

ASYMPTOTICALLY OPTIMAL TESTS WHEN PARAMETERS ARE ESTIMATED

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The main purpose of this paper is to provide an asymptotically optimal test. The proposed statistic is of Neyman-Pearson-type when the parameters are estimated with a particular kind of estimators. It is shown that the proposed estimators enable us to achieve this end. Two particular cases, AR(1) and ARCH models were studied and the asymptotic power function was derived.

1. Introduction. Local asymptotic normality (LAN) for the log likelihood ratio was studied for a several classes of nonlinear time series model, from a LAN the contiguity property follows, for more details the interested reader may refer to [2], [12], and [4]. Applying the contiguity property, we construct a statistic for testing a null hypothesis H_0 against the alternative hypothesis $H_1^{(n)}$, often a various classical test statistics depends on the central sequence which appears in the expression of the log likelihood ratio, in the case when the parameter of the time series model is known we obtain good properties of the test, precisely, the optimality, see for instance [9, Theorem 3]. However, in a general case, particularly in practice, the parameter is unspecified, in the expression of the estimate central sequence appears an additional term which is non degenerate asymptotically. The latter, alters the power function of the constructed test.

In order to solve this very problem, and on a basis of an estimator of the unknown parameter, we introduce and define another estimator which does not effects asymptotically the power function of the test, more precisely the additional term is absorbed. The principle of this construction is to modify one of the component of the first estimator in order to avoid the additional term, the details of this method are expanded further in the section 2.

The main purpose of this paper is to investigate the problem of testing two hypothesis corresponding to a stochastic model which is described in the following way. Let $\{(Y_i, X_i)\}$ be a sequence of stationary and ergodic random vectors with finite second moment such that for all $i \in \mathbb{Z}$, where Y_i is a univariate random variable and X_i is a d -variate random vector. We

Keywords and phrases: Local asymptotic normality, Contiguity, Efficiency, Stochastic models, Le Cam's third lemma, Time series models, ARCH models

consider the class of stochastic models

$$(1.1) \quad Y_i = T(Z_i) + V(Z_i) \epsilon_i, \quad i \in \mathbb{Z},$$

where, for given non negative integers q and s , the random vectors Z_i is equal to $(Y_{i-1}, Y_{i-2}, \dots, Y_{i-s}, X_i, X_{i-1}, \dots, X_{i-q})$, the ϵ_i 's are centered i.i.d. random variables with unit variance and density function $f(\cdot)$, such that for each $i \in \mathbb{Z}$, ϵ_i is independent of the filtration $\mathcal{F}_i = \sigma(Z_j, j \leq i)$, the real-valued functions $T(\cdot)$ and $V(\cdot)$ are assumed to be unknown. We consider the problem of testing whether the bivariate vector of functions $(T(\cdot), V(\cdot))$ belongs to a given class of parametric functions or not. More precisely, let

$$\mathcal{M} = \{(m(\rho, \cdot), \sigma(\theta, \cdot)), (\rho', \theta')' \in \Theta_1 \times \Theta_2\},$$

$\Theta_1 \times \Theta_2 \subset \mathbb{R}^\ell \times \mathbb{R}^p$, $\overset{\circ}{\Theta}_1 \neq \emptyset$, $\overset{\circ}{\Theta}_2 \neq \emptyset$, where for all set A , $\overset{\circ}{A}$ denotes the interior of the set A and the script " ' " denotes the transpose, ℓ and p are two positive integers, and each one of the two functions $m(\rho, \cdot)$ and $\sigma(\theta, \cdot)$ has a known form such that $\sigma(\theta, \cdot) > 0$. For a sample of size n , we derive a test of

$$(1.2) \quad H_0 : [(T(\cdot), V(\cdot)) \in \mathcal{M}] \quad \text{against} \quad H_1 : [(T(\cdot), V(\cdot)) \notin \mathcal{M}].$$

It is easy to see that the null hypothesis H_0 is equivalent to

$$(1.3) \quad H_0 : [(T(\cdot), V(\cdot)) = (m(\rho_0, \cdot), \sigma(\theta_0, \cdot))],$$

while the alternative hypothesis H_1 is equivalent to

$$H_1 : [(T(\cdot), V(\cdot)) \neq (m(\rho_0, \cdot), \sigma(\theta_0, \cdot))],$$

for some $(\rho'_0, \theta'_0)' \in \Theta_1 \times \Theta_2$.

In the sequel, our study will be focused on the following alternative hypotheses. For all integers $n \geq 1$, the alternative hypothesis $H_1^{(n)}$ is defined by the following equation

$$(1.4) \quad H_1^{(n)} : [(T(\cdot), V(\cdot)) = (m(\rho_0, \cdot) + n^{-\frac{1}{2}}G(\cdot), \sigma(\theta_0, \cdot) + n^{-\frac{1}{2}}S(\cdot))],$$

where $G(\cdot)$ and $S(\cdot)$ are two specified real functions. The situation is different in the case when the used statistic is the Neyman-Pearson test which is based on the log-likelihood ratio Λ_n defined as follows

$$(1.5) \quad \Lambda_n = \log \left(\frac{f_n}{f_{n,0}} \right) = \sum_{i=1}^n \log(g_{n,i}),$$

where $f_{n,0}(\cdot)$ and $f_n(\cdot)$ denote the probability densities of the random vector (Y_1, \dots, Y_n) corresponding to the null hypothesis and the alternative hypothesis, respectively.

The use of the Neyman-Pearson statistics needs to resort to the following conditions:

Under the hypothesis H_0 , there exists a random variable \mathcal{V}_n such that

$$\mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2),$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution and some constant $\tau > 0$ depending on the parameter $\phi_0 = (\rho'_0, \theta'_0)'$, such that

$$(1.6) \quad \Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1).$$

The equality (1.6) is a modified version of the LAN given by [9, Theorem 1]. We mention that there exist other versions of the LAN, we may refer to [11], [8], and the references therein. On the basis of the LAN, an efficient test of linearity based on Neyman-Pearson-type statistics was obtained in a class of nonlinear time series models contiguous to a first-order autoregressive process $AR(1)$ and its asymptotic power function is derived (see, [9, Theorem 1 and Theorem 3]). The expression of the obtained test depends on the central sequence $\mathcal{V}_n(\phi_0)$ which itself depends on the parameter ϕ_0 . In a general case the parameter ϕ_0 is unspecified, so, in order to estimate it, we introduce, under some assumptions, an estimate preserving, asymptotically, the power on Neyman-Pearson test when we replace, in the expression of the statistics, the parameter ϕ_0 by an appropriate estimator, $\bar{\phi}_n$. Say, this estimator will be constructed on the tangent space with the direction of the partial derivatives of the central sequences in $\hat{\phi}_n$, where $\hat{\phi}_n$ is a \sqrt{n} -consistent estimator of ϕ_0 . In the sequel, $\bar{\phi}_n$ will be called a *modified estimate* (M.E.).

This paper is organized as follows: Section 2 describes the methodology used to construct the M.E. In Section 3, we give the asymptotic properties of the proposed estimate. In Section 4, we conduct a simulation in order to evaluate the power of the proposed test. All mathematical developments are relegated to the Section 5.

2. Estimation with modifying one component . Consider the problem of testing the two hypothesis H_0 against $H_1^{(n)}$ which are given in (1.3) and (1.4) respectively and corresponding to the stochastic model (1.1). We assume that the LAN (1.6) of the model (1.1) is established, for example refer to [9].

Let $\hat{\phi}_n = (\hat{\rho}'_n, \hat{\theta}'_n)'$ a \sqrt{n} -consistent estimate of the parameter $\phi_0 = (\rho'_0, \theta'_0)'$, where

$$\begin{aligned}\hat{\rho}'_n &= (\hat{\rho}_{n,1}, \dots, \hat{\rho}_{n,\ell}), & \hat{\theta}'_n &= (\hat{\theta}_{n,1}, \dots, \hat{\theta}_{n,p}), \\ \rho_0' &= (\rho_1, \dots, \rho_\ell) & \text{and } \theta_0' &= (\theta_1, \dots, \theta_p).\end{aligned}$$

Our purpose is to construct another estimate $\bar{\phi}'_n$ of the parameter $(\rho'_0; \theta'_0)'$, such that the following fundamental equality is fulfilled

$$(2.1) \quad \mathcal{V}_n(\bar{\phi}_n) - \mathcal{V}_n(\hat{\phi}_n) = D_n,$$

where D_n is a specified bounded random function. In the sequel, the functions

$(\rho, \cdot) \rightarrow m(\rho, \cdot)$ and $(\theta, \cdot) \rightarrow \sigma(\theta, \cdot)$ are assumed to be twice differentiable. Our goal, is to find an estimate $\bar{\phi}_n$ satisfying (2.1) pertaining to the tangent space Γ_n , such that, for $(X', Y')' \in \mathbb{R}^\ell \times \mathbb{R}^p$, the following equation holds

$$\Gamma_n : \mathcal{V}_n((X, Y)) - \mathcal{V}_n(\hat{\phi}_n) = \partial \mathcal{V}'_n(\hat{\phi}_n) \cdot \left((X - \hat{\rho}_n)', (Y - \hat{\theta}_n)' \right)',$$

where

$$\partial \mathcal{V}_n(\hat{\phi}_n)' = \left(\frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_1}, \dots, \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_\ell}, \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \theta_1}, \dots, \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \theta_p} \right),$$

and the script " \cdot " denotes the inner product.

With the connection with the equality (2.1), the new estimate is then given by imposing that the value $(X', Y')'$ satisfied the following identity

$$(2.2) \quad D_n = \partial \mathcal{V}_n(\hat{\phi}_n)' \cdot \left((X - \hat{\rho}_n)', (Y - \hat{\theta}_n)' \right)'$$

Clearly, the equation (2.2) has $\ell + p$ unknown values, so it has an infinity of solutions, after modification of the j_n -th component of the first estimate $\hat{\rho}_n$, we shall propose an element in tangent space Γ_n which satisfies the equality (2.2). We obtain then a new estimate $\bar{\phi}'_n = \phi_n^{(1, j_n)'} = (\bar{\rho}'_n, \hat{\theta}'_n)'$ of the unknown parameter ϕ_0 , where

$$\bar{\rho}'_n = (\bar{\rho}_{n,1}, \dots, \bar{\rho}_{n,\ell}),$$

and such that: for $s \in \{1, \dots, \ell\}$, $\bar{\rho}_{n,s} = \hat{\rho}_{n,s}$ if $s \neq j_n$ and $\bar{\rho}_{n,j_n} \neq \hat{\rho}_{n,j_n}$.

The use of the notation $\phi_n^{(1, j_n)}$ explains that we obtain the new estimate $\bar{\phi}_n$ of the parameter ϕ_0 when we change in the expression of the estimate $\hat{\phi}_n$

the j_n component with respect to the first estimate $\hat{\rho}_n$ corresponding to the step n of the estimation. It follows from the equality (2.1) combined with the constraint (2.2) that

$$\begin{aligned} \mathcal{V}_n(\phi_n^{(1,j_n)}) - \mathcal{V}_n(\hat{\phi}_n) &= \sum_{s=1}^{\ell} \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_s} (\bar{\rho}_{n,s} - \hat{\rho}_{n,s}) + \sum_{t=1}^p \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \theta_t} (\bar{\theta}_{n,t} - \hat{\theta}_{n,t}), \\ (2.3) \qquad \qquad \qquad &= \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} (\bar{\rho}_{n,j_n} - \hat{\rho}_{n,j_n}). \end{aligned}$$

By imposing the following condition

$$(2.4) \qquad \qquad \qquad \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \neq 0,$$

and with the use of the equality (2.2) combined with (2.4), we deduce that

$$(2.5) \qquad \qquad \bar{\rho}_{n,j_n} = \frac{D_n}{\frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} + \hat{\rho}_{n,j_n}.$$

In summary, we define the modified estimate by

$$\bar{\phi}'_n = \phi_n^{(1,j_n)'} = \left(\hat{\rho}_{n,1}, \dots, \hat{\rho}_{n,j_n-1}, \bar{\rho}_{n,j_n}, \hat{\rho}_{n,j_n+1}, \dots, \hat{\rho}_{n,\ell}, \hat{\theta}_{n,1}, \dots, \hat{\theta}_{n,p} \right)'.$$

With a same reasoning as the previous case and after modifying the k_n -th component with respect to the second estimate, we shall define a new estimate

$$\bar{\phi}'_n = \phi_n^{(2,k_n)'} = (\hat{\rho}'_n, \bar{\theta}'_n)',$$

such that for $t \in \{1, \dots, p\}$

$$\bar{\theta}_{n,t} = \hat{\theta}_{n,t} \text{ if } t \neq k_n \text{ and } \bar{\theta}_{n,k_n} \neq \hat{\theta}_{n,k_n}.$$

we obtain

$$(2.6) \qquad \mathcal{V}_n(\phi_n^{(2,k_n)}) - \mathcal{V}_n(\hat{\phi}_n) = \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \theta_{k_n}} (\bar{\theta}_{n,k_n} - \hat{\theta}_{n,k_n}).$$

Under the following condition

$$(2.7) \qquad \qquad \qquad \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \theta_{k_n}} \neq 0,$$

it follows from the equality (2.2) combined with (2.7), that

$$(2.8) \quad \bar{\theta}_{n,k_n} = \frac{D_n}{\frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \theta_{k_n}}} + \hat{\theta}_{n,k_n}.$$

In summary, we obtain the modified estimate

$$\bar{\phi}'_n = \phi_n^{(2,k_n)'} = \left(\hat{\rho}_{n,1}, \dots, \hat{\rho}_{n,\ell}, \hat{\theta}_{n,1}, \dots, \hat{\theta}_{n,k_n-1}, \bar{\theta}_{n,k_n}, \hat{\theta}_{n,k_n+1}, \dots, \hat{\theta}_{n,p} \right)'.$$

The estimate $\phi_n^{(1,j_n)}$ (respectively, $\phi_n^{(2,k_n)}$) is called a modified estimate in j_n -th component with respect to the first estimate (respectively, in k_n -th component with respect to second estimate), we denote this estimate by (M.E.).

REMARK 2.1. *For each step n of the estimation corresponding a value of the position j_n or k_n of the component where the estimate was modified.*

3. Properties of the (M.E.).

3.1. *Consistency.* Throughout, $\hat{\phi}_n$ is a \sqrt{n} -consistent estimate of the unknown parameter ϕ_0 . The conditions (2.4) and (2.7) are not sufficient to get the consistency of the modified estimate (M.E.). In order to get its consistency, we need to resort to one of the following additional conditions .

(C.1)

$$\frac{1}{\sqrt{n}} \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \xrightarrow{P} c_1 \quad \text{as } n \rightarrow \infty,$$

(C.2)

$$\frac{1}{\sqrt{n}} \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \theta_{k_n}} \xrightarrow{P} c_2 \quad \text{as } n \rightarrow \infty,$$

where c_1 and c_2 are two constantes, such that $c_1 \neq 0$ and $c_2 \neq 0$.

Our first result concerning the consistency of the proposed estimate is summarized in the following proposition.

PROPOSITION 3.1. *Under (2.4) and (C.1) ((2.7) and (C.2), respectively), the estimate $\phi_n^{(1,j_n)}$ ($\phi_n^{(2,k_n)}$, respectively) is a \sqrt{n} -consistent estimator of the unknown parameter ϕ_0 .*

In practice, it is not easy to verify the condition (C.1) (respectively, (C.2)), in the case when the unknown parameter ϕ_0 is univariate, a sufficient condition will be stated in Lemma (3.1), in this case, we need the following assumption:

(C.3) : For all real sequence $(\eta_n)_{n \geq 1}$ with values in the interval $[0, 1]$, we have:

$$\frac{1}{\sqrt{n}} \ddot{\mathcal{V}}_n(\eta_n \phi_0 + (1 - \eta_n) \hat{\phi}_n) = o_P(1),$$

where $\ddot{\mathcal{V}}_n$ is a second derivative of \mathcal{V}_n .

REMARK 3.1. *In a problem of testing the two hypothesis H_0 against $H_1^{(n)}$, and when the error ϵ_i 's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$, a large classe of time series model satisfied the condition (C.3), for instance, we cite the nonlinear time series contiguous to AR(1) processes, the details are expanded further later in the proofs of the Propositions (3.3) and (3.4).*

Now, we may state the sufficient condition which implies assumptions (C.1) corresponding to the case when the parameter of the time series model is univariate.

LEMMA 3.1. *Let $\hat{\phi}_n$ be a \sqrt{n} -consistent estimate of the parameter ϕ_0 . Let c_1 be a constant, such that $c_1 \neq 0$, then we have:*

(i) *Under (C.3), if $\frac{1}{\sqrt{n}} \dot{\mathcal{V}}_n(\phi_0) \xrightarrow{P} c_1$, as $n \rightarrow \infty$, then $\forall A > 0$,*

$$P \left(\left| \frac{1}{\sqrt{n}} \dot{\mathcal{V}}_n(\hat{\phi}_n) - c_1 \right| > A \right) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

3.2. *Absorbtion of the error.* Consequently, with the modified estimate and in the case when the error between two central sequences is bounded, it is possible to absorb this error, this result is stated and proved in the following proposition.

PROPOSITION 3.2. *Let $\hat{\phi}'_n$ be an estimate (\sqrt{n} consistency) of the parameter $(\rho', \theta)'$. We assume that there exists a known bounded function D_n , such that*

$$(3.1) \quad \mathcal{V}_n(\hat{\phi}_n) = \mathcal{V}_n(\phi_0) - D_n + o_P(1).$$

Then, there exists an estimate $\bar{\phi}'_n$ of $(\rho', \theta)'$ such that

$$\mathcal{V}_n(\bar{\phi}_n) = \mathcal{V}_n(\phi_0) + o_P(1).$$

REMARK 3.2. *The equality (3.1) gives the link between the estimated central sequences $\mathcal{V}_n(\hat{\phi}_n)$ and the central sequence $\mathcal{V}_n(\phi_0)$. Sometimes it is not easy to establish the form of the function D_n , in the next section, we*

propose, under some assumptions, how to specify this function in the two cases, *i.e.*, the case when the problem of testing the linearity and nonlinearity of the s -th order and the case of time series model with conditional heteroscedasticity respectively corresponding to the equalities

$$Y_i = \rho_0 Y_{i-1} + \alpha G(Y(i-1)) + \epsilon_i \quad \text{and} \\ Y_i = \rho_0 Y_{i-1} + \alpha G(Y(i-1)) + \sqrt{1 + \beta B(Y(i-1))} \epsilon_i,$$

respectively, where $Y(i-1) = (Y_{i-1}, Y_{i-2}, \dots, Y_{i-s})$, α and β are real parameters and the ϵ_i 's are centered *i.i.d.* random variables with unit variance and density function $f(\cdot)$.

Throughout, we assume that the function $f(\cdot)$ is positive with a third derivative, we denote by $\dot{f}(\cdot)$, $\ddot{f}(\cdot)$ and $f^{(3)}(\cdot)$ the first, the second and the third derivative respectively. For all $x \in \mathbb{R}$, let

$$M_f(x) = \frac{\dot{f}(x)}{f(x)}.$$

According to the notation (1.5), we suppose that the three following conditions are satisfied :

- (L.1): $\max_{1 \leq i \leq n} |g_{n,i} - 1| = o_P(1)$,
- (L.2): there exists a positive constant τ^2 such that $\sum_{i=1}^n (g_{n,i} - 1)^2 = \tau^2 + o_P(1)$,
- (L.3): there exists a \mathcal{F}_n measurable \mathcal{V}_n satisfying $\sum_{i=1}^n (g_{n,i} - 1) = \mathcal{V}_n + o_P(1)$, where $\mathcal{V}_n \xrightarrow{D} \mathcal{N}(0, \tau^2)$.

Conditions (L.1), (L.2) and (L.3) imply under H_0 the local asymptotic normality LAN corresponding to the equality (1.6), for more details see ([9, Theorem 1]). This last theorem is the fundamental tool used later to aim to establish the LAN for the considering models.

3.3. *Link between central sequences in nonlinear time series contiguous to AR(1) processes.* Consider the s -th order (nonlinear) time series

$$(3.2) \quad Y_i = \rho_0 Y_{i-1} + \alpha G(Y(i-1)) + \epsilon_i, \quad |\rho_0| < 1.$$

In this case and with the comparison to the equality (1.1), we have

$$Z_i = Y_i \quad , \quad T(Z_i) = \rho_0 Y_{i-1} + \alpha G(Y(i-1)) \quad \text{and} \quad V(Z_i) = 1.$$

In the sequel, it will be assumed that the model is a stationary and ergodic time series with finite second moment. We consider the problem of testing

the null hypothesis $H_0 : \alpha = 0$ against the alternative hypothesis $H_1^{(n)} : \alpha = n^{-\frac{1}{2}}$, with the comparison to (1.3) and (1.4), we have

$$\begin{aligned} \left(m(\rho_0, Y_{i-1}), \sigma(\theta_0, Y_{i-1}) \right)' &= \left(\rho_0 Y_{i-1}, 1 \right)', \quad \mathcal{M} = \{ m(\rho, \cdot), \rho \in \Theta_1 \}, \\ Z_i' &= \left(Y_{i-1}, \dots, Y_{i-s} \right) \quad \text{and} \quad S(\cdot) = 0. \end{aligned}$$

Note that this problem of testing is equivalent to test the linearity of the s -th $AR(1)$ time series model when $(\alpha = 0)$ against the nonlinearity of the s -th $AR(1)$ time series model when $(\alpha = n^{-\frac{1}{2}})$.

Throughout, the scripts " $\|\cdot\|_{\ell+p}$ ", " $\|\cdot\|_{\ell}$ " and " $\|\cdot\|_p$ " denote the euclidian norms in $\mathbb{R}^{\ell+p}$, \mathbb{R}^{ℓ} and \mathbb{R}^p respectively. It will be assumed that the conditions (A.1) and (A.2) are satisfied, where

- (A.1): There exists positive constants η and c such that for all u with $\|u\|_{\ell+p} > \eta$, $G(u) \leq c\|u\|_{\ell+p}$.
- (A.2): for a location family $\{f(\epsilon_i - c), -\infty < c < \infty\}$, there exist a square integrable functions Ψ_1, Ψ_2 and a constant δ such that for all ϵ_i and $|c| < \delta$, such that :

$$\left| \frac{d^k f(\epsilon_i - c)}{f(\epsilon_i) dc^k} \right| \leq \Psi_k(\epsilon_i), \quad \text{for } k = 1, 2.$$

Under the conditions (A.1) and (A.2) the LAN of the time series model (3.2) was established in ([9, Theorem 2]), the proposed test T_n is the Neyman-Pearson statistic which is given by the following equality

$$T_n = I \left\{ \frac{\mathcal{V}_n(\rho_0)}{\tau(\rho_0)} \geq Z(\alpha) \right\}, \quad \text{where } \tau^2 = \mathbf{E}(M_f^2(\epsilon_0))\mathbf{E}(G^2(Y(0))),$$

and $Z(\alpha)$ is the $(1 - \alpha)$ -quantile of a standard normal distribution $\Phi(\cdot)$. In this case, the central sequence is given by the following equality

$$\mathcal{V}_n(\rho_0) = -\frac{1}{\sqrt{n}} \sum_{i=1}^n M_f(\epsilon_i) G(Y(i-1)), \quad \text{where } \tau^2 = \mathbf{E}(M_f^2(\epsilon_0))\mathbf{E}(G^2(Y(0))),$$

and such that under H_0 , $\mathcal{V}_n(\rho_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2)$. The asymptotic power of the test is derived and equal to $1 - \Phi(Z(\alpha) - \tau^2)$, recall that when ρ_0 is known, this test is asymptotically optimal, for more details see [9, Theorem 3].

Our aim is to specify the form of the function D_n which is defined in (3.1), the parameter ρ_0 is estimated by the \sqrt{n} -consistent estimator $\hat{\rho}_n$ and the residual ϵ_i is estimated by $\hat{\epsilon}_{i,n} = Y_i - Y_{i-1}\hat{\rho}_n$. We have the following statement:

PROPOSITION 3.3. *Assume that the conditions (A.1) and (A.2) hold and ϵ_i 's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$. We have*

$$(3.3) \quad \mathcal{V}(\hat{\rho}_n) = \mathcal{V}_n(\rho_0) - D_n + o_P(1),$$

where

$$(3.4) \quad D_n = -c_1 \sqrt{n}(\hat{\rho}_n - \rho_0),$$

$$(3.5) \quad \bar{\rho}_n = \frac{D_n}{\dot{\mathcal{V}}_n(\phi_n)} + \hat{\rho}_n \quad \text{and} \quad c_1 = -\mathbb{E}\left[Y_0 G(Y(0))\right].$$

REMARK 3.3. • *The use of the ergodicity of the model imposes to require the condition $\mathbb{E}\left[Y_{-1}G(Y_0)\right] < \infty$, therefore we choose the function $G(\cdot)$ in order to get this condition. For instance, we shall choose $G(Y(i-1)) = \frac{2a}{1+Y_{i-1}^2}$, where $a \neq 0$.*

- *With this choice of the function G , the condition (A.1) remains satisfied, in fact, we can remark that $|G(u)| \leq 2|a|$, then for all u with $\|u\|_{\ell+p} \geq \eta$ we have $G(u) \leq 2a \times \|u\|_{\ell+p} \times \frac{1}{\|u\|_{\ell+p}} \leq \frac{2a}{\eta} \times \|u\|_{\ell+p}$, therefore, we shall choose $c = \frac{2a}{\eta}$.*

3.4. *An extension to ARCH processes.* Consider the following time series model with conditional heteroscedasticity

$$(3.6) Y_i = \rho_0 Y_{i-1} + \alpha G(Y(i-1)) + \sqrt{1 + \beta B(Y(i-1))} \epsilon_i, \quad i \in \mathbb{Z}.$$

It is assumed that the model (3.6) is ergodic and stationary. It will be assumed that the conditions (B.1), (B.2) and (B.3) are satisfied, where

- (B.1): The fourth order moment of the stationary distributions of (3.6) exists.
- (B.2): There exists a positive constants η and c such that for all u with $\|u\|_{\ell+p} > \eta$, $B(u) \leq c\|u\|_{\ell+p}^2$.
- (B.3): for a location family $\{b^{-1}f(\frac{\epsilon_i - a}{b}), -\infty < a < -\infty, b > 0\}$, there exists a square integrable function $\varphi(\cdot)$, and a strictly positive real ς , where $\varsigma > \max(|a|, |b-1|)$, such that,

$$\left| \frac{\partial^2 b^{-1} f\left(\frac{\epsilon_i - a}{b}\right)}{f(\epsilon_i) \partial a^j \partial b^k} \right| \leq \varphi(\epsilon_i),$$

where j and k are two positive integers such that $j + k = 2$.

We consider the problem of testing the null hypothesis H_0 against the alternative hypothesis $H_1^{(n)}$ such that

$$\begin{aligned} H_0 &: m(\rho, Z_i) = \rho_0 Y_{i-1} \quad \text{and} \quad \sigma(\theta_0, \cdot) = 1, \\ H_1^{(n)} &: m(\rho, Z_i) = \rho_0 Y_{i-1} + n^{-\frac{1}{2}} G(Y(i-1)) \quad \text{and} \quad \sigma(\theta_0, Z_i) = \sqrt{1 + n^{-\frac{1}{2}} B(Y(i-1))}. \end{aligned}$$

Remark that $H_0, H_1^{(n)}$ correspond to $\alpha = \beta = 0$ (linearity of (3.6)) and $\alpha = \beta = n^{-\frac{1}{2}}$ (non linearity of (3.6)) with the comparison to the equality (1.1), we have

$$Z_i = Y_i, \quad T(Z_i) = \rho_0 Y_{i-1} + \alpha G(Y(i-1)) \quad \text{and} \quad V(Z_i) = \sqrt{1 + \beta B(Y(i-1))}.$$

Note that when n is large, we have

$$\sigma(\theta_0, Z_i) = \sqrt{1 + n^{-\frac{1}{2}} B(Y(i-1))} \sim 1 + \frac{n^{-\frac{1}{2}}}{2} B(Y(i-1)) = 1 + n^{-\frac{1}{2}} S(Y(i-1)).$$

Under the conditions (A.1), (B.1), (B.2), and (B.3), the LAN was established in [9, Theorem 4], an efficient test is obtained and its power function is derived. In this case, the central sequence is given by the following equality

$$\mathcal{V}_n(\rho_0) = -\frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^n M_f(\epsilon_i) G(Y(i-1)) + \sum_{i=1}^n (1 + \epsilon_i M_f(\epsilon_i)) B(Y(i-1)) \right\},$$

such that under H_0 ,

$$\mathcal{V}_n(\rho_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2),$$

where

$$\begin{aligned} \tau^2 &= I_0 \mathbf{E} (G(Y(0)))^2 + \frac{(I_2 - 1)}{4} \mathbf{E} (B(Y(0)))^2 + I_1 \mathbf{E} (G(Y(0)) B(Y(0))) \\ &\quad \text{where } I_j = \mathbf{E} (\epsilon_0^j M_f^2(\epsilon_0)) \text{ and } j = 0, 1, 2. \end{aligned}$$

The proposed test is then given by

$$(3.7) \quad T_n = I \left\{ \frac{\mathcal{V}_n(\hat{\rho}_n)}{\tau(\hat{\rho}_n)} \geq Z(\alpha) \right\}.$$

By the substituting ρ_0 by its \sqrt{n} -consistent estimator $\hat{\rho}_n$ in the expression of the central sequence, we shall state the following proposition:

PROPOSITION 3.4. *Suppose that the conditions (A.1), (B.1), (B.2) and (B.3) hold and ϵ_i 's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$. We have*

$$(3.8) \quad \mathcal{V}(\hat{\rho}_n) = \mathcal{V}_n(\rho_0) - D_n + o_P(1),$$

where

$$(3.9) \quad D_n = -c_1 \sqrt{n}(\hat{\rho}_n - \rho_0),$$

$$(3.10) \quad \bar{\rho}_n = \frac{D_n}{\dot{\mathcal{V}}_n(\phi_n)} + \hat{\rho}_n \quad \text{and} \quad c_1 = -\mathbb{E}\left[Y_0 G(Y(0))\right].$$

3.5. *Optimality of the proposed test.* Throughout, \bar{T}_n and $\bar{\tau}$ are the statistics test and the constant respectively obtained with the subsisting of the unspecified parameter ϕ_0 by its modified estimate $\bar{\phi}_n$ in the expression of the test (3.7) and the constant τ appearing in the expression of the log likelihood ratio (1.6) respectively.

We assume in the problem of testing the two hypothesis H_0 against $H_1^{(n)}$ that the LAN of the the model (1.1) is established, in order to prove the optimality of the proposed test. To this end, we need the following assumption :

- (E.1) There exists a \sqrt{n} -estimate $\hat{\phi}_n$ of the unknown parameter ϕ_0 and a random bounded function D_n , such that

$$\mathcal{V}_n(\hat{\phi}_n) = \mathcal{V}_n(\phi_0) - D_n + o_P(1).$$

It is now obvious from the previous definitions that we can state the following theorem:

THEOREM 3.1. *Under LAN and the conditions (2.4) (respectively, (2.7)), (C.1) ((C.2), respectively) and (E.1) the asymptotic power of \bar{T}_n under H_1^n is equal to to*

$$1 - \Phi(Z(\alpha) - \bar{\tau}^2).$$

Furthermore, \bar{T}_n is asymptotically optimal.

We shall now apply this last theorem in order to conduct simulations corresponding to the representation of the derived asymptotic power function. The concerned model is the Nonlinear time series contiguous to AR(1) processes with an extension to ARCH processes.

4. Simulations. In this section, we assume that ϵ_i 's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$, in this case, we have $\mathbb{E}(\epsilon_i) = 0$, $\mathbb{E}(\epsilon_i^2) = 1$, and $\mathbb{E}(\epsilon_i^4) = 3$. We treat the case when the unknown parameter $\phi_0 = \rho_0 \in \Theta_1 \subset \mathbb{R}$, under H_0 , the considering time series model can also rewritten

$$(4.1) \quad Y_i = \rho_0 Y_{i-1} + \epsilon_i \quad \text{where} \quad |\rho_0| < 1.$$

4.1. *Nonlinear time series contiguous to AR(1) processes.* To evaluate the performance of our estimator, we provide simulations with comment in this section. In the case when the parameter ρ_0 is known, the test T_n is optimal and its power is asymptotically equal to $1 - \Phi(Z(\alpha) - \tau^2)$, for more details see [9, Theorem 3]. In a general case, when the parameter ρ_0 is unspecified, firstly, we estimate it with the least square estimates $\hat{\rho}_n = \frac{\sum_{i=1}^n Y_i Y_{i-1}}{\sum_{i=1}^n Y_{i-1}^2}$, secondly, with the use of the (M.E.) under the conditions (2.4) and (C.1), the modified estimate $\bar{\rho}_n$ exists and remains \sqrt{n} -consistent, making use of (2.5) in connection with the Proposition (3.3) it follows:

$$(4.2) \quad \bar{\rho}_n = \frac{D_n}{\dot{\mathcal{V}}_n(\hat{\rho}_n)} + \hat{\rho}_n = \frac{-c_1(\hat{\rho}_n - \rho_0)}{\frac{\dot{\mathcal{V}}_n(\hat{\rho}_n)}{\sqrt{n}}} + \hat{\rho}_n,$$

with the substitution of the parameter ρ_0 by its estimator $\bar{\rho}_n$ in (3.7), we obtain the following statistics test

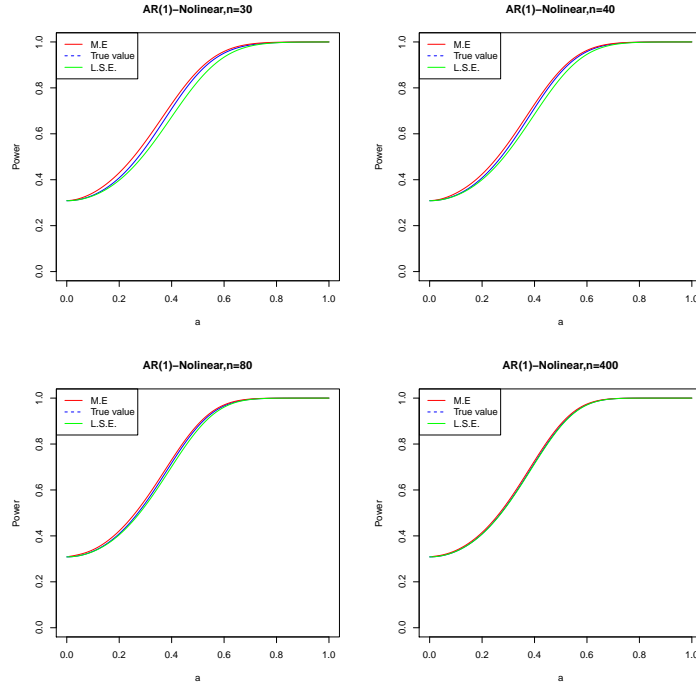
$$\bar{T}_n = \left\{ \frac{\mathcal{V}_n(\bar{\rho}_n)}{\tau(\bar{\rho}_n)} \geq Z(\alpha) \right\} \quad \text{where} \quad \bar{\tau}^2 = \mathbf{E}(M_f^2(\bar{\epsilon}_{0,n}))\mathbf{E}(G^2(Y_0)),$$

$$\text{and} \quad \bar{\epsilon}_{0,n} = Y_0 - Y_{-1}\bar{\rho}_n.$$

It follows from Theorem (3.1) that \bar{T}_n is optimal with an asymptotic power function equal to $1 - \Phi(Z(\alpha) - \tau^2(\bar{\rho}_n))$.

We choose the function G like this $G : (x_1, x_2, \dots, x_s, x_{s+1}, x_{s+2}, \dots, x_{s+q}) \rightarrow \frac{5a}{1+x_1^2}$ where $a \neq 0$.

In our simulations, the true value of the parameter ρ_0 is fixed at 0.1 and the sample sizes are fixed at $n = 30, 40, 80$ and 400, for a level $\alpha = 0.05$, the power relative for each test estimated upon $m = 1000$ replicates, we represent simultaneously the power test with a true parameter ρ_0 , the empirical power test which is obtained with the replacing the true value ρ_0 by its estimate (M.E.) $\bar{\rho}_n$ corresponding to the equality (4.2), and the empirical power test which is obtained with the subsisting the true value ρ_0 by its least square estimator LSE $\hat{\rho}_n$ (an estimator with no correction), we remark that, the two representations with the true value and the modified estimate M.E. are close for large n .



4.2. *ARCH processes.* With the substitution of the parameter ρ_0 by its modified estimate $\bar{\rho}_n$, in (3.7), we obtain the following test

$$\bar{T}_n = I \left\{ \frac{\mathcal{V}_n(\bar{\rho}_n)}{\tau(\bar{\rho}_n)} \geq Z(\alpha) \right\},$$

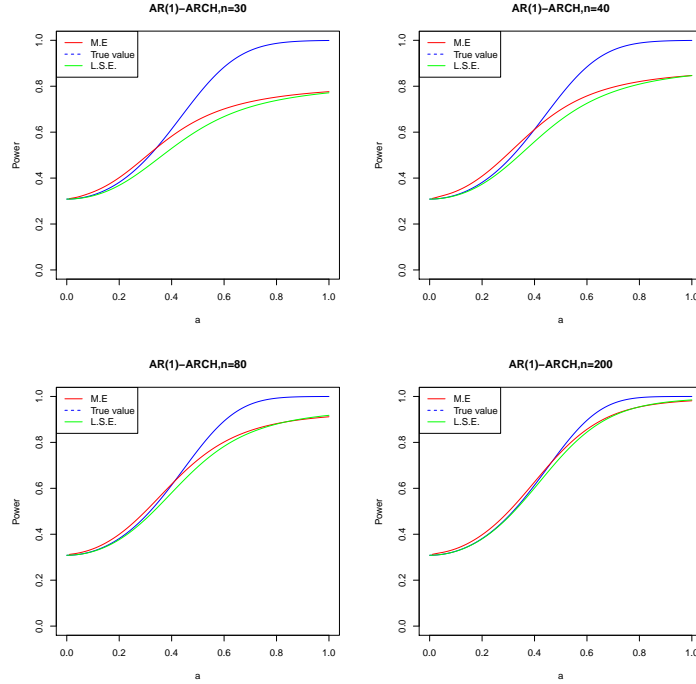
such that

$$\begin{aligned} \bar{\tau}^2 &= \bar{I}_{0,n} \mathbf{E} (G(Y(0)))^2 + \frac{(\bar{I}_{2,n} - 1)}{4} \mathbf{E} (B(Y(0)))^2 + \bar{I}_{1,n} \mathbf{E} (G(Y(0))B(Y(0))), \\ \bar{I}_{j,n} &= \mathbf{E} \left(\bar{\epsilon}_{0,n}^j M_f^2(\bar{\epsilon}_{0,n}) \right), \quad j = 0, 1, 2, \quad \text{and} \quad \bar{\epsilon}_{0,n} = Y_0 - Y_{-1} \bar{\rho}_n. \end{aligned}$$

In our simulations, the true value of the parameter ρ_0 is fixed at 0.1 and the sample sizes are fixed at $n = 30, 40, 80$ and 200, for a level $\alpha = 0.05$, the power relative for each test estimated upon $m = 1000$ replicates. We choose the functions G and B like this $G = B : (x_1, x_2, \dots, x_s, x_{s+1}, x_{s+2}, \dots, x_{s+q}) \rightarrow \frac{3.5a}{1+x_1^2}$ where $a \neq 0$.

We represent simultaneously the power test with a true parameter ρ_0 and the empirical power test which is obtained with the subsisting the true value ρ_0 by its estimate (M.E.) $\bar{\rho}_n$ corresponding to the equality (4.2), we represent

simultaneously the power test with a true parameter ρ_0 , the empirical power test which is obtained with the subsisting the true value ρ_0 by its estimate (M.E.) $\bar{\rho}_n$ corresponding to the equality (4.2), and the empirical power test which is obtained with the subsisting the true value ρ_0 by its least square estimator LSE $\hat{\rho}_n$ (estimator with no correction), we remark that, when n is large, we have a similar conclusion as the previous case .



REMARK 4.1. We mention that the limiting distributions appearing in Proposition (3.3) and Proposition (3.4) depend on the unknown quantity $b_n = (\hat{\rho}_n - \rho_0)$, i.e., in practice ρ_0 is not specified, in general. To circumvent this difficulty, we use the Efron's Bootstrap in order to evaluate b_n , more precisely, the interested reader may refer to the following references : [6] for the description of the Bootstrap methods, [1], [10] for the Bootstrap methods in AR(1) time series models and [7] for the ARCH models.

5. Proof of the results.

Proof of the Proposition 3.1. Consider the following fundamental decomposition:

$$(5.1) \quad (\phi_n^{(1,j_n)})' = (\hat{\phi}_n)' + (O_{j_n})',$$

where $O'_{j_n} = (O_{j_n,i})'_{i \in \{1, \dots, \ell+p\}}$, such that $O_{j_n,i} = 0$ when $i \neq j_n$,
and $O_{j_n,j_n} = \bar{\rho}_{n,j_n} - \hat{\rho}_{n,j_n}$.

Firstly, we have $\hat{\phi}_n \xrightarrow{P} \phi_0$, secondly we can deduce from (2.5) that:

$$(5.2) \quad O_{j_n,j_n} = \frac{D_n}{\frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} = \frac{1}{\sqrt{n}} D_n \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}}.$$

Since D_n is bounded, we can remark that $\frac{1}{\sqrt{n}} D_n \xrightarrow{P} 0$, from (C.1), there exists some constant $c_1 \neq 0$, such that $\frac{1}{\sqrt{n}} \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \xrightarrow{P} c_1$, from (2.4) and since the function $x \rightarrow \frac{1}{x}$ is continuous on $\mathbb{R} - \{0\}$, it follows that the random variable $\frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} \xrightarrow{P} \frac{1}{c_1}$, then the couple $\left(\frac{1}{\sqrt{n}} D_n; \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} \right)$ converges in probability to the couple $\left(0; \frac{1}{c_1} \right)$, since the function $(x, y) \rightarrow xy$ is continuous on $\mathbb{R} \times \mathbb{R}$, it result from (5.2), that the random variable $O_{j_n,j_n} \xrightarrow{P} \frac{0}{c_1} = 0$, therefore

$$(5.3) \quad O_{j_n}' = (0, \dots, 0, O_{j_n,j_n}, 0 \dots 0)' \xrightarrow{P} (0, \dots, 0, 0, 0 \dots 0)'.$$

Consider again the equality (5.1), since the function $(x, y) \rightarrow x + y$ is continuous on $\mathbb{R}^{\ell+p} \times \mathbb{R}^{\ell+p}$, it results from (5.3) that $\phi_n^{(1,j_n)}$ converges in probability to ϕ_0 as

$n \rightarrow \infty$. Notice that the last previous convergences in probability follow immediately with the use of the continuous mapping theorem, for more details, see [3] or [13]. By following the same previous reasoning, we shall prove the consistency of the estimate $\phi_n^{(2,k_n)}$. Note that $\phi_n^{(1,j_n)}$ is \sqrt{n} -consistent estimate of the parameter ϕ_0 and

$$\sqrt{n}(\phi_n^{(1,j_n)} - \phi_0) = O_P(1),$$

where $O_P(1)$ is bounded in probability in $\mathbb{R}^{\ell+p}$. In fact, it follows from (5.1) that

$$(5.4) \quad \sqrt{n}(\phi_n^{(1,j_n)} - \phi_0) = \sqrt{n}(\hat{\phi}_n - \phi_0) + \sqrt{n}O_{j_n} = O_P(1) + \sqrt{n}O_{j_n}.$$

Since $\sqrt{n}O_{j_n, j_n} = D_n \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{V}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}}$ and using the condition (C.1), it results

that

$\sqrt{n}O_{j_n} = O_{P_1}(1)$, where $O_{P_1}(1)$ is bounded in probability in \mathbb{R} .

We deduce that

$$(5.5) \quad \sqrt{n}(\phi_n^{(1, j_n)} - \phi_0) = O_P(1).$$

Notice that with a similar argument and with changing $\phi_n^{(1, j_n)}$, (C.1) and (2.4) by $\phi_n^{(2, k_n)}$, (C.2) and (2.7) respectively, we obtain

$$(5.6) \quad \sqrt{n}(\phi_n^{(2, k_n)} - \phi_0) = O_P(1).$$

In order to prove Lemma 3.1, we need to stated the following classical lemmas:

LEMMA 5.1. *Let $(X_i)_{i \in \{1, \dots, l\}}$ be a sequence of a positive random variables on the probability space (Ω, \mathcal{F}, P) , $(\alpha_i)_{i \in \{1, \dots, l\}}$ a sequence of a positive (strictly) reals such that $\sum_{i=1}^l \frac{1}{\alpha_i} = 1$, then we have, for each $\epsilon > 0$,*

$$P\left(\sum_{i=1}^l X_i > \epsilon\right) \leq \sum_{i=1}^l P\left(X_i > \frac{\epsilon}{\alpha_i}\right).$$

LEMMA 5.2. *Let $(X_n)_{n \geq 0}$ be a sequence of a random variables on the probability space (Ω, \mathcal{F}, P) , such that $X_n = O_P(1)$, then $X_n^2 = O_P(1)$.*

Proof of the Lemma 5.1. Firstly, we remark that, $\forall \epsilon > 0$, we have

$$\left\{\sum_{i=1}^l X_i > \epsilon\right\} \subset \bigcup_{i=1}^l \left\{X_i > \frac{\epsilon}{\alpha_i}\right\}.$$

In fact, we suppose there exists

$$\omega \in \left\{\sum_{i=1}^l X_i > \epsilon\right\} \quad \text{and} \quad \omega \notin \bigcup_{i=1}^l \left\{X_i > \frac{\epsilon}{\alpha_i}\right\},$$

then for each $i \in \{1, \dots, l\}$, we have $X_i(\omega) \leq \frac{\epsilon}{\alpha_i}$, which implies that

$$\sum_{i=1}^l X_i(\omega) \leq \epsilon,$$

hence a contradiction. With the use of the σ -additivity, we obtain

$$P\left(\sum_{i=1}^l X_i > \epsilon\right) \leq P\left(\bigcup_{i=1}^l \left\{X_i > \frac{\epsilon}{\alpha_i}\right\}\right) \leq \sum_{i=1}^l P\left(X_i > \frac{\epsilon}{\alpha_i}\right).$$

Proof of the Lemma 3.1. In this case $\phi_0 = \rho_0 \in \Theta_1 \subset \mathbb{R}$, we denote by $\hat{\rho}_n$ the \sqrt{n} -consistent estimator of ρ_0 .

Let $A > 0$, from the triangle inequality combined with the Lemma (5.1), we obtain:

$$\begin{aligned} & P \left(\left| \frac{1}{\sqrt{n}} \dot{V}_n(\hat{\rho}_n) - c_1 \right| > A \right) \\ &= P \left(\left| \frac{1}{\sqrt{n}} \dot{V}_n(\hat{\rho}_n) - \frac{1}{\sqrt{n}} \dot{V}_n(\rho_0) \right| + \left| \frac{1}{\sqrt{n}} \dot{V}_n(\rho_0) - c_1 \right| > A \right) \\ &\leq P \left(\left| \frac{1}{\sqrt{n}} \dot{V}_n(\hat{\rho}_n) - \frac{1}{\sqrt{n}} \dot{V}_n(\rho_0) \right| > \frac{A}{2} \right) + P \left(\left| \frac{1}{\sqrt{n}} \dot{V}_n(\rho_0) - c_1 \right| > \frac{A}{2} \right). \end{aligned}$$

Firstly, we have

$$(5.7) \quad P \left(\left| \frac{1}{\sqrt{n}} \dot{V}_n(\rho_0) - c_1 \right| > \frac{A}{2} \right) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

Secondly, we have

$$(5.8) \quad \left| \frac{1}{\sqrt{n}} \dot{V}_n(\hat{\rho}_n) - \frac{1}{\sqrt{n}} \dot{V}_n(\rho_0) \right| = \frac{1}{\sqrt{n}} \left| \ddot{V}_n(\tilde{\rho}_n) \right| \left| \hat{\rho}_n - \rho_0 \right|$$

$$(5.9) \quad = \frac{1}{\sqrt{n}} \left| \frac{1}{\sqrt{n}} \ddot{V}_n(\tilde{\rho}_n) \right| \left| \sqrt{n}(\hat{\rho}_n - \rho_0) \right|,$$

where $\tilde{\rho}_n$ is a point between ρ_0 and $\hat{\rho}_n$, then there exists a sequence η_n with values in the interval $[0, 1]$, such that $\tilde{\rho}_n = \eta_n \rho_0 + (1 - \eta_n) \hat{\rho}_n$, this implies that

$|\tilde{\rho}_n - \rho_0| \leq (1 - \eta_n) |\hat{\rho}_n - \rho_0| \leq |\hat{\rho}_n - \rho_0|$, this last inequality enable us to concluded that $\tilde{\rho}_n$ is \sqrt{n} -consistency estimator of ρ_0 , it follows from (C.3) applied on the equality (5.9) that

$$(5.10) \quad P \left(\left| \frac{1}{\sqrt{n}} \dot{V}_n(\hat{\rho}_n) - \frac{1}{\sqrt{n}} \dot{V}_n(\rho_0) \right| > \frac{A}{2} \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus we obtain (i).

Proof of Proposition 3.2. It suffices to choose under (2.4) and (C.1) the estimate $\bar{\phi}_n = \phi_n^{(1, j_n)}$, or under (2.7) and (C.2) the estimate $\bar{\phi}_n = \phi_n^{(2, k_n)}$.

In order to prove the Proposition (3.3), we need a following classical result.

LEMMA 5.3. *Let (Ω, \mathcal{F}, P) be a probability space, $(X_n)_{n \geq 1}$ is a sequence of real random variables on Ω . If X_n converges in probability to a constant c , then, there exists a sequence of random variable $(Y_n)_n$, with $X_n = c + Y_n$, such that, Y_n converges in probability to 0.*

Proof of Lemma 5.3. For all $A > 0$, we have:

$$P(|Y_n| > A) = P(|X_n - c| > A) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Proof of Lemma 5.2. For all $\epsilon > 0$, $\exists M_1 > 0$ such that:
 $\sup_\alpha \left((P(|X_\alpha| > M_1)) \right) < \epsilon$, this implies that $\sup_\alpha \left((P(|X_\alpha|^2 > M_1^2)) \right) < \epsilon$,
 therefore with the choice of $M = M_1$, we obtain the result.

Proof of Proposition 3.3. ϵ_i 's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$, making use of the results of [9, Theorem 2], we have

$$\mathcal{V}_n(\rho_0) = -\frac{1}{\sqrt{n}} \sum_{i=1}^n M_f(\epsilon_i) G(Y(i-1)).$$

The estimated central sequence is

$$\mathcal{V}_n(\hat{\rho}_n) = -\frac{1}{\sqrt{n}} \sum_{i=1}^n M_f(\hat{\epsilon}_{i,n}) G(Y(i-1)).$$

By Taylor expansion with order 2, we have :

$$(5.11) \quad \mathcal{V}_n(\hat{\rho}_n) - \mathcal{V}_n(\rho_0) = \dot{\mathcal{V}}_n(\hat{\rho}_n)(\hat{\rho}_n - \rho_0) + \frac{1}{2} \ddot{\mathcal{V}}_n(\tilde{\rho}_n)(\hat{\rho}_n - \rho_0)^2,$$

where $\tilde{\rho}_n$ is a point between ρ_0 and $\hat{\rho}_n$ and

$$\dot{\mathcal{V}}_n(\tilde{\rho}_n) = \frac{-1}{\sqrt{n}} \sum_{i=1}^n Y_{i-1} G(Y(i-1)).$$

Note that

$$R_n = \frac{1}{2} \ddot{\mathcal{V}}_n(\tilde{\rho}_n)(\hat{\rho}_n - \rho_0)^2 = \frac{1}{2\sqrt{n}} \frac{1}{\sqrt{n}} \ddot{\mathcal{V}}_n(\tilde{\rho}_n) \left(\sqrt{n}(\hat{\rho}_n - \rho_0) \right)^2.$$

Since the estimator $\hat{\rho}_n$ is \sqrt{n} -consistent and with the use of Lemma (5.2), it results that

$$\left(\sqrt{n}(\hat{\rho}_n - \rho_0) \right)^2 = O_P(1),$$

from the assumption (C.3), it follows that

$$R_n = o_P(1),$$

finally we deduce that,

$$(5.12) \quad \mathcal{V}_n(\hat{\rho}_n) - \mathcal{V}_n(\rho_0) = \dot{\mathcal{V}}_n(\hat{\rho}_n)(\hat{\rho}_n - \rho_0) + o_P(1).$$

This implies that

$$\frac{\dot{\mathcal{V}}_n(\hat{\rho}_n)}{\sqrt{n}} - \frac{\dot{\mathcal{V}}_n(\rho_0)}{\sqrt{n}} = \frac{\ddot{\mathcal{V}}_n(\check{\rho}_n)}{\sqrt{n}}(\hat{\rho}_n - \rho_0) + o_P(1) = \frac{1}{\sqrt{n}} \frac{\ddot{\mathcal{V}}_n(\check{\rho}_n)}{\sqrt{n}} \sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1), \quad (5.13)$$

where $\check{\rho}_n$ is between $\hat{\rho}_n$ and ρ_0 , and $\ddot{\mathcal{V}}_n$ is the second derivative of \mathcal{V}_n . From the assumption (C.3), we have

$$\frac{1}{\sqrt{n}} \frac{\ddot{\mathcal{V}}_n(\check{\rho}_n)}{\sqrt{n}} = o_P(1),$$

since the estimator $\hat{\rho}_n$ is \sqrt{n} -consistent, it result that

$$\frac{\dot{\mathcal{V}}_n(\hat{\rho}_n)}{\sqrt{n}} - \frac{\dot{\mathcal{V}}_n(\rho_0)}{\sqrt{n}} = o_P(1),$$

this implies that

$$(5.14) \quad \frac{\dot{\mathcal{V}}_n(\hat{\rho}_n)}{\sqrt{n}} = \frac{\dot{\mathcal{V}}_n(\rho_0)}{\sqrt{n}} + o_P(1).$$

With the use of (5.14), the equality (5.12) can also rewritten

$$(5.15) \quad \begin{aligned} \mathcal{V}_n(\hat{\rho}_n) - \mathcal{V}_n(\rho_0) &= \frac{\dot{\mathcal{V}}_n(\hat{\rho}_n)}{\sqrt{n}} \sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1), \\ &= \frac{\dot{\mathcal{V}}_n(\rho_0)}{\sqrt{n}} \sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1). \end{aligned}$$

It follows from the assumption (C.1) combined with the ergodicity and the stationarity of the model that, the random variable $\frac{1}{\sqrt{n}} \dot{\mathcal{V}}_n(\rho_0)$ converges in probability to the constant c_1 , as $n \rightarrow +\infty$, where

$$c_1 = -\mathbb{E}\left[Y_0 G(Y(0))\right],$$

therefore from the Lemma (5.3), there exists a random variable X_n , $X_n \xrightarrow{P} 0$ such that

$$\frac{1}{\sqrt{n}} \dot{\mathcal{V}}_n(\rho_0) = c_1 + X_n.$$

We deduce from the equality (5.15) and the \sqrt{n} -consistence of the estimator $\hat{\rho}_n$, that

$$(5.16) \quad \mathcal{V}_n(\hat{\rho}_n) - \mathcal{V}_n(\rho_0) = c_1 \sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1) = -D_n + o_P(1),$$

where $D_n = -c_1 \sqrt{n}(\hat{\rho}_n - \rho_0)$. Recall that the second derivative $\ddot{\mathcal{V}}_n$ is equal to 0, this implies that the assumption (C.3) is satisfied.

Proof of Proposition 3.4. The assumption (C.1) remains satisfied and the proof is similar as the proof of Proposition (3.3), in this case, for all $\rho \in \Theta_1$, we have

$$\ddot{\mathcal{V}}_n(\rho) = \frac{-1}{\sqrt{n}} \sum_{i=1}^n Y_{i-1}^2 B(Y(i-1)) 2\dot{M}_f(\rho).$$

By a simple calculus and since the the function f is the density of the standard normal distribution, it is easy to prove that the quantity $2\dot{M}_f(\rho)$ is bounded, therefore, there exists a positive constant w such that $2\dot{M}_f(\rho) \leq w$, then

$$\left| \frac{1}{\sqrt{n}} \ddot{\mathcal{V}}_n(\rho) \right| \leq w \frac{1}{n} \sum_{i=1}^n Y_{i-1}^2 |B(Y(i-1))|.$$

With the choice $B(Y(i-1)) = \frac{2a}{1+Y_{i-1}^2}$ with $a \neq 0$, it results that

$$\left| \frac{1}{\sqrt{n}} \ddot{\mathcal{V}}_n(\rho) \right| \leq 2w|a| \frac{1}{n} \sum_{i=1}^n Y_{i-1}^2.$$

By the use of the ergodicity of the model and since the model is with finite second moments, it follows that the random variable $\frac{1}{n} \sum_{i=1}^n Y_{i-1}^2 \xrightarrow{a.s.} k$, where k is some constant, this implies that the condition (C.3) is straightforward .

Proof of the Theorem 3.1. From the conditions (2.4) ((2.7), respectively), (C.1) ((C.2), respectively), it results the existence and the \sqrt{n} -consistency of the modified estimate estimate $\bar{\phi}_n$ corresponding to the equation (2.5) ((2.8), respectively). The combinaison of the condition (E₁) and the Proposition (3.2) enable us to get under H_0 the following equality

$$\mathcal{V}_n(\bar{\phi}_n) = \mathcal{V}_n(\phi_0) + o_P(1).$$

This last equation implies that with $o_P(1)$, the estimate central and central sequences are equivalent, in the expression of the test (3.3), the replacing of the central sequence by the estimate central sequence has no effect. LAN implies the contiguity of the two hypothesis (see, [5, Corrolary 4.3]), by Le Cam third lemma's (see for instance, [8, Theorem 2]), under $H_1^{(n)}$, we have

$$\mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(\tau^2, \tau^2).$$

It follows from the convergence in probability of the estimate $\bar{\phi}_n$ to ϕ_0 , the continuity of the function $\tau : \cdot \rightarrow \tau(\cdot)$ and the application of the continuous

mapping theorem see, for instance ([13]) or [3], that asymptotically, the power of the test is not effected when we replace the unspecified parameter ϕ_0 by its estimate, $\bar{\phi}_n$, hence the optimality of the test. The power function of the test is asymptotically equal to $1 - \Phi(Z(\alpha) - \tau^2(\bar{\phi}_n))$, the proof is similar as [9, Theorem 3].

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