Self-stabilizing processes in multi-wells landscape in $\mathbb{R}^d$ - Convergence*

Julian Tugaut
Fakultät für Mathematik
Universität Bielefeld
D-33615 Bielefeld
Germany
Email: jtugaut@math.uni-bielefeld.de

Abstract

Self-stabilizing processes are non-markovian diffusions. The own law of the process intervenes in the drift. When the non-interacting part of the drift corresponds to the gradient of a convex potential, it has been proved in some different ways that such processes converge weakly towards the unique stationary measure when the time goes to infinity. However, in the one-dimensional case, it has been pointed out that there are several stationary measures under easy to verify conditions. The convergence is then much more difficult to get. It has been obtained in a previous paper by using the free-energy. In superior dimension, we also have the non-uniqueness of the stationary measures. The aim of this paper is to study the long-time behaviour of the self-stabilizing processes even if the set of stationary measures is not finite.

Key words and phrases: Self-interacting diffuion, Free-energy, McKean-Vlasov stochastic differential equations, Multi-wells potential, Granular media equation

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Introduction

The aim of this work is the weak convergence in long time of the following non-markovian diffusion:

$$X_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t \nabla W_s(X_s) ds \tag{I}$$

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where W_s is a potential which depends on the time. Moreover, we assume that W_s is directly linked to the own law of the diffusion: u_s. In this paper, the potential W_s is given by

$$W_s(x) := V(x) + \int_{\mathbb{R}^d} F(x-y)u_s(dy) = (V + F * u_s)(x) .$$

The notation $*$ is used for denoting the convolution. V is the so-called confining potential. It corresponds to a classical drift and it permits to justify the existence of a unique strong solution. F is called the interacting potential because the term $\nabla F * u_s (X_s(\omega_0))$ corresponds to an attraction between all the trajectories $\omega \mapsto X_s(\omega)$. We can write (I) in this way:

$$\begin{cases} X_t = X_0 + \sqrt{\epsilon}B_t - \int_0^t \nabla V(X_s)\, ds - \int_0^t \nabla F * u_s (X_s)\, ds \\ u_s = \mathcal{L}(X_s) \end{cases} . \tag{I}$$

This equation is nonlinear in the sense of McKean, see [McK67, McK66]. The quantities X_t and u_t depend on ϵ but we skip this dependence in the notation for the comfort of the reading.

The convergence of the diffusion (I) towards one of its invariant probability is the subject of this article.

In this paper, we will make some smoothness assumptions on the interaction potential F. Let just note that there exist some previous works with non smooth F. If $d = 1$, if F is the Heaviside step function and with $V := 0$, (I) is the Burgers' equation:

$$X_t(\omega_0) = \sqrt{\epsilon}B_t(\omega_0) - \int_0^t \mathbb{P}\Big\{\omega \in \Omega \ \mid \ X_s(\omega) = X_s(\omega_0)\Big\} ds ,$$

see [SV79]. Still in the one-dimensional case, if $F := \delta_0$ and without confining potential, it is the Oelschläger equation:

$$X_t(\omega_0) = \sqrt{\epsilon}B_t(\omega_0) - \int_0^t \frac{d}{dx}\Big|_{x=X_s(\omega_0)} \mathbb{P}\Big\{\omega \in \Omega \ \mid \ X_s(\omega) = X_s(\omega_0)\Big\} ds$$

studied in [Oel85].

The diffusion X_t corresponds to the thermodynamical limit of a particle in a continuous mean-field system when the number of particles tends towards infinity. The mean-field system associated to the self-stabilizing process (I) is a classical diffusion in $\left(\mathbb{R}^d\right)^N$ like

$$\begin{cases} dX_t^1 = \sqrt{\epsilon}dB_t^1 - \nabla V\left(X_t^1\right)dt - \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_t^1 - X_t^j\right)dt \\ \vdots \\ dX_t^i = \sqrt{\epsilon}dB_t^i - \nabla V\left(X_t^i\right)dt - \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_t^i - X_t^j\right)dt \\ \vdots \\ dX_t^N = \sqrt{\epsilon}dB_t^N - \nabla V\left(X_t^N\right)dt - \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_t^N - X_t^j\right)dt \end{cases} \tag{II}$$

where the N brownian motions $\left(B_t^i\right)_{t\in\mathbb{R}_+}$ are independent. The link between (I) and (II) is called the propagation of chaos, see [Szn91, BRTV98, Mél96, BAZ99, CGM08]. Let us note also some works about the propagation of chaos with different hypotheses about the dynamic or the phase space: [Gra90, Gra92, Der03, JM08].

The existence problem of self-stabilizing processes (I) has been investigated by two different methods. The first one consists in the application of a fixed point theorem, see [McK67, BRTV98, HIP08]. The other method consists in using the propagation of chaos ([Mél96]).

The law of the unique strong solution u_t admits a $\mathcal{C}^\infty$-continuous density with respect to the Lebesgue measure for all $t > 0$, see [McK67]. We write it u_t. Furthermore, u_t satisfies a nonlinear partial differential equation of the following type:

$$\frac{\partial}{\partial t}u_t = \mathrm{div}\left\{\frac{\epsilon}{2}\nabla u_t + u_t\nabla W_t\right\} = \mathrm{div}\left\{\frac{\epsilon}{2}\nabla u_t + u_t\left(\nabla V + \nabla F * u_t\right)\right\}. \qquad \text{(III)}$$

The granular media equation (III) is a useful tool for characterizing the stationary measures and the long-time behavior of the McKean-Vlasov diffusion (I), see [BRTV98, BRV98, Tam84, Tam87, Ver06].

When the confining potential V is not convex - in the one-dimensionnal case - the work in [HT10a, HT10b, HT09, Tug11a, Tug11b] provides the exact number of invariant probabilities and their small-noise behavior under easy to verify assumptions. If the phase space is $\mathbb{R}^d$, it has been proved in [Tug11c] that the diffusion (I) admits several stationary measures if the interaction is sufficiently strong. Consequently, the weak convergence for t going to infinity will be more tedious than when V is convex.

In [BRV98], if V is identically equal to 0, the authors proved the convergence towards the stationary measure by using an ultracontractivity property, a Poincaré inequality and a comparison lemma for stochastic processes. The ultracontractivity property still holds if V is not convex by using the results in [KKR93]. It is possible to conserve the Poincaré inequality by using the theorem of Muckenhoupt (see [ABC$^+$00]) instead of the Bakry-Emery theorem in dimension one. In general dimension, Poincaré inequality can be obtained by using the results in [BBCG08]. But, the comparison lemma needs some convexity properties.

Another method consists in using the propagation of chaos in order to derive the convergence of the self-stabilizing process from the one of the mean-field system. However, we shall use it independently of the time and the classical result which is on a finite interval of time is not sufficiently strong. Cattiaux, Guillin and Malrieu proceeded a uniform propagation of chaos in [CGM08] and obtained the convergence in the convex case, including the non-uniformly strictly convex case. See also [Mal03]. Nevertheless, the non-uniqueness of the stationary measures pointed out in [Tug11c] implies that uniform propagation of chaos does not hold.

The method used in [Tug10b] is based on the one of [BCCP98]. See also [Mal03, Tam84, Mal01, HS87, AMTU01] for the convex case. In the non-convex case, Carrillo, McCann and Villani provided the convergence in [CMV03] under two strong restrictions: the center of mass is fixed and $V''(0)+F''(0)>0$ (that means it is the synchronized case).

However, by combining the results in [HT10a, HT10b, HT09] with the work of [BCCP98] (and the more rigorous proofs in [CMV03] about the free-energy), we have been able to prove the convergence without these two assumptions. The principal tool of the paper was the monotonicity of the free-energy along the trajectories of (III).
We will use similar method in this paper in a more general setting. Indeed, we will not need that there is a finite number of stationary measures.

The convergence plays an important role in the exit problem for the McKean-Vlasov diffusion since the exit time is strongly linked to the drift (according to the Kramers' law, see [DZ10] or [HIP08]) which converges towards a homogeneous function if the law of the process converges towards a stationary measure.

As noted previously, the diffusion (I) is similar to the particle X_t^1 defined in (II). However, Equation (II) can be rewritten:

$$d\mathcal{X}_t=\sqrt{\epsilon}\mathcal{B}_t-N\nabla\Upsilon^N\left(\mathcal{X}_t\right)dt \tag{II}$$

where the i-th coordinate of $\mathcal{X}_t$ (resp. $\mathcal{B}_t$) is X_t^i (resp. B_t^i) and

$$\Upsilon^N\left(\mathcal{X}\right):=\frac{1}{N}\sum_{j=1}^N V\left(\mathcal{X}_j\right)+\frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F\left(\mathcal{X}_i-\mathcal{X}_j\right)$$

for all $\mathcal{X}\in\left(\mathbb{R}^d\right)^N$. It yields

$$\frac{d}{dt}\mathbb{E}\left\{\Upsilon^N\left(X_t^1,\cdots,X_t^N\right)\right\}=-\mathbb{E}\left\{\left|\left|\nabla\Upsilon^N\left(X_t^1,\cdots,X_t^N\right)\right|\right|^2\right\}$$

if $\epsilon=0$. The equivalent of this potential Υ^N for the flow (III) is Υ defined as

$$\Upsilon(u):=\int_{\mathbb{R}^d}V(x)u(dx)+\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}F(x-y)u(dx)u(dy)\,.$$

However, $\epsilon>0$. Consequently, we add a term which corresponds to the entropy:

$$\Upsilon_\epsilon(u):=\frac{\epsilon}{2}\int_{\mathbb{R}^d}u\log(u)+\Upsilon(u) \tag{IV}$$

for all measures u which are absolutely continuous with respect to the Lebesgue measure. As noted previously, the law u_t satisfies this hypothesis for all $t>0$. The functionnal Υ_ϵ is called the free-energy and will play the role of a Lyapunov function that is to say that the free-energy is nonincreasing along the trajectories of the flow $(u_t)_{t\in\mathbb{R}_+}$. See [CMV03] for a proof of this statement. It has been proved in [Tug11c] that the family $\{u_t\,;\,t\in\mathbb{R}_+\}$ admits an adherence value which is an invariant probability of (I). Some elementary results are presented in the first section. Then, the convergence is proved. Finally, we postpone two technical propositions in the appendix.

Assumptions

We present the properties of the confining potential V:

Assumption (V-1): *V is a polynomial function on each coordinate $x_1, \cdots, x_d$. And, the total degree of V is* $\deg(V) =: 2m \geq 4$.

Assumption (V-2): *The equation $\nabla V(x) = 0$ admits a finite number of solution. We do not specify anything about the nature of these critical points. However, the wells will be denoted by a_0.*

Assumption (V-3): *$V(x) \geq C_4 ||x||^4 - C_2 ||x||^2$ for all $x \in \mathbb{R}^d$ with $C_2, C_4 > 0$. $||\,.\,||$ denotes the euclidian norm.*

Assumption (V-4): $\lim_{||x|| \to \pm\infty} \mathrm{Hess}\, V(x) = +\infty$ *and* $\mathrm{Hess}\, V(x) > 0$ *for all $x \notin K$ where K is a compact of $\mathbb{R}^d$ which contains all the critical points of V.*

These conditions ensure that the confining potential V confines the diffusion. It is used for proving the existence of a solution to (I), see [Tug10a].

An important constant associated to V is ϑ:

$$\vartheta := 2 \sup_{x \in \mathbb{R}^d} \sup_{z \in \mathbb{R}^d} \lim_{t \to 0} \frac{V(x) + t \langle \nabla V(x)\,;\, z \rangle - V(x + tz)}{t^2}\,.$$

In dimension one, $\vartheta = \sup_{z \in \mathbb{R}} -V''(z)$.

Let us present now the assumptions on the interaction potential F:

Assumption (F-1): *There exists an even polynomial function G on $\mathbb{R}$ such that $F(x) = G(||x||)$. And,* $\deg(G) =: 2n \geq 2$.

Assumption (F-2): *G and G'' are convex.*

Assumption (F-3): $G(0) = 0$.

An important constant will be used subsequently:

$$\alpha := G''(0) = \inf_{z \in \mathbb{R}_+} G''(z) \geq 0\,.$$

We present now the assumptions on the initial law u_0:

Assumption (ES): *The $8q^2$-th moment of the measure u_0 is finite with $q := \max\{m, n\}$.*

By Theorem 2.12 in [HIP08], we deduce that (I) admits a unique strong solution. Moreover, there exists $M_0 > 0$ such that the following inequality holds:

$$\max_{1 \leq j \leq 8q^2} \sup_{t \in \mathbb{R}_+} \mathbb{E}\left[||X_t||^j \right] \leq M_0\,. \tag{V}$$

If the $2p$-th moment of u_0 is finite, the previous inequality holds with $2p$ instead of $8q^2$. We deduce immediately that the family $(u_t)_{t \in \mathbb{R}_+}$ is tight.

Definition: *Let us introduce $\mathcal{A}_\epsilon$ the set of all the limiting value of the family $\{u_t\,;\, t \in \mathbb{R}_+\}$.*

Assumption (FE): *The measure u_0 admits a $\mathcal{C}^\infty$-continuous density u_0 with respect to the Lebesgue measure. And, the entropy $\int_{\mathbb{R}^d} u_0 \log(u_0)$ is finite.*

We call $\mathcal{S}_\epsilon$ the set of all the stationary measures for (I). We assume the following hypothesis in this work:

(D) $\mathcal{S}_\epsilon$ is discret.

This hypothesis is reasonible according to the exact countation in dimension one made in [HT10a, HT10b, HT09, Tug11a] and according to the finite number of possible limit of stationary measures pointed out in [Tug11c].

For concluding the introduction, we write the statement of the main theorem:
Theorem: *Let a probability measure u_0 which is absolutely continuous with respect to the Lebesgue measure, which admits a finite $8q^2$-th moment and a finite free-energy. We admit that the set of the invariant probabilities of the diffusion* (I) *is discret. Then u_t converges weakly towards a stationary measure.*

1 Preliminaries

We present the tools that we will use for proving the main result of the paper.In the following, we will need two particular functions: the free-energy of the system at time t and the function η_t which satisfies

$$\frac{\partial}{\partial t}u_t = \mathrm{div}\,\eta_t\,.$$

Definition 1.1. *For all $t \in \mathbb{R}_+$, we introduce the functions:*

$$\xi(t) := \Upsilon_\epsilon\left(u_t\right) \quad \textit{and} \quad \eta_t := \frac{\epsilon}{2}\nabla u_t + u_t\left(\nabla V + \nabla F * u_t\right)\,.$$

According to (III), we remark that if η_t is identically equal to 0 then u_t is a stationary measure for (I).

We recall the following well-known entropy dissipation:

Proposition 1.2. *Let a probability measure u_0 which verifies (FE) and (ES). Then, for all $t, s \geq 0$, we have*

$$\xi(t+s) \leq \xi(t) \leq \xi(0) < +\infty\,.$$

Furthermore, ξ is derivable and we have:

$$\xi'(t) \leq -\int_{\mathbb{R}^d}\frac{1}{u_t}\left|\left|\eta_t\right|\right|^2 dx\,.$$

See [CMV03] for a proof. Let us introduce the functionnal space

$$\mathcal{M}_{8q^2} := \left\{ f \in \mathcal{C}_0^2\left(\mathbb{R}^d, \mathbb{R}_+\right) \;\middle|\; \int_{\mathbb{R}^d} f = 1\,;\, \int_{\mathbb{R}^d}||x||^{8q^2} f < \infty \right\}\,.$$

We can remark that $u_t \in \mathcal{M}_{8q^2}$ for all $t > 0$, see [McK67]. The first tool is the Proposition 1.2 (that is to say the fact that the free-energy is nonincreasing along the orbits of (III)). The second one is its lower-bound. We recall now the following results:

Lemma 1.3. *There exists $L_0 \in \mathbb{R}$ such that $\Upsilon_\epsilon(u_t)$ converges towards L_0 as time goes to infinity.*

Proposition 1.4. *There exists a stationary measure u_ϵ and a sequence $(t_k)_k$ which converges towards infinity such that u_{t_k} converges weakly towards u_ϵ as k goes to infinity.*

See [Tug11c] for proofs.
An essential property on the stationary measure is the following:

Lemma 1.5. *A stationary measure of the diffusion* (I) *is uniquely determinated by its moments.*

Proof. Let a stationary measure u_ϵ. By Lemma 2.1 in [Tug11c], it satisfies the following implicit equation:

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + F * u_\epsilon(x)\right)\right]}{\int_{\mathbb{R}^d} \exp\left[-\frac{2}{\epsilon}\left(V(y) + F * u_\epsilon(y)\right)\right] dy}\,.$$

Consequently, for all $r > 0$, we have:

$$\int_{\mathbb{R}^d} e^{\frac{2}{\epsilon}r||x||} u_\epsilon(x)dx = \frac{\int_{\mathbb{R}^d} \exp\left[-\frac{2}{\epsilon}\left(V(x) + F * u_\epsilon(x) - r||x||\right)\right]}{\int_{\mathbb{R}^d} \exp\left[-\frac{2}{\epsilon}\left(V(x) + F * u_\epsilon(x)\right)\right] dy}\,.$$

Since F is convex and since $\mathrm{Hess}\, V(x) > 0$ if $||x||$ is sufficiently large, we deduce that $\int_{\mathbb{R}^d} e^{\frac{2}{\epsilon}||r||x} u_\epsilon(x)dx < \infty$. Consequently, the serie $\sum_{k=0}^{\infty} \frac{\nu_k}{k!} x^k$ has a positive convergence radius with $\nu_k := \int_{\mathbb{R}^d} ||x||^k u_\epsilon(x)dx$. After applying the Fourier criteria, the statement holds. □

2 Convergence

We write the main result of the paper:

Theorem 2.1. *Under (D), the law u_t converges weakly towards a stationary measure.*

Proof. **Plan:** The tightness of the family $\{u_t\,;\,t \in \mathbb{R}_+\}$ implies that it is sufficient to prove the uniqueness of the adherence values. We already know that there exists an invariant probability u_ϵ which is in $\mathcal{A}_\epsilon$. We proceed a *reductio ad absurdum* by assuming the existence of another adherence value v_∞. Since u_ϵ is determinated uniquely by its moments, we provide a function φ with compact support which separates u_ϵ and v_∞. By definition of $\mathcal{A}_\epsilon$ and since $\mathcal{S}_\epsilon \bigcap \mathcal{A}_\epsilon$ is discret, we find a contradiction.

Step 1: The inequality (V) implies that the family of probability measures $\{u_t\,;\,t \in \mathbb{R}_+\}$ is tight. Prohorov Theorem permits to conclude that each extracted sequence of this family is relatively compact with respect to the weak convergence. So, in order to prove the statement of the theorem, it is sufficient to prove $\#\mathcal{A}_\epsilon = 1$. We proceed a *reductio ad absurdum*. We assume in the following that the family admits at least two adherence values.

Step 2: By Proposition 1.4, we know that at least one of these adherence value is a stationary measure: u_ϵ. The assumption (D) implies that the set $\mathcal{S}_\epsilon \bigcap \mathcal{A}_\epsilon$ is discret.

Step 3: Let another adherence value v_∞. The hypothesis (ES) and Proposition A.2 implies that for all $k \in \mathbb{N}$, we have $\int_{\mathbb{R}^d} ||x||^k v_\infty(x)dx < \infty$. According to Lemma 1.5, there exists $k \in \mathbb{N}$ such that

$$\lambda_1 := \int_{\mathbb{R}^d} ||x||^k u_\epsilon(x)dx \neq \int_{\mathbb{R}^d} ||x||^k v_\infty(x)dx =: \lambda_2 \,.$$

Step 4: Consequently, there exists a polynomial function φ such that:

$$0 = \int_{\mathbb{R}^d} \varphi\left(||x||\right) u_\epsilon(x)dx < \int_{\mathbb{R}^d} \varphi\left(||x||\right) v_\infty(x)dx \,.$$

We consider now the following function:

$$\begin{aligned}\tau_R(x) :=& \mathbb{1}_{\{||x||\leq R\}} \\ &+ Z^{-1}\mathbb{1}_{\{R\leq||x||\leq R+1\}} \int_{||x||}^{R+1} \exp\left[-\frac{1}{(y-R)^2} - \frac{1}{(y-R-1)^2}\right] dy \\ \text{with} \quad Z :=& \int_0^1 \exp\left[-\frac{1}{z^2} - \frac{1}{(z-1)^2}\right] dz \,.\end{aligned}$$

Step 5: We introduce the function with compact support: $\xi_R(x) := \tau_r(x) \times \varphi\left(||x||\right)$. By construction, $\xi_R \in \mathcal{C}^\infty\left(\mathbb{R}^d\,;\mathbb{R}\right)$. Furthermore, $|\xi_R(x)| \leq ||x||^l$ where $l \in \mathbb{N}$. The same inequality holds for $||\nabla\xi_R||^2$ with an other integer l'. By using the triangular inequality, for all $u_\infty \in \mathcal{A}_\epsilon$, we have:

$$\left|\int_{\mathbb{R}^d} \xi_R u_\infty - \int_{\mathbb{R}^d} \varphi u_\infty\right| \leq \left|\int_{||x||\geq R} ||x||^l u_\infty(x)dx\right| \leq \frac{1}{R^3} C_0$$

where $C_0 := \sup_{t\geq 1} \int_{\mathbb{R}^d} ||x||^{l+3} u_t(x)dx < +\infty$. Consequently, for R large enough, we obtain the existence of a smooth function ξ with compact support such that

$$0 = \int_{\mathbb{R}^d} \xi(x)u_\epsilon(x)dx < \int_{\mathbb{R}^d} \xi(x)v_\infty(x)dx \,.$$

Moreover, since $\mathcal{S}_\epsilon$ is discret, there exists $\kappa > 0$ such that for all element $v_\epsilon \in \mathcal{S}_\epsilon \bigcap \mathcal{A}_\epsilon$, we have which verifies $\int_{\mathbb{R}^d} \xi(x)v_\epsilon(x)dx \notin [\kappa\,;2\kappa]$ where $0 < \kappa < \frac{1}{3}\int_{\mathbb{R}^d} \xi(x)v_\infty(x)dx$.

Step 6: By definition of $\mathcal{A}_\epsilon$, there exist an increasing sequences $(t_k^{(1)})_k$ (respectively $(t_k^{(2)})_k$) which goes to infinity such that $u_{t_k^{(1)}}$ (respectively $u_{t_k^{(2)}}$) converges

weakly towards u_ϵ (respectively v_∞).
We deduce that there exist two increasing sequences $(r_k)_k$ and $(s_k)_k$ such that $\int_{\mathbb{R}^d}\xi(x)u_{r_k}(x)dx = \kappa$ and $\int_{\mathbb{R}^d}\xi(x)u_{s_k}(x)dx = 2\kappa$. Then, for all $k\in\mathbb{N}$, we put $\widehat{r_k} := \sup\left\{t\in[0;s_k]\ \mid\ \int_{\mathbb{R}^d}\xi(x)u_t(x)dx=\kappa\right\}$ and we define the sequence $\widehat{s_k} := \inf\left\{s\in[\widehat{r_k};s_k]\ \mid\ \int_{\mathbb{R}^d}\xi(x)u_s(x)dx=2\kappa\right\}$. For simplicity, we write r_k (respectively s_k) instead of $\widehat{r_k}$ (respectively $\widehat{s_k}$). And, we have:

$$\kappa=\int_{\mathbb{R}^d}\xi(x)u_{r_k}(x)dx\leq\int_{\mathbb{R}^d}\xi(x)u_t(x)dx\leq\int_{\mathbb{R}^d}\xi(x)u_{s_k}(x)dx=2\kappa$$

for all $t\in[r_k;s_k]$.
Step 7: By applying Proposition A.1, we deduce the existence of $w_\epsilon\in\mathcal{S}_\epsilon$ verifying $\int_{\mathbb{R}^d}\xi(x)w_\epsilon(x)dx\in[\kappa;2\kappa]$. This is impossible. □

We provide now a result without assuming the hypothesis (D). It holds in the small-noise limit:

Theorem 2.2. *Let us admit that $\alpha-\vartheta>0$. Let a law u_0 which satisfies (ES) and (FE). Then, there exists a critical point of V a_0 such that:*

$$\limsup_{\epsilon\to0}\limsup_{t\to\infty}\int_{\mathbb{R}^d}\left|\left|x-a_0\right|\right|^2u_t(x)dx=0\,.$$

Proof. We proceed a *reductio ad absurdum* by assuming the existence of a constant $\kappa>0$ and two sequences $(\epsilon_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ which verify

- $\epsilon_k\to0$ as $k\to\infty$.
- $t_k\to\infty$ as $k\to\infty$.
- $\int_{\mathbb{R}^d}\left|\left|x-a_0\right|\right|^2u_{t_k}(x)dx\geq\kappa$ for all $k\in\mathbb{N}$ and for all the critical points a_0 of the potential V.

According to Proposition 3.8 in [Tug11c], since $\alpha>\vartheta$, we deduce that the only possible limits of stationary measures when ϵ goes to 0 are Dirac measures δ_{A_0}. Moreover, A_0 satisfies $\nabla V(A_0)=0$ so A_0 is a critical point of V. We deduce that for all $\rho>0$, there exists $\epsilon_0>0$ sufficiently small such that for all $\epsilon<\epsilon_0$, there is no element $v\in\mathcal{S}_\epsilon$ which would satisfy $\int_{\mathbb{R}^d}\left|\left|x-a_0\right|\right|^2v(x)dx\geq\rho$ for all the critical points a_0 of V. We take $\rho:=\kappa$. By proceeding exactly like in Theorem 2.1, the initial hypothesis would permit to construct a stationary measure $v\in\mathcal{S}_{\epsilon_k}$ such that $\int_{\mathbb{R}^d}\left|\left|x-a_0\right|\right|^2v(x)dx\geq\rho$ for all the critical points a_0 of V. This is impossible. □

A Useful technical results

The central idea in the proof of Theorem 2.1 is the fact that we can construct an element of $\mathcal{S}_\epsilon\bigcap\mathcal{A}_\epsilon$ such that its integration against a function $\varphi\in\mathcal{C}^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$ with compact support takes a value between κ and 2κ if for all $\tau\in[\kappa;2\kappa]$, there exists $v_\epsilon\in\mathcal{A}_\epsilon$ such that $\int\varphi v_\epsilon=\tau$. Let us prove now this statement.

Proposition A.1. *We assume the existence of two polynomial functions $\mathcal{P}$ and $\mathcal{Q}$, a smooth function φ from $\mathbb{R}^d$ to $\mathbb{R}$ with compact support such that $|\varphi(x)| \leq \mathcal{P}(||x||)$ and $||\nabla\varphi(x)||^2 \leq \mathcal{Q}(||x||)$, $\kappa > 0$ and two sequences $(r_k)_k$ and $(s_k)_k$ which go to ∞ such that for all $r_k \leq t \leq s_k < r_{k+1}$:*

$$\kappa = \int_{\mathbb{R}^d} \varphi(x) u_{r_k}(x) dx \leq \int_{\mathbb{R}^d} \varphi(x) u_t(x) dx \leq \int_{\mathbb{R}^d} \varphi(x) u_{s_k}(x) dx = 2\kappa\,.$$

Then, there exists $v_\epsilon \in \mathcal{A}_\epsilon \bigcap \mathcal{S}_\epsilon$ which verifies $\int_{\mathbb{R}^d} \varphi(x) v_\epsilon(x) dx \in [\kappa; 2\kappa]$.

Proof. **Plan:** We proceed exactly like in the proof of Proposition 2.1 in [Tug11c]. We just neeed to find a sequence $(q_k)_k$ which tends towards infinity as k goes to infinity and which verifies the two following conditions:

- $\xi'(q_k)$ converges towards 0.
- $r_k \leq q_k \leq s_k$.

For doing this, we remark that the inequality $\xi'(s) \leq 0$ and the convergence of $\int_t^\infty \xi'(s) ds$ towards 0 implies that $\sum_{p=k}^\infty \int_{r_p}^{s_p} \xi'(s) ds$ elapses when k is going to infinity. *However, for obtaining the existence of such a sequence $(q_k)_k$, we shall prove that* $\liminf\limits_{k\longrightarrow+\infty} (s_k - r_k) > 0$. This will be done like in the proof of Proposition A.1 in [Tug10b].

Step 1: We begin to prove that $\liminf\limits_{k\longrightarrow+\infty} (s_k - r_k) > 0$. We introduce the function :

$$\Phi(t) := \int_{\mathbb{R}^d} \varphi(x) u_t(x) dx\,.$$

This function is well-defined since $|\varphi|$ is bounded by a polynomial function of the norm of x. The derivation of Φ, the use of equation (III) and an integration by parts lead to:

$$\begin{aligned}\Phi'(t) &= -\int_{\mathbb{R}^d} \left\langle \nabla\varphi\,;\, \left\{\frac{\epsilon}{2}\nabla u_t + u_t\left(\nabla V + \nabla F * u_t\right)\right\}\right\rangle \\ &= -\int_{\mathbb{R}^d} \langle \nabla\varphi\,;\, \eta_t\rangle\,.\end{aligned}$$

The Cauchy-Schwarz inequality implies

$$|\Phi'(t)| \leq \sqrt{-\xi'(t)}\sqrt{\int_{\mathbb{R}^d} ||\nabla\varphi(x)||^2 u_t(x) dx}$$

where we recall that $\xi(t) = \Upsilon_\epsilon(u_t)$. The function $||\nabla\varphi||^2$ is bounded by a polynomial function and $\int_{\mathbb{R}^d} ||x||^{2N} u_t(x)$ is uniformly bounded with respect to $t \geq 1$ for all $N \in \mathbb{N}$. So, there exists $C > 0$ such that $\int_{\mathbb{R}^d} ||\nabla\varphi(x)||^2 u_t(x) dx \leq C^2$ for all $t \geq 1$. We deduce

$$|\Phi'(t)| \leq C\sqrt{|\xi'(t)|}\,. \tag{A.1}$$

By definition of the two sequences $(r_k)_k$ and $(s_k)_k$, we have

$$\Phi(s_k) - \Phi(r_k) = \kappa \,.$$

Combining this identity with (A.1), it yields

$$C \int_{r_k}^{s_k} \sqrt{|\xi'(t)|} dt \geq \kappa \,.$$

We apply the Cauchy-Schwarz inequality and we obtain:

$$C\sqrt{s_k - r_k}\sqrt{\xi(r_k) - \xi(s_k)} \geq \kappa$$

since ξ is nonincreasing (see Proposition 1.2). Moreover, $\xi(t)$ converges as t goes to ∞ (see Lemma 1.3). It implies the convergence of $\xi(r_k) - \xi(s_k)$ towards 0 when k goes to $+\infty$. Consequently, $s_k - r_k$ converges towards $+\infty$ so

$$\liminf_{k \longrightarrow +\infty} s_k - r_k > 0 \,.$$

Step 2: By Lemma 1.3, $\Upsilon_\epsilon(u_t) - \Upsilon_\epsilon(u_\epsilon) = \int_t^\infty \xi'(s) ds$ converges towards 0. As ξ' is nonpositive, we deduce that $\sum_{k=N}^\infty \int_{r_k}^{s_k} \xi'(s) ds$ converges also towards 0 when N goes to $+\infty$. As $\liminf\limits_{k \longrightarrow +\infty} s_k - r_k > 0$, we deduce that there exists an increasing sequence $q_k \in [r_k; s_k]$ which goes to ∞ and such that $\xi'(q_k)$ converges towards 0 when k goes to ∞. Furthermore, $\int_{\mathbb{R}} \varphi u_{q_k} \in [\kappa; 2\kappa]$ for all $k \in \mathbb{N}$.
Step 3: By proceeding similarly as in the proof of Theorem 2.7 in [Tug11c], we extract a subsequence of $(q_k)_k$ (we continue to write it q_k for simplifying the reading) such that u_{q_k} converges weakly towards a stationary measure v_ϵ. Moreover, v_ϵ verifies $\int_{\mathbb{R}^d} \varphi(x) v_\epsilon(x) dx \in [\kappa; 2\kappa]$. □

Now, we explain why all the moments of u_t are finite for all $t > 0$. This was essential for making the separation between an element of $\mathcal{S}_\epsilon \bigcap \mathcal{A}_\epsilon$ and any other elements of $\mathcal{A}_\epsilon$.

Proposition A.2. *For all $t > 0$, for all $n \in \mathbb{N}^*$, we have $\mathbb{E}\left[||X_t||^n\right] < \infty$. Moreover, $\sup_{t \geq 1} \mathbb{E}\left[||X_t||^n\right] < \infty$*

Proof. **Step 1:** If $\mathbb{E}\left[||X_0||^n\right] < \infty$ then $\mathbb{E}\left[||X_t||^n\right] < \infty$ for all $t > 0$, see Theorem 2.12 in [HIP08]. We assume now that there exists $n \in \mathbb{N}^*$ such that $\mathbb{E}\left[||X_0||^{2n}\right] = \infty$. Let us introduce $l_0 := \min\left\{ l \geq 0 \;|\; \mathbb{E}\left[||X_0||^{2l}\right] \right\} = +\infty$. We know that $\mathbb{E}\left[||X_t||^{2l_0 - 2}\right] < +\infty$ for all $t \geq 0$, see [HIP08].
Step 2: Let $t_0 > 0$. We proceed a *reductio ad absurdum* by assuming that $\mathbb{E}\left[||X_{t_0}||^{2l_0}\right] = +\infty$. This implies directly $\mathbb{E}\left[||X_t||^{2l_0}\right] = +\infty$ for all $t \in [0, t_0]$. We recall that $2m$ (respectively $2n$) is the degree of the confining (respectively interacting) potential V (respectively F). Also, $q := \max\{m \;;\; n\}$. For all $t \in [0, t_0]$, the application $\nabla F * u_t$ is a polynomial function in each coordinate

with parameters depending on u_t. And, these parameters are uniformly bounded with respect to t by the inequality (V):

$$\sup_{1\leq j\leq 8q^2}\sup_{t\in[0,t_0]}\mathbb{E}\left\{||X_t||^j\right\}\leq M_0\,.$$

Consequently, the application $\nabla V+\nabla F*u_t$ is a polynomial function in each coordinate $x_1,\cdots,x_d$ with global degree $2q-1$. Furthermore, the principal term does not depend on the moments of the law u_t so we can write:

$$\nabla V(x)+\nabla F*u_t(x)=\kappa_{2q-1}\,||x||^{2q-2}\,x+\mathcal{P}_t(x)$$

where $\kappa_{2q-1}\in\mathbb{R}_+^*$ is a constant, and $\mathcal{P}_t$ is a polynomial function with degree at most $2q-2$. Moreover, $\mathcal{P}_t$ is parametrized by the moments of degree equal to (or less than) $2n$ only.

Let $l\in\mathbb{N}$. Let us introduce: $\varphi(x):=||x||^{2l}$. Some computations lead to:

$$\langle\nabla\varphi(x)\,;\,\nabla V(x)+\nabla F*u_t(x)\rangle-\frac{\epsilon}{2}\Delta\varphi(x)\geq C_l\left(||x||^{2l+2q-2}-1\right)$$

where C_l is a positive constant. Consequently, for all $\omega\in\Omega$, we have the inequality:

$$||X_{t_0}(\omega)||^{2l}\leq||X_0(\omega)||^{2l}+\mathcal{M}_{t_0}(\omega)+C_lt_0-C_l\int_0^{t_0}||X_t(\omega)||^{2l+2q-2}\,dt$$

where $(\mathcal{M}_t)_{t\geq0}$ is a martingale. We take $l:=l_0+1-q<l_0$. By taking the expectation, it yields $\mathbb{E}\left[||X_{t_0}||^{2l}\right]\leq-\infty$. This is absurd. Consequently, for all $t_0>0$, $\mathbb{E}\left[||X_{t_0}||^{2l_0}\right]<+\infty$.

Step 3: Let $T>0$ and $l_1\in\mathbb{N}$ such that $l_1\geq l_0$ where the integer l_0 is defined as previously: $l_0:=\min\left\{l\geq0\;|\;\mathbb{E}\left[||X_0||^{2l}\right]=+\infty\right\}$. If $l_1=l_0$, the application of Step 2 leads to $\mathbb{E}\left[||X_T||^{2l_1}\right]<+\infty$. If $l_1>l_0$, we put $t_i:=\frac{i}{l_1+1-l_0}T$ for all $1\leq i\leq l_1+1-l_0$. We apply Step 2 to t_1 and we deduce $\mathbb{E}\left[||X_{t_1}||^{2l_0}\right]<+\infty$. By recurrence, we deduce $\mathbb{E}\left[||X_{t_i}||^{2l_0+2i}\right]<+\infty$ for all $1\leq i\leq l_1+1-l_0$, in particular $\mathbb{E}\left[||X_{t_{l_0-l_1}}||^{2l_0+2(l_1-l_0)}\right]<+\infty$ that means $\mathbb{E}\left[||X_T||^{2l_1}\right]<+\infty$. This inequality holds for all $l_1\geq l_0$ so the probability measure u_T satisfies $\int_{\mathbb{R}^d}||x||^{2n}\,u_T(x)dx<\infty$ for all $n\in\mathbb{N}^*$.

Step 4: In particular, it holds with $T:=1$. Thank to Theorem 2.12 in [HIP08], we obtain directly $\sup_{t\geq1}\mathbb{E}\left[||X_t||^n\right]<\infty$. □

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