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ON THE SPECTRUM OF STOCHASTIC PERTURBATIONS OF THE SHIFT AND JULIA SETS

E. H. EL ABDALAOUI AND A. MESSAOUDI

Abstract. We extend the Killeen-Taylor study in [2] by investigating in different Banach spaces ($\ell^\alpha(\mathbb{N}), c_0(\mathbb{N}), c_c(\mathbb{N})$) the point, continuous and residual spectra of stochastic perturbations of the shift operator associated to the stochastic adding machine in base 2 and in Fibonacci base. For the base 2, the spectra are connected to the Julia set of a quadratic map. In the Fibonacci case, the spectra involve the Julia set of an endomorphism of $\mathbb{C}^2$.

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1. Introduction

In this paper, we study in detail the spectrum of some stochastic perturbations of the shift operator introduced by Killeen and Taylor in [2]. We focus our study on large Banach spaces for which we complete the Killeen-Taylor study. We investigate also the case of Fibonacci base, but in this case, we are not able to compute the residual and continuous spectra exactly.
We recall that in [2], Killeen and Taylor defined the stochastic adding machine as a stochastic perturbation of the shift in the following way: let N be a nonnegative integer number written in base 2 as $N=\sum_{i=0}^{k(N)}\varepsilon_i(N)2^i$ where $\varepsilon_i(N)=0$ or 1 for all i. It is known that there exists an algorithm that computes the digits of $N+1$. This algorithm can be described by introducing an auxiliary binary "carry" variable $c_i(N)$ for each digit $\varepsilon_i(N)$ by the following manner:

Put $c_{-1}(N+1)=1$ and

$$\varepsilon_i(N+1)=\varepsilon_i(N)+c_{i-1}(N+1) \mod (2)$$
$$c_i(N+1)=\left[\frac{\varepsilon_i(N)+c_{i-1}(N+1)}{2}\right]$$

where $i\geq 0$ and $[z]$ denote the integer part of $z\in\mathbb{R}_+$.

Let $\{e_i(n): i\geq 0, n\in\mathbb{N}\}$ be an independent, identically distributed family of random variables which take the value 0 with probability $1-p$ and the value 1 with probability p. Let N be an integer. Given a sequence $(r_i(N))_{i\geq 0}$ of 0 and 1 such that $r_i(N)=1$ for finitely many indices i, we consider the sequences $(r_i(N+1))_{i\geq 0}$ and $(c'_i(N+1))_{i\geq -1}$ defined by $c'_{-1}(N+1)=1$ and for all $i\geq 0$

$$r_i(N+1)=r_i(N)+e_i(N)c'_{i-1}(N+1) \mod (2)$$
$$c'_i(N+1)=\left[\frac{r_i(N)+e_i(N)c'_{i-1}(N+1)}{2}\right],$$

With this we have that a number $\sum_{i=0}^{+\infty}r_i(N)2^i$ transitions to a number $\sum_{i=0}^{+\infty}r_i(N+1)2^i$. In particular, an integer N having a binary representation of the form $\varepsilon_n\ldots\varepsilon_{k+1}0\underbrace{11\ldots11}_{k}$ transitions to $\varepsilon_n\ldots\varepsilon_{k+1}1\underbrace{00\ldots00}_{k}$ with probability p^{k+1} and a number having binary representation of the form $\varepsilon_n\ldots\varepsilon_k\underbrace{11\ldots11}_{k}$ transitions to $\varepsilon_n\ldots\varepsilon_k\underbrace{00\ldots00}_{k}$ with probability $p^k(1-p)$. Equivalently, we obtain a Markov process $\psi(N)$ with state space $\mathbb{N}$ by $\psi(N)=\sum_{i=0}^{+\infty}r_i(N)2^i$. The corresponding transition operator is denoted by S_p and given in Figure 2.

For $p=1$ the transition operator equals the shift operator (cf. Figure 2), hence the stochastic adding machine can be seen as a stochastic perturbation of the shift operator. It is also a model of Weber law in the context of counter and pacemarker errors. This law is used in biology and psychophysiology [3].

In [KT], P.R. Killeen and J. Taylor studied the spectrum of the transition operator S_p (of $\psi(N)$) on ℓ^∞. They proved that the spectrum $\sigma(S_p)$ is equal to the filled Julia set of the quadratic map $f:\mathbb{C}\mapsto\mathbb{C}$ defined by: $f(z)=(z-(1-p))^2/p^2$, i.e: $\sigma(S_p)=\{z\in\mathbb{C},\ (f^n(z))_{n\geq 0}$ is bounded $\}$ where f^n is the n-th iteration of f.

In [5], Messaoudi and Smania defined the stochastic adding machine in the Fibonacci base. The corresponding transition operator is given in Figure 6. Their procedure can be extended to a large class of adding machine and is given by the following manner. Consider the Fibonacci sequence $(F_n)_{n\geq 0}$ given by the relation

$$F_0 = 1, F_1 = 2, \ F_n = F_{n-1} + F_{n-2} \ \ \forall n \geq 2.$$

Using the greedy algorithm, we can write every nonnegative integer N in a unique way as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N) F_i$ where $\varepsilon_i(N) = 0$ or 1 and $\varepsilon_i(N)\varepsilon_{i+1}(N) \neq 11$, for all $i \in \{0, \cdots, k(N) - 1\}$ (see [10]). It is known that the addition of 1 in the Fibonacci base (adding machine) is recognized by a finite state automaton (transductor). In [5], the authors defined the stochastic adding machine by introducing a " probabilistic transductor ". They also computed the point spectrum of the transition operator acting in ℓ^∞ associated to the stochastic adding machine with respect to the base $(F_n)_{n\geq 0}$. In particular, they showed that the point spectrum $\sigma_{pt}(S_p)$ in ℓ^∞ is connected to the filled Julia set $J(g)$ of the function $g : \mathbb{C}^2 \mapsto \mathbb{C}^2$ defined by:

$$g(x, y) = (\frac{1}{p^2}(x - 1 + p)(y - 1 + p), x).$$

Precisely, they proved that

$$\sigma_{pt}(S_p) = \mathcal{K}_p = \{\lambda \in \mathbb{C} \ (q_n(\lambda))_{n\geq 1} \text{ is bounded } \},$$

where $q_{F_0}(z) = z, \ q_{F_1}(z) = z^2, \ q_{F_k}(z) = \frac{1}{p} q_{F_{k-1}}(z) q_{F_{k-2}}(z) - \frac{1-p}{p}$, for all $k \geq 2$ and for all nonnegative integers n, we have $q_n(z) = q_{F_{k_1}} \dots q_{F_{k_m}}$ where $F_{k_1} + \cdots + F_{k_m}$ is the Fibonacci representation of n.

In particular, $\sigma_{pt}(S_p)$ is contained in the set

$$\begin{aligned} \mathcal{E}_p &= \{\lambda \in \mathbb{C} \mid (q_{F_n}(\lambda))_{n\geq 1} \text{ is bounded } \} \\ &= \{\lambda \in \mathbb{C} \mid (\lambda_1, \lambda) \in J(g)\} \end{aligned}$$

where $\lambda_1 = 1 - p + \frac{(1-\lambda-p)^2}{p}$.

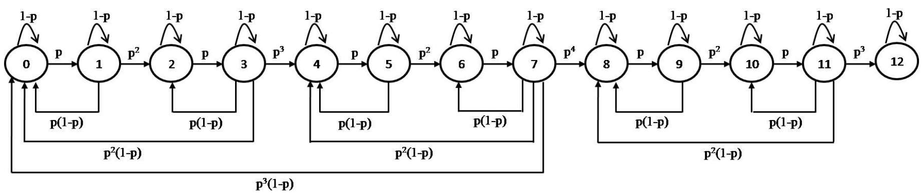

Fig.1. Transition graph of stochastic adding machine in base 2

Here we investigate the spectrum of the stochastic adding machines in base 2 and in the Fibonacci base in different Banach spaces. In particular, we compute exactly the point, continuous and residual spectra of the stochastic adding machine in base 2 for the Banach spaces c_0, c, ℓ^α, $\alpha \geq 1$.

For the Fibonacci base, we improve the result in [5] by proving that the spectrum of S_p acting on ℓ^∞ contain $\mathcal{E}_p$. The same result will be proven for the Banach spaces c_0, c and $\ell^\alpha, \alpha \geq 1$.

The paper is organized as follows. In section 2, we give some basic facts on spectral theory. In section 3, we state our main results (Theorems 1, 2 and 3). Section 4 contains the proof in the case of the base 2 and finally, in section 5, we present the proof in the case of the Fibonacci base.

2. Basic facts from the spectral theory of operators (see for instance [1], [7],[8], [9])

Let E be a complex Banach space and T a bounded operator on it. The spectrum of T, denoted by $\sigma(T)$, is the subset of complex numbers λ for which $T - \lambda Id_E$ is not an isomorphism (Id_E is the identity maps).

As usual we point out that if λ is in $\sigma(T)$ then one of the following assertions hold:

(1) $T - \lambda Id_E$ is not injective. In this case we say that λ is in the point spectrum denoted by $\sigma_{pt}(T)$.
(2) $T - \lambda Id_E$ is injective, not onto and has dense range. We say that λ is in the continuous spectrum denoted by $\sigma_c(T)$.
(3) $T - \lambda Id_E$ is injective and does not have dense range. We say that λ is in residual spectrum of T denoted by $\sigma_r(T)$.

It follows that $\sigma(T)$ is the disjoint union

$$\sigma(T) = \sigma_{pt}(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

It is well known and it is an easy consequence of Liouville Theorem that the spectrum of any bounded operator is a non empty compact set of $\mathbb{C}$. There is a connection between the spectrum of T and the spectrum of the dual operator T' acting on the dual space E' by $T' \quad : \quad \phi \mapsto \phi \circ T$. In particular, we have

Proposition 2.1 (Phillips Theorem)**.** *Let E be a Banach space and T a bounded operator on it, then $\sigma(T) = \sigma(T')$.*

We also have a classical relation between the point and residual spectra of T and the point spectrum of T'.

Proposition 2.2. *For a bounded operator T we have*

$$\sigma_r(T) \subset \sigma_{pt}(T') \subset \sigma_r(T) \cup \sigma_{pt}(T).$$

In particular, if $\sigma_{pt}(T)$ is an empty set then

$$\sigma_r(T) = \sigma_{pt}(T').$$

3. Main results.

Our main results are stated in the following three theorems.

Theorem 1. The spectrum of the operator S_p acting on c_0, c and ℓ^α, $\alpha \geq 1$ is equal to the filled Julia set $J(f)$ of the quadratic map $f(z) = (z-(1-p))^2/p^2$. Precisely, in c_0 (resp. ℓ^α, $\alpha > 1$), the continuous spectrum of S_p is equal to $J(f)$ and the point and residual spectra are empty. In c, the point spectrum is equal $\{1\}$, the residual spectrum is empty and the continuous spectrum equals $J(f)\backslash\{1\}$.

Theorem 2. In ℓ^1, the point spectrum of S_p is empty. The residual spectrum of S_p is not empty and contains a dense and countable subset of the Julia set $\partial(J_f)$, i.e. $\bigcup_{n=0}^{+\infty} f^{-n}\{1\} \subset \sigma_r(S_p)$. The continuous spectrum is equal to the relative complement of the residual spectrum with respect to the filled Julia set J_f.

Theorem 3. The spectra of S_p acting respectively in $\ell^\infty,\ c_0,\ c$ and $\ell^\alpha,\ \alpha \geq 1$, associated to the stochastic Fibonacci adding machines contain the set $\mathcal{E}_p = \{\lambda \in \mathbb{C} \mid (\lambda_1, \lambda) \in J(g)\}$ where $J(g)$ is the filled Julia set of the function g and $\lambda_1 = 1 - p + \frac{(1-\lambda-p)^2}{p}$.

Conjecture: We conjecture that in the case of ℓ^1, the residual spectrum of the transition operator associated to the stochastic adding machine in base 2 is $\sigma_r(S_p) = \bigcup_{n=0}^{+\infty} f^{-n}\{1\}$. For Fibonacci stochastic adding machine, we conjecture that the spectra of S_p in the Banach spaces cited in Theorem 3 are equals to the set $\mathcal{E}_p$.

Remark 3.1. *The methods used for the proof of our results can be adapted for a large class of stochastic adding machine given by transductors.*

Remark 3.2. *We point out that from Killeen and Taylor method one may deduce in the case of ℓ^∞ that the residual and continuous spectrum is empty. On the contrary here we compute directly the residual and continuous spectrum in ℓ^α, c_0 and c.*

$$S_p = \begin{pmatrix}
1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
p(1-p) & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
p^2(1-p) & 0 & p(1-p) & 1-p & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
0 & 0 & 0 & 0 & p(1-p) & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
p^3(1-p) & 0 & 0 & 0 & p^2(1-p) & 0 & p(1-p) & 1-p & p^4 & 0 & 0 & 0 & 0 & 0\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p(1-p) & 1-p & p^2 & 0 & 0 & 0\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2(1-p) & 0 & p(1-p) & 1-p & p^2 & 0\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p(1-p) & 1-p\ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & &
\end{pmatrix}$$

Fig.2. Transition operator of stochastic adding machine in base 2

4. **Proof of our main results for stochastic adding machine in base 2.**

We are interested in the spectrum of S_p on three Banach spaces connected by duality. The space c_0 is the space of complex sequences which converge to zero, in other words, the continuous functions on $\mathbb{N}$ vanishing at infinity. The dual space of c_0 is by Riesz Theorem the space of bounded Borel measures on $\mathbb{N}$ with total variation norm. This space can be identified with ℓ^1, the space of summable row vectors. Finally, the dual space of ℓ^1 is ℓ^∞ the space of bounded complex sequences.

We are also interested in the spectrum of S_p as operator on the space ℓ^α with $\alpha > 1$ and also in the space c of complex convergent sequences.

Proposition 4.1. *The operator S_p (acting on the right) is well defined on the space X where $X \in \{c_0,\ c,\ l^\alpha,\ \alpha \geq 1\}$, moreover $||S_p|| \leq 1$.*

Since the operator S_p is bi-stochastic, the proof of this proposition is a straightforward consequence of the following more general lemma.

Lemma 4.2. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ be an infinite matrix with nonnegative coefficients. Assume that there exists a positive constant M such that*

(1) $\displaystyle\sup_{i\in\mathbb{N}}\left(\sum_{j=0}^{\infty} a_{i,j}\right) \leq M,$

(2) $\displaystyle\sup_{j\in\mathbb{N}}\left(\sum_{i=0}^{\infty} a_{i,j}\right) \leq M.$

Then A defines a bounded operator on the spaces $c_0,\ c,\ l^\infty$ and $\ell^\alpha(\mathbb{N})$ with $\alpha \geq 1$. In addition the norm of A is less than M.

Proof. By the assumption (1) it is easy to get that A is well defined on $\ell^\infty(\mathbb{N})$ and its norms is less than M.
Now, let $v = (v_n)_{n\geq 0},\ v \neq 0$ such that $\displaystyle\lim_{n\longrightarrow+\infty} v_n = l \in \mathbb{C}$, then for any $\varepsilon > 0$ there exists a positive integer j_0 such that for any $j \geq j_0$, we have $|v_j - l| \leq \dfrac{\varepsilon}{2M}$.
Let $d = \displaystyle\sum_{j=0}^{+\infty} a_{n,j}$, then from the assumption (1), we have that for any $n \in \mathbb{N}$,

$$|(Av)_n - d\cdot l| = \left|\sum_{j=0}^{+\infty} a_{n,j}(v_j - l)\right| \leq \sum_{j=0}^{j_0-1} a_{n,j}|v_j - l| + \frac{\varepsilon}{2}. \tag{1}$$

But by the assumption (2), for any $j \in \{0,\ldots,j_0-1\}$, we have $\displaystyle\sum_{n=0}^{+\infty} a_{n,j} < \infty$. Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and for any $j \in \{0,\cdots,j_0 - 1\}$, we have

$$|a_{n,j}| \leq \frac{\varepsilon}{2j_0(\delta+1)} \text{ where } \delta = sup\{|v_j - l|,\ j \in \mathbb{N}\} \tag{2}$$

Combined (1) with (2) we get that

$$|(Av)_n - d\cdot l| \leq \varepsilon,\ \forall n \geq n_0.$$

Hence $AX \subset X$ if $X = c_0$ or c.

Now take $\alpha > 1$ and $v \in l^\alpha$. For any integer integer $i \in \mathbb{N}$, we have

$$|(Av)_i|^\alpha \leq \left(\sum_{j=0}^{+\infty} a_{i,j}|v_j|\right)^\alpha.$$

Let α' be a conjugate of α, i.e, $\dfrac{1}{\alpha} + \dfrac{1}{\alpha'} = 1$. Then, by Hölder inequality we get

$$\left(\sum_{j=0}^{+\infty} a_{i,j}|v_j|\right)^\alpha \leq \left(\sum_{j=0}^{+\infty} a_{i,j}\right)^{\frac{\alpha}{\alpha'}} \left(\sum_{j=0}^{+\infty} a_{i,j}|v_j|^\alpha\right).$$

Hence

$$\left(\sum_{j=0}^{+\infty} a_{i,j}|v_j|\right)^\alpha \leq \left(\sup_{l\in\mathbb{N}} \sum_{j=0}^{+\infty} a_{l,j}\right)^{\frac{\alpha}{\alpha'}} \left(\sum_{j=0}^{+\infty} a_{i,j}|v_j|^\alpha\right). \tag{3}$$

Thus

$$\begin{aligned}
||Av||_\alpha^\alpha &\leq M^{\frac{\alpha}{\alpha'}} \sum_{i=0}^{\infty} \left(\sum_{j=0}^{+\infty} a_{i,j}|v_j|^\alpha\right) \\
&= M^{\frac{\alpha}{\alpha'}} \sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{i,j}\right) |v_j|^\alpha \\
&\leq M^{\frac{\alpha}{\alpha'}} \sup_{j\in\mathbb{N}} \left(\sum_{i=0}^{\infty} a_{i,j}\right) ||v||_\alpha^\alpha \\
&\leq M^{1+\frac{\alpha}{\alpha'}} ||v||_\alpha^\alpha.
\end{aligned}$$

Then

$$||Av||_\alpha \leq M||v||_\alpha.$$

Hence A is a continuous operator and $||A|| \leq M$.

The case $\alpha = 1$ is an easy exercise and it is left to the reader.

□

From Proposition 4.1, we deduce that S_p is a Markov operator and its spectrum is contained in the unit disc of complex numbers.

Consider the map $f : z \in \mathbb{C} \longmapsto \left(\frac{z-(1-p)}{p}\right)^2$ and denote by $J(f)$ the associated filled Julia set defined by:

$$J(f) = \left\{z \in \mathbb{C},\ |f^{(n)}(z)| \not\longrightarrow \infty\right\}.$$

Killeen and Taylor investigated the spectrum of S_p acting on ℓ^∞. They proved that the point spectrum of S_p is equal to the filled Julia set of f. In addition, they showed that the spectrum is invariant under the action of f. As a consequence, one may deduce that the continuous and residual spectra in this case are empty.

Here we will compute exactly the residual part and the continuous part of the spectrum of S_p acting on the spaces $c_0,\ c$ and $\ell^\alpha,\ \alpha \geq 1$.

Theorem 4.3. *The spectrum of the operator S_p acting on X where $X \in \{c_0,\ c,\ l^\alpha,\ \alpha \geq 1\}$ is equal to the filled Julia set of $f,\ J(f)$. Precisely, in c_0 (resp. $\ell^\alpha,\ \alpha > 1$), the continuous spectrum of S_p is equal to $J(f)$ and the point and residual spectra are empty. In c, the point spectrum is the singleton $\{1\}$, the residual spectrum is empty and the continuous spectrum is $J(f)\backslash\{1\}$.*

For the proof of Theorem 4.3 we shall need the following proposition.

Proposition 4.4. *The spectrum of S_p in X, where $X \in \{c_0,\ c,\ l^\alpha,\ 1 \leq \alpha \leq +\infty\}$, is contained in the filled Julia set of f.*

The main idea of the proof of Proposition 4.4 can be found in the Killen-Taylor proof. The key argument is that the $\widetilde{S_p}^2$ is similar to the operator $ES_p \oplus OS_p$, where $\widetilde{S_p} = \dfrac{S_p - (1-p)Id}{p}$ and E, O denote the even and odd operators acting on X by

$$E(h_0, h_1, \ldots) = (h_0, 0, h_1, 0, h_2, \ldots),$$

and

$$O(h_0, h_1, \ldots) = (0, h_0, 0, h_1, 0, h_2, \ldots),$$

for any $h = (h_0, h_1, \ldots)$ in X. Precisely, for all $v = (v_i)_{i\geq 0} \in X$, we have

$$\widetilde{S_p}^2(v) = ES_p(v_0, v_2, \ldots v_{2n}, \ldots) + OS_p(v_1, v_3, \ldots v_{2n+1}, \ldots).$$

As a consequence we deduce from the mapping spectral theorem [8] that the spectrum of S_p is invariant under f.

Let us start the proof of Theorem 4.3 by proving the following result.

Proposition 4.5. *The point spectrum of S_p acting on X where $X \in \{c_0,\ l^\alpha,\ \alpha \geq 1\}$ is empty, and the point spectrum of S_p on c is equal to $\{1\}$.*

For the proof, we need the following lemma from [2] .

Lemma 4.6. [2]. *Let n be a nonnegative integer and $X_n = \{m \in \mathbb{N} :\ (S_p)_{n,m} \neq 0\}$, then the following properties are valid.*

(1) *For all nonnegative integers n, we have $n \in X_n$ and $(S_p)_{n,n} = 1 - p$.*
(2) *If $n = \varepsilon_k \ldots \varepsilon_1 0,\ k \geq 2$, is an even integer then $X_n = \{n, n+1\}$ and $(S_p)_{n,n+1} = p$.*
(3) *If $n = \varepsilon_k \ldots \varepsilon_t 0 \underbrace{1\ldots 1}_{s}$ is an odd integer with $s \geq 1$ and $k \geq t \geq s+1$, then $X_n = \{n, n+1, n-2^m+1,\ 1 \leq m \leq s\}$ and n transitions to $n+1 = \varepsilon_k \ldots \varepsilon_t 1 \underbrace{0\ldots 00}_{s}$ with probability $(S_p)_{n,n+1} = p^{s+1}$, and n transitions to $n - 2^m + 1 = \varepsilon_k \ldots \varepsilon_t 0 \underbrace{1\ldots 1}_{s-m}\underbrace{0\ldots 0}_{m},\ 1 \leq m \leq s$ with probability $(S_p)_{n,n-2^m+1} = p^m(1-p)$.*

Proof of Proposition 4.5. Let λ be an eigenvalue of S_p associated to the eigenvector $v = (v_n)_{n \geq 0}$ in X where where $X \in \{c_0, c, l^\alpha, \ \alpha \geq 1\}$ Let λ be an eigenvalue of S_p associated to the eigenvector $v = (v_i)_{i \geq 0}$ in X. By Lemma 4.6, we see that the operator S_p satisfies $(S_p)_{i,i+k} = 0$ for all $i, k \in \mathbb{N}$ with $k \geq 2$. Therefore, for all integers $k \geq 1$, we have

$$\sum_{i=0}^{k} (S_p)_{k-1,i} v_i = \lambda v_{k-1}. \tag{4}$$

Then, one can prove by induction on k that for all integers $k \geq 1$, there exists a complex number $q_k = q_k(p, \lambda)$ such that

$$v_k = q_k v_0 \tag{5}$$

By Lemma 4.6 and the fact that $(S_p - \lambda I)v)_{2^n} = 0$ for all nonnegative integers n, we get

$$p^{n+1} v_{2^n} + (1 - p - \lambda) v_{2^n - 1} + \sum_{i=1}^{n} p^i (1 - p) v_{2^n - 2^i} = 0, \ \forall n \geq 0. \tag{6}$$

Hence

$$v_{2^n} = \frac{1}{p} A - (\frac{1}{p} - 1) v_0,$$

where $A = -\frac{1}{p^n}((1 - p - \lambda) v_{2^{n-1} + (2^{n-1} - 1)} + \sum_{i=1}^{n-1} p^i (1 - p) v_{2^{n-1} + (2^{n-1} - 2^i)}$.

On the other hand, by the self similarity structure of the transition matrix S_p, one can prove that if i and j are two integers such that for some positive integer n we have $2^{n-1} \leq i, j < 2^n$, then the transition probability from i to j is equal to the transition probability from $i - 2^{n-1}$ to $j - 2^{n-1}$. Using this last fact and (6), it follows that

$$v_{2^n} = \frac{1}{p} q_{2^{n-1}} v_{2^{n-1}} - \left(\frac{1}{p} - 1 \right) v_0.$$

This gives

$$q_{2^n} = \frac{1}{p} q_{2^{n-1}}^2 - \left(\frac{1}{p} - 1 \right), \tag{7}$$

where

$$q_{2^0} = q_1 = -\frac{1 - p - \lambda}{p}.$$

Case 1: $v \in c_0$ **or** ℓ^α, $\alpha \geq 1$**.**

We have $\lim_{n \to \infty} q_{2^n} = 0$. Thus by (7), we get $p = 1$, which is absurd, then the point spectrum is empty.

Case 2: $v \in c$**.** Assume that $\lim q_n = l \in \mathbb{C}$, then by (7), we deduce that $l = 1$ or $l = p - 1$. On the other hand, for any $n \in \mathbb{N}$, there exist k nonnegative integers $n_1 < n_2 \ldots < n_k$ such that $n = 2^{n_1} + \cdots 2^{n_k}$. We can prove (see [2]) that

$$q_n = q_{2^{n_1}} \ldots q_{2^{n_k}}. \tag{8}$$

Then $\lim q_{2^{n-2}+2^n} = l^2 = l$, thus $l = p-1$ is excluded. Since S_p is stochastic, we conclude that $l = 1$ and $\sigma_{pt,c}(S_p) = \{1\}$.

□

Remark 4.7. *By the same arguments as above, Killeen and Taylor in* [2] *proved that the point spectrum of S_p acting on ℓ^∞ is equal to the filled Julia set of the quadratic map f. In fact, it is easy to see from the arguments above that $\sigma_{pt,l^\infty}(S_p) = \{\lambda \in \mathbb{C},\ q_n(\lambda)$ bounded $\}$. Indeed, (7) implies that if $(q_{2^n})_{n\geq 0}$ is bounded, then for all $n \geq 0$, $|q_{2^n}| \leq 1$. This clearly forces $\sigma_{pt,l^\infty}(S_p) = \{\lambda \in \mathbb{C},\ q_{2^n}(\lambda)$ bounded $\}$ by (8). Now, since*

$$q_{2^n} = h \circ f^{n-1} \circ h^{-1}(q_1) = h \circ f^{n-1}(\lambda), \forall n \in \mathbb{N},$$

where $h(x) = \dfrac{x}{p} - \dfrac{1-p}{p}$, we conclude that $\sigma_{pt,l^\infty}(S_p) = J(f)$. It follows from Proposition 4.4 that $\sigma_{l^\infty}(S_p) = J(f)$ and the residual and continuous spectra are empty.

Proposition 4.8. *The residual spectrum of S_p acting on $X \in \{c_0, c, \ell^\alpha,\ \alpha > 1\}$ is empty.*

Proof. Let λ be an element of the residual spectrum of S_p acting on c_0 (resp. c). Then, by Proposition 2.2, we deduce that there exists a sequence $u = (u_k)_{k\geq 0} \in l^1(\mathbb{N})$ such that $u(S_p - \lambda Id) = 0$.

Claim. $u_k = \dfrac{1}{q_k} u_0,\ \forall k \in \mathbb{N}$.

We have

$$\forall k \in 2\mathbb{N},\ (u(S_p - \lambda Id))_{k+1} = p u_k + (1 - p - \lambda) u_{k+1} = 0.$$

Hence

$$\forall k \in 2\mathbb{N},\ u_k = q_1 u_{k+1}. \tag{9}$$

If k is odd, then $k = 2^n - 1 + t$ where $t = 0$ or $t = \sum_{j=2}^{s} 2^{n_j}$ where $1 \leq n < n_2 < n_3, \ldots < n_s$. Since $(u(S_p - \lambda Id))_{k+1} = 0$, then we have

$$p^{n+1} u_k + (1 - p - \lambda) u_{k+1} + \sum_{i=1}^{n} p^i (1-p) u_{k+2^i} = 0. \tag{10}$$

Observe that the relation (10) between u_k and u_{k+2^n} is similar to the relation (6) between v_{2^n} and v_0. Hence, by induction on n, we obtain

$$u_k = q_{2^n} u_{k+2^n}. \tag{11}$$

Indeed, if $n = 1$ then by (10) and (9), we get

$$p^2 u_k + (q_1(1 - p - \lambda) + p(1-p)) u_{k+2} = 0.$$

Therefore

$$u_k = \left(\frac{q_1^2}{p} - \frac{1-p}{p} \right) u_{k+2} = q_2 u_{k+2}.$$

Then (11) is proved for $n = 1$.

Now, assume that (11) holds for the numbers $1, 2, \cdots, m-1$.

Take $n = m$ and $1 \le i < m$, then $k+2^i = 1+2+\cdots+2^{m-1}+2^i+t = 2^i-1+t'$ where $t' = 2^m + t$. Applying the induction hypothesis, we get

$$u_{k+2^i} = q_{2^i} u_{k+2^{i+1}} = q_{2^i} q_{2^{i+1}} \dots q_{2^{m-1}} u_{k+2^m}.$$

On the other hand, since $2^i + \cdots + 2^{m-1} = 2^m - 2^i$, we have

$$u_{k+2^i} = q_{2^m-2^i} u_{k+2^m}. \tag{12}$$

Considering (10) with $n = m$ and (12) yields

$$u_k = -\frac{1}{p^{m+1}} \left((1-p-\lambda)\, q_{2^m-1} + \sum_{i=1}^{m} p^i (1-p) q_{2^m-2^i} u_{k+2^m} \right).$$

Combined (5) and (6), we obtain (11) for $n = m$. Then (11) holds for all integers $n \ge 1$.

In particular, we have $u_{2^{n-1}-1} = q_{2^{n-1}} u_{2^n-1}$, for all integers $n \ge 1$. Thus

$$u_{2^n-1} = \frac{1}{q_{2^0} q_2 \dots q_{2^{n-1}}} u_0 = \frac{1}{q_{2^n-1}} u_0, \ \forall n \ge 1. \tag{13}$$

On the other hand, for all integers $n \ge 1$, by (9) we have $u_{2^n} = q_{2^0} u_{2^n+2^0}$. and from (11), we see that

$$u_{2^n} = q_{2^0} q_{2^1} u_{2^2-1+2^n} = \dots = q_{2^0} q_{2^1} \dots q_{2^{n-1}} u_{2^{n+1}-1}. \tag{14}$$

Consequently from (13) and (14), we obtain

$$u_{2^n} = \frac{1}{q_{2^n}} u_0, \ \forall n \ge 1. \tag{15}$$

Now fix an integer $k \in \mathbb{N}$ and assume that $k = \sum_{i=1}^{s} 2^{n_i}$ where $0 \le n_1 < n_2 < \dots < n_s$. We will prove by induction on s that the following statement holds

$$u_k = \frac{1}{q_{2^{n_1}} q_{2^{n_2}} \dots q_{2^{n_s}}} u_0 = \frac{1}{q_k} u_0. \tag{16}$$

Indeed, it follows from (15), that (16) is true for $s = 1$.

Now assume that(16) is true for all integers $1 \le i < s$.

Case 1. k is odd.

In this case $k = \sum_{i=1}^{s} 2^{n_i} = 2^n - 1 + l = \sum_{j=0}^{n-1} 2^j + l$ where $l = 0$ if $n = s+1$ and $l = \sum_{i=n}^{s} 2^{n_i}$ if $n \le s$.

If $n \ge 2$, we use (11) to get $u_{k-2^{n-1}} = q_{2^{n-1}} u_k$ and by induction hypothesis, we have

$$u_k = \frac{1}{q_{2^{n-1}} q_{k-2^{n-1}}} u_0 = \frac{1}{q_k} u_0.$$

If $n = 1$, we consider (9) to write $u_k = \frac{1}{q_1} u_{k-1}$. Thus, we deduce, by induction hypothesis, that

$$u_k = \frac{1}{q_1 q_{k-1}} u_0 = \frac{1}{q_k} u_0.$$

Case 2. k is even.

In this case $n_1 > 0$. and by (9), we deduce that

$$u_k = q_{2^0} u_{k+2^0} = q_{2^0} u_{k+2^1-1}.$$

Applying (11), it follows that

$$\begin{aligned} u_k = q_{2^0} q_{2^1} u_{k+2^2-1} = \dots &= q_{2^0} q_{2^1} \dots q_{2^{n_1-1}} u_{k+2^{n_1}-1} \\ &= q_{2^0} q_{2^1} \dots q_{2^{n_1-1}} u_{(k-2^{n_1})+2^{n_1+1}-1}. \end{aligned}$$

Hence

$$\begin{aligned} u_k &= q_{2^0} \dots q_{2^{n_1-1}} q_{2^{n_1+1}} u_{(k-2^{n_1})+2^{n_1+2}-1} \\ &= q_{2^0} \dots q_{2^{n_1-1}} q_{2^{n_1+1}} \dots q_{2^{n_2-1}} u_{(k-2^{n_1}-2^{n_2})+2^{n_2+1}-1}. \end{aligned}$$

Thus

$$u_k = \frac{\prod_{i=0}^{n_s} q_{2^i}}{\prod_{i=1}^{s} q_{2^{n_i}}} u_{2^{n_s+1}-1}. \tag{17}$$

By (17) and (13) we get $u_k = \dfrac{1}{\prod_{i=1}^{s} q_{2^{n_i}}} u_0 = \dfrac{1}{q_k} u_0.$

Therefore we have proved that for all nonnegative integers

$$u_k = \frac{1}{q_k} u_0. \tag{18}$$

We conclude that u is in $\ell^1(\mathbb{N})$ if and only if $\displaystyle\sum_{k=1}^{+\infty} \left| \frac{1}{q_k(\lambda)} \right| < \infty.$

But this gives that the residual spectrum of S_p acting on c_0 or c satisfy

$$\sigma_{r,C_0}(S_p) \subset \left\{ \lambda \in \overline{\mathbb{D}(0,1)} : \sum_{k=1}^{+\infty} \left| \frac{1}{q_k(\lambda)} \right| < \infty \right\}. \tag{19}$$

We claim that $\displaystyle\sum_{k=1}^{+\infty} \left| \frac{1}{q_k(\lambda)} \right| < \infty$ implies $|q_{2^n-1}| \geq 1$, for all integers $n \geq 1$. Indeed, by D'Alembert's Theorem, we have

$$\lim sup \frac{|q_n|}{|q_{n+1}|} \leq 1. \tag{20}$$

Now assume that n is even. Then $n = 2^{k_0} + \cdots + 2^{k_m}$ where $1 \leq k_0 < k_1 < \ldots < k_m$ (representation in base 2). In this case $n+1 = 2^0 + 2^{k_0} + \cdots + 2^{k_m}$. Using (8), we obtain $\dfrac{|q_n|}{|q_{n+1}|} = \dfrac{1}{|q_1|}$ and by (20), we get

$$|q_1| \geq 1. \tag{21}$$

Since for all integers $n \geq 0$, we have $q_{2^n} = \frac{1}{p} q_{2^{n-1}}^2 - \left(\frac{1}{p} - 1\right)$. It follows, from the triangle inequality, that $|q_{2^n}| \geq 1$ for all integers $n \geq 1$. Let i be a positive integer. Since $2^i - 1 = \sum_{j=0}^{i-1} 2^j$, we obtain by (8) that $q_{2^i-1} = q_{2^{i-1}} q_{2^{i-2}} \cdots q_1$. Hence

$$|q_{2^i-1}| \geq 1, \quad \text{for any integer } i \geq 1.$$

On the other hand, consider the first coordinate of the vector $\mu(S_p - \lambda Id) = 0$. Then we have

$$(1 - p - \lambda)\mu_0 + \sum_{i=1}^{+\infty} p^i (1-p)\mu_{2^i-1} = 0.$$

Dividing the two members of the last equality by p, we obtain

$$q_1 = \sum_{i=1}^{+\infty} p^{i-1}(1-p)/q_{2^i-1}. \tag{22}$$

We claim that there exists an integer $i_0 \in \mathbb{N}$ such that $|q_{2^{i_0}-1}| > 1$. Indeed, if not the series $\sum_{i \in \mathbb{N}} \frac{1}{|q_{2^i-1}|}$ will diverge. Thus $|q_1| < \sum_{i \neq i_0}^{+\infty} p^i(1-p) + p^{i_0-1}(1-p) < 1$. Absurd. We conclude that the residual spectrum of S_p acting on c_0 (resp. c) is empty.

The same proof yields that the residual spectrum of S_p acting on ℓ^α, $\alpha > 1$, is empty and the proof of the proposition is complete. □

Remark 4.9. *By (19), it follows that λ belongs to $\sigma_{r,X}$ where $X = c_0$ or c or ℓ^α, $\alpha > 1$, implies $\lim |q_n(\lambda)| = +\infty$. But this contradicts Proposition 4.4, which forces $\sigma_{r,C_0}(S_p) = \sigma_{r,l^\alpha}(S_p) = \emptyset$.*

Proposition 4.10. *The following equalities are satisfied:*

$$\sigma_{c,c}(S_p) = J(f)\backslash\{1\},\ \sigma_{c,c_0}(S_p) = \sigma_{c,l^\alpha}(S_p) = J(f) \text{ for all } \alpha > 1.$$

Proof. Assume that $X \in \{c_0,\ c\}$. Then, by Phillips Theorem, we see that the spectrum of S_p in X is equal to the the spectrum of S_p in ℓ^∞ and from Propositions 4.5 and 4.8, we obtain the result.

Now, assume $X = l^\alpha$, $\alpha > 1$. According to Propositions 4.4, 4.5 and 4.8, it is enough to prove that $J(f) \subset \sigma(S_p)$. Consider $\lambda \in J(f)$. We will prove that λ belongs to the approximate point spectrum of S_p . For all integers $k \geq 2$, put $w^{(k)} = (1, q_1(\lambda), \ldots, q_k(\lambda), 0 \ldots 0, \ldots)^t \in l^\alpha$ where $(q_k(\lambda))_{k \geq 1} = (q_k)_{k \geq 1}$ is the sequence defined in (5) of the proof of Theorem 4.3 and let $u^{(k)} = \frac{w^{(k)}}{||w^{(k)}||_\alpha}$, then we have the following claim.

Claim: $\lim_{n \to +\infty} ||(S_p - \lambda Id)u^{(2^n)}||_\alpha = 0$.

Indeed, we have

$$\forall i \in \{0, \ldots, k-1\}, \quad \Big((S_p - \lambda Id)u^{(k)}\Big)_i = 0.$$

Thus

$$\sum_{i=0}^{+\infty}\Big|((S_p-\lambda Id)u^{(k)})_i\Big|^{\alpha}=\frac{\displaystyle\sum_{i=k}^{+\infty}\left|\sum_{j=0}^{k}(S_p-\lambda Id)_{i,j}w_j^{(k)}\right|^{\alpha}}{||w^{(k)}||_{\alpha}^{\alpha}}\ .$$

Putting $a_{i,j}=|(S_p-\lambda Id)_{i,j}|$ for all i,j and using (3), we get

$$\left|\sum_{j=0}^{k}(S_p-\lambda Id)_{i,j}w_j^{(k)}\right|^{\alpha}\leq C\sum_{j=0}^{k}|(S_p-\lambda Id)_{i,j}||w_j^{(k)}|^{\alpha}$$

where $C=\sup_{i\in\mathbb{N}}\left(\sum_{j=0}^{\infty}|(S_p-\lambda Id)_{i,j}|\right)^{\frac{\alpha}{\alpha'}}$ and α' is the conjugate of α.

Observe that C is a finite nonnegative constant because S_p is a stochastic matrix and λ belongs to $J(f)$ which is a bounded set.

In this way we have

$$\begin{aligned}\Big|\Big|(S_p-\lambda Id)u^{(k)}\Big|\Big|_{\alpha}^{\alpha} &\leq C\sum_{i=k}^{+\infty}\frac{\left(\sum_{j=0}^{k}|w_j^{(k)}|^{\alpha}|(S_p-\lambda Id)_{ij}|\right)}{||w^{(k)}||_{\alpha}^{\alpha}}\\ &= \frac{C}{||w^{(k)}||_{\alpha}^{\alpha}}\sum_{j=0}^{k}|w_j^{(k)}|^{\alpha}\sum_{i=k}^{+\infty}|(S_p-\lambda Id)_{ij}|.\end{aligned}$$

Now, for $k=2^n$, we will compute the following terms

$$A_{kj}=\sum_{i=k}^{+\infty}|(S_p-\lambda Id)_{ij}|,\ 0\leq j\leq k.$$

Assume that $0\leq j<k=2^n$. Then $(S_p-\lambda Id)_{ij}=(S_p)_{ij}$ for all $i\geq k$.

Case 1: j is odd. Then by Lemma 4.6, $(S_p)_{ij}\neq 0$ if and only $i=j-1$ or $i=j$. Hence $(S_p)_{ij}=0$ for all $i\geq k$. Thus

(23) $$A_{kj}=0.$$

Case 2: $j=0$. Then by Lemma 4.6, we have

(24) $$A_{kj}=\sum_{i=2^n}^{+\infty}(S_p)_{i0}=\sum_{i=n+1}^{+\infty}p^i(1-p)=p^{n+1}.$$

Case 3: j is even and $j>0$. Then $j=\varepsilon_{n-1}\dots\varepsilon_s\underbrace{0\dots0}_{s}=\sum_{i=s}^{n-1}\varepsilon_i 2^i$ with $s\geq 1$ and $\varepsilon_s=1$. But by Lemma 4.6, $(S_p)_{ij}\neq 0$ if and only if $i=2^m-1+j$ where $0\leq m\leq s$. Hence $i<2^n=k$.

Therefore, in this case

(25) $$A_{kj}=0.$$

Now assume $j = k = 2^n$. In this case, we have $A_{kj} = |1-p-\lambda| + \sum_{i=2^n+1}^{+\infty} (S_p)_{i,2^n}$. On the other hand, by Lemma 4.6, we deduce that $(S_p)_{i,2^n} \neq 0$ if and only if $i = 2^n + 2^m - 1$ where $0 \leq m \leq n$ and $(S_p)_{2^n+2^m-1,2^n} = p^m(1-p)$. Therefore

$$A_{kj} = \sum_{i=2^n}^{+\infty} |(S_p - \lambda Id)_{i,2^n}| = |1-p-\lambda| + \sum_{m=0}^{n} p^m(1-p). \tag{26}$$

By (23),(24),(25) and (26), we have for $k = 2^n$ and $0 \leq j \leq k$,

$$A_{kj} \neq 0 \Longleftrightarrow j = 0 \text{ or } j = k = 2^n.$$

Consequently

$$\begin{aligned} \left|\left|(S_p - \lambda Id)u^{(2^n)}\right|\right|_\alpha^\alpha &\leq C \ .\frac{|w_0^{(k)}|^\alpha A_{k0} + |w_k^{(k)}|^\alpha A_{kk}}{||w^{(k)}||_\alpha^\alpha} \\ &= C \ .\frac{p^{n+1} + |q_{2^n}|^\alpha \left(|1-p-\lambda| + \sum_{m=0}^{n} p^m(1-p)\right)}{||w^{(2^n)}||_\alpha^\alpha}. \end{aligned}$$

We claim that $||w^{(2^n)}||_\alpha$ goes to infinity as n goes to infinity. Indeed, if not since the sequence $||w^{(2^n)}||_\alpha$ is a increasing sequence, it must converge. Put $w = (q_i)_{i\geq 0}$ with $q_0 = 1$. It follows that the sequence $(w^{(2^n)})_{n\geq 0}$ converges to w in ℓ^α which means that there exists a nonzero vector $w \in l^\alpha$ such that $(S_p - \lambda Id)w = 0$. This contradicts Proposition 4.5. Now, since λ belongs to the filled Julia set which is a bounded set and $(q_n)_{n\geq 0}$ is a bounded sequence, it follows that $||((S_p - \lambda Id)u^{(2^n)})||_\alpha$ converge to 0, and the claim is proved. We conclude that λ belongs to the approximate point spectrum of S_p and the proof of Proposition 4.10 is complete. □

This ends the proof of Theorem 4.3.

Spectrum of S_p acting on the right on ℓ^1.

Here, we will study the spectrum of S_p acting (on the right) in ℓ^1. We deduce from Proposition 4.4 that the Spectrum of S_p on ℓ^1 is contained in the filled Julia set $J(f)$. On the other hand, using the same proof than Proposition 4.10, we obtain that $J(f)$ is contained in the approximate point spectrum of S_p. This yields that the spectrum of S_p acting on ℓ^1 is equal to $J(f)$.

Theorem 4.11. *In ℓ^1, the residual spectrum contains a dense and countable subset of the Julia set $\partial(J(f))$. The continuous spectrum is not empty and is equal to the relative complement of the residual spectrum with respect to the the filled Julia set $J(f)$.*

Proof. The proof of Proposition 4.8, shows that the residual spectrum of S_p in ℓ^1 is equal to the point spectrum of S_p (acting on right) in $l^{1'} = l^\infty$. By (18) and (22), we see that

$$\sigma_r(S_p) =$$

$$\left\{\lambda \in \mathbb{C},\ (q_n(\lambda)) \text{ and } (1/q_n(\lambda)) \text{ are bounded and } q_1 = \sum_{i=1}^{+\infty} \frac{p^{i-1}(1-p)}{q_{2^i-1}}\right\}$$

$$= J(f) \cap \left\{\lambda \in \mathbb{C},\ (1/q_n(\lambda)) \text{ is bounded and } q_1 = \sum_{i=1}^{+\infty} \frac{p^{i-1}(1-p)}{q_{2^i-1}}\right\}.$$

On the other hand we have

$$q_{2^n}^2 = f(q_{2^{n-1}}^2) = \dots f^n(q_1^2) = f^{n+1}(\lambda),\ \forall n \geq 0. \tag{27}$$

Let $n \in \mathbb{N}$ and $E_n = \{\lambda \in \mathbb{C},\ q_{2^n}(\lambda) = 1\}$

Claim 1: $\displaystyle\bigcup_{n=0}^{+\infty} E_n = \bigcup_{n=0}^{+\infty} f^{-n}\{1\}$.

Indeed, let $\lambda \in \mathbb{C}$ such that $f^n(\lambda) = 1$ for some nonnegative integer $n \geq 1$. Then, by (27), we have $q_{2^{n-1}} = 1$ or $q_{2^{n-1}} = -1$. From (7), we see that $q_{2^{n-1}} = -1$ implies $q_{2^n} = 1$. Hence $f^{-n}\{1\} \subset E_{n-1} \cup E_n$. Since $1 \in E_n$ for all integers $n \geq 0$, we conclude that, $\bigcup_{n=0}^{+\infty} f^{-n}\{1\} \subset \bigcup_{n=0}^{+\infty} E_n$. The other inclusion follows from (27).

Claim 2: $\displaystyle\bigcup_{n=0}^{+\infty} E_n \subset \sigma_r(S_p)$.

Indeed, assume that $n \in \mathbb{N}$ and $\lambda \in E_n$. Then by (7), we get that

$$q_{2^k} = 1,\ \forall k \geq n. \tag{28}$$

But from (28) and (8), we have that $(q_k(\lambda))_{k\geq 0}$ and $(1/q_k(\lambda))_{k\geq 0}$ are bounded . Moreover, we have

$$\begin{aligned} q_1 = \sum_{i=1}^{+\infty} \frac{p^{i-1}(1-p)}{q_{2^i-1}} \quad &\Longleftrightarrow \quad q_2 = \sum_{i=2}^{+\infty} \frac{p^{i-2}(1-p)q_1}{q_{2^i-1}} \\ &\Longleftrightarrow \quad q_{2^k} = \sum_{i=k+1}^{+\infty} p^{i-k-1}(1-p)\frac{q_{2^0}\dots q_{2^{k-1}}}{q_{2^i-1}}, \quad \forall k \geq 0 \\ &\Longleftrightarrow \quad q_{2^k} = \sum_{i=k+1}^{+\infty} \frac{p^{i-k-1}(1-p)}{q_{2^k}q_{2^{k+1}}\dots q_{2^{i-1}}},\ \forall k \geq 0. \end{aligned}$$

Thus

$$q_1 = \sum_{i=1}^{+\infty} \frac{p^{i-1}(1-p)}{q_{2^i-1}} \Longleftrightarrow 1 = \sum_{i=0}^{+\infty} p^i(1-p).$$

From this $\lambda \in \sigma_r(S_p)$ and the claim 2 is proved.

But 1 is a repulsor fixed point of f, it follows that $\bigcup_{n=0}^{+\infty} f^{-n}\{1\}$ is a dense subset of the Julia set $\partial J(f)$. By this fact combined with claims 1 and 2, we conclude that the residual spectrum contains a dense and countable subset of the Julia set $\partial(J(f))$.

On the other hand, $(p-1)^2 \in J(f)$ since $f((p-1)^2)) = (p-1)^2$, but $(p-1)^2 \notin \sigma_r(S_p)$ because for any positive integer n, $q_{2^n}((p-1)^2) = (p-1)$, which implies that $\lim q_n = 0$ and hence $1/q_n$ is not bounded. Thus $(p-1)^2 \in \sigma_c(S_p)$. This finishes the proof of the theorem.

□

Conjecture 4.12. *We conjecture that the residual spectrum in ℓ^1 equals the set $\bigcup_{n=0}^{+\infty} f^{-n}\{1\}$.*

Spectrum of S_p acting on the left.

Phillips Theorem combined with Proposition 2.2, Theorems 4.3 and 4.11, leads to the following result.

Theorem 4.13. *The spectrum of S_p (acting on the left) in the spaces c_0, c, l^α where $1 \leq \alpha \leq +\infty$ equals to the filled Julia set $J(f)$. Precisely:*

In c_0, l^α where $1 \leq \alpha < +\infty$, the spectrum of S_p equals to the continuous spectrum of S_p.

In c, the point spectrum of S_p equals $\{1\}$ and the continuous spectrum equals $J(f)\backslash\{1\}$.

In ℓ^∞, the point spectrum equals to the residual spectrum of S_p in ℓ^1.

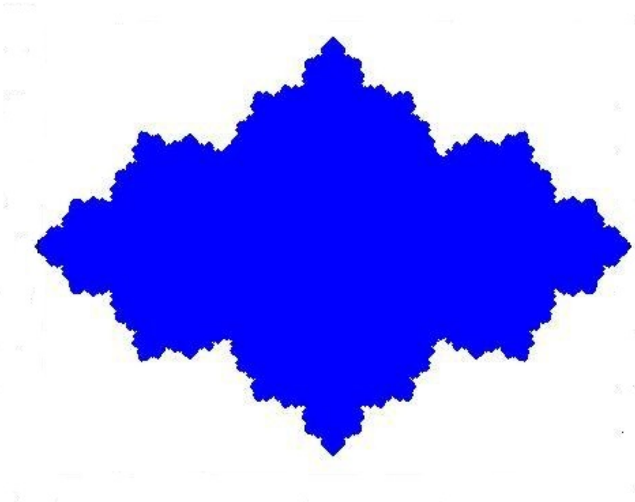

Fig.3. Filled Julia set: $p = 0.7$

5. Fibonacci Stochastic adding machine (see [5])

Let us consider the Fibonacci sequence $(F_n)_{n\geq 0}$ given by the relation

$$F_n = F_{n-1} + F_{n-2} \ \ \forall n \geq 2.$$

Using greedy algorithm, we can write (see [10]) every nonnegative integer N in a unique way as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N)F_i$ where $\varepsilon_i(N) = 0$ or 1 and $\varepsilon_i(N)\varepsilon_{i+1}(N) \neq 0,\ \forall 0 \leq i \leq k(N) - 1$.

It is known that the addition of 1 in base $(F_n)_{n\geq 0}$ (called Fibonacci adding machine) is given by a finite state automaton transductor on $A^* \times A^*$ where $A = \{0, 1\}$ (see Fig.4). This transductor is formed by two states (an initial state I and a terminal state T). The initial state is connected to itself by 2 arrows. One of them is labeled by $(10, 00)$ and the other by $(101, 000)$. There are also 2 arrows going from the initial state to the terminal one. One of these arrows is labeled by $(00, 01)$ and the other by $(001, 010)$. The terminal state is connected to itself by 2 arrows. One of them is labeled by $(0, 0)$ and the other by $(1, 1)$.

Assume that $N = \varepsilon_n \dots \varepsilon_0$. To find the digits of $N + 1$, we will consider the finite path $c = (p_{k+1}, a_k/b_k, p_k) \dots (p_2, a_1/b_1, p_1)(p_1, a_0/b_0, p_0)$ where $p_i \in \{I, T\},\ p_0 = I,\ p_{k+1} = T,\ a_i, b_i \in A^*$ where $A = \{0, 1\}$ and the words $a_k \dots a_0$ and $b_k \dots b_0$ have no two consecutive 1. Moreover $\dots 0 \dots 0a_k \dots a_0 = \dots 0 \dots 0\varepsilon_n \dots \varepsilon_0$.

Hence $N + 1 = \varepsilon'_n \dots \varepsilon'_0$, where

$$\dots 0 \dots 0b_k \dots b_0 = \dots 0 \dots 0\varepsilon'_n \dots \varepsilon'_0.$$

Example: If $N = 10 = 10010$ then

N corresponds to the path $(T, 1/1, T)\ (T, 00/01, I)\ (I, 10/00, I)$.

Hence $N + 1 = 10100 = 11$.

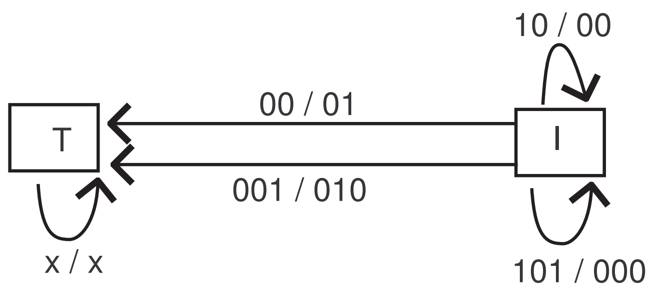

Fig.4. Transductor of Fibonacci adding machine

In [5], the authors define the stochastic adding machine by the following way:

Consider "probabilistic" transductor $\mathcal{T}_p$ (see Fig.5) where $0 < p < 1$, by the following manner.

The states of $\mathcal{T}_p$ are I and T. The labels are of the form $(0/0, 1), (1/1, 1)$, $(a/b, p)$ or $(a/a, 1 - p)$ where a/b is a label in $\mathcal{T}$.

The labeled edges in $\mathcal{T}_p$ are of the form $(T,(x/x,1),T)$ where $x \in \{0,1\}$ or of the form $(r,(a/b,p),q)$ or $(T,(a/a,1-p),q)$ where $(r,a/b,q)$ is a labeled edge in $\mathcal{T}$, with $q = I$.

The stochastic process $\psi(N)$ is defined by $\psi(N) = \sum_{i=0}^{+\infty} r_i(N)F_i$ where $(r_i(N))_{i\geq 0}$ is an infinite sequence of 0 or 1 without two 1 consecutive and with finitely many non zero terms.

The sequence $(r_i(N))_{i\geq 0}$ is defined by the following way:

Put $r_i(0) = 0$ for all i, and assume that we have defined $(r_i(N-1))_{i\geq 0},\ N \geq 1$. In the transductor $\mathcal{T}_p$, consider a path

$$\ldots(T,(0/0,1),T)\ldots(T,(0/0,1),T)(p_{n+1},(a_n/b_n,t_n),p_n)\ldots(p_1,(a_0/b_0,t_0),p_0)$$

where $p_0 = I$ and $p_{n+1} = T$, such that the words $\ldots r_1(N-1)r_0(N-1)$ and $\ldots 00a_n\ldots a_0$ are equal.

We define the sequence $(r_i(N))_{i\geq 0}$ as the infinite sequence whose terms are 0 or 1 such that $\ldots r_1(N)r_0(N) = \ldots 00b_n\ldots b_0$.

We remark that $\psi(N-1)$ transitions to $\psi(N)$ with probability of $p_{\psi(N-1)\psi(N)} = t_nt_{n-1}\ldots t_0$.

Example 1: If $N = 10 = 10010$, then, in the transductor of Fibonacci adding machine, N corresponds to the path $(T,1/1,T)$ $(T,00/01,I)$ $(I,10/00,I)$.

In the stochastic Fibonacci adding machine, we have the following paths (see Figure 2):

(1) $(T,(1/1,1),T)$ $(T,(0/0,1),T)(T,(0/0,1),T)$ $(T,(10/10,1-p),I)$. In this case $N = 10010$ transitions to 10010 with probability $1-p$.
(2) $(T,(1/1,1),T)$ $(T,(00/00,1-p),I)$ $(I,(10/00,p),I)$. In this case $N = 10$ transitions to $10000 = 8$ with probability $p(1-p)$.
(3) $(T,(1/1,1),T)$ $(T,(00/01,p),I)$ $(I,(10/00,p),I)$. In this case $N = 10$ transitions to $10100 = 11$ with probability p^2.

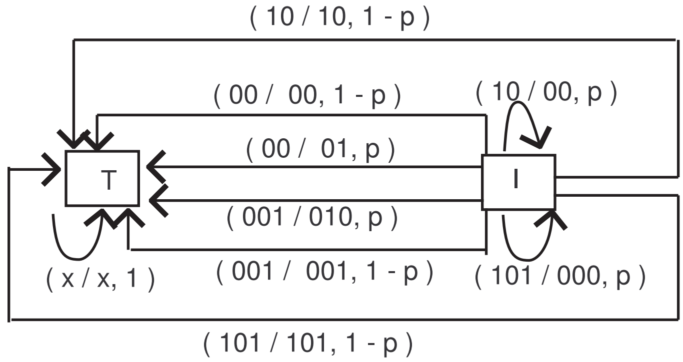

Fig.5. Transductor of Fibonacci fallible adding machine

By using the transductor $\mathcal{T}_p$, we can prove the following result (see [5]).

Proposition 5.1. *Let N be a nonnegative integer, then the following results are satisfied.*

(1) *N transitions to N with probability $1-p$.*
(2) *If $N=\varepsilon_k\dots\varepsilon_2 00$, $k\geq 2$, then N transitions to $N+1=\varepsilon_k\dots\varepsilon_2 01$ with probability p.*
(3) *If $N=\varepsilon_k\dots\varepsilon_t 00\underbrace{1010\dots1010}_{2s}$ with $s\geq 1$ and $k\geq t\geq 2s+2$, then N transitions to $N+1=\varepsilon_k\dots\varepsilon_t 01\underbrace{0\dots00}_{2s}$ with probability p^{s+1}, and N transitions to $N-\sum_{i=1}^{m}F_{2i-1}=N-F_{2m}+1=\varepsilon_k\dots\varepsilon_t 00\underbrace{10\dots10}_{2s-2m}\underbrace{0\dots00}_{2m}$, $1\leq m\leq s$ with probability $p^m(1-p)$.*
(4) *If $N=\varepsilon_k\dots\varepsilon_t 0\underbrace{0101\dots0101}_{2s}$, $s\geq 2$ and $k\geq t\geq 2s+1$, then N transitions to $N+1=\varepsilon_k\dots\varepsilon_t 0\underbrace{1000\dots000}_{2s}$ with probability p^s, and N transitions to $N-\sum_{i=0}^{m}F_{2i}=N-F_{2m+1}+1=\varepsilon_k\dots\varepsilon_t 00\underbrace{10\dots10}_{2s-2m}\underbrace{0\dots00}_{2m-1}$, $2\leq m\leq s$ with probability $p^{m-1}(1-p)$.*
(5) *If $N=\varepsilon_k\dots\varepsilon_3 001$, $k\geq 3$, then N transitions to $N+1=\varepsilon_k\dots\varepsilon_3 010$ with probability p.*

By Proposition 5.1, we construct the transition graph. We also find the transition operator S_p associated to the transition graph.

$$\begin{pmatrix}
1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots\\
0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots\\
p(1-p) & 0 & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots\\
0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots\\
p(1-p) & 0 & 0 & 0 & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots\\
0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0\dots\\
0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0\dots\\
p^2(1-p) & 0 & 0 & 0 & 0 & p(1-p) & 0 & 1-p & p^3 & 0 & 0 & 0 & 0\dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0\dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0\dots\\
0 & 0 & 0 & 0 & 0 & p(1-p) & 0 & 0 & p(1-p) & 0 & 1-p & p^2 & 0\dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}$$

Fig.6. Transition graph of stochastic adding machine in Fibonacci base.

Remark 5.2. *In [5], the authors prove that the point spectrum of S_p in ℓ^∞ is equal to the set $\mathcal{K}_p=\{\lambda\in\mathbb{C},\ (q_n(\lambda))_{n\geq 1}$ is bounded $\}$, where $q_{F_0}(z)=z$, $q_{F_1}(z)=z^2$, $q_{F_k}(z)=\dfrac{1}{p}q_{F_{k-1}}(z)q_{F_{k-2}}(z)-\dfrac{1-p}{p}$, for all $k\geq 2$ and for all nonnegative integers n, we have $q_n(z)=q_{F_{k_1}}\dots q_{F_{k_m}}$ where $F_{k_1}+\dots+F_{k_m}$ is the Fibonacci representation of n. In particular, $\sigma_{pt(S_p)}$ is contained in the set*

$$\begin{aligned}
\mathcal{E}_p &= \{\lambda\in\mathbb{C}\mid (q_{F_n}(\lambda))_{n\geq 1} \text{ is bounded }\}\\
&= \{\lambda\in\mathbb{C}\mid (\lambda_1,\lambda)\in J(g)\}
\end{aligned}$$

where $J(g)$ is the filled Julia set of the function $g:\mathbb{C}^2\mapsto\mathbb{C}^2$ defined by: $g(x,y)=(\frac{1}{p^2}(x-1+p)(y-1+p),x)$ and $\lambda_1=1-p+\frac{(1-\lambda-p)^2}{p}$. They also investigated the topological properties of $\mathcal{E}_p$.

Proposition 5.3. *The operator S_p is well defined in the Banach spaces c_0, c and ℓ^α, $\alpha \geq 1$. The point spectra of S_p acting in the spaces c_0, and ℓ^α associated to the stochastic Fibonacci adding machines are empty sets. In c, the point spectrum equals $\{1\}$.*

Proof. By Proposition 5.1, we can prove that the sum of coefficients of every column of S_p is bounded by a fixed constant $M > 0$.

Indeed, let $n \in \mathbb{N}$ and $s_n = \sum_{i=0}^{+\infty} p_{i,n}$ be the sum of coefficients of the n-th column.

If $n = \varepsilon_k \ldots \varepsilon_2 01$ or $n = \varepsilon_k \ldots \varepsilon_3 010$ (Fibonacci representation), then by 1), 2) and 5) of Proposition 5.1, we have $s_n = 1$

If $n = \varepsilon_k \ldots \varepsilon_t 01 \underbrace{0 \ldots 00}_{s}$, $s \geq 2$, then for all integers $i \in \mathbb{N}$, $p_{i,n} > 0$ implies that $i = n$ or $i = n - 1$ or $i = \varepsilon_k \ldots \varepsilon_t 01 \underbrace{0 \ldots 0}_{s-2m} \underbrace{01 \ldots 01}_{2m}$, $s \geq 2m$ or $i = \varepsilon_k \ldots \varepsilon_t 01 \underbrace{0 \ldots 0}_{s-2m} \underbrace{10 \ldots 10}_{2m}$, $s \geq 2m$. Hence $s_n \leq 1 - p + p^{\lceil 2 \rceil} + 2 \sum_{m=1}^{\infty} p^m (1-p) \leq 1 + 2p$.

If $n = 0$, then $s_n \leq 1 + p$.

On the other hand, since S_p is a stochastic matrix, then by Proposition 4.1, S_p is well defined in the spaces c_0, c (resp. in ℓ^α, $\alpha \geq 1$).

Now, let λ be an eigenvalue of S_p in X where $X \in \{c_0, c, \ell^\alpha, \ \alpha \geq 1\}$ associated to the eigenvector $v = (v_i)_{i \geq 0} \in X$. Since the transition probability from any nonnegative integer i to any integer $i + k$, $k \geq 2$ is $p_{i,i+k} = 0$ (see Proposition 5.1), the operator S_p satisfies $(S_p)_{i,i+k} = 0$ for all $i, k \in \mathbb{N}$ with $k \geq 2$. Thus for every integer $k \geq 1$, we have

$$\sum_{i=0}^{k} p_{k-1,i} v_i = \lambda v_{k-1}. \tag{29}$$

Then, we can prove by induction on k that for any integer $k \geq 1$, there exists a complex number $c_k = c_k(p, \lambda)$ such that

$$v_k = c_k v_0 \tag{30}$$

Using the fact that the matrix S_p is auto-similar, we can prove that $c_k = q_k$ for all integers $k \in \mathbb{N}$ (see Theorem 1, page 303, [5]). Since

$$q_{F_n}(z) = \frac{1}{p} q_{F_{n-1}}(z) q_{F_{n-2}}(z) - \frac{1-p}{p}, \ \forall n \in \mathbb{N},$$

and (q_{F_n}) converges to 0 when n goes to infinity, we obtain that the point spectrum of S_p acting in c_0 (resp. in ℓ^α, $\alpha \geq 1$) is empty. Using the same idea than proposition 4.5, we see that $\sigma_{pt,c} = \{1\}$.

□

Remark 5.4. *By Phillips Theorem and duality, it follows that the spectra of S_p acting in X where $X \in \{c_0, \ c, \ l^1, \ l^\infty\}$ associated to the stochastic Fibonacci adding machine are equals.*

Theorem 5.5. *The spectra of S_p acting in X where $X \in \{l^\infty, c_0, c, l^\alpha, \alpha \geq 1\}$ contain the set $\mathcal{E}_p = \{\lambda \in \mathbb{C}, \ (q_{F_n}(\lambda))_{n \geq 0}$ is bounded $\}$.*

Proof. The proof is similar to the proof of Proposition 4.10 and will be done in case $\ell^\alpha,\ \alpha > 1$. Let $\lambda \in \mathcal{E}_p$ and let us prove that λ belongs to the approximate point spectrum of S_p in $\ell^\alpha,\ \alpha > 1$.
For every integer $k \geq 2$, consider $w^{(k)} = (1, q_1(\lambda), \ldots, q_k(\lambda), 0 \ldots 0, \ldots)^t \in l^\alpha$ where $(q_k(\lambda))_{k\geq 1} = (q_k)_{k\geq 1}$ is the sequence defined in the proof of Theorem 5.5. Let $u^{(k)} = \dfrac{w^{(k)}}{||w^{(k)}||_\alpha}$, then we have the following claim.

Claim: $\lim\limits_{n\to+\infty} ||(S_p - \lambda Id)u^{(F_n)}||_\alpha = 0.$
By using the same proof than Proposition 4.10, we have

$$\left|\left|(S_p - \lambda Id)u^{(F_n)}\right|\right|_\alpha^\alpha \leq \frac{D}{||w^{(F_n)}||_\alpha^\alpha} \sum_{j=0}^{F_n} |w_j^{(F_n)}|^\alpha B_{F_n,j}$$

where D is a positive constant and $B_{F_n,j} = \sum_{i=F_n}^{+\infty} |(S_p - \lambda Id)_{ij}|$.

We can prove by the same manner done in Proposition 4.10 that for $0 \leq j \leq F_n$,

$$B_{F_n,j} \neq 0 \Longleftrightarrow j = 0 \text{ or } j = F_n.$$

Indeed, if $j \in \{1, \ldots F_n - 1\}$, then since $i \geq F_n$, we have $(S_p - \lambda Id)_{ij} = p_{i,j}$. If the Fibonacci representation of j is $j = \varepsilon_k \ldots \varepsilon_2 01$ or $j = \varepsilon_k \ldots \varepsilon_t 10 \ldots 0$, it is easy to see by Proposition 5.1 that $p_{i,j} \neq 0$ implies $i < F_n$.

On the other hand, if $j = 0$ then $B_{F_n,j} = \sum_{l=F_n}^{+\infty} p_{l,0}$. Since $p_{l,0} \neq 0$ if and only $l = F_i - 1$ and $p_{F_i-1,0} = p^{\lceil i/2\rceil}(1-p)$, we have $B_{F_n,j} \leq 2\sum\limits_{i=m}^{+\infty} p^i(1-p) = 2p^m$ where $m = \lceil (n+1)/2\rceil$.

Now assume $j = F_n$. In this case, we have $B_{F_n,j} = |1 - p - \lambda| + \sum\limits_{i=F_n+1}^{+\infty} p_{i,F_n}$. On the other hand, by Proposition 5.1, we deduce that $p_{i,F_n} \neq 0$ if and only if $i = F_n + F_m - 1$ where $0 \leq m \leq n$ and $p_{F_n+F_m-1,F_n} = p^{[m/2]}(1 - p)$. Therefore

$$B_{F_n,F_n} = |1 - p - \lambda| + \sum_{m=0}^{n} p^{\lceil m/2\rceil}(1 - p) \leq |1 - p - \lambda| + 2 \tag{31}$$

Hence

$$\left|\left|(S_p - \lambda Id)u^{(F_n)}\right|\right|_\alpha^\alpha \leq D\frac{2p^m + |q_{F_n}|^\alpha\,(|1 - p - \lambda| + 2)}{||w^{(F_n)}||_\alpha^\alpha}.$$

Since $||w^{(F_n)}||_\alpha$ goes to infinity as n goes to infinity and $(q_{F_n})_{n\geq 0}$ is bounded, it follows that $||((S_p - \lambda Id)u^{(F_n)})||_\alpha$ converge to 0. Therefore λ belongs to the approximate point spectrum of S_p. Thus the spectrum of S_p acting on ℓ^α $\alpha > 1$, contains $\mathcal{E}_p$.

This finishes the proof for ℓ^α $\alpha > 1$. The case ℓ^1 can be handled in the same way, the details being left to the reader.

□

Open questions. We are not yet able to compute the residual and continuous spectrum of S_p acting in the Banach spaces $\ell^\infty,\ c_0,\ c$ or in $\ell^\alpha,\ \alpha \geq 1$. We conjecture that $\sigma(S_p) = \mathcal{E}_p$. Moreover, in the case of ℓ^∞ we conjecture that the residual spectrum is empty and the continuous spectrum is the set $\mathcal{E}_p \setminus \mathcal{K}_p$. The difficulty here is that the matrix S_p is not bi-stochastic. One may also look for a characterization of all real numbers $0 < p < 1$ for which $\mathcal{E}_p \neq \mathcal{K}_p$.

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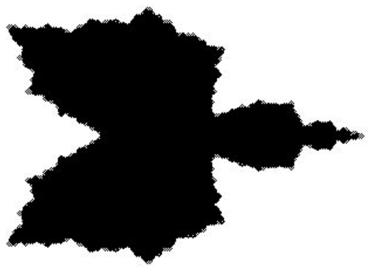

Fig.7. $p = 0.625$

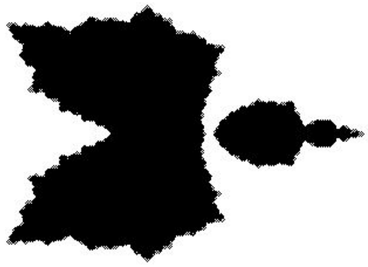

Fig.8. $p = 0.621$

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Department of Mathematics, University of Rouen, LMRS, UMR 60 85, Avenue de l'Université, BP.12, 76801 Saint Etienne du Rouvray - France
E-mail address: elhoucein.elabdalaoui@univ-rouen.fr

Departamento de Matemática, IBILCE-UNESP, Rua Cristovõ Colombo, 2265, CEP 15054-0000, São José de Rio Preto-SP, Brasil
E-mail address: messaoud@ibilce.unesp.br