Circle Diffeomorphisms: Quasi-reducibility and Commuting Diffeomorphisms

Mostapha Benhenda*

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Abstract

We show two related results on circle diffeomorphisms. The first result is on quasi-reducibility: for a Baire-dense set of α, for any diffeomorphism f of rotation number α, it is possible to accumulate R_α with a sequence $h_n f h_n^{-1}$, h_n being a diffeomorphism. The second result is: for a Baire-dense set of α, given two commuting diffeomorphisms f and g, such that f has α for rotation number, it is possible to approach each of them by commuting diffeomorphisms f_n and g_n that are differentiably conjugated to rotations.

1 Introduction

It is well-known that there are circle diffeomorphisms with Liouville rotation numbers (i.e. non-Diophantine) that are not smoothly conjugated to rotations [1, 5, 6, 7]. A natural question arises, namely, the problem of smooth quasi-reducibility: *given a smooth diffeomorphism f of rotation number α, is it possible to accumulate R_α in the C^∞-norm, with a sequence $h_n^{-1} f h_n$, h_n being a smooth diffeomorphism?* In this case, we say that f is smoothly *quasi-reducible* to R_α. Quasi-reducibility is a question that has been studied by Herman [5, pp.93-99], who showed that for any C^2-diffeomorphism f of irrational rotation number α, it is possible to accumulate R_α in the C^{1+bv}-norm, with a sequence $h_n^{-1} f h_n$, h_n being a C^2-diffeomorphism. Quasi-reducibility is also related to a problem solved by Yoccoz [8], who showed that it is possible to accumulate a smooth diffeomorphism f in the C^∞-norm with a sequence $h_n R_\alpha h_n^{-1}$, h_n being a smooth diffeomorphism. However, these two problems are not the same, and the method used by Yoccoz does not directly yield our result. In our case, we determine a Baire-dense set of rotation numbers α such that for any smooth diffeomorphism f of rotation number α, f is smoothly quasi-reducible.

Connected to the problem of quasi-reducibility is the following question, raised by Mather: *given two commuting C^∞-diffeomorphisms f and g, is it possible to approach each of them in the C^∞-norm by commuting smooth diffeomorphisms that are smoothly conjugated to rotations?* In this paper, we determine a Baire-dense set of rotation numbers α such that if f and g are commuting C^∞-diffeomorphisms, with f of rotation number α, then f and g are accumulated in the C^∞ norm by commuting C^∞-diffeomorphisms that are C^∞-conjugated to a rotation.

*Laboratoire d'Analyse, Geometrie et Applications, Paris 13 University, 99 Avenue J.B. Clement, 93430 Villetaneuse, France. Contact: mostaphabenhenda@gmail.com

Moreover, for Diophantine rotation numbers, which are of full Lebesgue measure, the question of quasi-reducibility and Mather's problem are trivial, because in this case, the diffeomorphism f is smoothly conjugated to a rotation. Therefore, these two questions remain open for a meagre set of rotation numbers of zero Lebesgue measure.

In order to derive our results, we use estimates of the conjugacy to rotations of diffeomorphisms having rotation numbers of Diophantine constant type. These estimates were obtained in [2].

The circle is noted $\mathbb{T}^1$. For $r \in \mathbb{R}_+ \cup \{+\infty\}$, we work in the universal cover $D^r(\mathbb{T}^1)$, which is the group of diffeomorphisms f of class C^r of the real line such that $f - Id$ is $\mathbb{Z}$-periodic. For $\alpha \in \mathbb{R}$, we denote $R_\alpha \in D^\infty(\mathbb{T}^1)$ the map $x \mapsto x + \alpha$.

Let $f \in D^0(\mathbb{T}^1)$ be a homeomorphism and $x \in \mathbb{R}$. The sequence $((f^n(x) - x)/n)_{n\geq 1}$ admits a limit independent of x, noted $\rho(f)$. This limit is called the *rotation number* of f. This is a real number invariant by conjugacy.

1.1 Statement of the results

Theorem 1.1. *There is a Baire-dense set $A_1 \subset \mathbb{R}$ such that for any $f \in D^\infty(\mathbb{T}^1)$ of rotation number $\alpha \in A_1$, there is a sequence $h_n \in D^\infty(\mathbb{T}^1)$ such that $h_n^{-1} f h_n \to R_\alpha$ in the C^∞-topology.*

Theorem 1.2. *There is a Baire-dense set $A_2 \subset \mathbb{R}$ such that for any $f \in D^\infty(\mathbb{T}^1)$ of rotation number $\alpha \in A_2$ and any g of class C^∞ with $fg = gf$, f and g are accumulated in the C^∞-topology by commuting C^∞-diffeomorphisms that are C^∞-conjugated to rotations.*

Remark 1.3. The proof of theorem 1.1 also gives that $h_n R_\alpha h_n^{-1} \to f$ in the C^∞-topology if $\alpha \in A_1$.

2 Preliminaries

When the rotation number α of f is irrational, and if f is of class C^2, Denjoy showed that f was topologically conjugated to R_α. However, this conjugacy is not always differentiable. It depends on the Diophantine properties of the rotation number α.

Let $\alpha = a_0 + 1/(a_1 + 1/(a_2 + ...))$ be the development of $\alpha \in \mathbb{R}$ in continued fraction (see [4]). It is noted $\alpha = [a_0, a_1, a_2, ...]$. Let $p_{-2} = q_{-1} = 0$, $p_{-1} = q_{-2} = 1$. For $n \geq 0$, we define integers p_n and q_n by:

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n q_{n-1} + q_{n-2}$$

We have $q_0 = 1$, $q_n \geq 1$ for $n \geq 1$. The rationals p_n/q_n are called the convergents of α. Remember that $q_{n+2} \geq 2q_n$, for $n \geq -1$.

For any real number $\beta \geq 0$, $\alpha \in \mathbb{R} - \mathbb{Q}$ is Diophantine of order β and constant C_d (a set noted $DC(C_d, \beta)$) if there is a constant $C_d > 0$ such that for any $p/q \in \mathbb{Q}$, we have:

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C_d}{q^{2+\beta}}$$

One of the following relations characterizes $DC(C_d, \beta)$:

1. $|\alpha - p_n/q_n| > C_d/q_n^{1+\beta}$ for any $n \geq 0$
2. $a_{n+1} < \frac{1}{C_d} q_n^{\beta}$ for any $n \geq 0$
3. $q_{n+1} < \frac{1}{C_d} q_n^{1+\beta}$ for any $n \geq 0$
4. $\alpha_{n+1} > C_d \alpha_n^{1+\beta}$ for any $n \geq 0$

$DC(C_d, 0)$ is the set of irrational numbers of *constant type* C_d. The first derivative of $f \in D^1(\mathbb{T}^1)$ is noted Df. For u integer, we note $\|f\|_u = \max_{0 \leq j \leq u} \max_{x \in [0,1]} |D^j f(x)|$.

For any n integer, let $\alpha_n = [a_0, ..., a_n, 1, ...]$.

Let $V_\alpha(n) = \max_{0 \leq i \leq n} a_i$. Observe that $\alpha_n \in DC(1/V_\alpha(n), 0)$. We will need the lemma:

Lemma 2.1. *We have:*

$$|\alpha_n - \alpha| \leq \frac{2}{q_n^2} \leq \frac{4}{2^n}$$

Proof. Let $\tilde{\alpha}_n = [a_0, ..., a_n, 0, ...]$. By induction, we can show that $\tilde{\alpha}_n = p_n/q_n$. Moreover, $\tilde{\alpha}_n$ is also the n^{th} convergent of α_n. Therefore, by the best rational approximation theorem, $|\alpha - p_n/q_n| \leq 1/q_n^2$ and $|\alpha_n - p_n/q_n| \leq 1/q_n^2$. Moreover, since $q_{n+2} \geq q_n$, then $q_n \geq (\sqrt{2})^{n-1}$. □

We need the lemma:

Lemma 2.2. *Let* $\phi :\to \mathbb{R}_+$ *such that* $\phi(n) \to_{n \to +\infty} +\infty$. *Let*

$$A = \{\alpha \in \mathbb{R} \,/ V_\alpha(n) < \phi(n) \textit{for an infinity of} \, n\}$$

is Baire-dense.

Proof. First, we show that for any positive integers n and i,

$A_{i,n} = \{\alpha \text{ such that } a_i < \phi(n)\}$ is open. Let $u(x) = \lfloor x \rfloor$, $v(x) = \frac{1}{x}$ and $w(x) = v(x) - u(v(x))$. We have: $a_{k+1} = v(w^k(x)) - w^{k+1}(x)$. Since v is continuous and u is upper semi-continuous and non-negative, then w is lower semi-continuous. Moreover, w is non-negative. Therefore, w^k and w^{k+1} are also lower semi-continuous and non-negative. Since v is decreasing, then $v \circ w^k - w^{k+1}$ is upper semi-continuous. We conclude that $A_{i,n}$ is open.

Moreover, for any $p \geq 0$,

$$\cup_{n \geq p} \cap_{i \leq n} A_{i,n}$$

is dense. Indeed, since $\phi(n) \to +\infty$, then it contains all numbers of constant type, which are dense. This set is also open and therefore,

$$A = \cap_{p \geq 0} \cup_{n \geq p} \cap_{i \leq n} A_{i,n}$$

is Baire-dense. □

- For any real numbers a and b, $a \vee b$ denotes $\max(a, b)$.

- For ϕ a real $\mathbb{Z}$-periodic C^r function, $0 \leq r < +\infty$, we define:

$$\|\phi\|_r = \max_{0\leq j\leq r} \max_{x\in\mathbb{R}} |D^j\phi(x)|$$

  Note that for $f, g \in D^r(\mathbb{T}^1)$, $f - g$ is $\mathbb{Z}$-periodic, and for $1 \leq j \leq r$, $D^j f$ is $\mathbb{Z}$-periodic. For $f \in D^r(\mathbb{T}^1)$, we also define:

$$\|f\|_r = \max\left(\|f - id\|_0, \max_{1\leq j\leq r} \|D^j f\|_0\right)$$

  Note that the notation $\|f\|_r$ is not a norm when $f \in D^r(\mathbb{T}^1)$, since $D^r(\mathbb{T}^1)$ is not a vector space.

- In all the paper, C denotes a constant depending on u. $W(f)$ denotes the total variation of $\log Df$, and Sf denotes the Schwartzian derivative of f.

The following theorem gives an estimation of the norm of the linearization of a diffeomorphism having a rotation numbers of Diophantine constant type. This estimation, obtained in [2], is necessary to derive our results.

Theorem 2.3. *Let $l \geq 3$ an integer and $\eta > 0$. Let $f \in D^l(\mathbb{T}^1)$ of rotation number α, such that α is of constant type C_d. There exists a diffeomorphism $h \in D^{l-1-\eta}(\mathbb{T}^1)$ conjugating f to R_α, and a function B of $C_d, l, \eta, W(f), \|Sf\|_{l-3}$, which satisfy the estimation:*

$$\max\left(\frac{1}{\min Dh}, \|h\|_{l-1-\eta}\right) \leq B(C_d, l, \eta, W(f), \|Sf\|_{l-3}) \tag{1}$$

In particular, we remark that if f_n is a sequence of diffeomorphisms of rotation number α_n, if the sequences $W(f_n)$ and $\|Sf_n\|_{l-3}$ are bounded (this will hold in our case, because we will take $f_n = \lambda_n + f$ for a properly chosen $\lambda_n \in \mathbb{R}$), if $V_\alpha(n) \to +\infty$ and if h_n is the conjugacy to a rotation associated with f_n, then there is a real function $E(V_\alpha(n))$ such that, for n sufficiently large, we have:

$$\max\left(\frac{1}{\min Dh_n}, \|h_n\|_{l-1-\eta}\right) \leq E(V_\alpha(n))$$

3 Quasi-Reducibility

Theorem 3.1. *Let $l \geq 3$ an integer, $f \in D^l(\mathbb{T}^1)$ of rotation number $\alpha \in \mathbb{T}^1$. Let $\eta > 0$ a real number. There exists a numerical sequence $F(n)$, going to $+\infty$ as $n \to +\infty$, such that, if*

$$\liminf \frac{V_\alpha(n)}{F(n)} = 0$$

then there is a sequence $h_n \in C^{l-1-\eta}$ such that $h_n^{-1} f h_n \to R_\alpha$ in the $C^{l-2-\eta}$-topology.

By applying lemma 2.2, we obtain the corollary:

Corollary 3.2. *There is a Baire-dense set $A_1 \subset \mathbb{R}$ such that if $l \geq 3$ is an integer, $f \in D^l(\mathbb{T}^1)$ of rotation number $\alpha \in A_1$ and $\eta > 0$, then f is $C^{l-2-\eta}$-quasi-reducible: there is a sequence $h_n \in D^{l-1-\eta}(\mathbb{T}^1)$ such that $h_n^{-1} f h_n \to R_\alpha$ in the $C^{l-2-\eta}$-topology.*

Proof of theorem 1.1. We let $\eta = l/3$ in corollary 3.2. Since f is smooth, then there is a sequence $(h_{n,l})_{n\geq 0} \in D^\infty(\mathbb{T}^1)$ such that, for any integer $l \geq 3$ fixed,

$$||h_{n,l}^{-1} f h_{n,l} - R_\alpha||_{2(\frac{l}{3}-1)} \to_{n\to+\infty} 0$$

In particular, there is $n(l)$ such that:

$$||h_{n(l),l}^{-1} f h_{n(l),l} - R_\alpha||_{2(\frac{l}{3}-1)} \leq \frac{1}{l}$$

Let $h_l = h_{n(l),l}$. Let $\epsilon > 0$ and $k > 0$ an integer. There is $l_0 \geq 0$ such that for any $l \geq l_0$, we have: $\epsilon \geq 1/l$, $k \leq 2\left(\frac{l}{3} - 1\right)$ and:

$$||h_l^{-1} f h_l - R_\alpha||_k \leq ||h_l^{-1} f h_l - R_\alpha||_{2(\frac{l}{3}-1)} \leq \frac{1}{l} \leq \epsilon$$

Therefore, $h_l^{-1} f h_l \to_{l\to+\infty} R_\alpha$ in the C^k-topology, for any k, and therefore, this convergence holds in the C^∞-topology.

□

To prove theorem 3.1, we need to consider the one-parameter family $R_\lambda f$ [5, p.31]. We have the lemma:

Lemma 3.3. *Let $\alpha = \rho(f)$. Let $\tilde{\alpha}$ an irrational Diophantine number. There exists $\lambda_0 \in \mathbb{R}$ and a C^1-diffeomorphism h such that $h^{-1} R_{\lambda_0} f h = R_{\tilde{\alpha}}$. Moreover,*

$$\frac{|\lambda_0|}{\min Dh} \geq |\tilde{\alpha} - \alpha| \geq \frac{|\lambda_0|}{||Dh||_0}$$

Proof. Let $\mu(\lambda) = \rho(R_\lambda f)$. μ is continuous, non-decreasing and $\mu(\mathbb{R}) = \mathbb{R}$ (see [5, p. 31]). Therefore, there exists $\lambda_0 \in \mathbb{R}$ such that $\tilde{\alpha} = \rho(R_{\lambda_0} f)$. Since $\tilde{\alpha}$ is Diophantine, there exists a C^1-diffeomorphism h such that $h^{-1} R_{\lambda_0} f h = R_{\tilde{\alpha}}$ and that satisfies estimation (1) of theorem 2.3. By the mean value theorem, for any x, there is $c(x)$ such that:

$$\tilde{\alpha} + x - h^{-1} f h(x) = R_{\tilde{\alpha}}(x) - h^{-1} f h(x) = h^{-1} R_{\lambda_0} f h(x) - h^{-1} f h(x) = D(h^{-1})(c(x))\lambda_0$$

By integrating this equation on an invariant measure of f, we get lemma 3.3. Note that since $h \in D^1(\mathbb{T}^1)$, then $Dh(x) > 0$ for any x, and $\min Dh > 0$.

□

Proof. The proof of theorem 3.1 is also based on the lemma:

Lemma 3.4. *Let $\alpha = \rho(f)$. Let $\tilde{\alpha}$ an irrational Diophantine number, and $\lambda_0 \in \mathbb{R}$ and h the C^1-diffeomorphism given by lemma 3.3. Recall that C denotes a constant that only depends on u, $0 \leq u \leq l - 2 - \eta$. We have the estimation:*

$$||h^{-1} f h - R_\alpha||_u \leq C||f||_u^C ||h||_{u+1}^C \frac{1}{(\min Dh)^C} |\tilde{\alpha} - \alpha|$$

Before proving lemma 3.4, we show how theorem 3.1 is derived from it.

If α is of constant type, then f is reducible and there is nothing to prove. Therefore, we can suppose that $V_\alpha(n) \to_{n\to+\infty} +\infty$. By applying theorem 2.3, there exists a real function $\tilde{F}$ strictly increasing with $V_\alpha(n)$, such that, for n sufficiently large:

$$||h_n^{-1} f h_n - R_\alpha||_{l-2-\eta} \leq \exp\left(\tilde{F}\left(V_\alpha(n)\right)\right)|\alpha_n - \alpha|$$

Let $F(n) = \tilde{F}^{-1}(n^{1/2})$. By extracting, we can suppose that $\lim \frac{V_\alpha(n)}{F(n)} = 0$. Therefore, $V_\alpha(n) \leq F(n)$ for n sufficiently large and therefore,

$$\tilde{F}\left(V_\alpha(n)\right) \leq n^{1/2}$$

We get, for n sufficiently large,

$$||h_n^{-1} f h_n - R_\alpha||_{l-2-\eta} \leq e^{-\frac{n \log 2}{4}} \to_{n\to+\infty} 0$$

Hence theorem 3.1.

Now, we show lemma 3.4. We need the Faa-di-Bruno formula (see e.g. [3]):

Lemma 3.5. *For every integer $u \geq 0$ and functions ϕ and ψ of class C^u, we have:*

$$D^u\left[\phi(\psi(x))\right] = \sum_{j=0}^{u} D^j\phi(\psi(x)) B_{u,j}\left(D\psi(x), D^2\psi(x), \ldots, D^{(u-j+1)}\psi(x)\right)$$

The $B_{u,j}$ are the Bell polynomials, defined by $B_{u,0} = 1$ and, for $j \geq 1$:

$$B_{u,j}(x_1, x_2, \ldots, x_{u-j+1}) = \sum \frac{u!}{l_1! l_2! \cdots l_{u-j+1}!}\left(\frac{x_1}{1!}\right)^{l_1}\left(\frac{x_2}{2!}\right)^{l_2}\cdots\left(\frac{x_{u-j+1}}{(u-j+1)!}\right)^{l_{u-j+1}}$$

The sum extends over all sequences $l_1, l_2, l_3, ..., l_{u-j+1}$ of non-negative integers such that:$l_1 + l_2 + ... = j$ and $l_1 + 2l_2 + 3l_3 + ... = u$.

Therefore, for any x, we have the estimation:

$$\left|B_{u,j}\left(D\psi(x), D^2\psi(x), \ldots, D^{(u-j+1)}\psi(x)\right)\right| \leq C\left(1 \vee ||\psi||_u^j\right) \quad (2)$$

Combining this estimation with lemma 3.5, we obtain the corollary:

Corollary 3.6.

$$||\phi \circ \psi||_u \leq C \max_{0\leq j\leq u} ||D^j\phi \circ \psi||_0 \left(1 \vee ||\psi||_u^u\right)$$

We apply this corollary to estimate $||h^{-1}||_u$. We let $\phi(x) = 1/x$ and $\psi = Dh \circ h^{-1}$. We observe that $D(h^{-1}) = \frac{1}{Dh \circ h^{-1}} = \phi \circ \psi$. Since there is x_0 such that $Dh(x_0) = 1$, then $||Dh||_0 \geq 1$ (and we also have $1 \geq \min Dh > 0$). Therefore, we get:

$$||D(h^{-1})||_u \leq C \max_{0\leq j\leq u} \frac{1}{||\left(Dh \circ h^{-1}\right)^{j+1}||_0} ||Dh \circ h^{-1}||_u^C$$

By corollary 3.6, we also have:

$$||Dh \circ h^{-1}||_u \leq C||Dh||_u ||h^{-1}||_u^C$$

By combining these two estimations, we get:

$$||D(h^{-1})||_u \leq C\frac{1}{(\min Dh)^C}||Dh||_u^C||h^{-1}||_u^C$$

We iterate this estimation to estimate $||h^{-1}||_u$, for $u \geq 1$. We get:

$$||h^{-1}||_{u+1} \leq C\frac{1}{(\min Dh)^C}||h||_{u+1}^C||h^{-1}||_1^C \tag{3}$$

Now, we estimate the C^u-distance of $h^{-1}fh$ to R_α. Let $\tilde{\alpha}, \lambda_0$ as in lemma 3.3. We have:

$$h^{-1}fh - R_\alpha = h^{-1}fh - h^{-1}R_{\lambda_0}fh + R_{\tilde{\alpha}} - R_\alpha$$

Therefore,

$$||h^{-1}fh - R_\alpha||_u \leq ||h^{-1}fh - h^{-1}R_{\lambda_0}fh||_u + |\tilde{\alpha} - \alpha| \tag{4}$$

On the other hand, by the Faa-di-Bruno formula, we have:

$$D^u\left[h^{-1}fh - h^{-1}R_{\lambda_0}fh\right](x) = \sum_{j=0}^{u} B_{u,j}\left(D(fh)(x), ..., D^{u-j+1}(fh)(x)\right)$$
$$\left[D^j(h^{-1})(fh(x)) - D^j(h^{-1})(fh(x) + \lambda_0)\right]$$

Since $|D^j(h^{-1})(fh(x)) - D^j(h^{-1})(fh(x) + \lambda_0)| \leq ||D^{j+1}(h^{-1})||_0|\lambda_0|$, then by applying estimation (2), we get:

$$||h^{-1}fh - h^{-1}R_{\lambda_0}fh||_u \leq C||f \circ h||_u^C||h^{-1}||_{u+1}|\lambda_0|$$

By applying corollary 3.6, we get:

$$||h^{-1}fh - h^{-1}R_{\lambda_0}fh||_u \leq C||f||_u^C||h||_u^C||h^{-1}||_{u+1}|\lambda_0|$$

By applying (3), we obtain:

$$||h^{-1}fh - h^{-1}R_{\lambda_0}fh||_u \leq C||f||_u^C||h||_u^C\frac{1}{(\min Dh)^C}||h||_{u+1}^C||h^{-1}||_1^C|\tilde{\alpha} - \alpha|||Dh||_0$$

$$||h^{-1}fh - h^{-1}R_{\lambda_0}fh||_u \leq C||f||_u^C||h||_{u+1}^C\frac{|\tilde{\alpha} - \alpha|}{(\min Dh)^C}$$

By estimation (4), we obtain:

$$||h^{-1}fh - R_\alpha||_u \leq C||f||_u^C||h||_{u+1}^C\frac{1}{(\min Dh)^C}|\tilde{\alpha} - \alpha| \tag{5}$$

Hence lemma 3.4.

□

4 Application to commuting diffeomorphisms

Theorem 4.1. *There exists a numerical sequence $G(n)$, going to $+\infty$ as $n \to +\infty$, such that, for any $l \geq 3$ an integer, $f \in D^l(\mathbb{T}^1)$ of rotation number $\alpha \in \mathbb{R}$, $\eta > 0$ and g of class C^l such that $fg = gf$, if*

$$\liminf \frac{V_\alpha(n)}{G(n)} = 0$$

then there exists two sequences of diffeomorphisms f_n and g_n that are $C^{l-1-\eta}$-conjugated to rotations, such that $f_n g_n = g_n f_n$, and with f_n and g_n converging respectively towards f and g in the $C^{l-2-\eta}$-norm.

Corollary 4.2. *There is a Baire-dense set $A_2 \subset \mathbb{R}$ such that if $l \geq 3$ is an integer, $f \in D^l(\mathbb{T}^1)$ has a rotation number $\alpha \in A_2$, g is of class C^l such that $fg = gf$ and $\eta \in \mathbb{R}_+$, then there exists two sequences of diffeomorphisms f_n and g_n that are $C^{l-1-\eta}$-conjugated to rotations, such that $f_n g_n = g_n f_n$ and with f_n and g_n converging respectively towards f and g in the $C^{l-2-\eta}$-norm.*

We derive theorem 1.2 from lemma 4.2 by following the same argument as in the proof of theorem 1.1.

To prove theorem 4.1, we consider $(h_n)_{n\geq 0}$, the sequence of conjugating diffeomorphisms constructed in the proof of theorem 3.1, $(\lambda_n)_{n\geq 0}$ the associated sequence of real numbers such that $f_n = R_{\lambda_n} f = h_n R_{\alpha_n} h_n^{-1}$. We also consider $g'_n = h_n^{-1} g h_n$ and $g_n = h_n R_{g'_n(0)} h_n^{-1}$. The diffeomorphisms f_n and g_n commute, and $f_n \to f$ in the $C^{l-2-\eta}$-norm. To prove theorem 4.1, it suffices to show that $g_n \to g$ in the $C^{l-2-\eta}$-norm. This convergence is based on the lemma:

Lemma 4.3. *Let $l \geq 3$ an integer, $f \in D^l(\mathbb{T}^1)$ of rotation number $\alpha \in \mathbb{R}$, $\eta > 0$, $0 \leq u \leq l - 2 - \eta$, and $g \in D^l(\mathbb{T}^1)$ such that $fg = gf$. Let $(q_t)_{t\geq 0}$ the sequence of denominators of the convergents of α, and $r \geq 0$ an integer. Let $\tilde{\alpha}$ an irrational Diophantine number, $\lambda_0 \in \mathbb{R}$ the associated number and h the associated diffeomorphism given by lemma 3.3. Let $f' = h^{-1} f h$ and $g' = h^{-1} g h$. We have the estimation:*

$$||g - hR_{g'(0)}h^{-1}||_u \leq C||h||_{u+1}^C ||f||_u^C ||g||_{u+1}^C \left(\frac{1}{q_r} + |\tilde{\alpha} - \alpha| \left(\frac{(C||h||_{u+1}||f||_{u+1})^{Cq_r}}{(\min Dh)^C} \right) \right)$$

Proof of theorem 4.1. Assuming lemma 4.3, we show theorem 4.1.

Let $\tilde{\alpha} = \alpha_n$ and h_n the associated diffeomorphism given by lemma 3.3. Since $V_\alpha(n) \to +\infty$, by applying the estimation for the conjugacy h_n, there exists $\tilde{G}(x)$ strictly increasing with x such that, for n sufficiently large:

$$||g - h_n R_{g'_n(0)} h_n^{-1}||_{l-2-\eta} \leq e^{C\tilde{G}(V_\alpha(n))} \left(\frac{1}{q_r} + \frac{e^{C\tilde{G}(V_\alpha(n))q_r}}{2^n} \right)$$

Moreover, since $q_n = a_n q_{n-1} + q_{n-2}$, and $q_{n-2} \leq q_{n-1}$, then

$$(\sqrt{2})^{n-1} \leq q_n \leq \prod_{k=1}^{n} (a_k + 1) \tag{6}$$

Therefore, we get:

$$\|g - h_n R_{g_n'(0)} h_n^{-1}\|_{l-2-\eta} \leq e^{C\tilde{G}(V_\alpha(n)) - \frac{1}{2}(r-1)\log 2} + e^{C\tilde{G}(V_\alpha(n)) + C\tilde{G}(V_\alpha(n))(V_\alpha(r)+1)^r - n\log 2} \tag{7}$$

Let $G(n) = \tilde{G}^{-1}((\log n)^{1/2})$. By extracting in the sequence $V_\alpha(n)/G(n)$, we can suppose that:

$$\frac{V_\alpha(n)}{G(n)} \to 0$$

Therefore, for n sufficiently large, we have:

$$\tilde{G}(V_\alpha(n)) \leq (\log n)^{1/2}$$

Moreover, for n sufficiently large, we can take an integer r_n such that:

$$(\log n)^{3/4} \leq r_n \leq (\log n)^{7/8}$$

we get:

$$(V_\alpha(r_n) + 1)^{r_n} = e^{r_n \log(V_\alpha(r_n)+1)} \leq e^{(\log n)^{15/16}}$$

The first term in estimation (7) tends towards 0. Moreover, since,for n sufficiently large,

$$(\log n)^{1/2} e^{(\log n)^{15/16}} \leq \frac{n}{2}\log 2$$

then the second term also tends towards 0. Hence theorem 4.1.

□

Proof of lemma 4.3. We need two higher-order analogous of the mean value theorem. The first one is:

Lemma 4.4. *Let $u \geq 0$, $s, t \in D^u(\mathbb{T}^1)$. Let $\delta \in \mathbb{R}$. We have:*

$$\|st - R_\delta t\|_u \leq C\|s\|_{u+1}\|s - R_\delta\|_u \|t\|_u^u$$

Proof. If $u = 0$, the estimate is trivial. We suppose $u \geq 1$. For any $x \in \mathbb{R}$, the Faa-di-Bruno formula gives:

$$D^u(st)(x) - D^u(R_\delta t)(x) = \sum_{j=0}^{u} \left((D^j s)(t(x)) - (D^j R_\delta)(t(x))\right) B_{u,j}\left(Dt(x), ..., D^{u-j+1}t(x)\right)$$

Therefore, by estimation (2), and since $\|t\|_u \geq 1$,

$$|D^u(st)(x) - D^u(R_\delta t)(x)| \leq C\|s\|_{u+1}\|s - R_\delta\|_u \|t\|_u^u$$

Hence lemma 4.4.

□

The second higher-order analogous of the mean value theorem is:

Lemma 4.5. *Let $u \geq 0$, $s \in D^{u+1}(\mathbb{T}^1)$, $t \in D^u(\mathbb{T}^1)$, $\delta \in \mathbb{R}$. We have:*

$$\|st - sR_\delta\|_u \leq C\|s\|_{u+1}\|t\|_u^u\|t - R_\delta\|_u$$

Proof. If $u = 0$, the estimate holds. We suppose $u \geq 1$. We use the following lemma:

Lemma 4.6. *Let $u \geq 1$, $j \leq u$ be integers and $a_1, ..., a_{u-j+1}, x_1, ..., x_{u-j+1} \geq 0$. Let $x \geq \max\{|x_k| \vee 1;\ 1 \leq k \leq u - j + 1\}$ and let $a \geq \max\{|a_k|;\ 1 \leq k \leq u - j + 1\}$. Let $B_{u,j}$ be a Bell polynomial. We have:*

$$|B_{u,j}(x_1 + a_1, ..., x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, ..., x_{u-j+1})| \leq Ca(x + a)^u$$

Proof. Let $p \geq 1$ and $l_1, ..., l_p$ be integers. Then we have:

$$(x_1+a_1)^{l_1}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_p^{l_p} = \sum_{i=1}^{p} x_1^{l_1}...x_{i-1}^{l_{i-1}}(x_i+a_i)^{l_i}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_i^{l_i}(x_{i+1}+a_{i+1})^{l_{i+1}}...(x_p+a_p)^{l_p}$$

$$(x_1+a_1)^{l_1}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_p^{l_p} = \sum_{i=1}^{p} x_1^{l_1}...x_{i-1}^{l_{i-1}}(x_{i+1}+a_{i+1})^{l_{i+1}}...(x_p+a_p)^{l_p}\left[(x_i + a_i)^{l_i} - x_i^{l_i}\right]$$

(with the conventions $x_1^{l_1}...x_0^{l_0} = 1$ and $x_{p+1}^{l_{p+1}}...x_p^{l_p} = 1$).

Since $(x_i + a_i)^{l_i} - x_i^{l_i} \leq l_i|a_i|(|x_i| + |a_i|)^{l_i-1} \leq l_i a(|x_i| + a)^{l_i-1}$, $1 \leq l_i \leq u$ and $x + a \geq 1$ (because $x \geq 1$), we obtain:

$$|B_{u,j}(x_1 + a_1, ..., x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, ..., x_{u-j+1})| \leq a(u - j + 1)uB_{u,j}(x + a, ..., x + a)$$

By the formula giving the Bell polynomials, we have:

$$B_{u,j}(x + a, ..., x + a) \leq C(x + a)^u$$

□

To show lemma 4.5, For any $0 \leq v \leq u$, we write:

$$D^v(st)(x) - D^v(sR_\delta)(x) = \sum_{j=0}^{v} D^j s(t(x))\left[B_{v,j}\left(Dt(x), ..., D^{v-j+1}t(x)\right) - B_{v,j}\left(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)\right)\right] +$$

$$\left[D^j s(t(x)) - D^j s(R_\delta(x))\right] B_{v,j}\left(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)\right)$$

We apply lemma 4.6 with $a = \|t - R_\delta\|_u$ and $x = \|R_\delta\|_u \geq 1$. Since $t \in D^u(\mathbb{T}^1)$, then $\|t\|_u \geq 1$. We get:

$$\left|B_{v,j}\left(Dt(x), ..., D^{v-j+1}t(x)\right) - B_{v,j}\left(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)\right)\right| \leq C\|t - R_\delta\|_u(1 + \|t - R_\delta\|_u)^u$$

$$\left|B_{v,j}\left(Dt(x), ..., D^{v-j+1}t(x)\right) - B_{v,j}\left(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)\right)\right| \leq C\|t - R_\delta\|_u(2 + \|t\|_u)^u \leq C\|t - R_\delta\|_u\|t\|_u^u$$

□

To prove lemma 4.3, we also need these successive estimations:

Lemma 4.7. *Let* $1 \leq u \leq k-2-\eta$. *We have:*

$$A_{1,u} = \|h^{-1}\|_u \leq C\|h\|_u^C \frac{1}{(\min Dh)^C} \tag{8}$$

$$A_{2,u} = \|f'\|_u \leq CA_{1,u}\|f\|_u^C\|h\|_u^C \tag{9}$$

$$A_{3,u}(m) = \|f'^m\|_u \leq C^m A_{2,u}^{mC} \tag{10}$$

$$A_{4,u} = \|f' - R_\alpha\|_u \leq C\|h\|_{u+1}^C\|f\|_u^C \frac{1}{(\min Dh)^C}|\tilde{\alpha} - \alpha| \tag{11}$$

$$A_{5,u}(m) = \|f'^m - R_{m\alpha}\|_u \leq mCA_{4,u}A_{2,u}^C \max_{k \leq m-1} A_{3,u+1}(k) \tag{12}$$

$$A_{6,u} = \|g'\|_u \leq CA_{1,u}\|g\|_u^C\|h\|_u^C \tag{13}$$

For any $r > 0$,

$$A_{7,u} = \|g' - R_{g'(0)}\|_u \leq \frac{A_{6,u+1}+1}{q_r} + \max_{m \leq 2q_r}\left(A_{6,u+1}A_{3,u}^C(m)A_{5,u}(m) + A_{6,u}^C A_{3,u+1}(m)A_{5,u}(m)\right) \tag{14}$$

$$A_{8,u} = \|g'h^{-1} - R_\alpha h^{-1}\|_u \leq CA_{6,u+1}A_{7,u}A_{1,u}^C \tag{15}$$

$$A_{9,u} = \|hg'h^{-1} - hR_{g'(0)}h^{-1}\|_u \leq C\|g\|_u^C A_{8,u}A_{1,u}^C\|h\|_{u+1} \tag{16}$$

The crucial estimate is (14), which is obtained by approaching modulo 1 each $x \in \mathbb{R}$ by a $m(x)\alpha$, with $m(x) \leq q_r$. If q_r increases, $x - m(x)\alpha$ is smaller modulo 1, but the bound on $A_{3,u}(m(x))$ and $A_{5,u}(m(x))$ increases. In the proof of theorem 4.1, a proper choice of r is made.

Estimation (11) corresponds to estimate (5) of the proof of the result of quasi-reducibility.

The other estimations, namely, estimations (8),(9),(10), (12),(13), (15) and (16) are derived from applications of the Faa-di-Bruno formula: either corollary 3.6, lemma 4.4 or lemma 4.5.

Proof of lemma 4.7. For $A_{1,u}$, by estimation (3), we have:

$$\|h^{-1}\|_u \leq C\|h\|_u^C \frac{1}{(\min Dh)^C}$$

Hence estimation (8).

For $A_{2,u}$, by applying corollary 3.6 twice, we have,

$$\|f'\|_u \leq CA_{1,u}\|f\|_u^C\|h\|_u^C$$

Hence estimation (9).

For $A_{3,u}$, by applying corollary 3.6 again, we have, for any m,

$$\|f'^{m+1}\|_u \leq C\|f'^m\|_u\|f'\|_u^C$$

and therefore, by iteration, we get:

$$\|f'^m\|_u \leq C^m\|f'\|_u^{mC}$$

Hence (10).

Estimation (11) is a direct application of estimation (5).

For estimation (12), we observe that for any $0 \leq v \leq u$:

$$D^v f'^m - D^v R_{m\alpha} = D^v \sum_{k=0}^{m-1} f'^{m-k} R_{k\alpha} - f'^{m-k-1} R_{(k+1)\alpha}$$

$$D^v f'^m - D^v R_{m\alpha} = \sum_{k=0}^{m-1} D^v \left(f'^{m-k-1} f'\right) R_{k\alpha} - D^v \left(f'^{m-k-1} R_\alpha\right) R_{k\alpha}$$

By applying lemma 4.5, and by noting that for any k, $\|f'^{m-k-1}\|_{u+1} \leq \max_{0\leq k\leq m-1} \|f'^k\|_{u+1}$, we get:

$$\|f'^m - R_{m\alpha}\|_u \leq mC\|f'\|_u^C \max_{0\leq k\leq m-1} \|f'^k\|_{u+1}\|f' - R_\alpha\|_u$$

Hence (12).

For $A_{6,u}$, estimation (13) is the same as (9):

$$\|g'\|_u \leq C\|h^{-1}\|_u\|g\|_u^C\|h\|_u^C$$

Hence (13).

For $A_{7,u}$, let $m \geq 0$ and $u \geq v \geq 1$. For any x, $D^v R_\alpha(x) = \int_0^1 D^v g'(y)\,dy$. Therefore,

$$|D^v g'(x) - D^v R_\alpha(x)| = \left|D^v g'(x) - \int_0^1 D^v g'(y)\,dy\right| =$$
$$\left|\int_0^1 (D^v g'(x) - D^v g'(y))\ dy\right| \leq \max_{x,y\in[0,1]} |D^v g'(x) - D^v g'(y)|$$

On the other hand, we have:

$$D^v g'(x) - D^v g'(y) = D^v g'(x) - D^v g'(y + m\alpha) + D^v g'(R_{m\alpha}(y)) - D^v(g' f'^m(y)) + D^v(f'^m g'(y)) - D^v g'(y)$$

Moreover, we have:

$$|D^v g'(x) - D^v g'(y + m\alpha)| \leq |D^{u+1} g'|_0 |x - y - m\alpha|$$

By lemma 4.5, we also have:

$$|D^v g'(R_{m\alpha}(y)) - D^v(g' f'^m(y))| \leq C\|g'\|_{u+1}\|f'^m\|_u^C\|f'^m - R_{m\alpha}\|_u$$

Finally, by lemma 4.4, we have:

$$|D^v(f'^m g'(y)) - D^v(R_{m\alpha} g'(y))| \leq C\|f'^m\|_{u+1}\|f'^m - R_{m\alpha}\|_u\|g'\|_u^C$$

By combining these estimations, we obtain:

$$|D^v g'(x) - D^v g'(y)| \leq \|g'\|_{u+1}|x - y - m\alpha| + C\|g'\|_{u+1}\|f'^m\|_u^C\|f'^m - R_{m\alpha}\|_u + C\|f'^m\|_{u+1}\|f'^m - R_{m\alpha}\|_u\|g'\|_u^C$$

Moreover, for any $r \geq 0$, any $x, y \in \mathbb{R}$, there is an integer $m(x, y) \leq 2q_r$, there are real numbers x', y' such that $x' - x \in \mathbb{Z}$, $y' - y \in \mathbb{Z}$ and such that $|x' - y' - m(x, y)\alpha| \leq 1/q_r$. Since $v \geq 1$, then $|D^v g'(x) - D^v g'(y)| = |D^v g'(x') - D^v g'(y')|$. We apply the former estimation with x' and y' and we get:

$$\max_{1\leq v\leq u} \left\|D^v g' - D^v R_{g'(0)}\right\|_0 \leq \frac{A_{6,u+1} + 1}{q_r} + \max_{m\leq 2q_r} \left(A_{6,u+1}A_{3,u}^C(m)A_{5,u}(m) + A_{6,u}^C A_{3,u+1}(m)A_{5,u}(m)\right)$$

If $v = 0$, we note that for any $r \geq 0$, any $x \in \mathbb{R}$, there is an integer $m(x) \leq q_r$ and a real number $x' \in \mathbb{R}$ such that $x' - x \in \mathbb{Z}$, and such that $|x' - m(x)\alpha| \leq 1/q_r$. Moreover, we have: $g'(x) - R_{g'(0)}(x) = g'(x') - R_{g'(0)}(x')$, and

$$g'(x') - R_{g'(0)}(x') = g'(x') - g'(m\alpha) + g'(m\alpha) - g' f'^m(0) + f'^m g'(0) - R_{m\alpha}(g'(0)) + R_{g'(0)}(m\alpha) - R_{g'(0)}(x')$$

Hence estimation (14).

For $A_{8,u}$, estimation (15) follows immediately from lemma 4.4.

For $A_{9,u}$, let $x \in \mathbb{R}$. Let $0 \leq v \leq u$. By the Faa-di-Bruno formula:

$$\begin{aligned}
&D^v\left(hg'h^{-1}\right)(x) - D^v\left(hR_{g'(0)}h^{-1}\right)(x) = \\
&\quad \sum_{j=0}^{v} D^j h(g'h^{-1}(x))B_{v,j}\left(D\left(g'h^{-1}\right)(x), ..., D^{v-j+1}\left(g'h^{-1}(x)\right)\right) - \\
&\quad D^j h(g'h^{-1}(x))B_{v,j}\left(D\left(R_{g'(0)}h^{-1}\right)(x), ..., D^{v-j+1}\left(R_{g'(0)}h^{-1}(x)\right)\right) \\
&= \sum_{j=0}^{v} D^j h(g'h^{-1}(x)) \\
&\quad \left[B_{v,j}\left(D\left(g'h^{-1}\right)(x), ..., D^{v-j+1}\left(g'h^{-1}(x)\right)\right) - B_{v,j}\left(D\left(R_{g'(0)}h^{-1}\right)(x), ..., D^{v-j+1}\left(R_{g'(0)}h^{-1}(x)\right)\right)\right] - \\
&\quad \left[D^j h(R_{g'(0)}h^{-1}(x)) - D^j h(g'h^{-1}(x))\right] B_{v,j}\left(D\left(R_{g'(0)}h^{-1}\right)(x), ..., D^{v-j+1}\left(R_{g'(0)}h^{-1}(x)\right)\right)
\end{aligned}$$

since $\|h^{-1}\|_u \geq 1$, then lemma 4.6 gives,

$$\left|B_{v,j}\left(D\left(g'h^{-1}\right)(x), ..., D^{v-j+1}\left(g'h^{-1}(x)\right)\right) - B_{v,j}\left(D\left(R_{g'(0)}h^{-1}\right)(x), ..., D^{v-j+1}\left(R_{g'(0)}h^{-1}(x)\right)\right)\right| \leq C\|g'h^{-1}\|_u^C\|g'h^{-1} - R_{g'(0)}h^{-1}\|_u$$

Since $g'h^{-1} = h^{-1}g$ and $||h^{-1}g||_u \leq C||h^{-1}||_u||g||_u^C$, we get,

$$\left|D^\nu\left(hg'h^{-1}\right)(x) - D^\nu\left(hR_{g'(0)}h^{-1}\right)(x)\right| \leq$$
$$C||g||_u^C||h||_u||h^{-1}||_u^C||g'h^{-1} - R_{g'(0)}h^{-1}||_u + C||h||_{u+1}||g'h^{-1} - R_{g'(0)}h^{-1}||_u||h^{-1}||_u^C$$

Hence estimation (16). This completes the proof of lemma 4.7.

□

By combining these estimations, we obtain:

$$A_{9,u} \leq CA_{1,u+1}^C||h||_{u+1}^C||g||_{u+1}^C\left(\frac{1}{q_r} + \max_{m\leq 2q_r}\left(A_{3,u+1}^C(m)A_{5,u}(m)\right)\right)$$

$$A_{9,u} \leq C||h||_{u+1}^C||f||_u^C||g||_{u+1}^C\left(\frac{1}{q_r} + |\tilde{\alpha} - \alpha|\left(\frac{(C||h||_{u+1}||f||_{u+1})^{Cq_r}}{(\min Dh)^C}\right)\right)$$

Hence lemma 4.3. Notice the loss of one derivative for h.

□

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