Equidistribution, counting and arithmetic applications

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Abstract This short note is an announcement of the results of [PP1] and [PP2].[1]

Let M be a finite volume hyperbolic manifold of dimension n at least 2. Let $T^1M \to M$ be the unit tangent bundle of M, where T^1M is endowed with its usual Riemannian metric, whose induced measure is the Liouville measure vol_{T^1M}. Let $(g^t)_{t\in\mathbb{R}}$ be the geodesic flow of M. Let C_0 be a finite volume immersed totally geodesic submanifold of M of dimension k with $0 < k < n$, and let $\nu^1 C_0$ be its unit normal bundle, so that $g^t\nu^1C_0$ is, for every $t \geq 0$, an immersed submanifold of T^1M.

Theorem 1 *The induced Riemannian measure of $g^t\nu^1C_0$ equidistributes to the Liouville measure as $t \to +\infty$:*

$$\mathrm{vol}_{g^t\nu^1C_0}/\|\,\mathrm{vol}_{g^t\nu^1C_0}\| \;\overset{*}{\rightharpoonup}\; \mathrm{vol}_{T^1M}/\|\,\mathrm{vol}_{T^1M}\|\;.$$

This theorem can be deduced from [EM, Theo. 1.2]. Our (short and direct) proof also uses, as in Margulis' equidistribution result for horospheres, the mixing property of the geodesic flow of M.

Let $\mathcal{H}_\infty$ be a small enough Margulis neighbourhood of an end of M, that is a connected component of the set of points of M at which the injectivity radius of M is at most ϵ_0, for some $\epsilon_0 > 0$ small enough. We use the above equidistribution theorem, and the fact that the submanifold $g^t\nu^1C_0$ is locally close to an unstable leaf in T^1M of the geodesic flow of M, to prove the following counting result.

Theorem 2 *The number of common perpendicular locally geodesic arcs between $\partial\mathcal{H}_\infty$ and C_0 with length at most t is equivalent, as t tends to $+\infty$, to*

$$\frac{\mathrm{Vol}(\mathbb{S}_{n-k-1})\,\mathrm{Vol}(\mathcal{H}_\infty)\,\mathrm{Vol}(C_0)}{\mathrm{Vol}(\mathbb{S}_{n-1})\,\mathrm{Vol}(M)}\;e^{(n-1)t}\;.$$

We refer to [PP1] for the proofs of the above theorems, as well as for references to other works and many geometric complements, and we now give a sample of their arithmetic applications, extracted from [PP1] except for the last corollary.

Counting quadratic irrationals. Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Endow the set of quadratic irrationals over K with the action by homographies of $\mathrm{PSL}_2(\mathcal{O}_K)$, and note that it is not transitive. We denote by α^σ the Galois conjugate over K of a quadratic irrational α over K. There are many works (see for instance [Bug]) on the approximation of real or complex numbers by algebraic numbers, and approximating them

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by elements in orbits of algebraic numbers under natural group actions for appropriate complexities seems to be interesting.

Starting with $K=\mathbb{Q}$, our first result is a counting result in orbits of real quadratic irrationals over $\mathbb{Q}$ for a natural complexity (see [PP1] for a more algebraic expression in terms of discriminants).

Corollary 1 *Let $\alpha_0 \in \mathbb{R}$ be a quadratic irrational over $\mathbb{Q}$, and let G be a finite index subgroup of* $\mathrm{PSL}_2(\mathbb{Z})$*. Then as s tends to $+\infty$,*

$$\mathrm{Card}\{\alpha \in G\cdot\{\alpha_0,\alpha_0^\sigma\} \bmod \mathbb{Z}\;:\;\frac{1}{|\alpha-\alpha^\sigma|}\leq s\} \sim \frac{24\;q_G\;\operatorname{argcosh}\frac{|\operatorname{tr}\gamma_0|}{2}}{\pi^2\;[\mathrm{PSL}_2(\mathbb{Z}):G]\;n_0}\;s\;,$$

where q_G is the smallest positive integer q such that $z\mapsto z+q$ belongs to G, $\gamma_0\in G-\{1\}$ fixes α_0 and n_0 is the index of ${\gamma_0}^{\mathbb{Z}}$ in the stabilizer of $\{\alpha_0,\alpha_0^\sigma\}$ in G (and note that q_G,γ_0,n_0 do exist).

For instance, if α_0 is the Golden ratio $\phi=\frac{1+\sqrt{5}}{2}$ (which is reciprocal in Sarnak's terminology) and $G=\mathrm{PSL}_2(\mathbb{Z})$, we get $\mathrm{Card}\{\alpha\in G\cdot\phi \mod \mathbb{Z} \;:\; \frac{1}{|\alpha-\alpha^\sigma|}\leq s\}\sim \frac{24\,\log\phi}{\pi^2}\;s$. With $\mathbb{H}^2_{\mathbb{R}}$ the upper halfplane model of the real hyperbolic plane, the proof applies Theorem 2 to M the orbifold $G\backslash\mathbb{H}^2_{\mathbb{R}}$, to C_0 the image in M of the geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ with endpoints α_0 and α_0^σ, and to $\mathcal{H}_\infty$ the image in M of the set of points in $\mathbb{H}^2_{\mathbb{R}}$ with Euclidean height at least 1. The trick is that if a and b are close enough distinct real numbers, then the hyperbolic length of the perpendicular arc between the horizontal line at Euclidean height 1 and the geodesic line with endpoints a and b is exactly $-\log|b-a|$.

Assume K is imaginary quadratic, with discriminant D_K. We proved a general statement analogous to the previous corollary, but we only give here a particular case for ϕ.

Corollary 2 *Let $\mathfrak{a}$ be a non zero ideal in $\mathcal{O}_K$ and $\Gamma_0(\mathfrak{a})=\Big\{\pm\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{PSL}_2(\mathcal{O}_K)\;:\;c\in\mathfrak{a}\Big\}$. Assume for simplicity that $D_K\neq-4$ and $\phi^\sigma\notin\Gamma_0(\mathfrak{a})\cdot\phi$. Then as s tends to $+\infty$, the cardinality of $\{\alpha\in\Gamma_0(\mathfrak{a})\cdot\{\phi,\phi^\sigma\}\bmod\mathcal{O}_K\;:\;\frac{1}{|\alpha-\alpha^\sigma|}\leq s\}$ is equivalent to*

$$\frac{8\pi^2\;k_{\mathfrak{a}}\;\log\phi}{|D_K|\;\zeta_K(2)\;N(\mathfrak{a})\prod_{\mathfrak{p}\text{ prime},\,\mathfrak{p}|\mathfrak{a}}\big(1+\frac{1}{N(\mathfrak{p})}\big)}\;s^2\;,$$

with $k_{\mathfrak{a}}$ the smallest $k\in\mathbb{N}-\{0\}$ such that the $2k$-th term of the standard Fibonacci sequence belongs to $\mathfrak{a}$ (and note that $k_{\mathfrak{a}}$ does always exist, contrarily to the odd case).

Counting representations of integers by binary forms. Recall that a binary quadratic form $Q(x,y)=ax^2+bxy+cy^2$ is primitive integral if $a,b,c\in\mathbb{Z}$ are relatively prime, and indefinite non product if its discriminant $D=b^2-4ac$ is positive and not a square. Using the well known correspondence between pairs of Galois conjugated quadratic irrationals over $\mathbb{Q}$ and the set of such Q's up to sign, we prove the following counting result for the number of values of a fixed such Q on couples of relatively prime integers satisfying some congruence relations. Let (t,u) be the minimal solution to the Pell-Fermat equation $t^2-Du^2=4$ and $\epsilon=\frac{t+u\sqrt{D}}{2}$ the corresponding fundamental unit.

Corollary 3 *Let Q be as above, and let n be an integer at least 3. Then the number of couples $(x, y) \in \mathbb{Z}^2$, relatively prime, with $x \equiv 1 \mod n$ and $y \equiv 0 \mod n$, such that $|Q(x, y)| \leq s$, modulo the linear action of $\mathrm{SL}_2(\mathbb{Z})$, is equivalent, as s tends to $+\infty$, to*

$$\frac{24 \log \epsilon}{\pi^2 n^2 \sqrt{D}} \prod_{p \text{ prime, } p|n} \left(1 - \frac{1}{p^2}\right)^{-1} s \; .$$

The final result, for a quadratic imaginary number field K, is proved in [PP2], along with extensions to representations satisfying congruence properties.

Corollary 4 *Let $f : (u, v) \mapsto a\,|u|^2 + 2\,\mathrm{Re}(b\,u\,\overline{v}) + c\,|v|^2$ be a binary Hermitian form, indefinite (that is $\Delta = |b|^2 - ac > 0$) and integral over K (that is $a, c \in \mathbb{Z}, b \in \mathcal{O}_K$). Let $\mathrm{SU}_f(\mathcal{O}_K) = \{g \in \mathrm{SL}_2(\mathcal{O}_K) \;:\; f \circ g = g\}$ be the group of automorphs of f. Then the number of orbits under $\mathrm{SU}_f(\mathcal{O}_K)$ of couples (u, v) of relatively prime elements of $\mathcal{O}_K$ such that $|f(u, v)| \leq s$ is equivalent, as s tends to $+\infty$, to*

$$\frac{\pi \;\mathrm{Covol}(\mathrm{SU}_f(\mathcal{O}_K))}{2\,|D_K|\,\zeta_K(2)\,\Delta} \; s^2 \; .$$

With $\mathbb{H}^3_{\mathbb{R}}$ the upper halfspace model of the real hyperbolic 3-space, the proof applies Theorem 2 to M the orbifold $\mathrm{PSL}_2(\mathcal{O}_K)\backslash\mathbb{H}^3_{\mathbb{R}}$, to C_0 the image in M of the unique hyperbolic plane $P(f)$ in $\mathbb{H}^3_{\mathbb{R}}$ preserved by $\mathrm{PSU}_f(\mathcal{O}_K)$, and to $\mathcal{H}_\infty$ the image in M of the set of points in $\mathbb{H}^3_{\mathbb{R}}$ with Euclidean height at least 1. The trick is that, for every $\gamma \in \mathrm{PSL}_2(\mathcal{O}_K)$, the hyperbolic plane $P(f \circ \gamma)$ is an Euclidean hemisphere whose diameter is $\frac{\sqrt{\Delta}}{f\circ\gamma(1,0)}$, hence whose perpendicular arc to the horizontal plane at Euclidean height 1 has (signed) hyperbolic length $\log \frac{f\circ\gamma(1,0)}{\sqrt{\Delta}}$, and that $\mathrm{SL}_2(\mathcal{O}_K)$ acts transitively on the couples of relatively prime elements of $\mathcal{O}_K$.

References

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