.2 *Suppose* $\lambda < \lambda_c^i$*, then there exists* $\beta_i > 0$ *such that for all* $n > 0$*,*

$$P_\lambda(\partial B_n \to o) \leq \exp(-\beta_i n).$$

It is worth noting that in the context of our paper, by Remark 3.1, [2, Theorem (3.1)] can be proved using the BK inequality instead of the Reimer inequality (see [6, Theorems (2.12), (2.19)]).

Theorems 3.1, 3.2 yield Proposition 2.1:

Proof of proposition 2.1. Suppose $\lambda < \lambda_c^i$. Then by translation invariance and Theorem 3.2 we have that for any $x \in \partial B_n$, $P_\lambda(o \to x) \leq \exp(-\beta_i n)$. Adding over all points of ∂B_n we get $P_\lambda(o \to \partial B_n) \leq K' n^d \exp(-\beta_i n)$ for some constant K', which implies that $\lim_{n\to+\infty} P_\lambda(o \to \partial B_n) = 0$. Therefore $\lambda \leq \lambda_c^o$ and $\lambda_c^i \leq \lambda_c^o$. The other inequality is obtained similarly. $\square$

From now on, we assume $\lambda > \lambda_c(\mathbb{Z}^d)$ and use the following notation: For $x, y \in \mathbb{Z}^d, A \subset \mathbb{Z}^d$,
(i) $\{x \to y \text{ within } A\}$ is the event on which there exists an open path $\Gamma_{x,y} = (x_0 = x, x_1, \ldots, x_n = y)$ from x to y such that $x_i \in A$ for all $i \in \{0, \ldots, n-1\}$. Note that the end point y may not belong to A.
(ii) $\{x \to y \text{ outside } A\}$ is the event on which there exists an open path $\Gamma_{x,y} = (x_0 = x, x_1, \ldots, x_n = y)$ from x to y such that none of the x_i's ($i \in \{0, \ldots, n\}$) belongs to A.

Definition 3.1 *For $x \in \mathbb{Z}^d, A \subset \mathbb{Z}^d$ let*

$$\begin{aligned} C_x^i(A) &= \{y \in A : y \to x \quad within\ A\} \qquad and \\ C_x^o(A) &= \{y \in A : x \to y \quad within\ A\}. \end{aligned}$$

Note that with this definition $C_x^i(A) \subset A$ and $C_x^o(A) \subset A$.

Next proposition on percolation on slabs follows from the methods of [7] or [6, Chapter 7].

Proposition 3.1 *For any $k \in \mathbb{N} \setminus \{0\}$, let $S_k = \mathbb{Z}^{d-1} \times \{0, 1, \ldots, k\}$ denote the slab of thickness k. Then for k large enough we have $\inf_{x \in S_k} P_\lambda(|C_x^i(S_k)| = +\infty) > 0$ and $\inf_{x \in S_k} P_\lambda(|C_x^o(S_k)| = +\infty) > 0$.*

More precisely, to adapt [6, Chapter 7], it is convenient to define the processes for different values of λ on the same probability space, whose probability is denoted by P. This is done as follows: Let $e_1(x, y)$ be a collection of independent exponential r.v.'s as before, but now with parameter 1. Then let $e_\lambda(x, y) = \lambda^{-1} e_1(x, y)$, and

$$X_\lambda(x, y) = \begin{cases} 1 & \text{if } e_\lambda(x, y) < T_x; \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The following lemma implies that given K and $\delta_1 > 0$, there exists $\varepsilon_1 > 0$ such that for any $\lambda \in [0, K]$ the random field $\{X_{\lambda+\delta_1}(u, v) : u, v \in \mathbb{Z}^d\}$ is stochastically above the random field $\{\max\{X_\lambda(u, v), Y(u, v)\} : u, v \in \mathbb{Z}^d\}$ where the random variables $Y(u, v)$ are i.i.d. Bernoulli with parameter ε_1 and are independent of the random variables $X_\lambda(u, v)$. This lemma justifies the use of this *sprinkling technique.* Its proof is elementary and will be omitted. Then, with Lemma 3.1 one can adapt the proof of [6, Theorem (7.2)] to get Proposition 3.1.

Lemma 3.1 *Let K and δ_1 be strictly positive, then there exists $\varepsilon > 0$ such that for any $\lambda \leq K$,$u \in \mathbb{Z}^d$,*

$$P(X_{\lambda+\delta_1}(u, v) = 1\ \forall v \sim u | \quad X_\lambda(x, y) : x, y \in \mathbb{Z}^d,\ x \sim y) > \varepsilon \quad a.s.$$

We introduce now some notation: For $A \subset \mathbb{Z}^d$ we define the *exterior vertex boundary* $\Delta_V A$ as:

$$\Delta_V A = \{x \in \mathbb{Z}^d : x \notin A, x \sim y \text{ for some } y \in A\}.$$

If $x \to y$ let $D(x, y)$ be the smallest number of bonds required to build an open path from x to y. For $A \subset \mathbb{Z}^d$, $x \in A, y \in \Delta_V A$, "$D(x, y) < m$ within A" means that there is an open path $\Gamma_{x,y}$ using less than m bonds

from x to y whose sites are all in A except for y.
The rest of this section provides some upper bounds for the tail of the conditional distribution of $D(x, y)$ given the event $\{x \to y\}$. These estimates are not optimal and better results can be obtained by adapting the methods of [1]. Instead of getting exponential decays in $\|x - y\|_1$ (or in n) we get exponential decays in $\|x-y\|_1^{1/d}$ (or in $n^{1/d}$). We have adopted this approach because the weaker results suffice for our purposes and are easier to obtain.

Lemma 3.2 *There exist $\delta > 0$, $\mathrm{C}_1 > 0$ and $k \in \mathbb{N} \setminus \{0\}$ such that*
(i) $\forall n > 0,\ x \in B_{n+k} \setminus B_n,\ y \in (B_{n+k} \setminus B_n) \cup \Delta_V(B_{n+k} \setminus B_n)$ we have :

$$P(x \to y \quad within \quad B_{n+k} \setminus B_n) > \delta.$$

(ii) Let

$$\begin{aligned} A_{n,m} \quad = \quad & \{z : -k + n \leq z_1 < n, -\infty < z_2 \leq k\} \cup \\ & \{z : -k + n \leq z_1 \leq m + k, 0 < z_2 \leq k\} \cup \\ & \{z : m < z_1 \leq m + k, -\infty < z_2 \leq k\}. \end{aligned}$$

$\forall n < m,\ \forall x \in A_{m,n}, \forall y \in A_{m,n} \cup \Delta_V A_{m,n}$, we have:

$$P(D(x, y) < \mathrm{C}_1(\|x - y\|_1 + |x_2| + |y_2|) \quad within \ A_{m,n}) > \delta.$$

Again, the proof of this lemma relies on the methods of [6, Chapter 7] and [7]. Since it is not an entirely straightforward adaptation we make some remarks that we believe will help the reader.

Remark 3.2 *In [6, Chapter 7], renormalised sites, which we will call r-sites, are introduced. These are the hypercubes $B_k = [-k, k]^d$ and their translates $B(2kx, k) = [-k, k]^d + 2kx,\ x \in \mathbb{Z}^d$. We will denote by x both the point in $\mathbb{Z}^d$ and the r-site centered at $2kx$ since we believe no confusion will arise from this. Loosely speaking, these r-sites are called occupied if they are well connected with their neighbors. Crucial for this method is the fact that for any given $p \in (0, 1)$ the set of occupied r-sites dominates a Bernoulli product measure of density p if k is large enough. In [6, Chapter 7] and [7], p is taken above the critical value for non-oriented Bernoulli percolation on some subset of $\mathbb{Z}^d$, but here it is convenient to take it above the critical value of 2-dimensional oriented bond percolation. This choice of p and the corresponding choice of k guarantee the following property:*

(P) *There exists $\gamma > 0$ such that for any pair of r-sites u and v the probability that there exists a path of occupied r-sites going from $u \in \mathbb{Z}^d$ to $v \in \mathbb{Z}^d$ which uses at most $2\|u - v\|_1$ r-sites is at least γ.*

From this property (P) *one deduces that there exist* $\delta > 0$ *and* C_1 *such that for any pair* $x, y \in \mathbb{Z}^2 \times [-k, k]^{d-2}$ *the probability that there is an open path from* x *to* y *contained in* $\mathbb{Z}^2 \times [-k, k]^{d-2}$ *that uses at most* $\mathrm{C}_1\|x - y\|_1$ *bonds is at least* δ*.*

Remark 3.3 *We have to add* $|x_2| + |y_2|$ *because if* $x \in \{z : -k + n \le z_1 < n, -\infty < z_2 \le 0\}$ *and* $y \in \{z : m < z_1 \le m + k, -\infty < z_2 \le 0\}$*, to move from* x *to* y *staying in* $A_{m,n}$ *we need to reach first the set* $\{z : -k + n \le z_1 \le m + k, 0 < z_2 \le k\}$ *(i.e. to increase the second coordinate until it is positive).*

Lemma 3.3 *Let* k *be given by Lemma 3.2 and let* x *and* y *be points in* $\mathbb{Z}^d$*. For* $n \in \mathbb{Z}$ *let* $H_n = \{x \in \mathbb{Z}^d : x_1 = n\}$ *and define the events*

$$
\begin{aligned}
J_n &= \{x \to H_{x_1-1-ik} \text{ within } B(x, nk), i = 0, \dots \lfloor n/2 \rfloor\} \cap \\
&\quad \{H_{y_1+1+ik} \to y \text{ within } B(y, nk), i = 0, \dots \lfloor n/2 \rfloor\}, \\
G_n &= \{x \to \partial B(x, kn),\ \partial B(y, kn) \to y\},
\end{aligned}
$$

where, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ *denotes the greatest integer not greater than* a*. Then, there exists* $\beta > 0$ *such that*

$$P(J_n | G_n) \ge 1 - \exp(-\beta n).$$

Proof of lemma 3.3. By translation invariance we may assume that x is the origin. When G_n occurs there are open paths from o to $B(o, ik) \setminus B(o, (i-1)k)$ and to y from $B(y, ik) \setminus B(y, (i-1)k)$ for $i = 2, \dots, n$. Hence we conclude by part *(i)* of Lemma 3.2. □

Lemma 3.4 *Let* k *be given by Lemma 3.2 and let* G_n *be as in Lemma 3.3. Then, there exist constants* C_2*,* C_3 *and* $\alpha_2 > 0$ *such that for all* $x, y \in \mathbb{Z}^d$*,* $n \in \mathbb{N}$ *we have*

$$P(D(x, y) > \mathrm{C}_2\|x - y\|_1 + \mathrm{C}_3(nk)^d |\ G_n) \le \exp(-\alpha_2 n).$$

Proof of lemma 3.4. Again, by translation invariance we may assume that x is the origin and without loss of generality, we also assume that $y_1 > 0$ and $y_2 \ge 0$. By Lemma 3.3 it suffices to show that

$$P(D(o, y) > \mathrm{C}_2\|x - y\|_1 + \mathrm{C}_3(nk)^d |\ J_n)$$

decays exponentially in n.

Let $r = \lfloor n/2 \rfloor$, and for $1 \le j \le r$ let

$$\begin{aligned} A_0 &= \{z : -k \le z_1 < 0,\ -\infty < z_2 \le y_2 + k\} \cup \\ &\quad \{z : -k \le z_1 \le y_1 + k,\ y_2 < z_2 \le y_2 + k\} \cup \\ &\quad \{z : y_1 < z_1 \le y_1 + k,\ -\infty < z_2 \le y_2 + k\}, \\ A_j &= \{z : -(j+1)k \le z_1 < -jk,\ -\infty < z_2 \le y_2 + k\} \cup \\ &\quad \{z : -(j+1)k \le z_1 \le y_1 + (j+1)k, \\ &\qquad y_2 + jk < z_2 \le y_2 + (j+1)k\} \cup \\ &\quad \{z : y_1 + jk < z_1 \le y_1 + (j+1)k,\ -\infty < z_2 \le y_2 + (j+1)k\}. \quad (8) \end{aligned}$$

Note that the sets $A_0, \ldots, A_r$ are disjoint. Figure 1 should help the reader

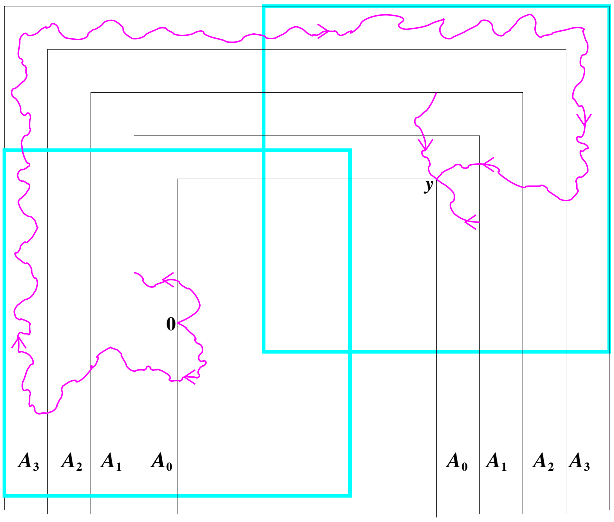

Figure 1: the event W_3

to visualize them. On the event J_n, we can reach from the origin each of these sets by means of an open path contained in B_{nk} and from each of these sets we can reach y by means of an open path contained in $B(y, nk)$. Hence, on J_n for each $i \in \{0, \ldots, r\}$ there exists a random point $U_i \in B_{nk} \cap A_i$ and there is an open path from the origin to U_i such that all its sites except U_i are in $B_{nk} \cap (\cap_{j=i}^{r} A_j^c)$. If there are many possible values of U_i we choose the first one in some arbitrary deterministic order. Similarly, there is a point $V_i \in B(y, nk) \cap \Delta A_i$ and an open path from V_i to y such that all its sites are in $B(y, nk) \cap (\cap_{j=i}^{r} A_j^c)$. Let u_i and v_i be possible values of U_i and V_i respectively. Then define

$$\begin{aligned} F_i(u_i, v_i) &= \{U_i = u_i, V_i = v_i\}, \\ E_i(u_i, v_i) &= \{D(u_i, v_i) < \mathrm{C}\|u_i - v_i\|_1 \text{ within } A_i\} \text{ and} \\ W_i &= \cup_{u_i, v_i} \left(F_i(u_i, v_i) \cap E_i(u_i, v_i)\right), \end{aligned}$$

where the union is over all possible values of U_i and V_i.

Now we define a subset of $\mathbb{Z}^d$

$$R_i = B_{nk} \cup B(y, nk) \cup \Big(A_0 \ldots \cup A_{i-1}\Big) \cap \Big(A_i^c \cup \ldots \cup A_{n-1}^c\Big), \tag{9}$$

and we denote by σ_i the σ-algebra generated by $\{T_x, e(x,y) : x \in R_i, x \sim y\}$. Then, noting that $\mathbf{1}_{F_i(u_i,v_i)}\Pi_{j=0}^{i-1}\mathbf{1}_{W_j^c}$ is σ_i-measurable write for $i = 1, \ldots r$:

$$\begin{aligned}
&P\left(W_i \cap J_n \cap (\cap_{j=0}^{i-1} W_j^c)\right) = \sum_{u_i,v_i} E\left(\mathbf{1}_{F_i(u_i,v_i)}\mathbf{1}_{E_i(u_i,v_i)}\mathbf{1}_{J_n}(\Pi_{j=0}^{i-1}\mathbf{1}_{W_j^c})\right) \\
= &\sum_{u_i,v_i} E\left(\mathbf{1}_{F_i(u_i,v_i)}(\Pi_{j=0}^{i-1}\mathbf{1}_{W_j^c})E(\mathbf{1}_{J_n}\mathbf{1}_{E_i(u_i,v_i)}|\sigma_i)\right) \\
\geq &\sum_{u_i,v_i} P(E_i(u_i,v_i))E\left(\mathbf{1}_{F_i(u_i,v_i)}(\Pi_{j=0}^{i-1}\mathbf{1}_{W_j^c})E(\mathbf{1}_{J_n}|\sigma_i)\right) \\
= &\sum_{u_i,v_i} P(E_i(u_i,v_i))E\left(\mathbf{1}_{F_i(u_i,v_i)}(\Pi_{j=0}^{i-1}\mathbf{1}_{W_j^c})\mathbf{1}_{J_n}\right) \\
\geq &\ \delta\sum_{u_i,v_i} P\left(F_i(u_i,v_i) \cap J_n \cap (\cap_{j=0}^{i-1} W_j^c)\right) = \delta P\left(J_n \cap (\cap_{j=0}^{i-1} W_j^c)\right),
\end{aligned}$$

where the sums are over all possible values of U_i and V_i, the first inequality follows from the facts that $E_i(u_i,v_i)$ is independent of σ_i and both J_n and $E_i(u_i,v_i)$ are increasing events, the second inequality from part *(ii)* of Lemma 3.2 and the last equality from the fact that J_n is contained in the union of the $F_i(u_i,v_i)$'s which are disjoint. Now, proceeding by induction one gets:

$$P\left(J_n \cap (\cap_{j=0}^{r-1} W_j^c)\right) \leq (1-\delta)^r P(J_n).$$

Since we can choose C_2 and C_3 in such a way that the event $\{D(o,y) > \mathrm{C}_2\|x-y\|_1 + \mathrm{C}_3(nk)^d\}$ does not occur if any of the W_i's occurs, the lemma follows. □

Noting that modifying the constant α_2 the statement of the above lemma holds for $\mathrm{C}_3 = 1/k^d$, we get:

Lemma 3.5 *(i) Let* C_2 *be as in Lemma 3.4. Then, there exists* $\alpha_3 > 0$ *such that* $P(D(x,y) \geq \mathrm{C}_2\|x-y\|_1 + n^d | x \to y) \leq \exp(-\alpha_3 n)$*;*

(ii) $P(x \to y|\ |C_x^o| = +\infty,\ |C_y^i| = +\infty) = 1$.

4 The shape theorem

Let

$$\widetilde{C} = \{x \in \mathbb{Z}^d : x \to \infty \text{ and } \infty \to x\}. \tag{10}$$

Remark 4.1 *As a consequence of Lemma 3.5 part (ii), if two sites x, y of $\mathbb{Z}^d$ belong to $\widetilde{C}$, then $x \to y$ and $y \to x$.*

4.1 Neighborhoods in $\widetilde{C}$

Definition 4.1 *For $x \in \mathbb{Z}^d$, let*

$$\begin{cases} R_x^o = \{y \in \mathbb{Z}^d : x \to y \text{ outside } \widetilde{C}\} & \text{(outgoing root from } x\text{);} \\ R_x^i = \{y \in \mathbb{Z}^d : y \to x \text{ outside } \widetilde{C}\} & \text{(incoming root to } x\text{).} \end{cases}$$

In particular x belongs to R_x^o and R_x^i if and only if $x \notin \widetilde{C}$.

Our next lemma says that the distribution of the radius of $R_o^o \cup R_o^i$ decreases exponentially.

Lemma 4.1 *There exists $\sigma_1 = \sigma_1(\lambda, d) > 0$ such that, for all $n \in \mathbb{N}$,*

$$P\left((R_o^o \cup R_o^i) \cap \partial B_n \neq \emptyset\right) \leq \exp(-\sigma_1 n).$$

Proof of lemma 4.1. For $n \in \mathbb{N} \setminus \{0\}$, $R_o^o \cap \partial B_{2n} \neq \emptyset$ means that there exists an open path $o \to \partial B_{2n}$ outside $\widetilde{C}$. This implies that there exists $x \in \partial B_n$ satisfying $o \to x \to \partial B_{2n}$ outside $\widetilde{C}$. Similarly, $R_o^i \cap \partial B_{2n} \neq \emptyset$ implies that there exists $x \in \partial B_n$ satisfying $\partial B_{2n} \to x \to o$ outside $\widetilde{C}$. Then for such a point, either the cluster C_x^o or the cluster C_x^i is finite, and has a radius larger than or equal to n. We adapt to our case [6, Theorems (8.18), (8.21)] to get the existence of $\sigma_0 = \sigma_0(\lambda, d) > 0$ such that:

$$\begin{cases} P(C_x^o \cap \partial B(x,n) \neq \emptyset, |C_x^o| < +\infty) \leq \exp(-\sigma_0 n); \\ P(C_x^i \cap \partial B(x,n) \neq \emptyset, |C_x^i| < +\infty) \leq \exp(-\sigma_0 n). \end{cases} \tag{11}$$

Hence

$$\begin{aligned} & P\left((R_o^o \cup R_o^i) \cap \partial B_{2n} \neq \emptyset\right) \\ \leq\ & P\left(R_o^o \cap \partial B_{2n} \neq \emptyset\right) + P\left(R_o^i \cap \partial B_{2n} \neq \emptyset\right) \\ \leq\ & 2 \sum_{x \in \partial B_n} P(|C_x^o| < +\infty, x \to \partial B(x,n)) \\ & +2 \sum_{x \in \partial B_n} P(|C_x^i| < +\infty, \partial B(x,n) \to x) \\ \leq\ & 4 |\partial B_n| \exp(-\sigma_0 n) \end{aligned}$$

which induces the result. □

To define the neighborhood $\mathcal{V}(x)$ on $\widetilde{C}$ of a site x, we introduce the smallest box whose interior contains R^o_x and R^i_x, which contains elements of $\widetilde{C}$, and is such that two elements of $\widetilde{C}$ in this box are connected by an open path which does not exit from a little larger box. For this last condition, which will enable to bound the passage time through $\mathcal{V}(x)$, we use the parameter C_2 obtained in Lemma 3.4.

Definition 4.2 *Let* $\mathrm{C}' = \mathrm{C}_2 d + 2$. *Let* $\kappa(x)$ *be the smallest* $l \in \mathbb{N} \setminus \{0\}$ *such that*

$$\begin{cases} (i) \quad \partial B(x,l) \cap (R^o_x \cup R^i_x) = \emptyset; \\ (ii) \;\; B(x,l) \cap \widetilde{C} \neq \emptyset; \\ (iii) \, \forall \, (y,z) \in (B(x,l) \cap \widetilde{C})^2, \; y \to z \textit{ within } B(x, \mathrm{C}'l). \end{cases}$$

Remark 4.2 *By (i) above,* $R^o_x \cup R^i_x \subset B(x, \kappa(x))$.

The random variable $\kappa(x)$ has a sub-exponential tail:

Lemma 4.2 *There exists a constant* $\sigma = \sigma(\lambda, d) > 0$ *such that, for any* $n \in \mathbb{N}$,

$$P(\kappa(x) \geq n) \leq \exp(-\sigma n^{1/d}).$$

Proof of lemma 4.2. We show that the probability that any of the 3 conditions in Definition 4.2 is not achieved for n decreases exponentially in $n^{1/d}$:
(i) By translation invariance, we have by Lemma 4.1,

$$P\left(\partial B(x,n) \cap \left(R^o_x \cup R^i_x\right) \neq \emptyset\right) \leq \exp(-\sigma_1 n). \tag{12}$$

(ii) There exist $m \in \mathbb{N}, \sigma_2 = \sigma_2(\lambda, d) > 0$ such that

$$P(B(x,n) \cap \widetilde{C} = \emptyset) \leq \exp(-\sigma_2 \lfloor n/(m+1) \rfloor). \tag{13}$$

Indeed, since $\{|C^i_x(S_m)| = +\infty\}$ and $\{|C^o_x(S_m)| = +\infty\}$ are increasing events, it follows from Proposition 3.1 and the FKG inequality (see Remark 3.1) that for $m = m(\lambda, d)$ large enough we have $\inf_{x \in S_m} P(|C^i_x(S_m)| = |C^o_x(S_m)| = +\infty) > 0$. Then, (13) follows from the facts that $B(x,n)$ intersects $\lfloor n/(m+1) \rfloor$ disjoint translates of S_m, and events in disjoint slabs are independent.

(iii) There exists $\sigma_3 = \sigma_3(\lambda, d) > 0$ such that

$$\begin{aligned} P\Big(\exists \, (y,z) \in (B(x,n) \cap \widetilde{C})^2, \; y \not\to z \text{ within } (B(x, \mathrm{C}'n)\Big) \\ \leq \exp(-\sigma_3 n^{1/d}). \end{aligned} \tag{14}$$

Indeed, if no open path from y to z (both in $B(x,n)\cap\widetilde{C}$) is contained in $B(x,\mathrm{C}'n)$, then $D(y,z)\geq 2(\mathrm{C}'-1)n$. Given our choice of C' this implies that $D(y,z)\geq \mathrm{C}_2\|y-z\|_1+n$. Therefore, (14) follows from part *(i)* of Lemma 3.5. □

We define the (site) neighborhood in $\widetilde{C}$ of x by

$$\mathcal{V}(x)=B(x,\kappa(x))\cap\widetilde{C}. \tag{15}$$

Remark 4.3 *By condition (ii) in Definition 4.2,* $\mathcal{V}(x)\neq\emptyset$.

By condition *(iii)* in Definition 4.2, for all y,z in $\mathcal{V}(x)$, there exists at least one open path from y to z, denoted by $\Gamma^*_{y,z}$ contained in $B(x,\mathrm{C}'\kappa(x))$. If there are several such paths we choose the first one according to some deterministic order. We finally define an "edge" neighborhood $\overline{\Gamma}(x)$ of x:

$$\begin{aligned}\overline{\Gamma}(x) &= \{(y',z')\subset B(x,\kappa(x)),(y',z')\text{ open}\}\cup\\ &\qquad \{(y',z')\in\Gamma^*_{y,z},y,z\in\mathcal{V}(x)\}.\end{aligned} \tag{16}$$

Those neighborhoods satisfy

$$\mathcal{V}(x)\subset B(x,\kappa(x));\qquad \overline{\Gamma}(x)\subset B(x,\mathrm{C}'\kappa(x)). \tag{17}$$

4.2 Radial limits

We now come back to the spatial epidemic model. Indeed, the underlying percolation model does not give any information on the time needed by the epidemic to cover $\widetilde{C}$. We first define an approximation for the passage time of the epidemic, then we prove the existence of radial limits for this approximation and for the epidemic. We will follow for this the construction in [5].

For $x,y\in\mathbb{Z}^d$, if $x\neq y, x\to y$ and $\Gamma_{x,y}=(x_0=x,x_1,\ldots,x_n=y)$ denotes an open path from x to y, we define the *passage time on* $\Gamma_{x,y}$ to be (see (1))

$$\overline{\tau}(\Gamma_{x,y})=\sum_{i=0}^{n-1}e(x_i,x_{i+1}) \tag{18}$$

and, if $x=y$, we put $\overline{\tau}(\Gamma_{x,x})=0$.

For $x, y \in \mathbb{Z}^d$, we define the *passage time from x to y* to be

$$\tau(x,y) = \begin{cases} \inf\limits_{\{\Gamma_{x,y}\}} \overline{\tau}(\Gamma_{x,y}) & \text{if } x \neq y, x \to y, \\ 0 & \text{if } x = y, \\ +\infty & \text{otherwise.} \end{cases} \tag{19}$$

where the infimum is over all possible open paths from x to y. By analogy with [4], [5], we also define

$$\begin{aligned} \widehat{\tau}(x,y) &= \inf_{x' \in \mathcal{V}(x), y' \in \mathcal{V}(y)} \tau(x', y'); \\ u(x) &= \begin{cases} \sum\limits_{(y',z') \in \overline{\Gamma}(x)} \tau(y', z') & \text{if } \overline{\Gamma}(x) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{20}$$

By Remarks 4.1, 4.3, $\widehat{\tau}(x,y)$ is well defined.

Remark 4.4 *If $\mathcal{V}(x) \cap \mathcal{V}(y) \neq \emptyset$, then $\widehat{\tau}(x,y) = 0$.*

We now show that if $y \in C_x^o \setminus R_x^o$, $\widehat{\tau}(x,y)$ approximates $\tau(x,y)$.

Lemma 4.3 *For $x \in \mathbb{Z}^d$, if $y \in C_x^o \setminus R_x^o$, we have*

$$\widehat{\tau}(x,y) \leq \tau(x,y) \leq u(x) + \widehat{\tau}(x,y) + u(y).$$

Proof of lemma 4.3. Let $\Gamma_{x,y}$ be an open path from x to y such that $\tau(x,y) = \overline{\tau}(\Gamma_{x,y})$. Since $y \notin R_x^o$ this path must intersect $\widetilde{C}$. Let c_1 and c_2 be the first and last points we encounter in $\widetilde{C}$ when moving from x to y along $\Gamma_{x,y}$. By condition *(i)* of Definition 4.2, $c_1 \in \mathcal{V}(x)$ and $c_2 \in \mathcal{V}(y)$: indeed (for instance for c_1), either $x \in \widetilde{C}$ and $c_1 = x$, or the point $a \in \partial B(x, \kappa(x)) \cap \Gamma_{x,y}$ does not belong to R_x^o and c_1 is the first point on $\Gamma_{x,y}$ between x and a; we might have $c_1 = c_2$, if $\mathcal{V}(x) \cap \mathcal{V}(y) \neq \emptyset$. We have

$$\Gamma_{x,y} = \Gamma_{x,c_1} \vee \Gamma_{c_1,c_2} \vee \Gamma_{c_2,y}$$

where $\vee$ denotes the concatenation of paths, Γ_{x,c_1} is an open path from x to c_1 contained in $B(x, \kappa(x))$, Γ_{c_1,c_2} is an open path from c_1 to c_2 and $\Gamma_{c_2,y}$ is an open path from c_2 to y contained in $B(y, \kappa(y))$. We then obtain the first inequality of the lemma since:

$$\widehat{\tau}(x,y) \leq \overline{\tau}(\Gamma_{c_1,c_2}) \leq \overline{\tau}(\Gamma_{x,y}) = \tau(x,y).$$

To prove the second inequality of the lemma let Γ_{d_1,d_2} be an open path from $d_1 \in \mathcal{V}(x)$ to $d_2 \in \mathcal{V}(y)$ such that $\overline{\tau}(\Gamma_{d_1,d_2}) = \widehat{\tau}(x,y)$. Since the open paths

Γ_{x,c_1} from x to c_1 and $\Gamma^*_{c_1,d_1}$ (which exists by Remark 4.1) from c_1 to d_1 have edges in $\overline{\Gamma}(x)$ (see (16)) the open path $\Gamma_{x,d_1} = \Gamma_{x,c_1} \vee \Gamma^*_{c_1,d_1}$ from x to d_1 satisfies $\overline{\tau}(\Gamma_{x,d_1}) \leq u(x)$. Similarly, there is an open path $\Gamma_{d_2,y}$ from d_2 to y such that $\overline{\tau}(\Gamma_{d_2,y}) \leq u(y)$. We conclude with

$$\tau(x,y) \leq \overline{\tau}(\Gamma_{x,d_1}) + \overline{\tau}(\Gamma_{d_1,d_2}) + \overline{\tau}(\Gamma_{d_2,y}) \leq u(x) + \widehat{\tau}(x,y) + u(y).$$

□

Lemma 4.4 *For all $x, y, z \in \mathbb{Z}^d$, we have the* subadditivity property

$$\widehat{\tau}(x,z) \leq \widehat{\tau}(x,y) + u(y) + \widehat{\tau}(y,z). \tag{21}$$

Proof of lemma 4.4. Let $\Gamma_{a,b}$ be an open path from $a \in \mathcal{V}(x)$ to $b \in \mathcal{V}(y)$ such that $\widehat{\tau}(x,y) = \overline{\tau}(\Gamma_{a,b})$. Similarly, let $\Gamma_{c,d}$ be an open path from $c \in \mathcal{V}(y)$ to $d \in \mathcal{V}(z)$ such that $\widehat{\tau}(y,z) = \overline{\tau}(\Gamma_{c,d})$ (we might have $a = b$, $c = d$ or $b = c$). Since both b and c are in $\mathcal{V}(y)$ there exists an open path $\Gamma^*_{b,c}$ from b to c such that $\overline{\tau}(\Gamma^*_{b,c}) \leq u(y)$ (see Remark 4.1 and (16)). The lemma then follows since the concatenation of these three paths is an open path from a point of $\mathcal{V}(x)$ to a point of $\mathcal{V}(z)$ and

$$\widehat{\tau}(x,z) \leq \overline{\tau}(\Gamma_{a,b}) + \overline{\tau}(\Gamma_{b,c}) + \overline{\tau}(\Gamma_{c,d}) \leq \widehat{\tau}(x,y) + u(y) + \widehat{\tau}(y,z).$$

□

Before stating our next lemma we introduce some notation. For $x, y \in \mathbb{Z}^d$, let

$$\overline{D}(x,y) = \inf_{x' \in \mathcal{V}(x), y' \in \mathcal{V}(y)} D(x', y').$$

Note that unlike $D(x,y)$, $\overline{D}(x,y)$ is always finite.

Lemma 4.5 *There exist constants* C_4 *and* $\alpha_4 > 0$ *such that*

$$P(\overline{D}(x,y) \geq \mathrm{C}_4 \|x-y\|_1 + n) \leq \exp(-\alpha_4 n^{1/d}), \qquad \forall\, x, y \in \mathbb{Z}^d, n \in \mathbb{N}.$$

Proof of lemma 4.5. Let K be a positive constant. Then

$$\begin{aligned} & P(\overline{D}(x,y) \geq K\|x-y\|_1 + (2d+1)Kn) \\ \leq \quad & P(\kappa(x) > n) + P(\kappa(y) > n) \\ + \quad & P(\overline{D}(x,y) \geq K\|x-y\|_1 + (2d+1)Kn, \kappa(x) \leq n, \kappa(y) \leq n) \\ \leq \quad & P(\kappa(x) > n) + P(\kappa(y) > n) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{x'\in B(x,n),y'\in B(y,n)} P(D(x',y') \geq K\|x-y\|_1 + (2d+1)Kn, x' \to y') \\
\leq\ & P(\kappa(x) > n) + P(\kappa(y) > n) \\
&+ \sum_{x'\in B(x,n),y'\in B(y,n)} P(D(x',y') \geq K\|x'-y'\|_1 + Kn, x' \to y').
\end{aligned}
$$

Taking $K = C_2$ (given in Lemma 3.5) the result follows from Lemmas 3.5 and 4.2. □

Of course, the random variables $u(x)$ and $\widehat{\tau}(x,y)$ are almost surely finite. But we will need a better control of their size, provided by our next lemma.

Lemma 4.6 *For all $x, y \in \mathbb{Z}^d$, $r \in \mathbb{N} \setminus \{0\}$, $u(x)$ and $\widehat{\tau}(x,y)$ have a finite r-th moment.*

Proof of lemma 4.6. By Lemma 4.2, $u(x)$ is bounded above by a sum of passage times $e(y,z)$ with y and z in the box $B(x,Y)$, where Y is a random variable whose moments are all finite. By Lemmas 4.2 and 4.5 the same happens to $\widehat{\tau}(x,y)$. Therefore it suffices to show that if $(X_i, i \in \mathbb{N})$ is a sequence of i.i.d. random variables and N is a random variable taking values in $\mathbb{N}$, then the moments of $\sum_{i=1}^N X_i$ are all finite if it is the case for both the X_i's and N. To prove this write:

$$
\begin{aligned}
E(|\sum_{i=1}^N X_i|^r) &= \sum_{n=1}^\infty E(|X_1 + \ldots + X_n|^r \mathbf{1}_{\{N=n\}}) \\
&\leq \sum_{n=1}^\infty [E(|X_1 + \ldots + X_n|^{2r}) P(N=n)]^{1/2} \\
&\leq \sum_{n=1}^\infty [E(|X_1| + \ldots + |X_n|)^{2r} P(N=n)]^{1/2} \\
&\leq \sum_{n=1}^\infty [n^{2r} C_{2r} P(N=n)]^{1/2}
\end{aligned}
$$

where the second line follows from Cauchy-Schwartz' inequality, the factor n^{2r} counts the number of terms in the development of $(|X_1| + \ldots + |X_n|)^{2r}$ and the constant C_{2r} depends on the distribution of the X_i's. As N has all its moments finite $P(N = n)$ decreases faster than n^{-2r-4} and the sum is finite. □

We now construct a process $(\vartheta.)$ which is subadditive in every direction, and has a.s., by Kingman's Theorem, a radial limit denoted by μ. We will then check that $\widehat{\tau}(o,\cdot)$ also has, in every direction, the same radial limit, and we will extend this conclusion to $\tau(o,\cdot)$ on the set C_o^o of sites that have ever been infected. Hence we first prove

Theorem 4.1 *For all $z \in \mathbb{Z}^d$, there exists $\mu(z) \in \mathbb{R}^+$ such that almost surely*

$$\lim_{n\to+\infty} \frac{\widehat{\tau}(o,nz)}{n} = \mu(z), \tag{22}$$

$$\lim_{n\to+\infty} \left[\frac{\tau(o,nz)}{n} - \mu(z)\right] \mathbf{1}_{\{nz\in C_o^o\}} = 0. \tag{23}$$

Proof of theorem 4.1. (i) For all $z \in \mathbb{Z}^d$, $(m,n) \in \mathbb{N}^2$, let

$$\vartheta_z(m,n) = \widehat{\tau}(mz,nz) + u(nz). \tag{24}$$

The process $(\vartheta_z(m,n))_{(m,n)\in\mathbb{N}^2}$ satisfies the hypotheses of Kingman' subadditive ergodic theorem (see [12, Theorem VI.2.6]) by (21). Hence there exists $\mu(z) \in \mathbb{R}^+$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\vartheta_z(0,n) = \mu(z) \text{ a.s. and in } L^1. \tag{25}$$

Since the random variables $(u(z) : z \in \mathbb{Z}^d)$ are identically distributed, it follows from Lemma 4.6 and Chebychev's inequality that $\sum_{n=0}^{\infty} P(u(nz) > n\varepsilon) < +\infty$ for all $\varepsilon > 0$, so that by Borel-Cantelli's Lemma

$$\lim_{n\to+\infty} \frac{u(nz)}{n} = 0, \text{ a.s.} \tag{26}$$

Thus by (25), (26) we have (22) for all $z \in \mathbb{Z}^d$.

(ii) Since R_o^o is a.s. finite, if $nz \in C_o^o$, then $nz \in C_o^o \setminus R_o^o$ for n large enough. Hence, from Lemma 4.3, for n large enough we have

$$\left|\frac{\tau(o,nz)}{n} - \mu(z)\right| \mathbf{1}_{\{nz\in C_o^o\setminus R_o^o\}} \le \frac{u(o)+u(nz)}{n} + \left|\frac{\widehat{\tau}(o,nz)}{n} - \mu(z)\right|$$

and we conclude by (26) and (22). □

4.3 Extending μ

We have proved the existence of a linear propagation speed in every direction of $\mathbb{Z}^d$. However, to derive an asymptotic shape result, in particular for the approximating passage times $(\widehat{\tau}(x,y), x, y \in \mathbb{Z}^d)$, we need to control this propagation speed uniformy in all directions. For this we study $\mu(z), z \in \mathbb{Z}^d$, and we follow [4] to construct a Lipschitz, convex and homogeneous function φ which extends μ to $\mathbb{R}^d$. The asymptotic shape of the epidemic will be given by the convex set D defined later on in (50).

Lemma 4.7 *The function g defined on $\mathbb{Z}^d$ by*

$$\forall z \in \mathbb{Z}^d, \quad g(z) = E(\vartheta_z(0,1)) \tag{27}$$

has a barycentric extension to $\mathbb{R}^d$.

Proof of lemma 4.7. By (25), we have a.s., for all $z \in \mathbb{Z}^d$,

$$\lim_{n\to+\infty} \frac{\vartheta_z(0,n)}{n} = \inf_{n\in\mathbb{N}} E\left(\frac{\vartheta_z(0,n)}{n}\right) = \inf_{n\in\mathbb{N}} \frac{g(nz)}{n} = \mu(z). \tag{28}$$

To do a barycentric extension of g, we decompose $[0,1]^d$ in simplexes: each of them having a unique barycentric decomposition, g will be defined on its elements as the barycenter of its values on extremal points. This (arbitrary) construction will be translation invariant.

1. Let M denote the center of $[0,1]^d$. We define $g(M)$ to be the mean of the values of g on the 2^d elements of $\mathbb{Z}^d \cap [0,1]^d$.

2. For the center c of each face $\mathbf{F}$ of $[0,1]^d$ (which is a cube of dimension $d-1$) we define $g(c)$ to be the mean of the values of g on the 2^{d-1} elements of $\mathbb{Z}^d \cap \mathbf{F}$. We proceed similarly on each sub-cube, up to sub-faces of dimension 2.

3. Now, on sub-faces F of dimension 2, we link the center to the 4 elements of $\mathbb{Z}^d \cap F$, to obtain 4 triangles, or simplexes, of dimension 2. On each of them, we define $g(x)$ for each point x to be the barycentric combination of the values of g on the 3 extremal points.

4. We deal with dimension 3 sub-faces by taking barycentric combinations between the dimension 2 simplexes and the center of the dimension 3 sub-face. This way we have decomposed each dimension 3 sub-face into 4×6 simplexes, on which for each point x we define $g(x)$ in a barycentric way. We go on in the same way until the dimension d cube, that is $[0,1]^d$.

The function g is continuous on $[0,1]^d$. Then, for $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, to each d-cube $\prod_{i=1}^d [z_i, z_i+1]$ we associate the $2^{d-1}d!$ simplexes translated from those described in $[0,1]^d$, and we define as previously $g(x)$ for each point $x \in \prod_{i=1}^d [z_i, z_i+1]$ to be the barycentric combination of the values of g on $d+1$ extremal points. □

Following [4, Lemma 3.2], we define a sequence of functions $(g_n)_{n\geq 0}$ by

$$\forall x \in \mathbb{R}^d, \quad g_n(x) = \frac{g(nx)}{n}. \tag{29}$$

Lemma 4.8 *The elements of the sequence $(g_n)_{n\geq 0}$ are Lipschitz functions with a common Lipschitz constant denoted by γ^*, hence the sequence is equicontinuous.*

Proof of lemma 4.8. It is enough to prove that g is a Lipschitz function. For all $x, y \in \mathbb{Z}^d$, by subadditivity and symmetry of $\vartheta_.$ on $\mathbb{Z}^d$ we get

$$\begin{aligned} g(x) + g(y) &\geq g(x+y), & (30)\\ g(x+y) + g(-y) &\geq g(x), & (31)\\ g(-y) = g(y) &\leq \|y\|_1 g(\mathrm{e}_1). & (32) \end{aligned}$$

Indeed,

$$\begin{aligned} g(x+y) &= E(\vartheta_{x+y}(0,1)) = E\left(\widehat{\tau}(o, n(x+y)) + u(n(x+y))\right) \\ &\leq E\left(\widehat{\tau}(o, nx) + u(nx)\right) + E\left(\widehat{\tau}(nx, n(x+y))\right) + E\left(u(n(x+y))\right) \\ &= g(x) + E\left(\widehat{\tau}(o, ny)\right) + E\left(u(o)\right) \\ &= g(x) + g(y) \end{aligned}$$

where we have used successively (27), (24), (21) and the translation invariance of the distributions of $\widehat{\tau}$ and u. Then, writing (30) for $x = (x+y)+(-y)$ gives (31). For (32), we first use the symmetry of $\vartheta_.$ on $\mathbb{Z}^d$ to get $g(-y) = g(y)$, that we then combine with (30) to write

$$g(y) \leq \sum_{i=1}^d g(|y_i|\mathrm{e}_i) \leq \sum_{i=1}^d |y_i| g(\mathrm{e}_i) = \|y\|_1 g(\mathrm{e}_1).$$

Therefore by (30), (31), (32),

$$|g(x+y) - g(y)| \leq g(y)\mathbf{1}_{\{g(x+y)\geq g(y)\}} + g(-y)\mathbf{1}_{\{g(x+y)<g(y)\}} \leq \|y\|_1 g(\mathrm{e}_1). \tag{33}$$

If we now take $x, y \in \mathbb{R}^d$, from the previous barycentric construction, let $(x_0 = x, \ldots, x_n = x + y)$ be the sequence of points on the simplexes crossed by $[x, x+y]$. Then

$$|g(x+y) - g(x)| \leq \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)|.$$

Thus, since $\|y\|_1 = \sum_{k=0}^{n-1} \|x_{k+1} - x_k\|_1$, we have to show that on a given simplex the Lipschitz constant of g does not depend on the simplex. Assuming now that x, y belong to the same simplex, they are written uniquely as a barycentric combination of the $d+1$ extremal points $(z_0, \ldots, z_d)$ of that simplex, z_0 being the center of the cube translated from $[0,1]^d$ containing the simplex. Similarly, each $z_i, 0 \leq i \leq d$, is the barycenter of 2^d extremal points $(c_i, 1 \leq i \leq 2^d)$, with coefficients given by the barycentric construction:

$$x = \sum_{i=0}^{d} \alpha_i z_i; \qquad y = \sum_{i=0}^{d} \beta_i z_i; \qquad \sum_{i=0}^{d} \alpha_i = \sum_{i=0}^{d} \beta_i = 1;$$
$$z_0 = \sum_{l=1}^{2^d} \kappa_l c_l; \, z_i = \sum_{l=1}^{2^d} \iota_l c_l, \; 1 \leq i \leq d; \, \sum_{l=1}^{2^d} \kappa_l = \sum_{l=1}^{2^d} \iota_l = 1. \tag{34}$$

As $(z_i - z_0, 1 \leq i \leq d)$ is a basis of the vector space $\mathbb{R}^d$, denoting by $\|.\|_1^*$ the l^1-norm w.r.t. this basis, there exists a constant $\gamma_0 > 0$ such that

$$\forall z \in \mathbb{R}^d, \quad \frac{1}{\gamma_0} \|z\|_1 \leq \|z\|_1^* \leq \gamma_0 \|z\|_1. \tag{35}$$

Since $[0,1]^d$ is decomposed in a finite number $2^{d-1}d!$ of simplexes, (35) is valid for all these simplexes, for a constant $\gamma > 0$ which is the infimum of all the γ_0's. We have, using (33), (34), (35),

$$\begin{aligned}
|g(x) - g(y)| \quad &= \left| \sum_{i=0}^{d} (\alpha_i - \beta_i) g(z_i) \right| = \left| \sum_{i=0}^{d} (\alpha_i - \beta_i)(g(z_i) - g(z_0)) \right| \\
&\leq \sum_{i=0}^{d} |\alpha_i - \beta_i| \left| \sum_{l=1}^{2^d} (\iota_l - \kappa_l)(g(c_l) - g(c_0)) \right| \\
&\leq \sum_{i=0}^{d} |\alpha_i - \beta_i| \sum_{l=1}^{2^d} |\iota_l - \kappa_l| \|c_l\|_1 g(\mathrm{e}_1) \\
&\leq \sum_{i=0}^{d} |\alpha_i - \beta_i| 2^d \times 2 \times 2d \times g(\mathrm{e}_1) = 2^{d+2} d g(\mathrm{e}_1) \|x - y\|_1^*
\end{aligned}$$

$$\leq 2^{d+2}\gamma d g(\mathrm{e}_1)\|x-y\|_1$$

hence there exists $\gamma^* > 0$ such that

$$\forall x, y \in \mathbb{R}^d, \quad |g(x) - g(y)| \leq \gamma^* \|x-y\|_1. \tag{36}$$

□

Lemma 4.9 *The sequence $(g_n)_{n\geq 0}$ converges uniformly on each compact subset of $\mathbb{R}^d$ to a function φ which extends μ to $\mathbb{R}^d$, is Lipschitz with Lipschitz constant γ^*, convex and homogeneous (that is which satisfies $\varphi(\alpha_1 x) = \alpha_1\varphi(x)$ for all $x \in \mathbb{R}^d$ and $\alpha_1 > 0$).*

Proof of lemma 4.9. (i) For $x \in \mathbb{Z}^d$,

$$g_n(x) = \frac{g(nx)}{n} = \frac{E(\vartheta_{nx}(0,1))}{n} = \frac{E(\vartheta_x(0,n))}{n}.$$

Hence by (25) we get

$$\lim_{m\to\infty} g_m(x) = \mu(x) \ \ \forall\, x \in \mathbb{Z}^d. \tag{37}$$

Let now $x \in \mathbb{Q}^d$, and

$$N_x = \min\{k \geq 1, k \in \mathbb{N} : kx \in \mathbb{Z}^d\}. \tag{38}$$

Then, $g_{nN_x}(x) = g(nN_x x)/(nN_x)$ converges to $\mu(N_x x)/N_x$ as n goes to infinity. To prove the convergence of $g_m(x)$ over the whole sequence, write $m = n(m)N_x + j(m)$ where $j(m) \in \{0, \ldots, N_x - 1\}$, so that

$$\begin{aligned} g_m(x) &= \frac{g(mx)}{m} = \frac{g(n(m)N_x x + j(m)x)}{n(m)N_x + j(m)} \\ &= \frac{g(n(m)N_x x)}{n(m)N_x} \times \frac{n(m)N_x}{n(m)N_x + j(m)} \\ &\quad + \frac{g(n(m)N_x x + j(m)x) - g(n(m)N_x x)}{n(m)N_x + j(m)}. \end{aligned}$$

By (36), the second term of the last right hand side above converges to 0 as m goes to infinity. Therefore,

$$\lim_{m\to\infty} g_m(x) = \lim_{m\to\infty} \frac{g(n(m)N_x x)}{n(m)N_x} \times \frac{n(m)N_x}{n(m)N_x + j(m)} = \frac{\mu(N_x x)}{N_x}, \qquad \forall x \in \mathbb{Q}^d. \tag{39}$$

It follows from Lemma 4.8 and Arzela-Ascoli's Theorem that any subsequence of $(g_m(x))_{m\geq 0}$ has a further subsequence that converges uniformly on compact subsets of $\mathbb{R}^d$ to a Lipschitz function φ with the same Lipschitz constant γ^* as the g_m's (cf. (36)). Since, by (39), $\varphi(x)$ must be equal to $\mu(N_x x)/N_x$ for all $x \in \mathbb{Q}^d$ and φ is Lipschitz, the limiting function does not depend on the subsequence and the whole sequence $(g_m)_{m\geq 0}$ converges uniformy on compact subsets of $\mathbb{R}^d$ to φ which extends μ by (37).

$$\forall\, x \in \mathbb{Q}^d, \quad \lim_{n\to+\infty} g_n(x) = \varphi(x). \tag{40}$$

This implies convergence on $\mathbb{R}^d$, since every subsequence of $(g_n)_{n\geq 0}$ has a subsequence which converges uniformly on each compact subset of $\mathbb{R}^d$ to a continuous function, equal to φ on $\mathbb{Q}^d$.

(ii) To prove that φ is homogeneous we start noting that for $z \in \mathbb{Z}^d$ and $k \in \mathbb{N}$ we have:

$$\varphi(z) = \mu(z) = \lim_{n\to+\infty} \frac{\vartheta_z(0,n)}{n} = \lim_{n\to+\infty} \frac{\vartheta_z(0,nk)}{nk} = \frac{\mu(kz)}{k} = \frac{\varphi(kz)}{k}. \tag{41}$$

Now let $x \in \mathbb{Q}^d$ and recall that $\varphi(x) = \mu(N_x x)/N_x = \varphi(N_x x)/N_x$. Then if n is a multiple of N_x, we let $k = n/N_x \in \mathbb{N}$ and write by (41),

$$\varphi(x) = \frac{\varphi(N_x x)}{N_x} = \frac{\varphi(kN_x x)}{kN_x} = \frac{\varphi(nx)}{n}. \tag{42}$$

Since N_x is a multiple of N_{kx}, (42) implies:

$$\varphi(kx) = \frac{\varphi(N_x kx)}{N_x} = \frac{\varphi(kN_x x)}{N_x} = \frac{k\varphi(N_x x)}{N_x} = k\varphi(x), \qquad \forall\, x \in \mathbb{Q}^d, k \in \mathbb{N}.$$

Hence, if $r = n/m$ and $x \in \mathbb{Q}^d$ we have:

$$\varphi(rx) = n\varphi((1/m)x) = (n/m)\varphi(x),$$

so that φ is homogeneous on $\mathbb{Q}^d$.

(iii) To prove that φ is convex on $\mathbb{Q}^d$, take $x, y \in \mathbb{Q}^d$ and $\alpha \in \mathbb{Q} \cap (0,1)$. Then let k_1, k_2 be elements in $\mathbb{N}$ such that $k_1\alpha \in \mathbb{N}$, $k_2 x \in \mathbb{Z}^d$ and $k_2 y \in \mathbb{Z}^d$. Using subadditivity of g and homogeneity of φ write:

$$\varphi(\alpha x + (1-\alpha)y) \quad = \quad \lim_{n\to\infty} \frac{g(n\alpha x + n(1-\alpha)y)}{n}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{g(nk_1\alpha k_2 x + nk_1(1-\alpha)k_2 y)}{nk_1k_2} \\
&\le \lim_{n\to\infty} \frac{g(nk_1\alpha k_2 x) + g(nk_1(1-\alpha)k_2 y)}{nk_1k_2} \\
&= \frac{\varphi(k_1k_2\alpha x) + \varphi(k_1k_2(1-\alpha)y)}{k_1k_2} \\
&= \alpha\varphi(x) + (1-\alpha)\varphi(y).
\end{aligned}
$$

Since φ is continuous it is also homogeneous and convex on $\mathbb{R}^d$. □

4.4 Behavior of $\widehat{\tau}$

Our next result says that for $z \in \mathbb{Z}^d$, $\widehat{\tau}(0,z)$ grows at most linearly in $\|z\|_\infty$.

Theorem 4.2 *There exist $K = K(\lambda, d) > 0$ and $\alpha > 0$ such that*

$$
\begin{aligned}
P(\widehat{\tau}(o,z) > K\|z\|_\infty) &\le \exp(-\alpha(\|z\|_\infty^{1/d}), \quad \forall\ z \in \mathbb{Z}^d, \\
P(\widehat{\tau}(o,z) > K(\|z\|_\infty + n)) &\le \exp(-\alpha n^{1/d}), \quad \forall\ z \in \mathbb{Z}^d, n \in \mathbb{N}, \\
\sum_{z\in\mathbb{Z}^d} P(\widehat{\tau}(o,z) > K\|z\|_\infty) &< +\infty.
\end{aligned}
$$

Proof of theorem 4.2. Let $K \ge 0, z \in \mathbb{Z}^d$. Then write:

$$
\begin{aligned}
& P(\widehat{\tau}(o,z) > K(\|z\|_\infty + n)) \\
\le\ & P(4\kappa(z) > \|z\|_\infty + n) + P(4\kappa(o) > \|z\|_\infty + n) + P(A) \qquad (43)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \{\widehat{\tau}(o,z) > K(\|z\|_\infty + n), 4\kappa(z) \le \|z\|_\infty + n, 4\kappa(o) \le \|z\|_\infty + n\} \quad (44) \\
&\subset \cup_{(x,y)\in B(o,(\|z\|_\infty+n)/4)\times B(z,(\|z\|_\infty+n)/4)}\{x \to y, \tau(x,y) > K(\|z\|_\infty + n)\}.
\end{aligned}
$$

Noting that if $(x,y) \in B(o,(\|z\|_\infty + n)/4) \times B(z,(\|z\|_\infty + n)/4)$ we have

$$
\begin{aligned}
\|z\|_\infty - n &\le 2\|x-y\|_\infty \le 3\|z\|_\infty + n \quad \text{and} \\
3(\|z\|_\infty + n) &= 3\|z\|_\infty + n + 2n \ge 2(\|x-y\|_\infty + n), \qquad (45)
\end{aligned}
$$

from (44), for C_2 given in Lemma 3.5, we get:

$$
\begin{aligned}
P(A) \le & \sum_{x\in B(o,(\|z\|_\infty+n)/4)} \quad \sum_{y\in B(z,(\|z\|_\infty+n)/4)} \\
& \Big(P(3\tau(x,y) > 2K(\|x-y\|_\infty + n), D(x,y) < (C_2+1)(\|x-y\|_1 + n)
\end{aligned}
$$

$$+P(x \to y, D(x,y) \geq (\mathrm{C}_2+1)(\|x-y\|_1+n))\Big). \tag{46}$$

It now follows from Lemma 3.5 part *(i)* that we have

$$\begin{aligned} & P(x \to y, D(x,y) \geq (\mathrm{C}_2+1)(\|x-y\|_1+n)) \\ \leq\ & \exp(-\alpha_3(\|x-y\|_1+n)^{1/d}) \\ \leq\ & \exp(-\alpha_3(\|x-y\|_\infty+n)^{1/d}). \end{aligned} \tag{47}$$

Then, taking K large enough, by large deviation results for exponential variables, we also have, for some $\alpha_5 > 0$,

$$\begin{aligned} & P(3\tau(x,y) > 2K(\|x-y\|_\infty+n), D(x,y) < (\mathrm{C}_2+1)(\|x-y\|_1+n)) \\ \leq\ & P(3\tau(x,y) > 2K(\|x-y\|_\infty+n), D(x,y) < (\mathrm{C}_2+1)d(\|x-y\|_\infty+n)) \\ \leq\ & \exp(-\alpha_5(\|x-y\|_\infty+n)). \end{aligned} \tag{48}$$

Hence, from (45)–(48), for some constants R and $\alpha_6 > 0$ we have:

$$P(A) \leq R(\|z\|_\infty+n)^{2d}\exp(-\alpha_6(\|z\|_\infty+n)^{1/d}),$$

which gives, by modifying the constants,

$$P(A) \leq R'\exp(-\alpha_7(\|z\|_\infty+n)^{1/d}). \tag{49}$$

All the statements of the Theorem now follow from (49), (43) and Lemma 4.2. □

4.5 Asymptotic shape for $\widehat{\tau}$

Theorem 4.3 *Let $\varepsilon > 0$, and*

$$\begin{aligned} \widehat{A}_t &= \{z \in \mathbb{Z}^d : \widehat{\tau}(o,z) \leq t\}, \\ D &= \{x \in \mathbb{R}^d : \varphi(x) \leq 1\}. \end{aligned} \tag{50}$$

Then, a.s. for t large enough,

$$(1-\varepsilon)tD \cap \mathbb{Z}^d \subset \widehat{A}_t \subset (1+\varepsilon)tD \cap \mathbb{Z}^d. \tag{51}$$

Remark 4.5 *The set D is bounded: indeed passage times along edges are bounded below by passage times of exponential distributions, hence the epidemic cannot propagate quicker than this first passage percolation process, whose passage times have exponential distribution of parameter $\lambda/(2d)$, and which, by [9, Theorem (1.15)], moves linearly following the boundary of a convex set.*

In the sequel K is a fixed constant satisfying the conclusions of Theorem 4.2, γ^* is the Lipschitz constant of φ (see (36)) and N_x was defined in (38) for any $x \in \mathbb{Q}^d \setminus \{o\}$.

Lemma 4.10 *Let $\rho > 0$ and let $\delta \leq \rho/(2K)$. Then, for all $x \in \mathbb{Q}^d \setminus \{o\}$,*

$$\sum_{k>0} P\Big(\sup_{z \in B(kN_x x, \delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(kN_x x, z) \geq kN_x \rho\Big) < \infty, \tag{52}$$

$$\sum_{k>0} P\Big(\sup_{z \in B(kN_x x, \delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(z, kN_x x) \geq kN_x \rho\Big) < \infty. \tag{53}$$

Proof of lemma 4.10. Let $k > 0, z \in B(o, \delta k N_x) \cap \mathbb{Z}^d$. By Theorem 4.2 we have:

$$\begin{aligned} P(\widehat{\tau}(o,z) \geq kN_x\rho) &\leq P(\widehat{\tau}(o,z) \geq K\|z\|_\infty + \lfloor kN_x\rho/2 \rfloor) \\ &\leq \exp(-\alpha \lfloor kN_x\rho/2 \rfloor^{1/d}). \end{aligned}$$

Therefore, for some constant C,

$$\begin{aligned} \sum_{k>0} P\Big(\sup_{z \in B(o, \delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(o,z) \geq kN_x\rho\Big) &\leq \sum_{k>0} \mathrm{C}(\delta k N_x)^d \exp(-\alpha \lfloor kN_x\rho/2 \rfloor^{1/d}) \\ &< \infty. \end{aligned}$$

Now (52) follows from the translation invariance of $\widehat{\tau}$. The proof of (53) is analogous. □

For $x = (x_1, \ldots, x_d) \in \mathbb{Q}^d \setminus \{o\}$ and $\delta > 0$, we define the cone associated to x of amplitude δ as

$$C(x, \delta) = \mathbb{Z}^d \cap \Big(\cup_{t \geq 0} B(xt, \delta t) \Big). \tag{54}$$

Lemma 4.11 *Let $x \in \mathbb{Q}^d \setminus \{o\}$. Then for any $0 < \delta' < \delta$ the set $C(x, \delta') \setminus \cup_{k \geq 0} B(kN_x x, \delta k N_x)$ is finite.*

The proof of this lemma is elementary and left to the reader.

Proof of theorem 4.3. Fix $\varepsilon \in (0,1)$ and let ρ, δ and ι be three small positive parameters such that $\delta \leq \rho/(2K)$, whose values will be determined later. The set $\mathcal{Y} = \{x \in \mathbb{Q}^d : 1 - 2\iota < \varphi(x) < 1 - \iota\}$ is a ring between two balls with the same center but with a different radius, because by Lemma 4.9, φ is homogeneous and positive except that $\varphi(o) = 0$. Hence the (compact) closure of $\mathcal{Y}$, which is recovered by balls of the same radius centered on the rational points of $\mathcal{Y}$, is in fact covered by a finite number of such balls. Thus there exists a finite subset Y of $\mathcal{Y}$ such that $\mathbb{Z}^d \subset \cup_{x\in Y} C(x, \delta/2)$ (if the balls recover the ring, the cones associated to them recover the whole space). Hence, to prove the first inclusion of (51) it suffices to show that for any $x \in Y$ and any sequences $(t_n)_{n>0}$ and $(z_n)_{n>0}$ such that $t_n \uparrow \infty$ in $\mathbb{R}^+$, $z_n \in C(x, \delta/2) \cap \mathbb{Z}^d$ with $\|z_n\|_\infty \geq n$ and $\varphi(z_n) \leq (1-\varepsilon)t_n$, we have $\widehat{\tau}(o, z_n) \leq t_n$ a.s. for n sufficiently large. So, let $(t_n)_{n>0}$ and $(z_n)_{n>0}$ be such sequences. Using Lemma 4.11, let $k_n \in \mathbb{N}$ be such that $z_n \in B(k_n N_x x, \delta k_n N_x)$, hence $k_n \geq \mathrm{C}n$ for some constant C. Since by Lemma 4.9, φ is Lipschitz with Lipschitz constant γ^*, write, for $\gamma = \gamma^* d$:

$$k_n N_x(1-2\iota) \leq \varphi(k_n N_x x) \leq \varphi(z_n) + \gamma\delta k_n N_x \leq (1-\varepsilon)t_n + \gamma\delta k_n N_x.$$

Therefore

$$k_n N_x \leq \Big(\frac{1-\varepsilon}{1-2\iota-\gamma\delta}\Big)t_n.$$

It now follows from this inequality and the subadditivity property (21) of $\widehat{\tau}$ that:

$$\frac{\widehat{\tau}(o,z_n)}{t_n} \leq \Big(\frac{1-\varepsilon}{1-2\iota-\gamma\delta}\Big)\Big(\frac{\widehat{\tau}(o,k_n N_x x)}{k_n N_x} + \frac{u(k_n N_x x)}{k_n N_x} + \frac{\widehat{\tau}(k_n N_x x, z_n)}{k_n N_x}\Big).$$

Therefore, by Theorem 4.1, Lemma 4.6 (the variables $u(.)$ are identically distributed, and $k_n \geq \mathrm{C}n$), Lemmas 4.9 and 4.10 we obtain:

$$\limsup_{n\to+\infty} \frac{\widehat{\tau}(o,z_n)}{t_n} \leq \Big(\frac{1-\varepsilon}{1-2\iota-\gamma\delta}\Big)\Big(\varphi(x)+\rho\Big) \qquad \text{a.s.}$$

Since $x \in Y$ this implies:

$$\limsup_{n\to+\infty} \frac{\widehat{\tau}(o,z_n)}{t_n} \leq \Big(\frac{1-\varepsilon}{1-2\iota-\gamma\delta}\Big)\Big(1-\iota+\rho\Big) \qquad \text{a.s.}$$

Taking ι, ρ and δ small enough, the right hand side is strictly less than 1 which proves that $\widehat{\tau}(o, z_n) \leq t_n$ a.s. for n sufficiently large.

Similarly, to prove the second inclusion of (51) it suffices to show that for any $x \in Y$ and any sequences $t_n \uparrow \infty$ in $\mathbb{R}^+$ and z_n in $C(x, \delta/2) \cap \mathbb{Z}^d$ such that $\varphi(z_n) \geq (1+\varepsilon)t_n$ we have $\widehat{\tau}(o, z_n) > t_n$ a.s. for n sufficiently large. As before, we let $(t_n)_{n>0}$ and $(z_n)_{n>0}$ be such sequences and we let $k_n \in \mathbb{N}$ be such that $z_n \in B(k_n N_x x, \delta k_n N_x)$. Then,

$$k_n N_x(1-\iota) \geq \varphi(k_n N_x x) \geq \varphi(z_n) - \gamma\delta k_n N_x \geq (1+\varepsilon)t_n - \gamma\delta k_n N_x.$$

Therefore,

$$k_n N_x \geq \Big(\frac{1+\varepsilon}{1-\iota+\gamma\delta}\Big)t_n.$$

Proceeding then as for the first inclusion, we get:

$$\frac{\widehat{\tau}(o,z_n)}{t_n} \geq \Big(\frac{1+\varepsilon}{1-\iota+\gamma\delta}\Big)\Big(\frac{\widehat{\tau}(o,k_nN_xx)}{k_nN_x} - \frac{u(z_n)}{k_nN_x} - \frac{\widehat{\tau}(z_n,k_nN_xx)}{k_nN_x}\Big),$$

and

$$\begin{aligned}\liminf_{n\to+\infty}\frac{\widehat{\tau}(o,z_n)}{t_n} &\geq \Big(\frac{1+\varepsilon}{1-\iota+\gamma\delta}\Big)\Big(\varphi(x)-\rho\Big) \qquad \text{a.s.}\\ &\geq \Big(\frac{1+\varepsilon}{1-\iota+\gamma\delta}\Big)\Big(1-2\iota-\rho\Big) \qquad \text{a.s.}\end{aligned}$$

Now, taking ι, ρ and δ small enough, the right hand side is strictly bigger than 1 and the second inclusion of (51) is proved. □

4.6 Asymptotic shape for the epidemic

We can now prove our main result:

Proof of theorem 2.1. *(i)* We first show that the infection grows at least linearly as t goes to infinity, that is, given $\varepsilon > 0$,

$$P\Big((\zeta_t \cup \xi_t) \supset ((1-\varepsilon)tD \cap C_o^o) \text{ for all } t \text{ large enough}\Big) = 1.$$

Since R_o^o is finite a.s. this will follow from:

$$P\Big((\zeta_t \cup \xi_t) \supset ((1-\varepsilon)tD \cap (C_o^o \setminus R_o^o)) \text{ for all } t \text{ large enough}\Big) = 1. \tag{55}$$

Let $z \in (1-\varepsilon)tD \cap (C_o^o \setminus R_o^o)$, then by Theorem 4.3,

$$\widehat{\tau}(o,z) \leq (1-\varepsilon/2)t, \text{ a.s. for } t \text{ large enough}, \tag{56}$$

and by Lemma 4.3, $\tau(o,z) \leq (1-\varepsilon/2)t + u(o) + u(z)$. Since $u(o) < \infty$ a.s. we have $u(o) < (\varepsilon/4)t$ a.s. for t large enough. Hence (55) will follow if we show that $\sup_{z\in tD} u(z) \leq (\varepsilon/4)t$ a.s. for t large enough: To derive this, it is enough to show that $\sup_{z\in(n+1)D} u(z) \leq (\varepsilon/4)n$ a.s. for $n(\in\mathbb{N})$ large enough. By Remark 4.5, D is bounded, hence the number of points in $(n+1)D$ with coordinates in $\mathbb{Z}$ is less than $\mathrm{C}_5(n+1)^d$ for some constant C_5. Then write

$$P\left(\sup_{z\in(n+1)D} u(z) \geq \frac{\varepsilon n}{4}\right) \;\leq\; \mathrm{C}_5(n+1)^d P\left(u(o) \geq \frac{\varepsilon n}{4}\right)$$

$$\leq \quad C_5(n+1)^d \frac{4^{d+2}}{(\varepsilon n)^{d+2}} E(u(o)^{d+2}).$$

Thus, by Lemma 4.6, $\sum_{n\in\mathbb{N}} P(\sup_{z\in(n+1)D} u(z) \geq \varepsilon n/4) < \infty$, and (55) follows from Borel-Cantelli's Lemma.

(ii) Next we show that

$$P\Big((\zeta_t \cup \xi_t) \subset ((1+\varepsilon)tD \cap C_o^o) \text{ for all } t \text{ large enough}\Big) = 1. \tag{57}$$

If z belongs to ξ_t or ζ_t, then $\tau(o,z) \leq t$ hence by Lemma 4.3, $\widehat{\tau}(o,z) \leq t$ for $z \in C_o^o \setminus R_o^o$, which implies $z \in (1+\varepsilon)tD$ for t large enough by Theorem 4.3. Since R_o^o is finite (57) follows.

(iii) Finally, assuming $E(|T_z|^d) < \infty$, we show that

$$P(\zeta_t \cap (1-\varepsilon)tD = \emptyset \text{ for } t \text{ large enough}) = 1. \tag{58}$$

Let $z \in (1-\varepsilon)tD \cap C_o^o$, then, by (56), $\tau(o,z) \leq (1-\varepsilon/2)t$ if t is large enough. Hence, (58) follows if we show that $T_z \geq (\varepsilon/2)\tau(o,z)$ occurs only for a finite number of z's. But from (57) we get that for some $\delta > 0$ we have $\tau(o,z) \geq \delta\|z\|_\infty$ except for a finite number of z's. Therefore, it suffices to show that for any $\delta' > 0$ the event $\{T_z \geq \delta'\|z\|_\infty\}$ can only occur for a finite number of z's. This will follow from Borel-Cantelli's Lemma once we prove that $\sum_{z\in\mathbb{Z}^d} P(T_z \geq \delta'\|z\|_\infty) < \infty$. To do so we write, since the T_z's are identically distributed:

$$\sum_{z\in\mathbb{Z}^d} P(T_z \geq \delta'\|z\|_\infty) = \sum_{n\in\mathbb{N}} \sum_{z:\|z\|_\infty=n} P(T_z \geq \delta' n) \leq c \sum_{n\in\mathbb{N}} n^{d-1} P(T_o \geq \delta' n)$$

for some constant c, and this last series converges because T_o has a finite moment of order d. □

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