

VOLUME ENTROPY RIGIDITY OF NON-POSITIVELY CURVED SYMMETRIC SPACES

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To Werner Ballmann for his 60th birthday

ABSTRACT. We characterize symmetric spaces of non-positive curvature by the equality case of general inequalities between geometric quantities.

1. INTRODUCTION

Let (M, g) be a closed connected Riemannian manifold, and $\pi : (\widetilde{M}, \widetilde{g}) \rightarrow (M, g)$ its universal cover endowed with the lifted Riemannian metric. We denote $p(t, x, y), t \in \mathbb{R}_+, x, y \in \widetilde{M}$ the heat kernel on \widetilde{M} , the fundamental solution of the heat equation $\frac{\partial u}{\partial t} = \text{Div } \nabla u$ on \widetilde{M} . Since we have a compact quotient, all the following limits exist as $t \rightarrow \infty$ and are independent of $x \in \widetilde{M}$:

$$\begin{aligned} \lambda_0 &= \inf_{f \in C_c^2(\widetilde{M})} \frac{\int |\nabla f|^2}{\int |f|^2} = \lim_t -\frac{1}{t} \ln p(t, x, x) \\ \ell &= \lim_t \frac{1}{t} \int d(x, y) p(t, x, y) d\text{Vol}(y) \\ h &= \lim_t -\frac{1}{t} \int p(t, x, y) \ln p(t, x, y) d\text{Vol}(y) \\ v &= \lim_t \frac{1}{t} \ln \text{Vol} B_{\widetilde{M}}(x, t), \end{aligned}$$

where $B_{\widetilde{M}}(x, t)$ is the ball of radius t centered at x in \widetilde{M} and Vol is the Riemannian volume on \widetilde{M} .

All these numbers are nonnegative. Recall λ_0 is the Rayleigh quotient of \widetilde{M} , ℓ the linear drift, h the stochastic entropy and v the volume entropy. There is the following relation:

$$(1) \quad 4\lambda_0 \stackrel{(a)}{\leq} h \stackrel{(b)}{\leq} \ell v \stackrel{(c)}{\leq} v^2.$$

See [L1] for (a), [Gu] for (b). Inequality (c) is shown in [L3] as a corollary of (b) and (2):

$$(2) \quad \ell^2 \leq h$$

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If (\widetilde{M}, g) is a locally symmetric space of nonpositive curvature, all five numbers $4\lambda_0, \ell^2, h, \ell v$ and v^2 coincide and are positive unless (\widetilde{M}, g) is $(\mathbb{R}^n, \text{Eucl.})$. Our result is a partial converse:

Theorem 1.1. *Assume (M, g) has nonpositive curvature. With the above notation, any of the equalities*

$$\ell = v, \quad h = v^2 \quad \text{and} \quad 4\lambda_0 = v^2$$

hold if, and only if, $(\widetilde{M}, \widetilde{g})$ is a symmetric space.

As recalled in [L3], Theorem 1.1 is known in negative curvature and follows from [K], [BFL], [FL], [BCG] and [L1]. The other possible converses are delicate: even for negatively curved manifolds, in dimension greater than two, it is not known that $h = \ell v$ holds only for locally symmetric spaces. This is equivalent to a conjecture of Sullivan (see [L2] for a discussion). Sullivan conjecture holds for surfaces of negative curvature ([L1], [Ka]). It is not known either whether $4\lambda_0 = h$ holds only for locally symmetric spaces. This would follow from the hypothetical $4\lambda_0 \stackrel{(d)}{\leq} \ell^2$ by the arguments of this note.

We assume henceforth that (M, g) has nonpositive sectional curvature. Given a geodesic γ in M , Jacobi fields along γ are vector fields $t \mapsto J(t) \in T_{\gamma(t)}M$ which describe infinitesimal variation of geodesics around γ . By nonpositive curvature, the function $t \mapsto \|J(t)\|$ is convex. Jacobi fields along γ form a vector space of dimension $2 \dim M$. The rank of the geodesic γ is the dimension of the space of Jacobi fields such that $t \mapsto \|J(t)\|$ is a constant function on \mathbb{R} . The rank of a geodesic γ is at least one because of the trivial $t \mapsto \dot{\gamma}(t)$ which describes the variation by sliding the geodesic along itself. The rank of the manifold M is the smallest rank of geodesics in M . Using rank rigidity theorem ([B1], [BS]), we reduce in section 2 the proof of Theorem 1.1 to proving that if (M, g) is rank one, equality in (2) implies that $(\widetilde{M}, \widetilde{g})$ is a symmetric space. For this, we show in section 3 that equality in (2) implies that $(\widetilde{M}, \widetilde{g})$ is asymptotically harmonic (see the definition below). This uses the Dirichlet property at infinity (Ballmann [B2]). Finally, it was recently observed by A. Zimmer ([Z]) that asymptotically harmonic universal covers of rank one manifolds are indeed symmetric spaces.

2. GENERALITIES AND REDUCTION OF THEOREM 1.1

We recall the notations and results from Ballmann's monograph [B3] about the Hadamard manifold $(\widetilde{M}, \widetilde{g})$ that we use. The space \widetilde{M} is homeomorphic to a ball. The covering group $G := \pi_1(M)$ satisfies the duality condition ([B3] page 45).

2.1. Boundary at infinity. Two geodesic rays γ, γ' in \widetilde{M} are said to be asymptotic if $\sup_{t \geq 0} d(\gamma(t), \gamma'(t)) < \infty$. The set of classes of asymptotic unit speed geodesic rays is called the boundary at infinity $\widetilde{M}(\infty)$. $\widetilde{M} \cup \widetilde{M}(\infty)$ is endowed with the topology of a compact space where $\widetilde{M}(\infty)$ is a sphere and where, for each unit speed geodesic ray γ , $\gamma(t) \rightarrow [\gamma]$

as $t \rightarrow \infty$. The action of the group G on $\widetilde{M}(\infty)$ is the continuous extension of its action on \widetilde{M} . For any $x, \xi \in \widetilde{M} \times \widetilde{M}(\infty)$, there is a unique unit speed geodesic $\gamma_{x,\xi}$ such that $\gamma_{x,\xi}(0) = x$ and $[\dot{\gamma}_{x,\xi}] = \xi$. The mapping $\xi \mapsto \dot{\gamma}_{x,\xi}(0)$ is a homeomorphism π_x^{-1} between $\widetilde{M}(\infty)$ and the unit sphere $S_x \widetilde{M}$ in the tangent space at x to \widetilde{M} . We will identify $S\widetilde{M}$ with $\widetilde{M} \times \widetilde{M}(\infty)$ by $(x, v) \mapsto (x, \pi_x v)$. Then the quotient SM is identified with the quotient of $\widetilde{M} \times \widetilde{M}(\infty)$ under the diagonal action of G .

Fix $x_0 \in \widetilde{M}$ and $\xi \in \widetilde{M}(\infty)$. The *Busemann function* b_ξ is the function on \widetilde{M} given by:

$$b_\xi(x) = \lim_{y \rightarrow \xi} d(y, x) - d(y, x_0).$$

Clearly, $b_{g\xi}(gx) = b_\xi(x) + b_{g\xi}(gx_0)$. Moreover, the function $x \mapsto b_\xi(x)$ is of class C^2 ([**HI**]). It follows that the function $\Delta_x b_\xi$ satisfies $\Delta_{gx} b_{g\xi} = \Delta_x b_\xi$ and therefore defines a function B on $G \setminus (\widetilde{M} \times \widetilde{M}(\infty)) = SM$. It follows from the argument of [**HI**] that the function B is continuous on SM (see [**B3**], Proposition 2.8, page 69).

2.2. Jacobi fields. Let (x, v) be a point in $T\widetilde{M}$. Tangent vectors in $T_{x,v}T\widetilde{M}$ correspond to variations of geodesics and can be represented by Jacobi fields along the unique geodesic $\gamma_{x,v}$ with initial value $\gamma(0) = x, \dot{\gamma}(0) = v$. A Jacobi field $J(t), t \in \mathbb{R}$ along $\gamma_{x,v}$ is uniquely determined by the values of $J(0)$ and $J'(0)$. We describe tangent vectors in $T_{x,v}T\widetilde{M}$ by the associated pair $(J(0), J'(0))$ of vectors in $T_x \widetilde{M}$. The metric on $T_{x,v}T\widetilde{M}$ is given by $\|(J_0, J'_0)\|^2 = \|J_0\|^2 + \|J'_0\|^2$. Assume $(x, v) \in SM$. A vertical vector in $T_{x,v}S\widetilde{M}$ is a vector tangent to $S_x \widetilde{M}$. It corresponds to a pair $(0, J'(0))$, with $J'(0)$ orthogonal to v . Horizontal vectors correspond to pairs $(J(0), 0)$. In particular, let X be the vector field on $S\widetilde{M}$ such that the integral flow of X is the geodesic flow. The geodesic spray $X_{x,v}$ is the horizontal vector associated to $(v, 0)$. The orthogonal space to X is preserved by the differential Dg_t of the geodesic flow. More generally, the Jacobi fields representation of $TT\widetilde{M}$ satisfies $D_{x,v}g_t(J(0), J'(0)) = (J(t), J'(t))$.

For any vector $Y \in T_x \widetilde{M}$, there is a unique vector $Z = S_{x,v}Y$ such that the Jacobi field J with $J(0) = Y, J'(0) = Z$ satisfies $\|J(t)\| \leq C$ for $t \geq 0$ ([**B3**] Proposition 2.8 (i)). The mapping $S_{x,v} : T_x \widetilde{M} \rightarrow T_x \widetilde{M}$ is linear and selfadjoint. The vectors (Y, SY) describe variations of asymptotic geodesics and the subspace $E_{x,v}^s \subset T_{x,v}T\widetilde{M}$ they generate corresponds to $TW_{x,v}^s$, where $W_{x,v}^s$, the set of initial vectors of geodesics asymptotic to $\gamma_{x,v}$, is identified with $\widetilde{M} \times \pi_x(v)$ in $\widetilde{M} \times \widetilde{M}(\infty)$. Observe that $S_{x,v} \dot{\gamma}_{x,v}(0) = 0$ and that the operator $S_{x,v}$ preserves $(\dot{\gamma}_{x,v}(0))^\perp$. Recall from [**B3**], Proposition 3.2 page 71, that, for $Y \in (\dot{\gamma}_{x,v}(0))^\perp$, with $\pi_x v = \xi$,

$$D_Y(\nabla b_\xi) = -S_{x,v}Y,$$

and therefore $\Delta_x b_\xi = -\text{Tr } S_{x,v}$ with $\pi_x(v) = \xi$.

Similarly, there is a selfadjoint linear operator $U_{x,v} : T_x \widetilde{M} \rightarrow T_x \widetilde{M}$ such that the Jacobi field J with $J(0) = Y, J'(0) = UY$ satisfies $\|J(t)\| \leq C$ for $t \leq 0$. The subspace $E_{x,v}^u \subset$

$T_{x,v}T\widetilde{M}$ they generate corresponds to $TW_{x,v}^u$, where $W_{x,v}^u$ is the set of opposite vectors to vectors in $W_{x,-v}^s$. By definition, $S_{\dot{\gamma}_{x,v}(0)} = -U_{\dot{\gamma}_{x,-v}(0)}$, so that we also have:

$$B(x,v) := -\text{Tr } S_{x,v} = \text{Tr } U_{x,-v}.$$

We have $\text{Ker } S = \text{Ker } U$ and $Y \in \text{Ker } S$ if, and only if, the Jacobi field $J(t)$ with $J(0) = Y, J'(0) = 0$ is bounded for all $t \in \mathbb{R}$. The rank of the geodesic $\gamma_{x,v}$ therefore is $\kappa = \text{Dim Ker } S$ and the geodesic $\gamma_{x,v}$ is of rank one only if $\text{Det}((U - S)|_{(\dot{\gamma}_{x,v}(0))^\perp}) = 0$.

Recall that SM is identified with the quotient of $\widetilde{M} \times \widetilde{M}(\infty)$ under the diagonal action of G . Clearly, for $g \in G$, $g(W_{x,v}^s) = W_{Dg(x,v)}^s$ so that the W^s define a foliation \mathcal{W}^s on SM . The leaves of the foliation \mathcal{W}^s are quotient of \widetilde{M} , they are naturally endowed with the Riemannian metric induced from \widetilde{g} .

2.3. Proof of Theorem 1.1. We continue assuming that $(\widetilde{M}, \widetilde{g})$ has nonpositive curvature. By the Rank Rigidity Theorem (see [B3]), $(\widetilde{M}, \widetilde{g})$ is of the form

$$(\widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_j \times \widetilde{M}_{j+1} \times \cdots \times \widetilde{M}_k, \widetilde{g})^1,$$

where \widetilde{g} is the product metric $\widetilde{g}^2 = (\widetilde{g}_0)^2 + (\widetilde{g}_1)^2 + \cdots + (\widetilde{g}_j)^2 + (\widetilde{g}_{j+1})^2 + \cdots + (\widetilde{g}_k)^2$, $(\widetilde{M}_0, \widetilde{g}_0)$ is Euclidean, $(\widetilde{M}_i, \widetilde{g}_i)$ is an irreducible symmetric space of rank at least two for $i = 1, \dots, j$ and a rank-one manifold for $i = j+1, \dots, k$. If the $(\widetilde{M}_i, \widetilde{g}_i), i = j+1, \dots, k$, are all symmetric spaces of rank one, then $(\widetilde{M}, \widetilde{g})$ is a symmetric space. Moreover in that case, all inequalities in (1) are equalities: this is the case for irreducible symmetric spaces (all numbers are 0 for Euclidean space; for the other spaces, $4\lambda_0$ and v^2 are classically known to coincide ([O]) and we have:

$$4\lambda_0(\widetilde{M}) = \sum_i 4\lambda_0(\widetilde{M}_i), \quad v^2(\widetilde{M}) = \sum_i v^2(\widetilde{M}_i).$$

To prove Theorem 1.1, it suffices to prove that if $\ell^2 = h$, all \widetilde{M}_i in the decomposition are symmetric spaces. This is already true for $i = 0, 1, \dots, j$. It remains to show that $(\widetilde{M}_i, \widetilde{g}_i)$ are symmetric spaces for $i = j+1, \dots, k$. Eberlein showed that each one of the spaces $(\widetilde{M}_i, \widetilde{g}_i)$ admits a cocompact discrete group of isometries (see [Kn], Theorem 3.3). This shows that the linear drifts ℓ_i and the stochastic entropies h_i exist for each one of the spaces $(\widetilde{M}_i, \widetilde{g}_i)$. Moreover, we clearly have

$$\ell^2 = \sum_i \ell_i^2, \quad h = \sum_i h_i.$$

Therefore Theorem 1.1 follows from

Theorem 2.1. *Assume (M, g) is a closed connected rank one manifold of nonpositive curvature and that $\ell^2 = h$. Then $(\widetilde{M}, \widetilde{g})$ is a symmetric space.*

¹With a clear convention for the cases when $\text{Dim } \widetilde{M}_0 = 0, j = 0$ or $k = j$.

A Hadamard manifold \widetilde{M} is called asymptotically harmonic if the function $B(= \Delta_x b)$ is constant on $S\widetilde{M}$. Theorem 2.1 directly follows from two propositions:

Proposition 2.2. *Assume (M, g) is a closed connected rank one manifold of nonpositive curvature and that $\ell^2 = h$. Then $(\widetilde{M}, \widetilde{g})$ is asymptotically harmonic.*

Proposition 2.3. *[Z], Theorem 1.1] Assume (M, g) is a closed connected rank one manifold of nonpositive curvature such that $(\widetilde{M}, \widetilde{g})$ is asymptotically harmonic. Then, $(\widetilde{M}, \widetilde{g})$ is a symmetric space.*

3. PROOF OF PROPOSITION 2.2

We consider the foliation \mathcal{W} of subsection 2.2. Recall that the leaves are endowed with a natural Riemannian metric. We write $\Delta^{\mathcal{W}}$ for the associated Laplace operator on functions which are of class C^2 along the leaves of \mathcal{W} . A probability measure m on SM is called harmonic if it satisfies, for any C^2 function f , we have:

$$\int_{SM} \Delta^{\mathcal{W}} f dm = 0.$$

Let M be a closed connected manifold such that $\ell^2 = h$. In [L3] it is shown that then, there exists a harmonic probability measure m on SM such that, at m -a.e. (x, v) , $B(x, v) = \ell$. Since B is a continuous function, Proposition 2.2 follows from

Theorem 3.1. *Let (M, g) be a closed connected rank one manifold of nonpositive curvature, \mathcal{W} the stable foliation on SM endowed with the natural metric as above. Then, there is only one harmonic probability measure m and the support of m is the whole space SM .*

Proof. Let m be a \mathcal{W} harmonic probability measure on SM . Then, there is a unique G -invariant measure \widetilde{m} on $S\widetilde{M}$ which coincide with m locally. Seen as a measure on $\widetilde{M} \times \widetilde{M}(\infty)$, we claim that \widetilde{m} is given, for any f continuous with compact support, by:

$$(3) \quad \int f(x, \xi) d\widetilde{m}(x, \xi) = \frac{1}{\text{Vol}M} \int_{\widetilde{M}} \left(\int_{\widetilde{M}(\infty)} f(x, \xi) d\nu_x(\xi) \right) dx,$$

where the family $x \mapsto \nu_x$ is a family of probability measures on $\widetilde{M}(\infty)$ such that, for all φ continuous on $\widetilde{M}(\infty)$, $x \mapsto \int \varphi(\xi) d\nu_x(\xi)$ is a harmonic function on \widetilde{M} and the measure dx is the Riemannian volume on \widetilde{M} . The claim follows from [Ga]. For convenience, let us reprove it: on the one hand, the measure \widetilde{m} projects on \widetilde{M} as a G -invariant measure satisfying $\int \Delta f dm = 0$. The projection of \widetilde{m} on \widetilde{M} is proportional to Volume, gives measure 1 to fundamental domains and formula (3) is the desintegration formula. On the other hand, if one projects \widetilde{m} first on $\widetilde{M}(\infty)$, there is a probability measure ν on $\widetilde{M}(\infty)$ such that

$$\int f(x, \xi) d\widetilde{m}(x, \xi) = \int_{\widetilde{M}(\infty)} \left(\int_{\widetilde{M}} f(x, \xi) dm_{\xi}(dx) \right) d\nu(\xi).$$

For ν -a.e. ξ , the measure m_ξ is a harmonic measure on \widetilde{M} ; therefore, for ν -a.e. ξ , there is a positive harmonic function $k_\xi(x)$ such that $m_\xi = k_\xi(x)\text{Vol}$. Comparing the two expressions for $\int f d\widetilde{m}$, we see that the measure ν_x is given by

$$\nu_x = k_\xi(x)\nu$$

and $x \mapsto \int_{\widetilde{M}(\infty)} \varphi(\xi) d\nu_x(\xi)$ is indeed a harmonic function.

The G -invariance of \widetilde{m} implies that, for all $g \in G$, $g_*\nu_x = \nu_{gx}$. In particular, the support of ν is G -invariant. By [E] (see [B3], page 48), the support of ν is the whole $\widetilde{M}(\infty)$ and therefore the support of m is the whole SM . This result would be sufficient for proving Proposition 2.2, but using discretization, we are going to identify the measure ν_x on $\widetilde{M}(\infty)$ as the hitting measure of the Brownian motion on \widetilde{M} starting from x . This shows Theorem 3.1.

Fix $x_0 \in \widetilde{M}$. The discretization procedure of Lyons and Sullivan ([LS]) associates to the Brownian motion on \widetilde{M} a probability measure μ on G such that $\mu(g) > 0$ for all g and that any bounded harmonic function F on \widetilde{M} satisfies

$$F(x_0) = \sum_{g \in G} F(gx_0)\mu(g).$$

Recall that for all φ continuous on $\widetilde{M}(\infty)$, $x \mapsto \nu_x(\varphi)$ is a harmonic function and that $\nu_{gx} = g_*\nu_x$. It follows that the measure ν_{x_0} is stationary for μ , i.e. it satisfies:

$$\nu_{x_0} = \sum_{g \in G} g_*\nu_{x_0}\mu(g).$$

Since the support of μ generates G as a semigroup (actually, it is already the whole G), there is only one stationary probability measure on $\widetilde{M}(\infty)$ (see [B3], Theorem 4.11 page 58). We know one already: the hitting measure m_{x_0} of the Brownian motion on \widetilde{M} starting from x_0 . This shows that $\nu_{x_0} = m_{x_0}$. Since x_0 was arbitrary in the above reasoning, we have $\nu_x = m_x$ for all $x \in \widetilde{M}$ and the measure \widetilde{m} is given by:

$$\int f(x, \xi) d\widetilde{m}(x, \xi) = \frac{1}{\text{Vol}M} \int_{\widetilde{M}} \left(\int_{\widetilde{M}(\infty)} f(x, \xi) dm_x(\xi) \right) dx.$$

□

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