

# NEW GLOBAL STABILITY ESTIMATES FOR THE CALDERÓN PROBLEM IN TWO DIMENSIONS

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ABSTRACT. We prove a new global stability estimate for the Gel'fand-Calderón inverse problem on a two-dimensional bounded domain or, more precisely, the inverse boundary value problem for the equation  $-\Delta\psi + v\psi = 0$  on  $D$ , where  $v$  is a smooth real-valued potential of conductivity type defined on a bounded planar domain  $D$ . The principal feature of this estimate is that it shows that the more a potential is smooth, the more its reconstruction is stable, and the stability varies exponentially with respect to the smoothness (in a sense to be made precise). As a corollary we obtain a similar estimate for the Calderón problem for the electrical impedance tomography.

## 1. INTRODUCTION

Let  $D \subset \mathbb{R}^2$  be a bounded domain equipped with a potential given by a function  $v \in L^\infty(D)$ . The corresponding Dirichlet-to-Neumann map is the operator  $\Phi : H^{1/2}(\partial D) \rightarrow H^{-1/2}(\partial D)$ , defined by

$$(1.1) \quad \Phi(f) = \frac{\partial u}{\partial \nu} \Big|_{\partial D},$$

where  $f \in H^{1/2}(\partial D)$ ,  $\nu$  is the outer normal of  $\partial D$ , and  $u$  is the  $H^1(D)$ -solution of the Dirichlet problem

$$(1.2) \quad (-\Delta + v)u = 0 \text{ on } D, \quad u|_{\partial D} = f;$$

here we assume that

$$(1.3) \quad 0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v \text{ in } D.$$

The following inverse boundary value problem arises from this construction:

**Problem 1.** *Given  $\Phi$ , find  $v$  on  $D$ .*

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [10]) and can also be seen as a generalization of the Calderón problem for the electrical impedance tomography (see [7]), in two dimensions.

We recall how our problem is related with the inverse conductivity problem proposed by Calderón. In that framework  $D$  is a body equipped with an isotropic conductivity  $\sigma(x) \in L^\infty(D)$  (with  $\sigma \geq \sigma_{\min} > 0$ ),

$$(1.4) \quad v(x) = \frac{\Delta \sigma^{1/2}(x)}{\sigma^{1/2}(x)}, \quad x \in D,$$

$$(1.5) \quad \Phi = \sigma^{-1/2} \left( \Lambda \sigma^{-1/2} + \frac{\partial \sigma^{1/2}}{\partial \nu} \right),$$

where  $\sigma^{-1/2}$ ,  $\partial \sigma^{1/2} / \partial \nu$  in (1.5) denote the multiplication operators by the functions  $\sigma^{-1/2}|_{\partial D}$ ,  $\partial \sigma^{1/2} / \partial \nu|_{\partial D}$ , respectively and  $\Lambda$  is the voltage-to-current map on  $\partial D$ , defined as

$$(1.6) \quad \Lambda f = \sigma \frac{\partial u}{\partial \nu} \Big|_{\partial D},$$

where  $f \in H^{1/2}(\partial D)$ ,  $\nu$  is the outer normal of  $\partial D$ , and  $u$  is the  $H^1(D)$ -solution of the Dirichlet problem

$$(1.7) \quad \operatorname{div}(\sigma \nabla u) = 0 \text{ on } D, \quad u|_{\partial D} = f.$$

Indeed, the substitution  $u = \tilde{u} \sigma^{-1/2}$  in (1.7) yields  $(-\Delta + v)\tilde{u} = 0$  in  $D$  with  $v$  given by (1.4). In this framework we have the Calderón problem:

**Problem 2.** *Given  $\Lambda$ , find  $\sigma$  on  $D$ .*

We would like to remark that Problems 1 and 2 are not overdetermined, in the sense that we consider the reconstruction of a real-valued function of two variables from real-valued inverse problem data dependent on two variables. In addition, the history of inverse problems for the two-dimensional Schrödinger equation at fixed energy goes back to [8].

There are several questions to be answered in these inverse problems: one would like to prove the uniqueness, i.e. the injectivity of the map  $v \rightarrow \Phi$  (for Problem 1, for example), then the reconstruction of  $v$  from  $\Phi$  and after the stability of the inverse  $\Phi \rightarrow v$ .

In this paper we study interior stability estimates, i.e. (for Problem 1 with a potential of conductivity type, for example) we want to prove that given two Dirichlet-to-Neumann operators  $\Phi_1$  and  $\Phi_2$ , corresponding to potentials  $v_1$  and  $v_2$  on  $D$ , we have that

$$\|v_1 - v_2\|_{L^\infty(D)} \leq \omega(\|\Phi_1 - \Phi_2\|_{H^{1/2} \rightarrow H^{-1/2}}),$$

where the function  $\omega(t) \rightarrow 0$  as fast as possible as  $t \rightarrow 0$ . We will study also stability estimates for Problem 2.

There is a wide literature on the Gel'fand-Calderón inverse problem. In the case of complex-valued potentials the global injectivity of the map  $v \rightarrow \Phi$  was firstly proved in [16] for  $D \subset \mathbb{R}^d$  with  $d \geq 3$  and in [6] for  $d = 2$  with  $v \in L^p$ : in particular, these results were obtained by the use of global reconstructions developed in the same papers. A global stability estimate for Problem 1 and 2 for  $d \geq 3$  was first found by Alessandrini in [1]; this result was recently improved in [19]. In the two-dimensional case the first global stability estimate for Problem 1 was given in [21].

Global results for Problem 2 in the two dimensional case have been found much earlier than for Problem 1. In particular, global uniqueness was first proved in [15] for conductivities in the  $W^{2,p}(D)$  class ( $p > 1$ ) and after in [2] for  $L^\infty$  conductivities. The first global stability result was given in [13], where a logarithmic estimate is obtained for conductivities with two continuous derivatives. This result was improved in [4], where the same kind of estimate is obtained for Hölder continuous conductivities.

This paper is not intended to pursue this direction of research, i.e. to prove stability estimates for potentials/conductivities with less regularity. Here, in fact, we focus on the opposite situation, i.e. smooth potentials/conductivities, and try to answer another question: how the stability estimates vary with respect to the smoothness of the potentials/conductivities.

The results obtained also constitutes progress for non-smooth potentials: indeed they indicate stability dependence of the smooth part of a singular potential with respect to boundary value data.

We will assume for simplicity that

$$(1.8) \quad \begin{aligned} D \text{ is an open bounded domain in } \mathbb{R}^2, \quad \partial D \in C^2, \\ v \in W^{m,1}(\mathbb{R}^2) \text{ for some } m > 2, \quad \text{supp } v \subset D, \end{aligned}$$

where

$$(1.9) \quad \begin{aligned} W^{m,1}(\mathbb{R}^2) &= \{v : \partial^J v \in L^1(\mathbb{R}^2), |J| \leq m\}, \quad m \in \mathbb{N} \cup \{0\}, \\ J &\in (\mathbb{N} \cup \{0\})^2, \quad |J| = J_1 + J_2, \quad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \partial x_2^{J_2}}. \end{aligned}$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^2)}.$$

The last (strong) hypothesis is that we will consider only potentials of conductivity type, i.e.

$$(1.10) \quad v = \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}}, \text{ for some } \sigma \in L^\infty(D), \text{ with } \sigma \geq \sigma_{\min} > 0.$$

The main result is the following.

**Theorem 1.1.** *Let the conditions (1.3), (1.8), (1.10) hold for the potentials  $v_1, v_2$ , where  $D$  is fixed, and let  $\Phi_1, \Phi_2$  be the corresponding Dirichlet-to-Neumann operators. Let  $\|v_j\|_{m,1} \leq N$ ,  $j = 1, 2$ , for some  $N > 0$ . Then, for any  $\alpha < m$  there exists a constant  $C = C(D, N, m, \alpha)$  such that*

$$(1.11) \quad \|v_2 - v_1\|_{L^\infty(D)} \leq C(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{-\alpha},$$

where  $\|\Phi_2 - \Phi_1\| = \|\Phi_2 - \Phi_1\|_{H^{1/2} \rightarrow H^{-1/2}}$ .

**Corollary 1.2.** *Let  $\sigma_1, \sigma_2$  be two isotropic conductivities such that  $\Delta(\sigma_j^{1/2})/\sigma_j^{1/2}$  satisfies conditions (1.8), where  $D$  is fixed and  $0 < \sigma_{\min} \leq \sigma_j \leq \sigma_{\max} < \infty$  for  $j = 1, 2$  and some constants  $\sigma_{\min}$  and  $\sigma_{\max}$ . Let  $\Lambda_1, \Lambda_2$  be the corresponding Dirichlet-to-Neumann operators and  $\|\Delta(\sigma_j^{1/2})/\sigma_j^{1/2}\|_{m,1} \leq N$ ,  $j = 1, 2$ , for some  $N > 0$ . We suppose, for simplicity, that  $\text{supp}(\sigma_j - 1) \subset D$  for  $j = 1, 2$ . Then, for any  $\alpha < m$  there exists a constant  $C = C(D, N, m, \alpha)$  such that*

$$(1.12) \quad \|\sigma_2 - \sigma_1\|_{L^\infty(D)} \leq C(\log(3 + \|\Lambda_2 - \Lambda_1\|^{-1}))^{-\alpha},$$

where  $\|\Lambda_2 - \Lambda_1\| = \|\Lambda_2 - \Lambda_1\|_{H^{1/2} \rightarrow H^{-1/2}}$ .

The main feature of these estimates is that, as  $m \rightarrow +\infty$ , we have  $\alpha = \alpha(m) \rightarrow +\infty$  (one can take  $\alpha(m) = m - 1$ ). In addition we would like to mention that, under the assumption of Theorem 1.1 and Corollary (1.2), according to instability estimates of Mandache [14] and Isaev [12], our results are almost optimal. Note that in the linear approximation near zero potential Theorem 1.1 (without condition (1.10) but with  $\alpha \leq 2$ ) was proved in [20]. In dimension  $d \geq 3$  a global stability estimate similar to our result (with respect to dependence on smoothness) was proved in [19].

The proof of Theorem 1.1 relies on the  $\bar{\partial}$ -techniques introduced by Beals-Coifman [5], Henkin-Novikov [11] and developed by Novikov [16] and Nachman [15] for solving the Calderón problem in two dimensions.

The Novikov-Nachman method starts with the construction of a special family of solutions  $\psi(x, \lambda)$  of equation (1.2), which was originally introduced by Faddeev in [9]. These solutions have an exponential behaviour depending on the complex parameter  $\lambda$  and they are constructed via some function  $\mu(x, \lambda)$  (see (2.5)). One of the most important property of  $\mu(x, \lambda)$  is that it satisfies a  $\bar{\partial}$ -equation with respect to the variable  $\lambda$  (see equation (2.8)), in which appears the so-called Faddeev generalized scattering amplitude  $h(\lambda)$  (defined in (2.6)). On the contrary, if one knows  $h(\lambda)$  for every  $\lambda \in \mathbb{C}$ , it is possible to recover  $\mu(x, \lambda)$  via this  $\bar{\partial}$ -equation. Starting from these arguments we will prove that the map  $h(\lambda) \rightarrow v(x)$  satisfies an Hölder condition (Proposition 4.2). This is done in Section 4.

The remaining part of the method relates the scattering amplitude  $h(\lambda)$  to the Dirichlet-to-Neumann operator  $\Phi$ . In the present paper this is done using the Alessandrini identity (see [1]) and an estimate of  $h(\lambda)$  for high values of  $|\lambda|$  given in [17]. We find that the map  $\Phi \rightarrow h$  has logarithmic stability in some natural norm (Proposition 3.3). This is explained in Section 3.

The composition of the two above-mentioned maps gives the result of Theorem 1.1, as showed in Section 5.

This work was fulfilled in the framework of researches under the direction of R. G. Novikov.

## 2. PRELIMINARIES

In this section we recall some definitions and properties of the Faddeev functions, the above-mentioned family of solutions of equation (1.2), which will be used throughout all the paper.

Following [15], we fix some  $1 < p < 2$  and define  $\psi(x, k)$  to be the solution (when it exists unique) of

$$(2.1) \quad (-\Delta + v)\psi(x, k) = 0 \text{ in } \mathbb{R}^2,$$

with  $e^{-ixk}\psi(x, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2) = \{u : \partial^J u \in L^{\tilde{p}}(\mathbb{R}^2), |J| \leq 1\}$ , where  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $k = (k_1, k_2) \in \mathcal{V} \subset \mathbb{C}^2$ ,

$$(2.2) \quad \mathcal{V} = \{k \in \mathbb{C}^2 : k^2 = k_1^2 + k_2^2 = 0\}$$

and

$$(2.3) \quad \frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}.$$

The variety  $\mathcal{V}$  can be written as  $\{(\lambda, i\lambda) : \lambda \in \mathbb{C}\} \cup \{(\lambda, -i\lambda) : \lambda \in \mathbb{C}\}$ . We henceforth denote  $\psi(x, (\lambda, i\lambda))$  by  $\psi(x, \lambda)$  and observe that, since  $v$  is real-valued, uniqueness for (2.1) yields  $\psi(x, (-\bar{\lambda}, i\bar{\lambda})) = \overline{\psi(x, (\lambda, i\lambda))} = \overline{\psi(x, \lambda)}$  so that, for reconstruction and stability purpose, it is sufficient to work on the sheet  $k = (\lambda, i\lambda)$ .

We now identify  $\mathbb{R}^2$  with  $\mathbb{C}$  and use the coordinates  $z = x_1 + ix_2$ ,  $\bar{z} = x_1 - ix_2$ ,

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right),$$

where  $(x_1, x_2) \in \mathbb{R}^2$ .

Then we define

$$(2.4) \quad \psi(z, \lambda) = \psi(x, \lambda),$$

$$(2.5) \quad \mu(z, \lambda) = e^{-iz\lambda}\psi(z, \lambda),$$

$$(2.6) \quad h(\lambda) = \int_D e^{i\bar{z}\lambda} v(z) \psi(z, \lambda) d\operatorname{Re}z d\operatorname{Im}z,$$

for  $z, \lambda \in \mathbb{C}$ .

Throughout all the paper  $c(\alpha, \beta, \dots)$  is a positive constant depending on parameters  $\alpha, \beta, \dots$

We now restate some fundamental results about Faddeev functions. In the following statement  $\psi_0$  denotes  $\sigma^{1/2}$ .

**Proposition 2.1** ([15]). *Let  $D \subset \mathbb{R}^2$  be an open bounded domain with  $C^2$  boundary,  $v \in L^p(\mathbb{R}^2)$ ,  $1 < p < 2$ ,  $\operatorname{supp} v \subset D$ ,  $\|v\|_{L^p(\mathbb{R}^2)} \leq N$ , be such that there exists a real-valued  $\psi_0 \in L^\infty(\mathbb{R}^2)$  with  $v = (\Delta\psi_0)/\psi_0$ ,  $\psi_0(x) \geq c_0 > 0$  and  $\psi_0 \equiv 1$  outside  $D$ . Then, for any  $\lambda \in \mathbb{C}$  there is a unique solution  $\psi(z, \lambda)$  of (2.1) with  $e^{-iz\lambda}\psi(\cdot, \lambda) - 1$  in  $L^\infty$ . Furthermore,  $e^{-iz\lambda}\psi(\cdot, \lambda) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2)$  ( $\tilde{p}$  is defined in (2.3)) and*

$$(2.7) \quad \|e^{-iz\lambda}\psi(\cdot, \lambda) - 1\|_{W^{s, \tilde{p}}} \leq c(p, s)N|\lambda|^{s-1},$$

for  $0 \leq s \leq 1$  and  $\lambda$  sufficiently large.

The function  $\mu(z, \lambda)$  defined in (2.5) satisfies the equation

$$(2.8) \quad \frac{\partial \mu(z, \lambda)}{\partial \bar{\lambda}} = \frac{1}{4\pi\lambda} h(\lambda) e_{-\lambda}(z) \overline{\mu(z, \lambda)}, \quad z, \lambda \in \mathbb{C},$$

in the  $W^{1, \tilde{p}}$  topology, where  $h(\lambda)$  is defined in (2.6) and the function  $e_{-\lambda}(z)$  is defined as follows:

$$(2.9) \quad e_\lambda(z) = e^{i(z\lambda + \bar{z}\bar{\lambda})}.$$

In addition, the functions  $h(\lambda)$  and  $\mu(z, \lambda)$  satisfy

$$(2.10) \quad \left\| \frac{h(\lambda)}{\lambda} \right\|_{L^r(\mathbb{R}^2)} \leq c(r, N), \text{ for all } r \in (\tilde{p}', \tilde{p}), \quad \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1,$$

$$(2.11) \quad \inf_{z, \lambda \in \mathbb{C}} |\mu(z, \lambda)| \geq c(D, N) > 0,$$

$$(2.12) \quad \sup_{z \in \mathbb{C}} \|\mu(z, \cdot) - 1\|_{L^r(\mathbb{C})} \leq c(r, D, N), \quad \text{for all } r \in (p', \infty]$$

and

$$(2.13) \quad |h(\lambda)| \leq c(p, D, N)|\lambda|^\varepsilon,$$

$$(2.14) \quad \|\mu(\cdot, \lambda) - \psi_0\|_{W^{1, \tilde{p}}} \leq c(p, D, N)|\lambda|^\varepsilon,$$

for  $\lambda \leq \lambda_0(p, D, N)$  and  $0 < \varepsilon < \frac{2}{p}$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

We recall that if  $v \in W^{m,1}(\mathbb{R}^2)$  with  $\text{supp } v \subset D$ , then  $\|\hat{v}\|_m < +\infty$ , where

$$(2.15) \quad \hat{v}(p) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ipx} v(x) dx, \quad p \in \mathbb{C}^2,$$

$$(2.16) \quad \|u\|_m = \sup_{p \in \mathbb{R}^2} |(1 + |p|^2)^{m/2} u(p)|,$$

for a test function  $u$ .

In addition, if  $v \in W^{m,1}(\mathbb{R}^2)$  with  $\text{supp } v \subset D$  and  $m \geq 1$ , we have

$$(2.17) \quad \|v\|_{L^\infty(D)} \leq \text{diam}(D) \|v\|_{m,1},$$

so, in particular, the hypothesis  $v \in L^p(\mathbb{R}^2)$ ,  $\text{supp } v \subset D$ , in the statement of Proposition 2.1 satisfied for every  $1 < p < 2$  (since  $D$  is bounded).

The following lemma is a variation of a result in [17]:

**Lemma 2.2.** *Under the assumption (1.8), there exists  $R = R(m, \|\hat{v}\|_m) > 0$  such that*

$$(2.18) \quad |h(\lambda)| \leq 8\pi^2 \|\hat{v}\|_m (1 + 4|\lambda|^2)^{-m/2}, \quad \text{for } |\lambda| > R.$$

*Proof.* We consider the function  $H(k, p)$  defined as

$$(2.19) \quad H(k, p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(p-k)x} v(x) \psi(x, k) dx,$$

for  $k \in \mathcal{V}$  (where  $\mathcal{V}$  is defined in (2.2)),  $p \in \mathbb{R}^2$  and  $\psi(x, k)$  as defined at the beginning of this section.

We deduce that  $h(\lambda) = (2\pi)^2 H(k(\lambda), k(\lambda) + \overline{k(\lambda)})$ , for  $k(\lambda) = (\lambda, i\lambda)$ . By [17, Corollary 1.1] we have

$$(2.20) \quad |H(k, p)| \leq 2 \|\hat{v}\|_m (1 + p^2)^{-m/2} \quad \text{for } |\lambda| > R,$$

for  $R = R(m, \|\hat{v}\|_m) > 0$  and then the proof follows.  $\square$

We restate [3, Lemma 2.6], which will be useful in section 4.

**Lemma 2.3** ([3]). *Let  $a \in L^{s_1}(\mathbb{R}^2) \cap L^{s_2}(\mathbb{R}^2)$ ,  $1 < s_1 < 2 < s_2 < \infty$  and  $b \in L^s(\mathbb{R}^2)$ ,  $1 < s < 2$ . Assume  $u$  is a function in  $L^{\tilde{s}}(\mathbb{R}^2)$ , with  $\tilde{s}$  defined as in (2.3), which satisfies*

$$(2.21) \quad \frac{\partial u(\lambda)}{\partial \bar{\lambda}} = a(\lambda) \bar{u}(\lambda) + b(\lambda), \quad \lambda \in \mathbb{C}.$$

*Then there exists  $c > 0$  such that*

$$(2.22) \quad \|u\|_{L^{\tilde{s}}} \leq c \|b\|_{L^s} \exp(c(\|a\|_{L^{s_1}} + \|a\|_{L^{s_2}})).$$

3. FROM  $\Phi$  TO  $h(\lambda)$ 

**Lemma 3.1.** *Let the condition (1.8) holds. Then we have, for  $p > 1$ ,*

$$(3.1) \quad \left\| \frac{h(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda| > R)} \leq c(p, m) \|\hat{v}\|_m \frac{1}{R^{m+1-2/p}},$$

where  $R$  is as in Lemma 2.2.

*Proof.* It's a corollary of Lemma 2.2. Indeed we have

$$\left\| \frac{h(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda| > R)}^p \leq c \|\hat{v}\|_m^p \int_{r>R} r^{1-mp-p} dr = \frac{c(p, m) \|\hat{v}\|_m^p}{R^{(m+1)p-2}}. \quad \square$$

**Lemma 3.2.** *Let  $D \subset \{x \in \mathbb{R}^2 : |x| \leq l\}$ ,  $v_1, v_2$  be two potentials satisfying (1.3), (1.8), (1.10), let  $\Phi_1, \Phi_2$  the corresponding Dirichlet-to-Neumann operator and  $h_1, h_2$  the corresponding generalised scattering amplitude. Let  $\|v_j\|_{m,1} \leq N$ ,  $j = 1, 2$ . Then we have*

$$(3.2) \quad |h_2(\lambda) - h_1(\lambda)| \leq c(D, N) e^{2l|\lambda|} \|\Phi_2 - \Phi_1\|_{H^{1/2} \rightarrow H^{-1/2}}, \quad \lambda \in \mathbb{C}.$$

*Proof.* We have the following identity:

$$(3.3) \quad h_2(\lambda) - h_1(\lambda) = \int_{\partial D} \overline{\psi_1(z, \lambda)} (\Phi_2 - \Phi_1) \psi_2(z, \lambda) |dz|,$$

where  $\psi_i(z, \lambda)$  are the Faddeev functions associated to the potential  $v_i$ ,  $i = 1, 2$ . This identity is a particular case of the one in [18, Theorem 1]: we refer to that paper for a proof.

From this identity we have:

$$(3.4)$$

$$|h_2(\lambda) - h_1(\lambda)| \leq \|\psi_1(\cdot, \lambda)\|_{H^{1/2}(\partial D)} \|\Phi_2 - \Phi_1\|_{H^{1/2} \rightarrow H^{-1/2}} \|\psi_2(\cdot, \lambda)\|_{H^{1/2}(\partial D)}.$$

Now take  $\tilde{p} > 2$  and use the trace theorem to get

$$\begin{aligned} \|\psi_j(\cdot, \lambda)\|_{H^{1/2}(\partial D)} &\leq C \|\psi_j(\cdot, \lambda)\|_{W^{1, \tilde{p}}(D)} \leq C e^{l|\lambda|} \|e^{-iz\lambda} \psi_i(\cdot, \lambda)\|_{W^{1, \tilde{p}}(D)} \\ &\leq C e^{l|\lambda|} \left( \|e^{-iz\lambda} \psi_i(\cdot, \lambda) - 1\|_{W^{1, \tilde{p}}(D)} + \|1\|_{W^{1, \tilde{p}}(D)} \right), \quad i = 1, 2, \end{aligned}$$

which from (2.7) and (2.12) is bounded by  $C(D, N) e^{l|\lambda|}$ . These estimates together with (3.4) give (3.2).  $\square$

The main result of this section is the following:

**Proposition 3.3.** *Let  $v_1, v_2$  be two potentials satisfying (1.3), (1.8), (1.10), let  $\Phi_1, \Phi_2$  the corresponding Dirichlet-to-Neumann operator and  $h_1, h_2$  the corresponding generalised scattering amplitude. Let  $0 < \varepsilon < 1$ ,  $1 < p < \frac{2}{1-\varepsilon}$*

and  $\|v_j\|_{m,1} \leq N$ ,  $j = 1, 2$ . Then for every  $\alpha < m + 1 - 2/p$  there exists a constant  $c = c(D, N, m, p, \alpha)$  such that

$$(3.5) \quad \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C})} \leq c \log(3 + \|\Phi_2 - \Phi_1\|_{H^{1/2} \rightarrow H^{-1/2}}^{-1})^{-\alpha}.$$

*Proof.* Let choose  $a, b > 0$ ,  $a$  close to 0 and  $b$  big to be determined and let

$$(3.6) \quad \delta = \|\Phi_2 - \Phi_1\|_{H^{1/2} \rightarrow H^{-1/2}}.$$

We split down the left term of (3.5) as follows:

$$\begin{aligned} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C})} &\leq \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda| < a)} + \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(a < |\lambda| < b)} \\ &\quad + \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda| > b)}. \end{aligned}$$

From (2.13) we obtain

$$(3.7) \quad \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda| < a)} \leq c(D, N, p) \left( \int_{|\lambda| < a} |\lambda|^{(\varepsilon-1)p} d\operatorname{Re}\lambda d\operatorname{Im}\lambda \right)^{\frac{1}{p}} \\ = c(D, N, p) a^{\varepsilon-1+2/p}.$$

From Lemma 3.2 and (3.6) we get

$$(3.8) \quad \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(a < |\lambda| < b)} \leq c(D, N) \left( \frac{\delta}{a^{1-2/p}} + \delta b^{1/p} e^{(2l+1)b} \right).$$

From Lemma 3.1

$$(3.9) \quad \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(|\lambda| > b)} \leq \frac{c(N)}{b^{m+1-2/p}}.$$

We now define, for  $0 < \alpha < m + 1 - \frac{2}{p}$ ,

$$(3.10) \quad a = \log(3 + \delta^{-1})^{-\frac{\alpha}{\varepsilon-1+2/p}}, \quad b = \log(3 + \delta^{-1})^{\frac{\alpha}{m+1-2/p}},$$

in order to have (3.7) and (3.9) of the order  $\log(3 + \delta^{-1})^{-\alpha}$ . We also choose  $\delta_\alpha < 1$  such that for every  $\delta \leq \delta_\alpha$ ,  $a$  is sufficiently small in order to have (2.13) (which yields (3.7)),  $b \geq R$  (with  $R$  as in Lemma 2.2) and also

$$(3.11) \quad \frac{\delta}{a^{1-2/p}} = \delta \log(3 + \delta^{-1})^{\left(\frac{\alpha}{\varepsilon-1+2/p}\right)(1-2/p)} < \log(3 + \delta^{-1})^{-\alpha}.$$

Thus we obtain

$$(3.12) \quad \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C})} \leq \frac{c(D, N, p)}{\log(3 + \delta^{-1})^\alpha} \\ + c(D, N) \delta \log(3 + \delta^{-1})^{\frac{\alpha}{m+1-2/p}} e^{(2l+1) \log(3 + \delta^{-1})^{\frac{\alpha}{m+1-2/p}}},$$

for  $\delta \leq \delta_\alpha$ . As

$$\delta \log(3 + \delta^{-1})^{\frac{\alpha}{m+1-2/p}} e^{(2l+1) \log(3+\delta^{-1})^{\frac{\alpha}{m+1-2/p}}} \rightarrow 0 \text{ for } \delta \rightarrow 0$$

more rapidly than the other term, we obtain that

$$(3.13) \quad \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C})} \leq \frac{c(D, N, m, p, \alpha)}{\log(3 + \delta^{-1})^\alpha},$$

for  $0 < \alpha < m + 1 - \frac{2}{p}$ ,  $\delta \leq \delta_\alpha$ .

Estimate (3.13) for general  $\delta$  (with modified constant) follows from (3.13) for  $\delta \leq \delta_\alpha$  and the property (2.10) of the scattering amplitude. This completes the proof of Proposition 3.3.  $\square$

#### 4. FROM $h(\lambda)$ TO $v(x)$

**Lemma 4.1.** *Let  $v_1, v_2$  be two potentials satisfying (1.3), (1.8), (1.10), with  $\|v_j\|_{m,1} \leq N$ ,  $h_1, h_2$  the corresponding scattering amplitude and  $\mu_1(z, \lambda), \mu_2(z, \lambda)$  the corresponding Faddeev functions. Let  $1 < s < 2$ , and  $\tilde{s}$  be as in (2.3).*

*Then*

$$(4.1) \quad \sup_{z \in \mathbb{C}} \|\mu_2(z, \cdot) - \mu_1(z, \cdot)\|_{L^{\tilde{s}}(\mathbb{C})} \leq c(D, N, s) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})},$$

$$(4.2) \quad \sup_{z \in \mathbb{C}} \|v_2 \mu_2(z, \cdot) - v_1 \mu_1(z, \cdot)\|_{L^{\tilde{s}}(\mathbb{C})} \leq c(D, N, s) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}.$$

*Proof.* Let

$$(4.3) \quad \nu(z, \lambda) = \mu_2(z, \lambda) - \mu_1(z, \lambda),$$

$$(4.4) \quad \tau(z, \lambda) = v_2(z) \mu_2(z, \lambda) - v_1(z) \mu_1(z, \lambda).$$

From the  $\bar{\partial}$ -equation (2.8) (and the fact that  $v_1$  and  $v_2$  are real-valued) we deduce that  $\nu, \tau$  satisfy the following non-homogeneous  $\bar{\partial}$ -equations:

$$(4.5) \quad \frac{\partial}{\partial \bar{\lambda}} \nu(z, \lambda) = \frac{e_{-\lambda}(z)}{4\pi} \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \overline{\mu_2(z, \lambda)} + \frac{e_{-\lambda}(z)}{4\pi} \frac{h_1(\lambda)}{\bar{\lambda}} \overline{\nu(z, \lambda)},$$

$$(4.6) \quad \frac{\partial}{\partial \bar{\lambda}} \tau(z, \lambda) = \frac{e_{-\lambda}(z)}{4\pi} \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \overline{v_2 \mu_2(z, \lambda)} + \frac{e_{-\lambda}(z)}{4\pi} \frac{h_1(\lambda)}{\bar{\lambda}} \overline{\tau(z, \lambda)},$$

for  $\lambda \in \mathbb{C}$ , where  $e_{-\lambda}(z)$  is defined in (2.9).

By Lemma 2.3 and (2.10) we obtain

$$\begin{aligned} \|\nu(z, \cdot)\|_{L^{\tilde{s}}} &\leq c(D, N) \left\| \overline{\mu_2(z, \lambda)} \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})} \\ &\leq c(D, N) \sup_{z \in \mathbb{C}} \|\mu_2(z, \cdot)\|_{L^\infty} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})} \\ &\leq c(D, N) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}, \end{aligned}$$

where we used the property (2.12) of  $\mu_2(z, \lambda)$ . With the same arguments (along with (2.17)) we also obtain

$$\begin{aligned} \|\tau(z, \cdot)\|_{L^{\bar{s}}} &\leq c(D, N) \left\| \frac{v_2 \mu_2(z, \lambda) h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})} \\ &\leq c(D, N) \sup_{z \in \mathbb{C}} \|v_2(z) \mu_2(z, \cdot)\|_{L^\infty} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})} \\ &\leq c(D, N) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})}, \end{aligned}$$

which ends the proof.  $\square$

The main result of this section is the following proposition.

**Proposition 4.2.** *Let  $v_1, v_2$  be two potentials satisfying (1.3), (1.8), (1.10), with  $\|v_j\|_{m,1} \leq N$ , and let  $h_1, h_2$  be the corresponding scattering amplitude. Let  $p, p'$  such that  $1 < p < 2 < p' < \infty$ ,  $1/p + 1/p' = 1$ . Then*

$$(4.7) \quad \|v_2 - v_1\|_{L^\infty(D)} \leq c(D, N, p) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})}.$$

*Proof.* We write

$$v_2(z) - v_1(z) = \frac{1}{\mu_2(z, 0)} (v_2 \mu_2(z, 0) - v_1 \mu_1(z, 0) - v_1(z) [\mu_2(z, 0) - \mu_1(z, 0)]),$$

that yields

$$(4.8) \quad |v_2(z) - v_1(z)| \leq \frac{1}{|\mu_2(z, 0)|} (|v_2 \mu_2(z, 0) - v_1 \mu_1(z, 0)| + |v_1(z)| |\mu_2(z, 0) - \mu_1(z, 0)|).$$

We claim that

$$(4.9) \quad \|v_2 \mu_2(\cdot, 0) - v_1 \mu_1(\cdot, 0)\|_{L^\infty(D)} \leq c(D, N, p) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})},$$

$$(4.10) \quad \|\mu_2(\cdot, 0) - \mu_1(\cdot, 0)\|_{L^\infty(D)} \leq c(D, N, p) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})},$$

for  $1 < p < 2 < p' < \infty$ ,  $1/p + 1/p' = 1$ . Suppose (4.9), (4.10) already proved; then estimate (4.7) follows from (4.8), (4.9), (4.10), property (2.11) and (2.17).

Before proving (4.9), (4.10), we would like to recall that if  $v \in W^{m,1}(\mathbb{R}^2)$ ,  $m \geq 1$ , with  $\text{supp } v \subset D$  then  $v \in L^p(D)$  for  $p \in [1, \infty]$ ; in particular, from Proposition 2.1, this yields  $h(\lambda)/\bar{\lambda} \in L^p(\mathbb{C})$ , for  $1 < p < \infty$ .

Now, in order to prove (4.9), (4.10) we write as before

$$(4.11) \quad \nu(z, \lambda) = \mu_2(z, \lambda) - \mu_1(z, \lambda),$$

$$(4.12) \quad \tau(z, \lambda) = v_2(z) \mu_2(z, \lambda) - v_1(z) \mu_1(z, \lambda),$$

which satisfy the non-homogeneous  $\bar{\partial}$ -equations (4.5) and (4.6), respectively. From these equations we obtain

$$\begin{aligned}
(4.13) \quad |\nu(z, 0)| &= \frac{1}{\pi} \left| \int_{\mathbb{C}} \frac{e_{-\lambda}(z)}{4\pi\lambda} \frac{h_1(\lambda)}{\bar{\lambda}} \overline{\nu(z, \lambda)} d\operatorname{Re}\lambda d\operatorname{Im}\lambda \right. \\
&\quad \left. + \int_{\mathbb{C}} \frac{e_{-\lambda}(z)}{4\pi\lambda} \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \overline{\mu_2(z, \lambda)} d\operatorname{Re}\lambda d\operatorname{Im}\lambda \right| \\
&\leq \frac{1}{4\pi^2} \sup_{z \in \mathbb{C}} \|\nu(z, \cdot)\|_{L^r} \left\| \frac{h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^{r'}} \\
&\quad + \frac{1}{4\pi^2} \sup_{z \in \mathbb{C}} \|\mu_2(z, \cdot)\|_{L^\infty} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^1}
\end{aligned}$$

and

$$\begin{aligned}
(4.14) \quad |\tau(z, 0)| &= \frac{1}{\pi} \left| \int_{\mathbb{C}} \frac{e_{-\lambda}(z)}{4\pi\lambda} \frac{h_1(\lambda)}{\bar{\lambda}} \overline{\tau(z, \lambda)} d\operatorname{Re}\lambda d\operatorname{Im}\lambda \right. \\
&\quad \left. + \int_{\mathbb{C}} \frac{e_{-\lambda}(z)}{4\pi\lambda} \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \overline{v_2\mu_2(z, \lambda)} d\operatorname{Re}\lambda d\operatorname{Im}\lambda \right| \\
&\leq \frac{1}{4\pi^2} \sup_{z \in \mathbb{C}} \|\tau(z, \cdot)\|_{L^r} \left\| \frac{h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^{r'}} \\
&\quad + \frac{\|v_2\|_{L^\infty(D)}}{4\pi^2} \sup_{z \in \mathbb{C}} \|\mu_2(z, \cdot)\|_{L^\infty} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^1},
\end{aligned}$$

where  $1/r + 1/r' = 1$ ,  $1 < r' < 2 < r < \infty$ . The number  $s = 2r/(r+2)$  can be chosen  $s < 2$  and as close to 2 as wanted, by taking  $r$  big enough.

Then

$$(4.15) \quad \left\| \frac{h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^{r'}(|\lambda| < R)} \leq \left\| \frac{h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p} \left\| \frac{1}{\lambda} \right\|_{L^q(|\lambda| < R)} \leq c(N, r),$$

where we have chosen  $p > 2$  such that  $\|h_1(\lambda)/\bar{\lambda}\|_{L^p} \leq c(N, p)$  from (2.10) and also, since  $1/q = 1/r' - 1/p = 1 - 1/r - 1/p$ ,  $q$  can be chosen less than 2 by taking  $r$  big enough depending on  $p$ . With the same choice of  $p, q$  we also obtain

$$(4.16) \quad \left\| \frac{h_1(\lambda)}{\lambda\bar{\lambda}} \right\|_{L^{r'}(|\lambda| > R)} \leq \left\| \frac{h_1(\lambda)}{\bar{\lambda}} \right\|_{L^q} \left\| \frac{1}{\lambda} \right\|_{L^p(|\lambda| > R)} \leq c(N, r).$$

From Lemma 4.1 with  $r = \tilde{s}$  we get

$$(4.17) \quad \sup_{z \in \mathbb{C}} \|\nu(z, \cdot)\|_{L^r} \leq c(D, N, r) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})},$$

$$(4.18) \quad \sup_{z \in \mathbb{C}} \|\tau(z, \cdot)\|_{L^r} \leq c(D, N, r) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^s(\mathbb{C})},$$

and from (2.12)

$$(4.19) \quad \sup_{z, \lambda \in \mathbb{C}} |\mu_2(z, \lambda)| \leq c(D, N).$$

Finally

$$(4.20) \quad \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\lambda \bar{\lambda}} \right\|_{L^1} \leq \left\| \frac{1}{\lambda} \right\|_{L^p(|\lambda| > R)} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^{p'}} + \left\| \frac{1}{\lambda} \right\|_{L^{p'}(|\lambda| < R)} \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p},$$

by taking  $p' = s$  and  $p$  such that  $1/p + 1/p' = 1$ .

Now (4.9) and (4.10) follows from (4.11)–(4.20); this finishes the proof of Proposition 4.2.  $\square$

## 5. PROOF OF THEOREM 1.1 AND COROLLARY 1.2

*Proof of Theorem 1.1.* Fix  $\alpha < m$  and take  $p$  such that  $\frac{2}{m-\alpha+1} < p < 2$ . From Proposition 4.2 we have

$$(5.1) \quad \|v_2 - v_1\|_{L^\infty(D)} \leq c(D, N, p) \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})},$$

where  $1/p + 1/p' = 1$ . From Proposition 3.3

$$(5.2) \quad \left\| \frac{h_2(\lambda) - h_1(\lambda)}{\bar{\lambda}} \right\|_{L^p(\mathbb{C}) \cap L^{p'}(\mathbb{C})} \leq c(D, N, p) \log(3 + \|\Phi_2 - \Phi_1\|_{H^{1/2} \rightarrow H^{-1/2}}^{-1})^{-\alpha},$$

as  $\alpha < m + 1 - \frac{2}{p}$ . Theorem 1.1 is proved.  $\square$

*Proof of Corollary 1.2.* We first extend  $\sigma$  on the whole plane by putting  $\sigma(x) = 1$  for  $x \in \mathbb{R}^2 \setminus D$  (this extension is smooth by our hypothesis on  $\sigma$ ). Now since  $\sigma_j|_{\partial D} = 1$  and  $\frac{\partial \sigma_j}{\partial \nu}|_{\partial D} = 0$  for  $j = 1, 2$ , from (1.5) we deduce that

$$(5.3) \quad \Phi_j = \Lambda_j, \quad j = 1, 2.$$

In addition, from (2.14) we get

$$(5.4) \quad \lim_{\lambda \rightarrow 0} \mu_j(z, \lambda) = \sigma_j^{1/2}(z), \quad j = 1, 2;$$

thus we obtain, using the fact that  $\sigma_j$  are bounded from above and below,

$$(5.5) \quad \begin{aligned} \|\sigma_2 - \sigma_1\|_{L^\infty(D)} &\leq c(N) \|\sigma_2^{1/2} - \sigma_1^{1/2}\|_{L^\infty(D)} \\ &= c(N) \|\mu_2(\cdot, 0) - \mu_1(\cdot, 0)\|_{L^\infty(D)}. \end{aligned}$$

Now the proof follows by repeating the proof of Theorem 1.1, using (5.5), (4.10) and (5.3).  $\square$

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