Captivity of mean-field systems*

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Abstract

The aim of this work is to establish the stability of mean-field system under non-convex confining potential. A mean-field system corresponds to a system of N particles in weak interaction and confined by an exterior force. With our hypotheses, it is a Kolmogorov diffusion with potential Υ^N. Exit time of these systems have been studied in details in the small-noise limit. Here, we will deal with the large-dimension limit with fixed noise. In one hand, we show that the meta-potential Υ^N admits a number of wells which tends to infinity when N goes to infinity. In the other hand, by using the convergence of McKean-Vlasov processes in long-time and the propagation of chaos, we prove that there exist traps such that the diffusion can not escape from. Furthermore, the traps do not coincide with the wells of Υ^N.

Key words and phrases: Interacting particle system ; Propagation of chaos ; Exit-time ; Non-convexity ; Free-energy ; Stationary measures.

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Introduction

We are interested in the asymptotic behavior - when N tends towards infinity - of a mean-field system of the following form:

$$\begin{cases} dX_t^1 = \sqrt{\epsilon}dB_t^1 - V'\left(X_t^1\right)dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^1 - X_t^j\right)dt \\ \vdots \\ dX_t^i = \sqrt{\epsilon}dB_t^i - V'\left(X_t^i\right)dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^i - X_t^j\right)dt \\ \vdots \\ dX_t^N = \sqrt{\epsilon}dB_t^N - V'\left(X_t^N\right)dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^N - X_t^j\right)dt \end{cases} \tag{I}$$

where the N Brownian Motions $\left(B_t^i\right)_{t\in\mathbb{R}_+}$ are independent. We write X_t^i instead of $X_t^{i,N,\epsilon}$ for simplifying the reading. In this paper, we will make some smoothness assumptions on the confining (resp. interacting) potential V (resp. F). Furthermore, we will consider the dimension one even if the results in this work can be adapted to a more general setting, under the same hypotheses than the ones in [Tug11c, Tug11d].

Let us note some applications of this kind of system: [CDPS10] deals with social interactions ; [CX10] studies the stochastic partial differential equations. Diffusion (I) is continuous but mean-field system in discret space have also been studied, particularly the Currie-Weiss model, see [BBI09] or [MP98] for example.

We introduce the notations: $\mathcal{X}_t := \left(X_t^1, \cdots, X_t^N\right)$ and $\mathcal{B}_t := \left(B_t^1, \cdots, B_t^N\right)$. Thereby, (I) corresponds to a Kolmogorov diffusion in $\mathbb{R}^N$:

$$d\mathcal{X}_t = \sqrt{\epsilon}d\mathcal{B}_t - N\,\nabla\Upsilon^N\left(\mathcal{X}_t\right)dt \tag{I}$$

where Υ^N is called the meta-potential and is defined by:

$$\Upsilon^N\left(\mathcal{X}\right) := \frac{1}{N}\sum_{i=1}^N V\left(X_i\right) + \frac{1}{2N^2}\sum_{1\leq i,j\leq N} F\left(X_i - X_j\right) \tag{II}$$

for all $\mathcal{X} = \left(X_1, \cdots, X_N\right) \in \mathbb{R}^N$. The motion of the process $\left(\mathcal{X}_t\right)_{t\in\mathbb{R}_+}$ is subject to three concurrent forces. The first one is the gradient of the diagonal potential $\frac{1}{N}\sum_{j=1}^N V(X_j)$. The second term represents the average tension of the interacting potential F between the coordinates. The third influence is a heat process $(\sqrt{\epsilon}\,\mathcal{B}_t)_{t\in\mathbb{R}_+}$ which allows the particle to escape from the stable domains of the meta-potential Υ^N. The first two forces generate the meta-potential Υ^N. The division by N in (II) aims to stress the influence of N in the line level. Indeed, Lemma 5.3 in [Tug10a] tells us that for all probability law μ on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure, the following limit holds:

$$\Upsilon^N\left(X^1, \cdots, X^N\right) \longrightarrow \Upsilon_0(\mu) := \int_{\mathbb{R}}\left\{V(x) + \frac{1}{2}F*\mu(x)\right\}\mu(x)dx \tag{III}$$

where $\left(X^i\right)_{i\in\mathbb{N}^*}$ is an iid sequence of random values with law μ. Here, $*$ denotes the convolution.

When N goes to infinity, each particle can be seen as a diffusion in $\mathbb{R}$ which satisfies the following non-linear stochastic differential equation:

$$\begin{cases} X_t = X_0 + \sqrt{\epsilon}B_t - \int_0^t V'\left(X_s\right)ds - \int_0^t F' * u_s\left(X_s\right)ds \\ u_s = \mathcal{L}\left(X_s\right) \end{cases} . \tag{IV}$$

Also, the law u_t can be seen as the limit of the whole system. The own law of the McKean-Vlasov process X_t intervenes in the drift. Consequently, it is non markovian, the nonlinearity appearing through the convolution with the law u_s. We call it a self-stabilizing process. Let us give briefly some of the previous works on these diffusions (IV). For the existence problem, see [McK67, BRTV98, CGM08, Mél96, HIP08, Tug10a]. In [McK67], it has been proved that the probability measure u_t admits a $\mathcal{C}^\infty$-continuous density - that we write u_t for simplicity - with respect to the Lebesgue measure for all $t > 0$ and the density satisfies the following non-linear partial differential equation:

$$\frac{\partial}{\partial t}u_t = \frac{\partial}{\partial x}\left\{\frac{\epsilon}{2}\frac{\partial}{\partial x}u_t + u_t\Big(V' + F' * u_t\Big)\right\} . \tag{V}$$

When V is a double-well potential, this permits to show in [HT10a] that there is non-uniqueness of the stationary measures for ϵ small enough. The same result has been stated in dimension d and with less hypotheses, see [Tug11c]. The exact number of stationary measures and their behavior in the small-noise limit has been the subject of [HT10b, HT09, Tug11a, Tug11b, Tug11c]. This permits to study the convergence in long time. See [BRV98, Mal03, Ver06, BCCP98, CMV03, CGM08] in the convex case ; when there is a unique stationary measure.

The long-time behavior of the law u_t in the non-convex case has been proved in [Tug10b, Tug11d]: u_t converges weakly towards a stationary measure under assumptions easy to verify. The main tool is the following so-called free-energy:

$$\Upsilon_\epsilon(u) := \int_{\mathbb{R}}\left\{\frac{\epsilon}{2}\log(u) + V + \frac{1}{2}F * u\right\}u \tag{VI}$$

for all the measures u which are absolutely continuous with respect to the Lebesgue measure. Let us recall that u_t satisfies this hypothesis for all $t > 0$. We can observe that the last two terms of Υ_ϵ correspond to Υ_0.

The link between the non-linear process and the mean-field system for N tending to ∞ is called the propagation of chaos, see [Szn91, BRTV98, Mal01, Mal03] under hypotheses different on V and F from the ones of this paper. Ben Arous and Zeitouni proved chaoticity for a non-finite number of coordinates in [BAZ99]: $\kappa(N)$ particles become independent when N tends to infinity provided that $\kappa(N) = o(N)$. A sharp estimate is provided in [BGV07, DPdH96]. Cattiaux, Guillin and Malrieu gived a uniform result with respect to the time in the non-uniformly convex case, see [CGM08].

Let us note also some works about the propagation of chaos with different hypotheses about the dynamic or the phase space: [Gra90, Gra92, Der03, JM08, DF99].

Exit time of diffusions have already been studied when N is fixed and when the coefficient diffusion $\sqrt{\epsilon}$ tends to 0. Indeed, for classical diffusions, Freidlin and Wentzell (see [DZ10, FW98]) proved a Kramer's type law theorem. In the case of the mean-field system (I), the exit time of an open set $\mathcal{O}\subset\mathbb{R}^N$ which contains at least one wells of the meta-potential Υ^N is exponentially equivalent to $\exp\left[\frac{2N}{\epsilon}H\right]$ with

$$H:=\inf_{x\in\partial\mathcal{O}}\Upsilon^N(x)-\inf_{y\in\mathcal{O}}\Upsilon^N(y)\,.$$

However, in this paper, we do not want ϵ to be small but N to be large and we can not apply this method even in the small-noise asymptotic since an interchange between the two limits is not possible.

In [DG87a], Dawson and Gärtner considered the empirical measure associated to the diffusion (I) as a small perturbation (with respect to N) of the law $(u_t)_{t\geq0}$ satisfying (V). And, by extending Freidlin-Wentzell theory to an infinite dimensional space, they provided links between exit-time results when N tends towards ∞ of $\mathcal{X}_t$ and convergence in long time of the self-stabilizing process. Indeed, it has been proved that the empirical law of the mean-field system satisfies a large deviations principle with a rate function which depends on the law u_t. Consequently, the long-time behavior of $\mathcal{L}\left(X_t\right)$ provides some consequences on the exit time for the particle system (I). See [DG87b] for a resume.

However, here, we will deal with more general settings since the confining potential V is not assumed to be even and since the interacting potential F is authorized to have a degree more than 2. Despite this, we will provide much simpler proof by using the propagation of chaos, the convergence established in [Tug10b, Tug11d] and the result about the phase transition proved in [Tug11a, Tug11c].

The paper is organized as follows. First, the assumptions and the notations are presented in first section with some of the results about self-stabilizing diffusions and classical lemmas which will be used subsequently. The second section deals with the potential geometry, particularly the number of wells. Propagation of chaos is proved in third section then used for obtaining the main results. Let us providing the statements of these main results:

Number of wells *If V is even and V'' is convex, let us call c the unique positive real such that $V''(c)=0$. Therefore, the inequality $V'(c)+\frac{1}{2}F'(2c)<0$ implies that Υ^N admits $2^N(1-o(1))$ wells.*
If there exists a wells a_0 of V and $b\neq a_0$ such that $V'(b)+F'(b-a_0)=0$ and $V''(b)+F''(b-a_0)>0$, the number of wells of Υ^N tends towards infinity hen N goes to infinity.

Stability of the balls *We assume here that V is even and V'' convex and that ϵ is sufficiently large such that the diffusion* (IV) *admits a unique stationary* *measure:* u_ϵ^0. *Let a law* u_0 *with finite free-energy and which verifies the classical hypotheses for the existence and the uniqueness of a solution for* (IV). *We consider a sequence of iid random values with law* u_0: $\left(X_0^i\right)_{i\geq 1}$. *For all* $N \geq 1$, *we call* $\mathcal{X}_0^N := \left(X_0^1, \cdots, X_0^N\right)$. *For all* $r > \sqrt{\mathrm{Var}\left(u_\epsilon^0\right)}$, *there exists* $T_r \geq 0$ *such that for all* $t \geq 0$, *we have*

$$\lim_{N\to+\infty} \mathbb{P}\left\{\sup_{T_r \leq s \leq T_r+t} \frac{1}{N}\sum_{i=1}^N \left(X_s^i\right)^2 < r^2\right\} = 1\,.$$

Stability of the positive half-space *Let us assume that* V *is even. Let* ϵ *small enough and a law* u_0 *such that* $\Upsilon_0(u_0) < \inf\left\{\Upsilon_0(\mu) \,:\, \int_{\mathbb{R}} x\mu(x)dx = 0\right\}$, $\mathbb{E}(u_0) > 0$ *and which verifies the classical hypotheses for the existence and the uniqueness of a solution for* (IV). *We consider a sequence of iid random values with law* u_0: $\left(X_0^i\right)_{i\in\mathbb{N}^*}$. *Then, for all* $t \geq 0$, *we have:*

$$\lim_{N\to+\infty} \mathbb{P}\left\{\inf_{0\leq s\leq t} \sum_{i=1}^N X_s^i > 0\right\} = 1\,.$$

Assumptions

We assume the following properties on the confining potential V:

(V-1) V is a polynomial function with $\deg(V) =: 2m \geq 4$.

(V-2) The equation $V'(x) = 0$ admits exactly three solutions: $a_-, 0, a_+$. The critical points will be denoted generally a_0.

(V-3) $V(x) \geq C_4x^4 - C_2x^2$ for all $x \in \mathbb{R}$ with $C_2, C_4 > 0$.

(V-4) $\lim_{x\to\pm\infty} V''(x) = +\infty$ and $V''(x) > 0$ for all $x \notin [a_-; a_+]$.

(V-5) V'' is convex.

We would like to stress that weaker assumptions could be considered but all the mathematical difficulties are present in the polynomial case and it permits to avoid some technical and tedious computations. Eventually, we will assume the following additional hypotheses:

(V-6) V is even. Then, we write a the positive wells and $-a$ the negative one. Also, we call c the unique positive point such that $V''(x) = 0$.

(V-7) V is even and for all $k \geq 2$, $V^{(2k)}(0) \geq 0$.

Let us present now the assumptions on the interaction potential F:

(F-1) F is an even polynomial function with $\deg(F) =: 2n \geq 2$.

(F-2) F and F'' are convex.

(F-3) Initialization: $F(0) = 0$.

In the subsequent, the initial law u_0 satisfies

(ES) The $8q^2$-th moment of the measure u_0 is finite with $q := \max\{m, n\}$.

(FE) The probability measure u_0 admits a $\mathcal{C}^\infty$-continuous density u_0 with respect to the Lebesgue measure. And, the entropy $\int_{\mathbb{R}} u_0 \log(u_0)$ is finite.

Under (ES), we know by Theorem 2.12 in [HIP08] that (IV) admits a strong solution. Moreover, there exists $M_0 > 0$ such that:

$$\max_{j \in [\![1;8q^2]\!]} \sup_{t \in \mathbb{R}_+} \mathbb{E}\left[|X_t|^j\right] \leq M_0 \,. \tag{VII}$$

We deduce immediately that the family $(u_t)_{t\in\mathbb{R}_+}$ is tight. The assumptions (FE) and (ES) ensure that the free-energy is finite.
In the following, we will need two constants introduced in [HT10a]:

$$\alpha := F''(0) = \inf_{z\in\mathbb{R}} F''(z) \geq 0 \quad \text{and} \quad \vartheta := \sup_{z\in\mathbb{R}} -V''(z)\,.$$

We shall use occasionnaly one of the following additional properties:

(LIN) F' is linear.

(SYN) $\alpha - \vartheta > 0$.

We call $\mathcal{S}_\epsilon$ the set of all the stationary measures for (IV). Here, we assume:

(D) $\mathcal{S}_\epsilon$ is discret.

We know by Theorem 2.1 in [Tug11d] that under (D), u_t converges weakly towards a stationary measure for ϵ small enough if u_0 which verifies (FE) and (ES). Under the assumptions (V-1)–(V-5), (F1)–(F-3) and (SYN), we know that there is a finite number of stationary measures for ϵ small enough, see [HT10a, HT10b, HT09, Tug11a].

1 Preliminaries

Let us now present the material which will be used in the following sections. First, let us note that for each $x \in \mathbb{R}$, $\mathcal{E}(x)$ denotes the unique integer such that $x - 1 < \mathcal{E}(x) \leq x$ and $\binom{n}{p} := \frac{n!}{p!(n-p)!}$ is the binomial coefficient.

Now, we present some notations linked to the spaces $\mathbb{R}^N$. By convention, we consider that $\mathbb{R}^\infty$ is the set of the measures on $\mathbb{R}$. In the following, $\mathcal{X}$ is an arbitrary element of $\mathbb{R}^N$ and the i-th coordinate is written X_i, when $N < \infty$. Let us introduce some definitions:

Definition 1.1. **1)** *For all* $\mathcal{X} := (X_1, \cdots, X_N) \in \mathbb{R}^N$*, we consider the norm* $||\mathcal{X}||_N := \sqrt{\frac{\sum_{i=1}^N X_i^2}{N}}$.

2) *For all* $r > 0$*,* $\mathbb{B}_r^N$ *denotes the set* $\left\{\mathcal{X} \in \mathbb{R}^N : ||\mathcal{X}||_N \leq r\right\}$*. And, for all* $\mathcal{X} \in \mathbb{R}^N$*:* $\mathbb{B}_r^N(\mathcal{X}) := \mathbb{B}_r^N + \mathcal{X}$.

3) *We consider also the half-space* $\mathbb{E}_+^N := \left\{\mathcal{X} \in \mathbb{R}^N : \sum_{i=1}^N X_i > 0\right\}$.

4) *For each* $\mathcal{X}_0 \in \mathbb{R}^N$*, we call* $\mathcal{X}_t$ *the mean-field system* (I) *starting by* $\mathcal{X}_0$ *and* X_t^i *the i-th coordinate of* $\mathcal{X}_t$.

5) *For all* $m \in \mathbb{R}$*, we note* $\mathcal{H}_m^N := \left\{\mathcal{X} \in \mathbb{R}^N : \frac{1}{N}\sum_{i=1}^N X_i = m\right\}$.

We give now the notion of signature.

Definition 1.2. *Let* $N \in \mathbb{N}$ *and* $\mathcal{X} \in \mathbb{R}^N$*. We say that* $\mathcal{X}$ *has the signature* $(p, 1-p)$ *with* $p \in \frac{1}{N}[\![0; N]\!]$ *if* $\#\{i \in [\![1; N]\!] : X_i > 0\} = pN$.
Let a sequence $\mathcal{X} \in \mathbb{R}^{\mathbb{N}}$*. We say that* $\mathcal{X}$ *has the signature* $(p, 1-p)$ *with* $p \in [0; 1]$ *if* $\lim_{N\to+\infty} \frac{\#\{i\in[\![1;N]\!] : X_i>0\}}{N} = p$*. Obviously, if* $\mu \in \mathbb{R}^\infty$ *is a measure, we say that it has the signature* $(p, 1-p)$ *if and only if* $\mu(\{0\}) = 0$ *and* $\mu(\mathbb{R}_+) = p$.

Immediately, if $\mathcal{X}$ is a sequence of iid random values with law u_0 absolutely continuous with respect to the Lebesgue measure, $\mathcal{X}$ has the signature $\left(\int_{\mathbb{R}_+} u_0(x)dx, \int_{\mathbb{R}_-} u_0(x)dx\right)$.

Definition 1.3. *Let* $p \in [0; 1]$ *and* $N \in \mathbb{N} \cup \{+\infty\}$*. We call* $\mathbb{S}_p^N$ *the set of the elements* $\mathcal{X} \in \mathbb{R}^N$ *which have the signature* $(p, 1-p)$.

Remark 1.4. *The particularity of the random dynamical system that we consider is its invariance for each element* $\sigma \in \mathcal{S}_N$ *where* $\mathcal{S}_N$ *is the set of all the permutations of the set* $[\![1; N]\!]$*. Consequently, we will not work on* $\mathbb{R}^N$ *but on* $\mathbb{R}^N/\mathcal{S}_N$*. For simplifying the reading, we will not specify that we consider class of equivalence instead of elements of* $\mathbb{R}^N$*. Particularly, for* $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^N$*, the expression* $\frac{1}{N}\sum_{i=1}^N (X_i - Y_i)^2$ *will be in fact a notation which corresponds to* $\frac{1}{N}\inf_{\sigma\in\mathcal{S}_N} \sum_{i=1}^N \left(X_{\sigma(i)} - Y_i\right)^2$.

Let us stress that this last expression corresponds to the Wasserstein distance between two measures with support containing N elements at most. Also, we can remark that $\mathbb{E}_+^N$, $\mathbb{B}_r^N$ and $\mathbb{S}_p^N$ are invariant under the actions of any permutations so we can use these three sets without paying attention on the difference between $\mathbb{R}^N$ and $\mathbb{R}^N/\mathcal{S}_N$.

Since we use the classes of equivalence of $\mathbb{R}^N$ with respect to the permutations instead of $\mathbb{R}^N$ itself, we introduce some particular classes in order to make the reading easier.

Definition 1.5. **1)** *Let* $x \in \mathbb{R}$*, we define the vector* $\overline{x} \in \mathbb{R}^N$ *as a pure state with coordinates equal to* x*. In other words:* $\overline{x} := (x, \cdots, x)$.

2) *Let* $a, b \in \mathbb{R}$ *and* $p \in]0; 1[$*. The class of vectors* (a, b, p) *denotes all the bi-state* *vectors with $\mathcal{E}(pN)$ coordinates equal to a and $N-\mathcal{E}(pN)$ coordinates equal to b. Let us note that this class contains exactly $\binom{N}{\mathcal{E}(pN)}$ elements of $\mathbb{R}^N$.*

We recall now some previous results about the stationary measures from [HT10a, HT10b, Tug11a].

Definition 1.6. *We say that a critical point $a_0\in\{a_-;0;a_+\}$ of V admits an outlying stationary measure if for all $\delta>0$, there exists $\epsilon_0>0$ such that for all $\epsilon<\epsilon_0$, the diffusion* (IV) *admits a stationary measure $u_\epsilon^{a_0}$ which verifies*

$$\left|\int_{\mathbb{R}} x^k u_\epsilon^{a_0}(x)dx - a_0^k\right| \leq \delta\,. \tag{1.1}$$

Furthermore, $u_\epsilon^{a_0}$ is a - non-necessary unique - outlying stationary measure.

If V is even, it has been proved in [Tug11c] that a (and $-a$) admits an outlying stationary measure under (V-1)–(V-5). Uniqueness of outlying stationary measure around $\pm a$ has been proved in [HT09].
The exact number of stationary measures and the phase transition have been studied in [Tug11a]. In particular, let us recall Theorem 2.1, under the hypothesis (V-7) and (LIN):

Theorem 1.7. *If $V^{(2n)}$ is convex for all $n\geq 1$ and if $F(x):=\alpha\frac{x^2}{2}$, there exists $\epsilon_c\in\mathbb{R}$ such that:*

- *For all $\epsilon\geq\epsilon_c$, Diffusion* (IV) *admits a unique stationary measure: u_ϵ^0.*
- *For all $\epsilon<\epsilon_c$, Diffusion* (IV) *admits exactly three stationary measures: u_ϵ^0, u_ϵ^+ and u_ϵ^- with $\pm\int_{\mathbb{R}} xu_\epsilon^\pm(x)dx>0$.*

Moreover, the critical value ϵ_c is the unique solution of the equation:

$$\int_{\mathbb{R}_+}\left(x^2-\frac{1}{2\alpha}\right)\exp\left[-\left(\alpha+V''(0)\right)x^2-\sum_{p=2}^{m}\frac{2\epsilon^{p-1}V^{(2p)}(0)}{(2p)!}x^{2p}\right]dx=0\,. \tag{1.2}$$

The asymmetrical case has also been studied:

Theorem 1.8. *Let $V(x):=\frac{x^4}{4}+\frac{\gamma}{3}x^3-\frac{\rho}{2}x^2$ with $\rho>0$ and $\gamma>0$. Let $V(x):=\frac{\alpha}{2}x^2+\frac{\beta}{4}x^4$ with $\alpha\beta\geq 0$ and $\alpha+\beta>0$. Then, there exists $\alpha_c>0$, $\epsilon_c>0$ and $\epsilon_0>0$ such that Diffusion* (IV) *admits exactly:*

- *one stationary measure if $\epsilon>\epsilon_0(\alpha,\beta)$.*
- *one stationary measure if $\epsilon<\epsilon_c(\alpha,\beta)$ and $\alpha\leq\alpha_c$.*
- *three stationary measures if $\epsilon<\epsilon_c(\alpha,\beta)$ and $\alpha>\alpha_c$.*

Let us recall Theorem 5.4 in [HT10b]:

Proposition 1.9. *Let us assume that V is even. The symmetric stationary measure u_ϵ^0 converges weakly in the small-noise limit towards $\frac{1}{2}\delta_{x_0}+\frac{1}{2}\delta_{-x_0}$ where x_0 is the unique solution of*

$$\begin{cases} V'(x)+\frac{1}{2}F'(2x)=0 \\ V''(x)+\frac{1}{2}F''(0)+\frac{1}{2}F''(2x)\geq 0 \end{cases}.$$

We also proved in [Tug11c] (Proposition 3.11):

Proposition 1.10. *Let us assume that a_0 admits an outlying stationary measures. Then, $u_\epsilon^{a_0}$ converges weakly towards δ_{a_0} in the small-noise limit.*

We recall Corollary 2.2 in [Tug11c]:

Lemma 1.11. *A stationary measure of the diffusion* (IV) *is uniquely determinated by its moments.*

In order to conclude the preliminaries, we put two lemmas which will be used in the next section. The first one is about linear algebra:

Lemma 1.12. *Let $a,b\in\mathbb{R}$, $c\in\mathbb{R}_-$, $N\geq 1$ and $k\in[\![1;N-1]\!]$. Let I_k the identity matrix with size k and J_k the matrix whose each coordinate is equal to 1 and with size k. We define in the same way I_{N-k} and J_{N-k}. Finally, we define the following matrix per blocks:*

$$M:=\begin{pmatrix} (a-c)I_k+cJ_k & (c) \\ (c) & (b-c)I_{N-k}+cJ_{N-k} \end{pmatrix}$$

where all the coefficients of the two blocks denoted by (c) (which are differents) are equal to c. If $a+(N-1)c>0$ and $b+(N-1)c>0$ then $M>0$.

The proof is left to the attention of the reader. We also recall de Moivre Theorem, see (The Doctrine of Chance, Pearson ed. London, 1718):

Lemma 1.13. *Let $\delta\in\left]0;\frac{1}{2}\right[$. Then* $\lim_{N\to+\infty}2^{-N}\sum_{n=\left(\frac{1}{2}-\delta\right)N}^{n=\left(\frac{1}{2}+\delta\right)N}\binom{N}{n}=1.$

2 Potential geometry

This section is devoted to the geometry of the meta-potential Υ^N. It is immediate that $\overline{a_0}$ is a critical point of Υ^N. First, we will assume the hypothesis (SYN) that is to say: $\alpha-\vartheta>0$.

Proposition 2.1. *Let $N\geq 2$ and $\alpha>\vartheta$. Then, the meta-potential Υ^N admits three critical points: $\overline{a_-}$, $\overline{a_+}$ and $\overline{0}$. The first two ones are wells and $\overline{0}$ is a saddle whose the signature of the Hessian is $(N-1,1)$.*

Proof. **Step 1.** Let us prove that there are exactly three critical points if $V''(0)+F''(0)\geq 0$. For all $1\leq i\leq N$, the derivative of the meta-potential with respect to x_i is

$$\frac{\partial}{\partial x_i}\Upsilon^N(x_1,\cdots,x_N)=\frac{1}{N}\left\{V'(x_i)+\frac{1}{N}\sum_{j=1}^N F'\left(x_i-x_j\right)\right\}\,.$$

Let a critical point $\mathcal{X}$. We deduce $\rho_{\mathcal{X}}(x_i)=\rho_{\mathcal{X}}(x_j)$ for all the indexes i and j with $\rho_{\mathcal{X}}(x):=V'(x)+\frac{1}{N}\sum_{j=1}^N F'\left(x-x_j\right)$. But, $\rho_{\mathcal{X}}'(x)=V''(x)+\frac{1}{N}\sum_{j=1}^N F''\left(x-x_j\right)\geq-\vartheta+\alpha>0$ for all $x\in\mathbb{R}$. We deduce directly $x_i=x_j$ for all the indexes i and j. Consequently, there exists $x\in\mathbb{R}$ such that $\mathcal{X}=\overline{x}$. We obtain $V'(x)=0$. Then $x\in\{a_-,0,a_+\}$.

Step 2. We compute the Hessian of Υ^N on the points $\overline{x}$:

$$\frac{\partial^2}{\partial x_i^2}\Upsilon^N\left(x,\cdots,x\right)=\frac{1}{N}\left\{V''(x)+F''(0)\left(1-\frac{1}{N}\right)\right\}$$

$$\text{and}\quad\frac{\partial^2}{\partial x_i\partial x_j}\Upsilon^N\left(x,\cdots,x\right)=-\frac{F''(0)}{N^2}\,.$$

We apply Lemma 1.12 and we deduce that the Hessian is strictly positive if it is in $\overline{a_-}$ or $\overline{a_+}$. And, a simple computation tells us that the two eigenvalues in $\overline{0}$ are $F''(0)+V''(0)>0$ associated to an eigenspace of dimension $N-1$ and $V''(0)<0$ which achieves the proof. ☐

According to Proposition 1.3 in [Tug11a], under (LIN), the diffusion (IV) admits an outlying stationary measure around 0 for ϵ small enough if we have

$$V(x)+\frac{F''(0)}{2}x^2>0\quad\text{for all}\quad x\neq 0\,.\tag{2.1}$$

If $V''(0)+F''(0)>0$, (2.1) holds which proves the existence of an outlying stationary measure near δ_0 for ϵ small enough. But, $\overline{0}$ is *never* a wells of Υ^N when $V''(0)+F''(0)>0$.

This points out the importance of the entropy and ϵ since there is no correspondance between the wells of Υ^N and the stationary measures of (IV).

Now we will study the critical points when $V''(0)+F''(0)<0$. We will split between the symmetric case and the asymmetric one.

Theorem 2.2. *Let us assume that V is even. If $V''(0)+F''(0)<0$, the meta-potential Υ^N admits $2^N(1-o(1))$ critical points.*
Furthermore, if $V'(c)+\frac{1}{2}F'(2c)<0$, Υ^N admits $2^N(1-o(1))$ wells.

Let us remark that the bifurcation $V'(c)+\frac{1}{2}F'(2c)=0$ already appeared in the proof of Theorem 3.2 in [HT10a], in the particular case (LIN).

Proof. Since now, we assume that N is even.

Step 1. First, we look at the critical points. The partial derivative with respect to the coordinate x_i is:

$$\frac{\partial}{\partial x_i}\Upsilon^N\left(x_1,\cdots,x_N\right)=\frac{1}{N}V'(x_i)+\frac{1}{N^2}\sum_{j=1}^N F'(x_i-x_j)\,.$$

Since the third derivative of the application $x\mapsto V'(x)+\frac{1}{N}\sum_{j=1}^N F'(x-x_j)$ is nonnegative, we deduce that for all $\mathcal{X}\in\mathbb{R}^N$ such that $\nabla\Upsilon^N\left(\mathcal{X}\right)=\overline{0}$, $\mathcal{X}$ has the form $\overline{x}$ or (a_1,a_2,p). Let $p\in\frac{1}{N}[\![1;N-1]\!]$. We have now to solve

$$\Psi_1(a_1,a_2):=V'(a_1)-V'(a_2)-F'(a_2-a_1)=0 \tag{2.2}$$

$$pV'(a_1)+(1-p)V'(a_2)=0\,. \tag{2.3}$$

In the case $a_1=a_2=x$, we find directly $x\in\{a,0,-a\}$ and p does not have any importance. If $a_2\neq a_1$, we assume $a_2>a_1$. Elementary remarks lead to $-a<a_1<0<a_2<a$.
According to Theorem 5.4 in [HT10b], we know that $(-x_0,x_0,\frac{1}{2})$ is a wells of Υ^N. Consequently $\Psi_1(-x_0,x_0)=0$.
We note that $\frac{\partial\Psi_1}{\partial a_1}(-x_0,x_0)=V''(x_0)+F''(2x_0)=\chi'(x_0)$ with $\chi(x):=V'(x)+\frac{1}{2}F'(2x)$. Theorem 5.4 in [HT10b] provides

$$V''(x_0)+F''(2x_0)+\frac{F''(2x_0)-F''(0)}{2}\geq 0$$

which implies $\chi'(x_0)\geq 0$.
Let us prove that $\chi'(x_0)>0$ by proceeding a *reductio ad absurdum*. We assume $\chi'(x_0)=0$. Hypotheses (V-5) and (F-3) imply the convexity of χ'. We deduce: $\chi'(z)\leq 0$ for all $z\in[-x_0;x_0]$. Since $\chi'(0)=V''(0)+F''(0)<0$ and since χ' is continuous, we deduce that $0=\chi(x_0)<\chi(-x_0)=0$. Therefore, we have the inequality $\chi'(x_0)>0$.
It leads to $\frac{\partial\Psi_1}{\partial a_1}(-x_0,x_0)>0$. We apply the implicit function theorem and we obtain the existence of a bijection a_2 from the interval $]-x_0-\rho_1;-x_0+\rho_1[$ to the interval $]x_0-\rho_2;x_0+\rho_2[$ such that $\Psi_1(a_1,a_2(a_1))=0$ for all $a_1\in\,]-x_0-\rho_1;-x_0+\rho_1[$. Moreover, $a_2(-x_0)=x_0$.
Step 2. We look at the equation (2.3). Let us introduce $\Psi_2(p,a_1):=pV'(a_1)+(1-p)V'(a_2(a_1))$. We already know that $\Psi_2(\frac{1}{2},-x_0)=0$. Since $x_0\in]0;a[$, we have $\frac{\partial\Psi_2}{\partial p}(\frac{1}{2},-x_0)=V'(-x_0)-V'(x_0)=-2V'(x_0)>0$.
By applying the implicit function theorem, we deduce the existence of two bijections a_1 and a_2 (we keep the same name for the comfort of the reading) from $\left]\frac{1}{2}-\rho_3;\frac{1}{2}+\rho_3\right[$ to $]-x_0-\rho_4;-x_0+\rho_4[$ and $]x_0-\rho_5;x_0+\rho_5[$ such that $(a_1(p),a_2(p),p)$ is a critical point of Υ^N if $pN\in\mathbb{N}$.
Then, for all the natural number N, for all $k\in\left[\!\left[\left(\frac{1}{2}-\rho_3\right)N;\left(\frac{1}{2}+\rho_3\right)N\right]\!\right]$, the point $\left(a_1\left(\frac{k}{N}\right)\,;\,a_2\left(\frac{k}{N}\right)\,;\,\frac{k}{N}\right)$ is a critical point of the meta-potential Υ^N. By applying Lemma 1.13, we deduce that the number of critical points is equivalent to 2^N when N tends to infinity.

Step 3. From now, we assume that $V'(c) + \frac{1}{2}F'(2c) = \chi(c) < 0$. By definition of x_0, $\chi(x_0) = 0$. Since χ' is convex, $\chi'(x_0) > 0$ and $\chi'(c) = F''(c) > 0$. Hence $x_0 > d$ and $c > d$ where d is the unique positive solution of the equation $\chi'(x) = 0$. As χ is increasing on $[d; +\infty[$, the hypothesis $\chi(c) < 0$ implies $c < x_0$. The convexity of V'' implies $V''(x_0) > 0$.

Step 4. Let us study the Hessian in $\left(a_1\left(\frac{k}{N}\right)\,;\,a_2\left(\frac{k}{N}\right)\,;\,\frac{k}{N}\right)$. For simplifying the reading, let us write - until the end of this proof - $(a_1(k)\,;\,a_2(k))$ instead of $\left(a_1\left(\frac{k}{N}\right)\,;\,a_2\left(\frac{k}{N}\right)\,;\,\frac{k}{N}\right)$.

$$\frac{\partial^2}{\partial x_i^2}\Upsilon^N\left(a_1(k)\,;\,a_2(k)\right) = \frac{V''\left(a_1(k)\right)}{N} + \frac{F''(0)}{N} - \frac{F''(0)}{N^2} \quad \forall i \in [\![1;k]\!],$$
$$\frac{\partial^2}{\partial x_i^2}\Upsilon^N\left(a_1(k)\,;\,a_2(k)\right) = \frac{V''\left(a_2(k)\right)}{N} + \frac{F''(0)}{N} - \frac{F''(0)}{N^2} \quad \forall i \in [\![k+1;N]\!]$$
$$\frac{\partial^2}{\partial x_i \partial x_j}\Upsilon^N\left(a_1(k)\,;\,a_2(k)\right) = -\frac{F''(0)}{N^2} \quad \forall i,j \in [\![1;N]\!] \quad i \neq j\,.$$

By applying Lemma 1.12, if $V''\left(a_1(k)\right) > 0$ and $V''\left(a_2(k)\right) > 0$ then Υ^N is strictly convex in $(a_1(k)\,;\,a_2(k))$.
The functions a_1 and a_2 are continuous, $V''(x_0) > 0$ and $a_1(\frac{1}{2}) = -a_2(\frac{1}{2}) = -x_0$. Consequently, by restricting a_1 and a_2 to a smaller interval $\left]\frac{1}{2} - \rho_6; \frac{1}{2} + \rho_6\right[$, the two functions $V''(a_1)$ and $V''(a_2)$ are positive. By using Lemma 1.13, we deduce that the number of wells is equivalent to 2^N when N tends to infinity. □

Let us look now at the asymmetric case.

Theorem 2.3. *Let us assume that there exists $b \neq a_\pm$ such that $V'(b) + F'(b - a_\pm) = 0$ and $V''(b) + F''(b - a_\pm) > 0$. Then the number of wells of Υ^N tends towards infinity when N goes to infinity.*

Proof. We will prove it for a_-. We proceed exactly like in the proof of Theorem 2.2. We first recover the fact that a critical point of Υ^N has the form (a_1, a_2, p) with $a_- < a_1 < 0 < a_2 < a_+$ and (2.2)–(2.3) are satisfied.
A simple study of function implies the existence of $\xi(a_-) \in]0; a_+[$ such that $V'(a_-) - V'(\xi(a_-)) = F'(\xi(a_-) - a_-)$. Consequently, $(a_-, \xi(a_-), 1)$ verifies (2.2)–(2.3). Since $V''(a_-) > 0$, $V''(a_-) + F''(a_- - \xi(a_-)) > 0$ and $V'(a_-) - V'(\xi(a_-)) = V'(\xi(a_-)) < 0$, we can apply two times the implicit function theorem and we obtain the existence of two bijections a_1 (respectively a_2) from $]1-\rho; 1]$ to $]a_-; 0[$ (respectively $]0; a_+[$) such that $(a_1(p), a_2(p), p)$ is a wells of Υ^N for all $p \in]1-\rho; 1]$ which verifies $pN \in \mathbb{N}$. The sum $\sum_{p=(1-\rho)N}^{p=N} \binom{N}{p}$ tends to infinity which ends the proof. □

Theorem 2.2 and Theorem 2.3 permit to obtain a result which was previously stated in [BFG07] for a near-neighbour system that is to say the convergence towards infinity of the number of wells when N goes to $+\infty$,

Remark 2.4. *In the proofs of Theorem 2.2 and Theorem 2.3, we recovered the family of equalities (3.11) in [HT10b]. Since we restricted ourself to points*

(a_1, a_2, p) such that $V''(a_1) > 0$ and $V''(a_2) > 0$, we also recovered the family of inequalities (3.13) in [HT10b]. However, there is no correspondance between the wells of Υ^N and the stationary measures of the non-linear diffusion since we do not have necessary the family of equalities (3.12) in [HT10b] that is to say $(V(a_2) - V(a_1)) F'(a_2 - a_1) = (V'(a_2) + V'(a_1)) F(a_2 - a_1)$ in this case. However, a discrete measure is the small-noise limit of a stationary measure only if it satisfies (3.11)–(3.13).

Nevertheless, even if the number of wells tends to infinity, we will state in the following that the number of classes of steady states for the dynamic in the mean-field system (I) does not depend on N.

3 Stability and instability of the wells

We will begin to state a classical result of propagation of chaos. In other words, we will prove on a finite interval of time $[0;T]$ that the behavior of each particle of (I) is closed to the one of a self-stabilizing process (IV) when N converges towards infinity. We recall their definition:

$$X_t^i = X_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V'(X_s^i) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N F'(X_s^i - X_s^j) ds \qquad \text{(I)}$$

$$\text{and} \quad \overline{X_t^i} = X_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V'(\overline{X_s^i}) ds - \int_0^t F' * u_s(\overline{X_s^i}) ds \,, \qquad \text{(IV)}$$

where $B_t^1, \cdots, B_t^N$ are N independent Brownian Motions. We will use a method similar to the one in [BRTV98].

Proposition 3.1. *Let a probability measure u_0 which satisfies the hypotheses (ES) and (FE). Let $X_0^1, \cdots, X_0^N$ N iid random values with law u_0. Let $T > 0$. Then, there exists $C, K > 0$ such that:*

$$\max_{1 \le i \le N} \sup_{t \in [0;T]} \mathbb{E}\left\{ \left| X_t^i - \overline{X_t^i} \right|^{2p} \right\} \le \frac{C^p}{N^p} \exp\left[K p T \right] \qquad (3.1)$$

for all $p \in \mathbb{N}^*$ such that $\int_{\mathbb{R}} |x|^{2p} u_0(x) dx < \infty$.

Proof. We will start to prove it for $p = 1$. By definition, we have

$$X_t^i - \overline{X_t^i} = -\int_0^t \left\{ V'(X_s^i) - V'(\overline{X_s^i}) \right\} ds - \int_0^t \left\{ \frac{1}{N} \sum_{j=1}^N F'(X_s^i - X_s^j) - F' * u_t(\overline{X_s^i}) \right\} ds \,.$$

We apply the Itô formula to $X_t^i - \overline{X_t^i}$ with the function $x \mapsto x^2$ and by putting $\xi_i(t) := \left| X_t^i - \overline{X_t^i} \right|^2$, we obtain:

$$
\begin{aligned}
d\sum_{i=1}^N \xi_i(t) &= -2\sum_{i=1}^N \Delta_1(i,t)dt - \frac{2}{N}\sum_{i=1}^N\sum_{j=1}^N \left(\Delta_2(i,j,t) + \Delta_3(i,j,t)\right)dt \\
\text{with} \quad \Delta_1(i,t) &:= \left(X_t^i - \overline{X_t^i}\right)\left(V'(X_t^i) - V'\left(\overline{X_t^i}\right)\right), \\
\Delta_2(i,j,t) &:= \left(X_t^i - \overline{X_t^i}\right)\left[F'(X_t^i - X_t^j) - F'\left(\overline{X_t^i} - \overline{X_t^j}\right)\right] \\
\text{and} \quad \Delta_3(i,j,t) &:= \left(X_t^i - \overline{X_t^i}\right)\left[F'\left(\overline{X_t^i} - \overline{X_t^j}\right) - F' * u_t\left(\overline{X_t^i}\right)\right].
\end{aligned}
$$

Since F is even and its derivative F' is convex on $\mathbb{R}_+$ (because F'' is even and convex), we have the inequality $(x-y)\,F'(x-y) \geq \alpha\,(x-y)^2 \geq 0$. Then, $\Delta_2(i,j,t) + \Delta_2(j,i,t) \geq 0$. Indeed:

$$
\begin{aligned}
&\Delta_2(i,j,t) + \Delta_2(j,i,t) \\
=&\left(F'(X_t^i - X_t^j) - F'(\overline{X_t^i} - \overline{X_t^j})\right) \times \left\{\left(X_t^i - \overline{X_t^i}\right) - \left(X_t^j - \overline{X_t^j}\right)\right\} \\
=&\left(F'(X_t^i - X_t^j) - F'(\overline{X_t^i} - \overline{X_t^j})\right) \times \left\{\left(X_t^i - X_t^j\right) - \left(\overline{X_t^i} - \overline{X_t^j}\right)\right\} \\
\geq&\alpha\left|\left(X_t^i - X_t^j\right) - \left(\overline{X_t^i} - \overline{X_t^j}\right)\right|^2 \geq 0\,.
\end{aligned}
$$

Consequently

$$
\mathbb{E}\left\{\sum_{i=1}^N\sum_{j=1}^N \Delta_2(i,j,t)\right\} = \mathbb{E}\left\{\sum_{1\leq i<j\leq N}\left(\Delta_2(i,j,t) + \Delta_2(j,i,t)\right)\right\} \geq 0\,. \tag{3.2}
$$

Since $V'' \geq -\vartheta$, we have $(x-y)\,(V'(x) - V'(y)) \geq -\vartheta\,(x-y)^2$. This implies

$$
-2\sum_{i=1}^N \Delta_1(i,t) \leq 2\vartheta\sum_{i=1}^N \xi_i(t)\,. \tag{3.3}
$$

Now, we will deal with the double sum containing $\Delta_3(i,j,t)$. We apply the Cauchy-Schwarz inequality:

$$
\begin{aligned}
-\mathbb{E}\left[\sum_{j=1}^N \Delta_3(i,j,t)\right] &\leq \left\{\mathbb{E}\left[\left|X_t^i - \overline{X_t^i}\right|^2\right]\right\}^{\frac{1}{2}}\left\{\sum_{j=1}^N\sum_{k=1}^N \mathbb{E}\left[\rho_j\rho_k\right]\right\}^{\frac{1}{2}} \\
\text{with} \quad \rho_j &:= F'\left(\overline{X_t^i} - \overline{X_t^j}\right) - F' * u_t\left(\overline{X_t^i}\right).
\end{aligned}
$$

By conditionning with respect to $\overline{X_t^i}$ then to $\overline{X_t^j}$, we obtain: $\mathbb{E}\left[\rho_j\rho_k\right] = 0$ for $j \neq k$. Consequently, it yields

$$
-\mathbb{E}\left[\sum_{j=1}^N \Delta_3(i,j,t)\right] \quad \leq \sqrt{N\mathbb{E}\left[\xi_i(t)\right]}\left\{\mathbb{E}\left[\left|F'(X_t - Y_t) - F' * u_t(X_t)\right|^2\right]\right\}^{\frac{1}{2}}
$$

where X_t and Y_t are two independent random values with law u_t. F' is a polynomial function with degree $2n-1$ according to the hypothesis (F-1). According to (VII), there exists $C>0$ such that

$$-\mathbb{E}\left[\sum_{j=1}^{N}\Delta_3(i,j,t)\right]\quad\leq C\sqrt{N\mathbb{E}\left[\xi_i(t)\right]}\,. \tag{3.4}$$

By combining (3.2), (3.3) and (3.4), we obtain

$$\frac{d}{dt}\sum_{i=1}^{N}\mathbb{E}\left[\xi_i(t)\right]\leq 2\sum_{i=1}^{N}\left\{\vartheta\mathbb{E}\left[\xi_i(t)\right]+\frac{C}{\sqrt{N}}\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\right\}\,.$$

The invariance of the dynamic under each permutation implies that $X_t^i-\overline{X_t^i}$ and $X_t^j-\overline{X_t^j}$ have the same law. Thereby, for all $1\leq i\leq N$, we have

$$\frac{d}{dt}\mathbb{E}\left\{\xi_i(t)\right\}\leq 2\vartheta\mathbb{E}\left\{\xi_i(t)\right\}+\frac{2C}{\sqrt{N}}\sqrt{\mathbb{E}\left[\xi_i(t)\right]}$$

As $\xi_i(0)=0$, we deduce after applying the Grönwall lemma:

$$\xi_i(t)\leq\frac{C}{N}\exp\left[KT\right]\,.$$

This achieves the proof of the inequality (3.1) with $p=1$ after taking the supremum. Let us now prove (3.1) for general p by similar way:

$$\begin{aligned}d\sum_{i=1}^{N}\xi_i(t)^p=&-2p\sum_{i=1}^{N}\xi_i(t)^{p-\frac{1}{2}}\Delta_1(i,t)dt\\&-\frac{2p}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i(t)^{p-\frac{1}{2}}\left(\Delta_2(i,j,t)+\Delta_3(i,j,t)\right)dt\,.\end{aligned}$$

We can prove exactly like previously that:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i(t)^{p-\frac{1}{2}}\Delta_2(i,j,t)\geq 0\,.$$

And, by using Hölder inequality, we have:

$$-\mathbb{E}\left[\xi_i(t)^{p-\frac{1}{2}}\sum_{j=1}^{N}\Delta_3(i,j,t)\right]\leq\left\{\mathbb{E}\left[\xi_i(t)^p\right]\right\}^{1-\frac{1}{2p}}\left\{\mathbb{E}\left[\left(\sum_{j=1}^{N}\rho_j\right)^{2p}\right]\right\}^{\frac{1}{2p}}\,.$$

By conditionning, we can prove easily that $\mathbb{E}\left[\rho_j\prod_{k\neq j}\rho_k^{l_k}\right]=0$. Consequently, the only terms which do not vanish after taking expectation in the expansion are the ones with the form $\prod_{k=1}^N \rho_k^{2l_k}$ with $\sum_{k=1}^N l_k = p$. Let us consider an arbitrary partition of p: $l_1 + \cdots + l_p = p$ with $p \geq l_1 \geq \cdots \geq l_p \geq 0$. By conditionning, we obtain $\mathbb{E}\left[\prod_{k=1}^p \rho_k^{2l_k}\right] = \prod_{k=1}^p \mathbb{E}\left[\rho_k^{2l_k}\right] \leq \mathbb{E}\left[\rho_j^{2p}\right]$ for some j. However, this quantity is bounded by a constant C which does not depend on t according to (VII). The number of terms which do not vanish is equal to N^p. We deduce:

$$-\mathbb{E}\left[\xi_i(t)^{p-\frac{1}{2}} \sum_{j=1}^N \Delta_3(i,j,t)\right] \leq \sqrt{CN}\left\{\mathbb{E}\left[\left(X_t^i - \overline{X}_t^i\right)^{2p}\right]\right\}^{1-\frac{1}{2p}}.$$

We obtain finally:

$$\frac{d}{dt}\tau_i(t) \leq 2p\vartheta\tau_i(t) + \frac{C}{\sqrt{N}}\tau_i(t)^{1-\frac{1}{2p}}$$

with $\tau_i(t) := \mathbb{E}\left[\xi_i(t)^p\right]$. Applying Grönwall lemma permits to achieve the proof of the inequality (3.1) for any $p \in \mathbb{N}^*$. □

We will go further by putting the supremum on the expectation. For doing this, we will apply the inequality (3.1) with $p = 2$ and with $p = 1$.

Proposition 3.2. *Let a probability measure u_0 which satisfies the hypothesis (ES). Let $X_0^1, \cdots, X_0^N$ N iid random values with law u_0. Let $T > 0$. Then, there exists $C, K > 0$ such that:*

$$\max_{1\leq i\leq N} \mathbb{E}\left\{\sup_{t\in[0;T]}\left|X_t^i - \overline{X}_t^i\right|^2\right\} \leq \frac{CT}{N}\exp\left[K\,T\right]. \tag{3.5}$$

Proof. We use the same notations than the ones in Proposition 3.1. So we have:

$$\xi_i(t) = -2\int_0^t \Delta_1(i,s)ds - \frac{2}{N}\sum_{j=1}^N \int_0^t \left[\Delta_2(i,j,s) + \Delta_3(i,j,s)\right]ds.$$

Consequently:

$$\begin{aligned}\sup_{0\leq t\leq T}\xi_i(t) \leq & 2\int_0^T \left|\Delta_1(i,s)\right|ds \\ & + \frac{2}{N}\sum_{j=1}^N\int_0^T \Big[\left|\Delta_2(i,j,s)\right| + \left|\Delta_3(i,j,s)\right|\Big]ds.\end{aligned}$$

The inequality $V'' \geq -\vartheta$ provides immediatly

$$\mathbb{E}\left\{\sup_{0\leq t\leq T} 2\int_0^T \left|\Delta_1(i,s)\right|ds\right\} \leq 2T\vartheta \sup_{0\leq t\leq T}\xi_i(t).$$

Also, by doing exactly like in the proof of Proposition 3.1, we have:

$$\mathbb{E}\left[\sum_{j=1}^N|\Delta_3(i,j,s)|\right]\leq C\sqrt{N\mathbb{E}\left[\xi_i(s)\right]}\leq C_2\exp\left[\frac{K}{2}T\right]$$

after using (3.1) with $p=1$.
Let us study the term Δ_2 now. Thanks to the hypotheses on F, we have:

$$\mathbb{E}\left\{|\Delta_2(i,j,t)|\right\}=\mathbb{E}\left\{\left|\left(X_t^i-\overline{X_t^i}\right)\right|\times\left|F'\left(X_t^i-X_t^j\right)-F'\left(\overline{X_t^i}-\overline{X_t^j}\right)\right|\right\}$$

$$\leq\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\sqrt{\mathbb{E}\left[\left(F'(X_t^i-X_t^j)-F'\left(\overline{X_t^i}-\overline{X_t^j}\right)\right)^2\right]}$$

$$\leq C_3\sqrt{\mathbb{E}\left[\xi_i(t)\right]}$$

$$\times\sqrt{\mathbb{E}\left[\left(X_t^i-X_t^j-\overline{X_t^i}+\overline{X_t^j}\right)^2\left(1+\left|X_t^i-X_t^j\right|^{2n}+\left|\overline{X_t^i}-\overline{X_t^j}\right|^{2n}\right)^2\right]}$$

$$\leq C_4\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\left\{\mathbb{E}\left[\xi_i^2(t)\right]+\mathbb{E}\left[\xi_j^2(t)\right]\right\}^{\frac{1}{4}}$$

$$\times\left\{\mathbb{E}\left[\left(1+\left|X_t^i-X_t^j\right|^{2n}+\left|\overline{X_t^i}-\overline{X_t^j}\right|^{2n}\right)^4\right]\right\}^{\frac{1}{4}}.$$

The factor $\sqrt{\mathbb{E}\left[\xi_i(t)\right]}$ is less than $\left\{\mathbb{E}\left[\xi_i^2(t)\right]\right\}^{1/4}$ by Jensen inequality. Consequently, by applying (3.1) with $p=2$, we have:

$$\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\left\{\mathbb{E}\left[\xi_i^2(t)\right]+\mathbb{E}\left[\xi_j^2(t)\right]\right\}^{\frac{1}{4}}\leq\frac{C'}{N}\exp\left[KT\right].$$

The last factor is bounded by a constant C'' which does not depend on T. This implies: $\mathbb{E}\left\{|\Delta_2(i,j,t)|\right\}\leq\frac{C_5}{N}\exp[KT]$ which achieves the proof. □

This result is not uniform with respect to the time. Indeed, the factor $\exp\left[KT\right]$ tends to infinity when T tends to infinity. We will now justify that there is no propagation of chaos uniform with respect to the time.

Definition 3.3. *There is uniform propagation of chaos of the system* (I) *to the system* (IV) *if there exists a positive function η which vanishes when N tends towards $+\infty$ such that*

$$\sup_{t\geq0}\mathbb{E}\left\{\left|X_t^i-\overline{X_t^i}\right|^2\right\}\leq\eta(N)\quad\textit{for all}\quad 1\leq i\leq N\,. \tag{3.6}$$

This uniform propagation of chaos has been stated and used in the convex case in [CGM08] for obtaining a convergence of the process. However, in this non-convex case, there is no possible uniformity:

Proposition 3.4. *Let us assume a uniform propagation of chaos with respect to time. Then the diffusion* (IV) *admits at most one stationary measure.*

Proof. The proof is similar to Step 2. and Step 4. in the proof of Proposition 2.2 in [Tug11a] with V and F instead of V_0 and F_0. The main idea is based on a coupling method. □

Remark 3.5. *It has been proved in [Tug11c] that there is non-uniqueness of stationary measures if ϵ is small enough. Consequently, we can not prove a general result of propagation of chaos if we do not take into account the diffusion coefficient $\sqrt{\epsilon}$.*

We begin by providing a general theorem and after, we will derive all the results from it. Before, we provide the following general lemma:

Lemma 3.6. *Let $T>0$. Let $\left(\overline{X_0^i}\right)_{i\in\mathbb{N}^*}$ a sequence of iid random variable with law u_0 which satisfies (FE) and (ES). We consider also a sequence of independent Brownian motions $\left(B^i\right)_{i\in\mathbb{N}^*}$. We introduce the self-stabilizing processes $\overline{X_t^i}$ starting from $\overline{X_0^i}$. Then the sequence of empirical measure $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{\overline{X^i}}$ converges in law and in probability towards $(u_t)_{t\in[0;T]}$ on the Skorokhod path space $\mathbb{D}\left([0;T]\,;\mathbb{R}\right)$.*

Proof. We proceed exactly like in the proof of Theorem 4.4 in [Mél96]. Let π^N be the law of the random measure μ^N. The three arguments are the following:

- The sequence $\left(\pi^N\right)_{N\in\mathbb{N}^*}$ is tight.
- Each adherence value of this sequence satisfies the martingale problem associated to (IV).
- There is a unique solution to the martingale problem.

For the last point, see [HIP08]. The tightness is a consequence of Proposition 4.6 in [Mél96]. Indeed, this proposition points out that a sequence of probability measure $\left(\pi^N\right)_{N\in\mathbb{N}^*}$ on $\mathcal{P}(E)$ where E is a polish space is tight if the sequence of intensity $\left(I(\pi^N)\right)_{N\in\mathbb{N}^*}$ is tight where $I(\mu)$ is a measure on E such that

$$\langle I(\mu)\,;\,f\rangle = \int_{\mathcal{P}(E)} \langle m\,;\,f\rangle\,\mu(dm)\,.$$

Here, the intensity measure is equal to $(u_t)_{t\in[0;T]}$ so it is tight. The identification between the limiting values and the solutions of the martingale problem is identic to the proof of Theorem 4.4 in [Mél96]. □

In order to simplify the writing, we introduce the following notation:

Definition 3.7. *Let a continuous function f from $\mathbb{R}$ to $\mathbb{R}$ and $\mathcal{X}\in\mathbb{R}^N$ with $N\in\mathbb{N}^*$. We define: $f(\mathcal{X}) := \frac{1}{N}\sum_{i=1}^N f(X_i)$. If μ is a measure absolutely continuous with respect to the Lebesgue measure, we define $f(\mu)$ as $\int_{\mathbb{R}} f(x)\mu(x)dx$.*

We are now able to provide the main result of the work:

Theorem 3.8. *Let a law u_0 which satisfies (FE) and (ES) such that u_t converges weakly towards a measure μ. We consider a sequence of iid random values with law u_0: $\left(X_0^i\right)_{i\geq 1}$. Let a $\mathcal{C}^\infty$-function f such that $|f(x)-f(y)| \leq C|x-y|\left(1+|x|+|y|\right)$ with $C>0$. For all $N\geq 1$, we put $\mathcal{X}_0^N := \left(X_0^1,\cdots,X_0^N\right)$. Then, for all $\delta>0$ and for all $t\geq 0$, we have*

$$\lim_{N\to+\infty}\mathbb{P}\left\{\sup_{0\leq s\leq t}\left|f\left(\mathcal{X}_s^N\right)-f(u_t)\right|<\delta\right\}=1\,. \tag{3.7}$$

Furthermore, there exists $T_\delta\geq 0$ deterministic such that for all $t\geq 0$, we have

$$\lim_{N\to+\infty}\mathbb{P}\left\{\sup_{T_\delta\leq s\leq T_\delta+t}\left|f\left(\mathcal{X}_s^N\right)-f(\mu)\right|<\delta\right\}=1\,. \tag{3.8}$$

Proof. We begin by proving the limit (3.7). For all $i\in[\![1;N]\!]$, let $\left(\overline{X_s^i}\right)_{s\in[0;t]}$ the diffusion (IV) starting with X_0^i. By definition, $\mathcal{L}\left(X_s^i\right)=u_s$. The triangular inequality provides:

$$\mathbb{P}\left\{\sup_{s\in[0;t]}\left|f\left(\mathcal{X}_s^N\right)-f(u_s)\right|\geq\delta\right\}\leq\mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^N\sup_{s\in[0;t]}\left|f\left(X_s^i\right)-f\left(\overline{X_s^i}\right)\right|\geq\frac{\delta}{2}\right\}$$
$$+\mathbb{P}\left\{\sup_{s\in[0;t]}\left|\frac{1}{N}\sum_{i=1}^N f\left(\overline{X_s^i}\right)-f(u_s)\right|\geq\frac{\delta}{2}\right\}.$$

The second term tends towards 0 by Lemma 3.6.
Let us focus on the first one. We take $f(x):=x$ and we apply Cauchy-Schwarz inequality and Proposition 3.2:

$$\mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^N\sup_{s\in[0;t]}\left|X_s^i-\overline{X_s^i}\right|\geq\frac{\delta}{2}\right\}\leq\frac{2}{\delta}\mathbb{E}\left\{\sup_{s\in[0;t]}\left|X_s^i-\overline{X_s^i}\right|\right\}$$
$$\leq\frac{2}{\delta}\sqrt{\frac{Ct}{N}}\exp\left[\frac{Kt}{2}\right]\,.$$

Let us consider now general function f. We will prove that for all $t>0$, we have the inequality:

$$\mathbb{E}\left[\sup_{s\in[0;t]}\left|X_s^i\right|\right]+\mathbb{E}\left[\sup_{s\in[0;t]}\left|\overline{X_s^i}\right|\right]<\infty\,. \tag{3.9}$$

Since $\mathbb{E}\left[\left|\overline{X_s^i}\right|^j\right]\leq M_0$ defined in (VII) for all $i\in[\![1;N]\!]$, $j\in[\![1;8q^2]\!]$ and $s\geq 0$, we deduce that the same holds for X_s^i by using the propagation of chaos stated in Proposition 3.1. Using the same argument than the one at the end of the proof of Proposition 3.2 permits to prove (3.9). The condition on f and C, the Cauchy-Schwarz inequality and Proposition 3.2 imply

$$\mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^N\sup_{[0;t]}\left|f\left(X_s^i\right)-f\left(\overline{X_s^i}\right)\right|\geq\frac{\delta}{2}\right\}\leq\frac{2}{\delta}\mathbb{E}\left\{\sup_{[0;t]}\left|f\left(X_s^i\right)-f\left(\overline{X_s^i}\right)\right|\right\}$$

$$\leq\frac{2}{\delta}\sqrt{\frac{Ct}{N}}\exp\left[\frac{Kt}{2}\right]\sqrt{1+\left\{\mathbb{E}\left[\sup_{[0;t]}|X_s^i|\right]+\mathbb{E}\left[\sup_{[0;t]}\left|\overline{X_s^i}\right|\right]\right\}}\longrightarrow 0\,.$$

In order to prove the second statement, it is sufficient to note that the tightness of the family $(u_t)_{t\in\mathbb{R}_+}$ and the convergence of u_t towards μ implies the convergence of $f(u_t)$ towards $f(\mu)$ so for all $\delta>0$, there exists $T_\delta\geq 0$ such that $|f(u_t)-f(\mu)|\leq\frac{\delta}{2}$ for all $t\geq T_\delta$ then we apply the first statement with $\frac{\delta}{2}$. □

The time T_δ is deterministic and linked to the rate of convergence towards the stationary measure μ so it depends on ϵ.

With this general theorem, we will obtain five corollaries. The first one states that the mean-field system is prisonner of a ball.

Corollary 3.9. *Let us assume that V is even and that the diffusion* (IV) *admits a unique stationary measure u_ϵ. Let a law u_0 which satisfies (ES) and (FE). We consider a sequence of iid random values with law u_0: $\left(X_0^i\right)_{i\geq 1}$. For all $N\geq 1$, we put $\mathcal{X}_0^N:=\left(X_0^1,\cdots,X_0^N\right)$. Then, for all $r>\sqrt{\mathrm{Var}\left(u_\epsilon\right)}$, there exists $T_r\geq 0$ such that for all $t\geq 0$, we have*

$$\lim_{N\to+\infty}\mathbb{P}\left\{\mathcal{X}_s^N\in\mathbb{B}_r^N\left(\overline{0}\right)\ ;\ \forall T_r\leq s\leq t+T_r\right\}=1\,.$$

Proof. Theorem 2.1 in [Tug10b] states that u_t converges towards a stationary measure u_ϵ^0 when t tends to ∞. We know by Theorem 4.5 in [HT10a] that (IV) admits a symmetric stationary measure u_ϵ^0. According to the hypotheses, there is a unique stationary measure. We deduce by Theorem 2.1 in [Tug10b] that u_t converges towards u_ϵ^0. We conclude by applying Theorem 3.8 (more precisely we use the limit (3.8)) with $\delta:=r^2-\mathrm{Var}\left(u_\epsilon^0\right)$ and $f(x):=x^2$. □

We can not find smaller radius since $f(u_\epsilon^0)=\mathrm{Var}\left(u_\epsilon^0\right)$. This result means that for all $t\geq 0$, we have:

$$\lim_{N\to+\infty}\mathbb{P}\left\{\tau_N\left(\mathbb{B}_r^N\left(\overline{0}\right)\right)\leq t\right\}=0$$

where $\tau_N\left(\mathbb{B}_r^N\left(\overline{0}\right)\right)$ is the first exit time of $\mathbb{B}_r^N\left(\overline{0}\right)$. We will now provide similar result when ϵ is small for the outlying stationary measures.

Corollary 3.10. *Let a_0 a wells of V which admits a unique outlying measure $u_\epsilon^{a_0}$ for ϵ small enough. Let a law u_0 which satisfies (ES) and (FE) such that u_t converges weakly towards $u_\epsilon^{a_0}$. We consider a sequence of iid random values* *with law u_0: $\left(X_0^i\right)_{i\geq 1}$. For all $N\geq 1$, we put $\mathcal{X}_0^N:=\left(X_0^1,\cdots,X_0^N\right)$. Then, for all $r>\sqrt{\mathrm{Var}\left(u_\epsilon^{a_0}\right)}$, there exists $T_r\geq 0$ such that for all $t\geq 0$, we have*

$$\lim_{N\to+\infty}\mathbb{P}\left\{\mathcal{X}_s^N\in\mathbb{B}_r^N\left(\overline{a_0}\right)\ ;\ \forall T_r\leq s\leq t+T_r\right\}=1\,.$$

The proof is similar to the one of Corollary 3.9 so the details are left to the attention of the reader. This result means that for all $t\geq 0$, we have:

$$\lim_{N\to+\infty}\mathbb{P}\left\{\tau_N\left(\mathbb{B}_r^N\left(\overline{a_0}\right)\right)\leq t\right\}=0$$

where $\tau_N\left(\mathbb{B}_r^N\left(\overline{a_0}\right)\right)$ is the first exit time of $\mathbb{B}_r^N\left(\overline{a_0}\right)$. This implies the existence of points $\mathcal{X}_0\in\mathbb{B}_r^N\left(\overline{a_0}\right)$ such that $\mathcal{X}_t$ never leaves $\mathbb{B}_r^N\left(\overline{a_0}\right)$.
The third corollary provides sufficient condition for forbiding to cross some hyperplane of the form $\left\{\mathcal{X}\in\mathbb{R}^N\,:\,\frac{1}{N}\sum_{i=1}^N X_i=m\right\}$.

Corollary 3.11. *Let a law u_0 which satisfies (ES) and (FE). Let assume the existence of m_0 such that $\Upsilon_\epsilon(u_0)<\inf\left\{\Upsilon_\epsilon(\mu)\,:\,\int_{\mathbb{R}}x\mu(x)dx=m_0\right\}$. We consider a sequence of iid random values with law u_0: $\left(X_0^i\right)_{i\geq 1}$. For all $N\geq 1$, we put $\mathcal{X}_0^N:=\left(X_0^1,\cdots,X_0^N\right)$. Then for all $t\geq 0$, we have:*

$$\lim_{N\to+\infty}\mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^N X_s^i\neq m_0\ ;\ \forall\, 0\leq s\leq t\right\}=1\,.$$

Proof. We recall that the free-energy is nonincreasing along the orbit $(u_t)_{t\in\mathbb{R}_+}$. Consequently, $\Upsilon_\epsilon\left(u_s\right)<\inf_{\{\mu:\,\int_{\mathbb{R}}x\mu(x)dx=m_0\}}\Upsilon_\epsilon(\mu)$ for all $s\in[0;t]$. This implies $\int_{\mathbb{R}}xu_s(x)dx\neq m_0$ for all $s\in[0;t]$. We conclude by applying Theorem 3.8, more precisely the limit (3.7) with $f(x):=x$ and

$$\delta:=\inf_{s\in[0;t]}\left|\int_{\mathbb{R}}xu_s(x)dx-m_0\right|>0\,.$$

□

This result means that for all $t\geq 0$, we have:

$$\lim_{N\to+\infty}\mathbb{P}\left\{T_N\left(\mathcal{H}_m^N\right)\leq t\right\}=0$$

where $T_N\left(\mathcal{H}_m^N\right)$ is the first hitting time of $\mathcal{H}_m^N$. We can remark that under the condition

$$\max\left\{V(a_-)\,;\,V(a_+)\right\}<\inf\left\{\Upsilon_0(\mu)\,:\,\int_{\mathbb{R}}x\mu(x)dx=m_0\right\}\,,$$

if a_- and a_+ admit outlying stationary measure, for ϵ sufficiently small, we can apply this previous result with $u_0=u_\epsilon^{a_-}$ or with $u_0=u_\epsilon^{a_+}$.
Finally, the last corollary stresses the fact that the steady states do not correspond to the wells of Υ^N.

Corollary 3.12. *Let us asssume that for all $\epsilon<\epsilon_0$, the diffusion* (IV) *admits exactly three stationary measures: u_ϵ^{a-}, u_ϵ^{a+} and u_ϵ^0. Also, we assume that u_ϵ^0 converges weakly towards $p_0\delta_{A_1}+(1-p_0)\delta_{A_2}$ with $p_0\in]0;1[$ and $a_-<A_1<0<A_2<a_+$. Let a law u_0 which satisfies (ES) and (FE). We consider a sequence of iid random values with law u_0: $\left(X_0^i\right)_{i\geq1}$. For all $N\geq1$, we put $\mathcal{X}_0^N:=\left(X_0^1,\cdots,X_0^N\right)$. Let $\kappa>0$. There exists $\epsilon_1>0$ such that for all $\epsilon\in]0;\epsilon_1[$, there exists $T_\kappa\geq0$ such that for all $t>0$, we have:*

$$\lim_{N\to+\infty}\mathbb{P}\left[\mathcal{X}_s\in\bigcup_{\rho=0}^{\kappa}\left(\mathbb{S}_\rho^N\cup\mathbb{S}_{1-\rho}^N\cup\mathbb{S}_{p_0+\rho}^N\cup\mathbb{S}_{p_0-\rho}^N\right)\,,\,\forall T_\kappa\leq s\leq T_\kappa+t\right]=1$$

where $\mathbb{S}_p^N$ is defined in Definition 1.1. Here, the union is taken for $\rho\in\mathbb{R}$ such that $N\rho\in\mathbb{N}$.

Proof. The law u_t converges towards u_ϵ^0, u_ϵ^{a-} or u_ϵ^{a+} according to Theorem 2.1 in [Tug11d] and $u_\epsilon^{a\pm}$ converges towards $\delta_{a_\pm}$ when ϵ tends to 0 according to Proposition 1.10. So, for ϵ small enough, we have $\int_{\mathbb{R}}\mathbb{1}_{]0;+\infty[}(x)u_\epsilon^{a+}(x)dx\geq1-\frac{\kappa}{3}$ and $\int_{\mathbb{R}}\mathbb{1}_{]0;+\infty[}(x)u_\epsilon^{a-}(x)dx\leq\frac{\kappa}{3}$. We apply Theorem 3.8, more precisely (3.8) with $\delta:=\frac{\kappa}{3}$ and with

$$f(x):=\mathbb{1}_{]\frac{\eta}{3};+\infty[}(x)+\mathbb{1}_{[0;\frac{\eta}{3}]}(x)Z^{-1}\int_0^x\exp\left[-\frac{1}{y^2}-\frac{1}{\left(y-\frac{\kappa}{3}\right)^2}\right]dy$$

where $Z:=\int_0^{\eta/3}\exp\left[-\frac{1}{y^2}-\frac{1}{(y-\kappa/3)^2}\right]dy$ is such that $f\left(\frac{\eta}{3}\right)=1$. We take η sufficiently small for having $\left|f(u_\epsilon^0)-p_0\right|\leq\frac{\kappa}{3}$ and $\left|f(u_\epsilon^{a\pm})-(\pm1)\right|\leq\frac{\kappa}{3}$. The proof is achieved by applying Theorem 3.8 with f and $\delta:=\frac{\kappa}{3}$. ☐

This result means that for all $\kappa>0$, for all $t\geq0$, we have:

$$\lim_{N\to+\infty}\mathbb{P}\Big\{S_N\left([0;\kappa]\cup[p_0-\kappa;p_0+\kappa]\cup[1-\kappa;1]\right)\leq T_\kappa+t\Big\}=1$$

where $S_N\left([0;\kappa]\cup[p_0-\kappa;p_0+\kappa]\cup[1-\kappa;1]\right)$ is the first hitting time of the following subset of $\mathbb{R}^N$: $\bigcup_{\rho=0}^{\kappa}\left(\mathbb{S}_\rho^N\cup\mathbb{S}_{1-\rho}^N\cup\mathbb{S}_{p_0+\rho}^N\cup\mathbb{S}_{p_0-\rho}^N\right)$.

We can prove some similar result in the case where there is a unique stationary measure u_ϵ^{a-}. The same holds with a_+. The proof is left to the attention of the reader since it is exactly the same than the one of the previous:

Corollary 3.13. *Let us asssume that for all $\epsilon<\epsilon_0$, the diffusion* (IV) *admits exactly one stationary measure: u_ϵ^{a-}. Let a law u_0 which satisfies (ES) and (FE). We consider a sequence of iid random values with law u_0: $\left(X_0^i\right)_{i\geq1}$. For all $N\geq1$, we put $\mathcal{X}_0^N:=\left(X_0^1,\cdots,X_0^N\right)$. Let $\kappa>0$. There exists $\epsilon_1>0$ such that for all $\epsilon\in]0;\epsilon_1[$, there exists $T_\kappa\geq0$ such that for all $t>0$, we have:*

$$\lim_{N\to+\infty}\mathbb{P}\left[\mathcal{X}_s\in\bigcup_{\rho=0}^{\kappa}\mathbb{S}_{1-\rho}^N\,,\,\forall T_\kappa\leq s\leq T_\kappa+t\right]=1\,.$$

Here, the union is taken for $\rho\in\mathbb{R}$ such that $N\rho\in\mathbb{N}$.

Corollary 3.12 and Corollary 3.13 prove that the most of the wells with signature $(p, 1-p)$ are not stable and even if it is possible to have $2^N(1-o(1))$ wells, these points do not intervene in the dynamic that achieves to prove that the meta-potential is not sufficient for understanding the behavior of the mean-field system (I) in the large-dimension limit.

Moreover, in the asynchronized and even case, the set of relevant points is not reduced to the set of the minima of Υ^N. Indeed, the point $(x_0, -x_0, \frac{1}{2})$ is not necessary a wells in this case. And, in the synchronized and even case, $\overline{0}$ is never a wells. Then, we can not simply study Υ^N for knowing the basins of attraction of the different stationary measures for the self-stabilizing process (IV).

Before concluding, let us make the following remark:

Remark 3.14. *The value of the meta-potential Υ^N in each point (a_1, a_2, p) is $pV(a_1)+(1-p)V(a_2)+p(1-p)F(a_2-a_1)$. This implies that the different wells do not have the same values. Particularly, if the hypotheses of Theorem 2.3 are verified, for all p sufficiently large, there exists a_1 and a_2 such that (a_1, a_2, p) is a wells of Υ^N. And, its value is closed to $V(a)$. Despite this, each point (a_1, a_2, p) with $p \notin \{0; 1; p_0\}$ is irrelevant. Consequently, we can not classify the relevant points by the values taken by the meta-potential in these points. However, Corollary 3.12 also show that the set which contains all the wells of the form (a_1, a_2, p) for $1-\delta \leq p < 1$ is relevant.*

When N is fixed, the Freidlin-Wentzell theory takes into account these microscopics wells for the computations of the exit-time in the small-noise limit. However, when ϵ is fixed and when N tends towards ∞, they do not intervene in the dynamic. Moreover, this dynamic depends on ϵ.

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