

GEODESICS IN MARGULIS SPACETIMES

WILLIAM M. GOLDMAN AND FRANÇOIS LABOURIE

Dedicated to the memory of Dan Rudolph

ABSTRACT. Let M^3 be a Margulis spacetime whose associated complete hyperbolic surface Σ^2 has compact convex core. Generalizing the correspondence between closed geodesics on M^3 and closed geodesics on Σ^2 , we establish an orbit equivalence between recurrent spacelike geodesics on M^3 and recurrent geodesics on Σ^2 . In contrast, no timelike geodesic recurs in neither forward nor backwards time.

CONTENTS

Introduction	1
1. Geodesics on affine manifolds	3
2. Flat Lorentz 3-manifolds	3
3. From geodesics in Σ^2 to geodesics in M^3	5
4. Appendix: Cohomology and positive functions	8
References	10

INTRODUCTION

A *Margulis spacetime* is a complete flat affine 3-manifold M^3 with free nonabelian fundamental group Γ . It necessarily carries a unique parallel Lorentz metric. Parallelism classes of timelike geodesics form a noncompact complete hyperbolic surface Σ^2 . This complete hyperbolic surface is naturally associated to the flat 3-manifold M^3 and we regard M^3 as an *affine deformation* of Σ^2 . This note relates the dynamics of the geodesic flow of the flat affine manifold M^3 to the dynamics of the geodesic flow on the hyperbolic surface Σ^2 .

We restrict to the case that Σ^2 has a compact convex core (that is, Σ^2 has finite type and no cusps). Equivalently, the Fuchsian group Γ_0 corresponding to $\pi_1(\Sigma^2)$ is *convex cocompact*. In particular Γ_0 is finitely generated and contains no parabolic elements. Under this assumption,

Date: January 24, 2011.

every free homotopy class of an essential closed curve in Σ^2 contains a unique closed geodesic. Since Σ^2 and M^3 are homotopy-equivalent, free homotopy classes of essential closed curves in M correspond to free homotopy classes of essential closed curves in Σ^2 . Every essential closed curve in M^3 is likewise homotopic to a unique closed geodesic in M^3 .

In her thesis [4, 7], Charette studied the next case of dynamical behavior: geodesics spiralling around closed geodesics both in forward and backward time. She proved bispiralling geodesics in M^3 exist, and correspond to bispiralling geodesics in Σ^2 .

This paper extends the above correspondence between geodesics on Σ^2 and M^3 to recurrent geodesics.

A geodesic (either in Σ^2 or in M^3) is *recurrent* if and only if it (together with its velocity vector) is recurrent in *both* directions. These correspond to recurrent points for the corresponding geodesic flows as in Katok-Hasselblatt [15], §3.3. Under our hypotheses on Σ^2 , a geodesic on Σ^2 is recurrent if and only if the corresponding orbit of the geodesic flow is precompact.

Theorem 1. *Let M^3 be a Margulis spacetime whose associated complete hyperbolic surface Σ has compact convex core.*

- *The recurrent part of the geodesic flow for Σ^2 is topologically orbit-equivalent to the recurrent spacelike part of the geodesic flow of M^3 .*
- *The set of recurrent spacelike geodesics in a Margulis spacetime is the closure of the set of periodic geodesics.*
- *No timelike geodesic recurs.*

A semiconjugacy between these flows was observed by D. Fried [11].

This note is the sequel to [13], which characterizes properness of affine deformations by positivity of a marked Lorentzian length spectrum, the *generalized Margulis invariant*. A crucial step in the proof that properness implies positivity is the construction of sections of the associated flat affine bundle, called *neutralized sections*. A further modification of neutralized sections produces an orbit equivalence between recurrent geodesics in Σ and recurrent geodesics in M .

It follows that the set of recurrent spacelike geodesics is a Smale hyperbolic set in TM .

We thank Mike Boyle, Virginie Charette, Todd Drumm, David Fried, and Gregory Margulis for helpful conversations.

1. GEODESICS ON AFFINE MANIFOLDS

An *affinely flat manifold* is a smooth manifold with a distinguished atlas of local coordinate systems whose charts map to an affine space \mathbb{E} such that the coordinate changes are restrictions of affine automorphisms of \mathbb{E} . Denote the group of affine automorphisms of \mathbb{E} by $\text{Aff}(\mathbb{E})$. This structure is equivalent to a flat torsionfree affine connection. The affine coordinate atlas globalizes to a *developing map*

$$\tilde{M} \xrightarrow{\text{dev}} \mathbb{E}$$

where $\tilde{M} \rightarrow M$ denotes a universal covering space of M . The coordinate changes globalize to an affine holonomy homomorphism

$$\pi_1(M) \xrightarrow{\rho} \text{Aff}(\mathbb{E})$$

where $\pi_1(M)$ denotes the group of deck transformations of $\tilde{M} \rightarrow M$. The developing map is equivariant respecting ρ .

Denote the vector space of translations $\mathbb{E} \rightarrow \mathbb{E}$ by \mathbb{V} . The action of \mathbb{V} by translations on \mathbb{E} defines a trivialization of the tangent bundle $TM \cong M \times \mathbb{V}$. In these local coordinate charts, a geodesic is a path

$$p \longmapsto p + tv$$

where $p \in \mathbb{E}$ and $v \in \mathbb{V}$ is a vector. In terms of the trivialization the geodesic flow is:

$$\begin{aligned} \mathbb{E} \times \mathbb{V} &\xrightarrow{\tilde{\psi}_t} \mathbb{E} \times \mathbb{V} \\ (p, v) &\longmapsto (p + tv, v) \end{aligned}$$

for $t \in \mathbb{R}$. Clearly this \mathbb{R} -action commutes with $\text{Aff}(\mathbb{E})$.

Geodesic completeness implies that dev is a diffeomorphism. Thus the universal covering \tilde{M} is affinely isomorphic to affine space \mathbb{E} and $M \cong \mathbb{E}/\Gamma$, where $\Gamma := \rho(\pi_1(M))$ is a discrete group of affine transformations acting properly and freely on \mathbb{E} .

2. FLAT LORENTZ 3-MANIFOLDS

Let $\text{Aff}(\mathbb{E}) \xrightarrow{L} \text{GL}(\mathbb{V})$ denote the homomorphism given by *linear part*, that is, $L(\gamma) = A$ where

$$p \xrightarrow{\gamma} L(\gamma)(p) + u(\gamma).$$

Any $L(\Gamma)$ -invariant nondegenerate inner product $\langle \cdot, \cdot \rangle$ on \mathbb{V} defines a Γ -invariant flat pseudo-Riemannian structure on \mathbb{E} which descends to $M = \mathbb{E}/\Gamma$. In particular affine manifolds with $L(\Gamma) \subset \text{O}(n-1, 1)$ are precisely the *flat Lorentzian manifolds*, and the underlying affine structures their Levi-Civita connections.

For this reason we henceforth fix the invariant Lorentzian inner product on \mathbb{V} , and hence the (parallel) flat Lorentzian structure on \mathbb{E} . The group $\mathbf{Isom}(\mathbb{E})$ of Lorentzian isometries is the semidirect product of the group \mathbb{V} of translations of \mathbb{E} with the orthogonal group $\mathbf{SO}(n-1, 1)$ of linear isometries. Linear part $\mathbf{Isom}(\mathbb{E}) \xrightarrow{L} \mathbf{SO}(n-1, 1)$ defines the projection homomorphism for the semidirect product. For $l \in \mathbb{R}$, define

$$S_l := \{v \in \mathbb{V} \mid \langle v, v \rangle = l\}.$$

When $l > 0$, S_l is a Riemannian submanifold of constant curvature $-l^{-2}$ and when $l < 0$, it is a Lorentzian submanifold of constant curvature l^{-2} . In particular S_{-1} is a disjoint union of two isometrically embedded copies of hyperbolic $n-1$ -space \mathbf{H}^{n-1} and S_1 is *de Sitter space*, a model space of Lorentzian curvature $+1$.

The subset $\mathbb{T}_l(M)$ consisting of tangent vectors v such that $\langle v, v \rangle = l$ is invariant under the geodesic flow. Indeed, using parallel translation, these bundles trivialize over the universal covering \mathbb{E} :

$$\mathbb{T}_l(\mathbb{E}) \xrightarrow{\cong} \mathbb{E} \times S_l$$

By Abels-Margulis-Soifer [2, 3], if a discrete group of Lorentz isometries acts properly on Minkowski space \mathbb{E} , and $L(\Gamma)$ is Zariski dense in $\mathbf{SO}(n-1, 1)$, then $n = 3$. Consequently every complete flat Lorentz manifold is a flat Euclidean affine fibration over a complete flat Lorentz 3-manifold. Thus we henceforth restrict to $n = 3$.

Let M^3 be a complete affinely flat 3-manifold. By Fried-Goldman [12], either Γ is solvable or $L \circ h$ embeds Γ as a discrete subgroup in (a conjugate of) the orthogonal group

$$\mathbf{SO}(2, 1) \subset \mathbf{GL}(3, \mathbb{R}).$$

The cases when Γ is solvable are easily classified (see [12]) and we assume we are in the latter case. In that case, M^3 is a complete flat Lorentz 3-manifold.

In the early 1980's Margulis, answering a question of Milnor [20]), constructed the first examples [17, 18], which are now called *Margulis spacetimes*. Explicit geometric constructions of these manifolds have been given by Drumm [8, 9] and his coauthors [5, 6, 10].

Since the hyperbolic plane \mathbf{H}^2 is the symmetric space of $\mathbf{SO}(2, 1)$, Γ acts properly and discretely on \mathbf{H}^2 . Since M^3 is aspherical, its fundamental group $\pi_1(M^3) \cong \Gamma$ is torsionfree, so Γ acts freely as well. Therefore the quotient $\mathbf{H}^2/L(\Gamma)$ is a complete hyperbolic surface Σ^2 . Furthermore Σ is noncompact (Mess [19], see also Goldman-Margulis [14] and Labourie [16] for alternate proofs), and every noncompact complete hyperbolic surface occurs for a Margulis spacetime (Drumm [8])

The points of Σ^2 correspond to parallelism classes of timelike geodesics on M^3 as follows. A timelike geodesic is given by an ψ -orbit in

$$\mathbb{T}_{-1}M \cong (\mathbb{E} \times \mathbb{H}^2)/\Gamma$$

Since $\Gamma \xrightarrow{\mathbb{L}} \mathrm{SO}(2, 1)$ is a discrete embedding [12], $\mathrm{SO}(2, 1)$ acting properly on \mathbb{H}^2 implies that Γ acts properly on \mathbb{H}^2 . Cartesian projection $\mathbb{E} \times \mathbb{H}^2 \rightarrow \mathbb{H}^2$ defines a projection $\mathbb{T}^{-1}M \rightarrow \mathbb{H}^2/\mathrm{L}(\Gamma) = \Sigma$ which is invariant under the restriction of the geodesic flow ψ to $\mathbb{T}^{-1}M$; and defines an \mathbb{E} -bundle. Its fiber over a fixed future-pointing unit-timelike vector \mathbf{v} is the union of geodesics parallel to \mathbf{v} . In particular properness of the $\mathrm{L}(\Gamma)$ -action on \mathbb{H}^2 implies nonrecurrence of timelike geodesics, the last statement in Theorem 1.

More generally, any $\mathrm{L}(\Gamma)$ -invariant subset $\Omega \subset \mathbb{V}$ defines a subset $\mathbb{T}_\Omega(M) \subset \mathbb{T}M$ invariant under the geodesic flow. If Ω is an open set upon which $\mathrm{L}(\Gamma)$ acts properly, then the geodesic flow defines a proper \mathbb{R} -action on $\mathbb{T}_\Omega(M)$. In particular every geodesic whose velocity lies in Ω is properly immersed and is neither positively nor negatively recurrent.

3. FROM GEODESICS IN Σ^2 TO GEODESICS IN M^3

While timelike directions correspond to points of Σ^2 , spacelike directions correspond to geodesics in \mathbb{H}^2 . The recurrent geodesics in Σ intimately relate to the recurrent spacelike geodesics on M^3 .

Denote the set of oriented spacelike geodesics in \mathbb{E} by \mathcal{S} . It identifies with the orbit space of the geodesic flow $\tilde{\psi}$ on $\mathbb{T}_{+1}\mathbb{E} \cong \mathbb{E} \times \mathbb{S}_{+1}$. The natural map $\mathcal{S} \xrightarrow{\mathbb{r}} \mathbb{S}_{+1}$ which associates to a spacelike vector its direction is equivariant respecting $\mathrm{Isom}(\mathbb{E}) \xrightarrow{\mathbb{L}} \mathrm{SO}(2, 1)$.

The identity component of $\mathrm{SO}(2, 1)$ acts simply transitively on the unit tangent bundle UH^2 , and therefore we identify $\mathrm{SO}(2, 1)^0$ with UH^2 by choosing a basepoint u_0 in UH^2 . Unit-spacelike vectors in \mathbb{S}_{+1} correspond to oriented geodesics in \mathbb{H}^2 . Explicitly, if $v \in \mathbb{S}_{+1}$, then there is a one-parameter subgroup $a(t) \in \mathrm{SO}(2, 1)$, having v as a fixed vector, and such that $\det(v, v^-, v^+) > 0$, where v^+ is an expanding eigenvector of $a(t)$ (for $t > 0$) and v^- is the contracting eigenvector. Choose a basepoint $v_0 \in \mathbb{S}_{+1}$ corresponding to the orbit of u_0 under the geodesic flow on $\mathrm{U}\Sigma$. Geodesics in \mathbb{H}^2 relate to spacelike directions by an equivariant mapping

$$\begin{aligned} \mathrm{UH}^2 &\xrightarrow{\tilde{v}} \mathbb{S}_{+1} \\ g(u_0) &\longmapsto g(v_0) \end{aligned}$$

The unit tangent bundle $U\Sigma$ of Σ identifies with the quotient

$$L(\Gamma)\backslash UH^2 \cong L(\Gamma)\backslash SO(2,1)^0,$$

where the geodesic flow ψ corresponds the right-action of $a(-t)$ (see, for example, [13], §1.2).

Observe that a geodesic in Σ^2 is recurrent if and only if the endpoints of any of its lifts to $\tilde{\Sigma} \approx H^2$ lie in the limit set Λ of $L(\Gamma)$. If the convex core of Σ^2 is compact, then the union $U_{\text{rec}}\Sigma$ of recurrent ϕ -orbits is compact.

Lemma 2. *There exists an orbit-preserving map*

$$U_{\text{rec}}\Sigma \xrightarrow{\tilde{N}} T_{+1}(M)$$

mapping ϕ -orbits injectively to recurrent ψ -orbits.

Proof. The associated flat affine bundle E over $U\Sigma$ associated to the affine deformation Γ is defined as follows. The affine representation of Γ defines a diagonal action of Γ on $\tilde{U}\Sigma \times \mathbb{E}$

Its total space is the quotient of the product $\tilde{U}\Sigma \times \mathbb{E}$ by the diagonal action of $\pi_1(\Sigma)$

$$\pi_1(U\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow \text{Isom}(\mathbb{E}).$$

Similarly the flat vector bundle V over $U\Sigma$ is the quotient of $\tilde{U}\Sigma \times \mathbb{V}$ by the diagonal action

$$\pi_1(U\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow \text{Isom}(\mathbb{E}) \xrightarrow{L} SO(2,1).$$

According to [13], the *neutral section* of V is a $SO(2,1)$ -invariant section which is parallel with respect to the geodesic flow on $U\Sigma$. and arises from the graph of the $SO(2,1)$ -equivariant map

$$U\tilde{\Sigma} \cong UH^2 \longrightarrow \mathbb{V}$$

with image S_{+1} , the space of unit-spacelike vectors in \mathbb{V} .

Here is the main construction of [13]. To every section σ of E continuously differentiable along ϕ , associate the function

$$F_\sigma := \langle \nabla_\phi \sigma, \nu \rangle$$

on $U\Sigma$. (Here the covariant derivative of a section of E along a vector field ϕ in the base is a section of the associated vector bundle V .) Different choices of section σ yield cohomologous functions F_σ . (Recall that two functions f_1, f_2 are *cohomologous*, written $f_1 \sim f_2$, if

$$f_1 - f_2 = \phi g$$

for a function g which is differentiable with respect to the vector field ϕ ([15], §2.2).

Restrict the affine bundle \mathbf{E} to $\mathbf{U}_{\text{rec}}\Sigma$. Lemma 8.4 of [13] guarantees the existence of a *neutralized section*, that is, a section N of $\mathbf{E}|_{\mathbf{U}_{\text{rec}}\Sigma}$ satisfying

$$\nabla_X N = f\nu,$$

for some function f .

Although the following lemma is well known, we could not find a proof in the literature. For completeness, we supply a proof in the appendix.

Lemma 3. *Let X be a compact space equipped with a flow ϕ . Let $f \in C(X)$, such that for all ϕ -invariant measures μ on X ,*

$$\int f \, d\mu > 0.$$

Then f is cohomologous to a positive function.

Since Γ acts properly, Proposition 8.1 of [13] implies that $\int F_\sigma d\mu \neq 0$ for all ϕ -invariant probability measures μ on $\mathbf{U}_{\text{rec}}\Sigma$. Since the set of invariant measures is connected, $\int F_\sigma d\mu$ is either positive for all ϕ -invariant probability measures μ on $\mathbf{U}_{\text{rec}}\Sigma$ or negative for all ϕ -invariant probability measures μ on $\mathbf{U}_{\text{rec}}\Sigma$. Conjugating by $-I$ if necessary we may assume that $\int F_\sigma d\mu > 0$. Lemma 3 implies $F_\sigma + L_\phi g > 0$ for some function g . Write

$$\widehat{N} = N + g\nu.$$

\widehat{N} remains neutralized, and $\nabla_X \widehat{N}$ vanishes nowhere.

Let $\widetilde{\mathbf{U}}_{\text{rec}}\Sigma$ be the preimage of $\mathbf{U}_{\text{rec}}\Sigma$ in \mathbf{UH}^2 . Then \widehat{N} determines a Γ -equivariant map

$$\widetilde{\mathbf{U}}_{\text{rec}}\Sigma \xrightarrow{\widehat{N}} \mathbb{E}.$$

Each $\tilde{\phi}$ -orbit injectively maps to a spacelike geodesic. The map

$$\begin{aligned} \mathbf{U}_{\text{rec}}\Sigma &\xrightarrow{\widehat{N}} (\mathbb{E} \times \mathbf{S}_{+1})/\Gamma = \mathbf{T}_{+1}(M) \\ \widehat{\mathbf{N}}(x) &:= [(\widehat{N}(x), \nu(x))]. \end{aligned}$$

is the desired orbit equivalence $\mathbf{U}_{\text{rec}}\Sigma \longrightarrow \mathbf{T}_{+1}(M)$. □

Lemma 4. *Any spacelike recurrent geodesic parallel to a geodesic γ in the image of $\widehat{\mathbf{N}}$ coincides with γ .*

Proof. Let $t \xrightarrow{g} \phi_t(v)$ be an orbit in $\mathbf{U}_{\text{rec}}\Sigma$. A geodesic ξ parallel to $\widehat{\mathbf{N}}(g)$ determines a parallel section u of \mathbb{V} along g . Since g recurs, the resulting parallel section is a bounded invariant parallel section along the closure of g . By the Anosov property, such a section is along ν , and therefore, up to reparametrization, $\gamma = \widehat{\mathbf{N}}(g)$. □

Proposition 5. $\widehat{\mathbf{N}}$ is injective and its image is the set of recurrent spacelike geodesics.

Proof. An orbit of the geodesic flow ϕ recurs if and only if the corresponding Γ -orbit in the space \mathcal{S} of spacelike geodesics in \mathbb{E} recurs. Similarly a ϕ -orbit in $\mathbb{T}_{+1}(M)$ recurs if and only if the corresponding $\mathbf{L}(\Gamma)$ -orbit in \mathbf{S}_{+1} recurs. The map $\mathcal{S} \xrightarrow{\Upsilon} \mathbf{S}_{+1}$ recording the direction of a spacelike geodesic is \mathbf{L} -equivariant. If the Γ -orbit of $g \in \mathcal{S}$ corresponds to a recurrent spacelike geodesic in M , then the $\mathbf{L}(\Gamma)$ -orbit of $\Upsilon(g)$ corresponds to a recurrent ϕ -orbit in $\mathbf{U}\Sigma$.

$\widehat{\mathbf{N}}$ is injective along orbits of the geodesic flow. Thus it suffices to prove that the restriction of Υ to the subset of Γ -recurrent geodesics in \mathcal{S} is injective. Since the fibers of Υ are parallelism classes of spacelike geodesics, Lemma 4 implies injectivity of $\widehat{\mathbf{N}}$.

Finally let g be a ψ -recurrent point in $\mathbb{T}_{+1}(M)$, corresponding to a spacelike recurrent geodesic γ in M . It corresponds to a recurrent Γ -orbit Γg in \mathcal{S} . Then $\Upsilon(\Gamma g)$ is a recurrent $\mathbf{L}(\Gamma)$ -orbit in \mathbf{S}_{+1} , and corresponds to a recurrent ϕ -orbit in $\mathbf{U}\Sigma$. The image of this ϕ -orbit under $\widehat{\mathbf{N}}$ is a spacelike recurrent geodesic in $\mathbb{T}_{+1}(M)$ parallel to γ . Now apply Lemma 4 again to conclude that g lies in the image of $\mathbf{h}N$. \square

The proof of Theorem 1 is complete.

4. APPENDIX: COHOMOLOGY AND POSITIVE FUNCTIONS

Let X be a smooth manifold equipped with a smooth flow ϕ . A function $g \in C(X)$ is *continuously differentiable along ϕ* if for each $x \in X$, the function

$$t \mapsto g(\phi_t(x))$$

is a continuously differentiable map $\mathbb{R} \rightarrow \mathbb{R}$. Denote the subspace of $C(X)$ consisting of functions continuously differentiable along ϕ by $C_\phi(X)$. For $g \in C_\phi(X)$, denote its directional derivative by:

$$\phi(g) := \left. \frac{d}{dt} \right|_{t=0} g \circ \phi_t.$$

The proof of Lemma 3 will be based on two lemmas.

Lemma 6. Let $f \in C_\phi(X)$. For any $T > 0$, define

$$f_T(x) := \frac{1}{T} \int_0^T f(\phi_s(x)) \, ds.$$

Then $f \sim f_T$.

Proof. We must show that there exists a function $g \in C_\phi(X)$ such that:

$$f_T - f = \phi g.$$

By the fundamental theorem of calculus,

$$f \circ \phi_t = f + \int_0^t (\phi f \circ \phi_s) \, ds.$$

Writing

$$g = \frac{1}{T} \int_0^T \int_0^t (f \circ \phi_s) \, ds \, dt,$$

then

$$\begin{aligned} f_T - f &= \frac{1}{T} \int_0^T (f \circ \phi_t - f) \, dt \\ &= \frac{1}{T} \int_0^T \int_0^t \phi(f \circ \phi_s) \, ds \, dt \\ &= \phi g. \end{aligned}$$

as desired. \square

Lemma 7. *Assume that for all ϕ -invariant measures μ ,*

$$\int f \, d\mu > 0.$$

Then $f_T > 0$ for some $T > 0$.

Proof. Otherwise sequences $\{T_m\}_{m \in \widehat{\mathbb{N}}}$ of positive real numbers and sequences $\{x_m\}_{m \in \widehat{\mathbb{N}}}$ of points in M exist such that

$$f_{T_m}(x_m) \leq 0.$$

Using the flow ϕ_t , push forward the (normalized) Lebesgue measure

$$\frac{1}{T_m} \mu_{[0, T_m]}$$

on the interval $[0, T_m]$ to X , to obtain a sequence of probability measures μ_n on X such that

$$\int f \, d\mu_n \leq 0.$$

As in [13], §7, a subsequence weakly converges to an ϕ -invariant measure μ for which

$$\int f \, d\mu \leq 0,$$

contradicting our hypotheses. \square

Proof of Lemma 3. By Lemma 6, $f \sim f_T$ for any $T > 0$, and Lemma 7 implies that $f_T > 0$ for some T . \square

REFERENCES

- [1] Abels, H., *Properly discontinuous groups of affine transformations, A survey*, Geom. Ded. **87** (2001) 309–333.
- [2] Abels, H., Margulis, G., and Soifer, G., *Properly discontinuous groups of affine transformations with orthogonal linear part.*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 3, 253–258.
- [3] Abels, H., Margulis, G., and Soifer, G., *The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant*, Geom. Ded. (to appear).
- [4] Charette, V., *Proper Actions of Discrete Groups on 2 + 1 Spacetime*, Doctoral Dissertation, University of Maryland (2000).
- [5] Charette, V., Drumm, T., and Goldman, W., *Affine deformations of a three-holed sphere*. Geom. Topol. **14** (2010), no. 3, 1355–1382,
- [6] Charette, V., and Goldman, W., *Affine Schottky groups and crooked tilings*. Crystallographic groups and their generalizations (Kortrijk, 1999), 69–97, Contemp. Math., 262, Amer. Math. Soc., Providence, RI, 2000.
- [7] Charette, V., Goldman, W., and Jones, C., *Recurrent Geodesics in Flat Lorentz 3-Manifolds*, Canad. Math. Bull. **47** (3), (2004), 332–342.
- [8] Drumm, T., *Linear holonomy of Margulis space-times*. J. Differential Geom. **38** (1993), no. 3, 676–690.
- [9] Drumm, T., *Fundamental polyhedra for Margulis space-times*. Doctoral dissertation, University of Maryland (1990); Topology **31** (1992), no. 4, 677–683.
- [10] Drumm, T. and Goldman, W., *Complete flat Lorentz 3-manifolds with free fundamental group*. Internat. J. Math. **1** (1990), no. 2, 149–161.
- [11] Fried, D., personal communication to W. Goldman (2002).
- [12] Fried, D. and Goldman, W., *Three-dimensional affine crystallographic groups*, Adv. Math. **47** (1983), 1–49.
- [13] Goldman, W., Labourie, F., and Margulis, G., *Proper affine actions and geodesic flows of hyperbolic surfaces*, Ann. Math. **170** No. 3, (2009), 1051–1083.
- [14] Goldman, W. and Margulis, G., *Flat Lorentz 3-manifolds and cocompact Fuchsian groups*, in “Crystallographic Groups and their Generalizations,” Contemp. Math. **262** (2000), 135–146, Amer. Math. Soc.
- [15] Katok, A. and Hasselblatt, B., “Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza.” Encyclopedia of Mathematics and its Applications, **54**. Cambridge University Press, Cambridge (1995).
- [16] Labourie, F., *Fuchsian affine actions of surface groups*, J. Diff. Geo. **59** (1), (2001), 15 – 31.
- [17] Margulis, G., *Free properly discontinuous groups of affine transformations*, Dokl. Akad. Nauk SSSR **272** (1983), 937–940.
- [18] Margulis, G., *Complete affine locally flat manifolds with a free fundamental group*, J. Soviet Math. **134** (1987), 129–134.
- [19] Mess, G., *Lorentz spacetimes of constant curvature*, Geom. Ded. **126**, no. 1 (2007), 3–45, in “New techniques in Lorentz manifolds : Proceedings of the BIRS 2004 workshop,” (V. Charette and W. Goldman, eds.)

- [20] Milnor, J., *On fundamental groups of complete affinely flat manifolds*, Adv. Math. **25** (1977), 178–187.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK,
MD 20742 USA

E-mail address: `wmg@math.umd.edu`

UNIV. PARIS-SUD, LABORATOIRE DE MATHÉMATIQUES, ORSAY F-91405 CEDEX;
CNRS, ORSAY CEDEX, F-91405

E-mail address: `francois.labourie@math.u-psud.fr`