

***Interconnection network with a shared whiteboard:
Impact of (a)synchronicity on computing power***

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Interconnection network with a shared whiteboard: Impact of (a)synchronicity on computing power *

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Abstract: In this work we study the computational power of graph-based models of distributed computing in which each node additionally has access to a global whiteboard. A node can read the contents of the whiteboard and, when activated, can write one message of $O(\log n)$ bits on it. A message is only based on the local knowledge of the node and the current content of the whiteboard. When the protocol terminates, each node computes the output based on the final contents of the whiteboard in order to answer some question on the network's topology. We propose a framework to formally define several scenarios modelling how nodes access the whiteboard, in a synchronous way or not. This extends the work of Becker *et al.* [IPDPS 2011] where nodes were imposed to create their messages only based on their local knowledge (i.e., with the whiteboard empty). We prove that the four models studied have increasing power of computation: any problem that can be solved in the weakest one can be solved in the the second, and so on. Moreover, we exhibit problems that *separate* models, i.e., that can be solved in one model but not in a weaker one. These problems are related to Maximal Independent Set and detection of cycles. Finally we investigate problems related to connectivity as the construction of spanning- or BFS-tree in our different models.

Key-words: Distributed Computing, Communication Complexity, Asynchronism, Maximal Independent Set.

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Réseaux d'interconnexion avec mémoire partagée: Impact de l'asynchronisme sur le pouvoir de calcul

Résumé : Dans ce travaux, nous étudions le pouvoir de calcul de modèles de calcul distribué basés sur des graphes. Dans ces modèle, chaque sommet a accès à une zone de mémoire partagée, un tableau. Un sommet peut lire le contenu du tableau et, lorsqu'il est actif, il peut y écrire un message de $O(\log n)$ bits. Un message est basé sur la connaissance locale du sommet et sur le contenu courant du tableau. Lorsque le protocole se termine, chaque sommet doit être capable de calculer la réponse à une question sur la topologie du réseau, uniquement en se basant sur le contenu final du tableau. Nous proposons un cadre d'étude en définissant formellement plusieurs scénarii qui modélisent la façon dont les sommets accèdent au tableau, de manière synchrone ou non. Cet article généralise les travaux de Becker *et al.* [IPDPS 2011] dans lesquels les sommets devaient créer leurs messages en se basant uniquement sur leurs connaissances locales (lorsque le tableau est vide). Nous prouvons que les quatre modèles étudiés forment une hiérarchie avec un pouvoir de calcul croissant: chaque problème qui peut être résolu par le plus faible modèle peut l'être par le second modèle et ainsi de suite. De plus, nous décrivons des problèmes qui *séparent* les modèles, c'est-à-dire qui peuvent être résolus dans un des modèles mais pas dans un modèle plus faible. Les problèmes sont liés aux ensembles indépendant maximaux et à la détection de cycles. Pour finir, nous étudions dans les différents modèles des problèmes liés à la connexité comme la construction d'arbres couvrants et d'arbres BFS.

Mots-clés : Calcul Distribué, Complexité de Communication, Asynchronisme, Ensemble indépendant maximal.

1 Introduction

A distributed system is a network where nodes correspond to agents or processors and links express the local knowledge of the nodes. To perform any calculation – like deciding some network’s property – nodes may exchange information by interacting locally, i.e., with their neighbors. Since nodes lack of global knowledge, new algorithmic and complexity notions arise. In contrast with classical algorithmic theory – where the Turing machine is the consensus formal model of algorithm – in distributed systems many different models are considered. Under the paradigm that communication is much slower and more costly than local computations, complexity analysis of distributed algorithms mainly focuses on message passing. That is, an important performance measure is the number and the size of messages that are exchanged by nodes for performing some computation. Theoretical models were conceived for studying particular aspects of protocols such as fault-tolerance, synchronism, locality, congestion, etc. One of the main questions arising is to determine the global properties of the network that can be computed locally.

In the model CONGEST [17], a network is represented by a graph whose nodes correspond to network processors and edges to inter-processor links. The communication is synchronous and occurs in discrete time rounds. In each round, each of the n processors can send a message of size $O(\log n)$ bits through each of its outgoing links. A restriction of the CONGEST model has been proposed by Grumbach and Wu to study *frugal* computation [9]. In this model, where the total amount of information traversing each link is bounded by $O(\log n)$ bits, they showed that any first order logic formula can be evaluated in any planar or bounded degree network [9].

In [3], Becker *et al.* investigated a variation of CONGEST inspired by the Simultaneous Message Model defined by Babai, Kimmel and Lokam [2]. In this model, the total amount of local information that each node may provide is bounded by $O(\log n)$ bits. However, to compensate the little amount of knowledge that is shared, the communication is *global*: each node directly transmits its message to a central authority, *the referee*, that collects and uses them to answer some question about the network. This model allows to abstract away from the cost of *transmitting* data throughout the network, and to look at *how much local information must be shared* in order to compute some property.

More precisely, each of the n nodes, knowing only its own ID, the IDs of its neighbors and the size of the network, is allowed to send one message of $O(\log n)$ bits to the referee. Then, the referee can use the information it received to answer some question. Becker *et al.* asked whether this small amount of local information provided by each node is sufficient for the referee to decide some basic structural properties of the network topology G [3]. For instance, simple questions like “Does G contain a square?” or “Is the diameter of G at most 3?” cannot be solved. On the other hand, the referee can decode the messages in order to have full knowledge of G when G belongs to one of many graph classes such as planar graphs, bounded treewidth graphs and, more generally, bounded degeneracy graphs [3].

In this paper, we define natural extensions of the model in [3] and investigate the computational power of these new models.

Computations using a shared whiteboard. The computational model in [3] can be stated equivalently in the following form. Given a question on the topology of the network, every node writes *simultaneously* one message (computed from its local knowledge) on a global zone of shared memory, a *whiteboard*, and then, each of the nodes must answer the question by using only the content of the whiteboard.

In this paper, we intend to give more power to the initial model of [3]. For this purpose, we relax in different ways the simultaneity constraint. Roughly, messages may be written sequentially on the whiteboard. This allows nodes to compute their message taking into account the content of the whiteboard, i.e., the messages, or part of the messages, that have previously been written. In other words, in the new models we propose, nodes have extra ways to share information. Basically, the four models we now present aim at describing how the nodes can access the shared medium, in particular, differentiating synchronous and asynchronous networks.

The time is divided into discrete steps and, at every step, each node can see the current content of the whiteboard and may perform some local computation according to this content and its local knowledge. Along the evolution of the system, the nodes may be in three states: *awake*, *active*, or *terminated*. Initially, all nodes are *awake*. A node becoming *active* means that this node would like to write a message on the whiteboard, metaphorically speaking, it “rises its hand to speak”. Finally, a node is in state *terminated* when its message has been written on the whiteboard. During one step, several awake nodes may become active and exactly one active node becomes terminated. To model the worst-case behavior of a model, the choice of the node that becomes terminated is done by an *adversary* among the set of active nodes. Note that a node may become active and terminated during one step. After the last step, when all nodes are terminated, all of them must be able to answer the question by using only the information stored on the whiteboard.

In this setting, we propose several scenarios leading to the definition of four computational models. A computational model is said *free* if, at any step, any awake node may decide to become active based on its knowledge and on its own protocol. On the other hand, the model is said *simultaneous* if all nodes are forced to become active during the first step. The other criterion we use to distinguish models is the state-transition during which a node must create the message it will eventually write on the whiteboard. In the *asynchronous* scenario, the nodes must create their message during the step when they become active. In the *synchronous* scenario, the nodes must create their message during the step when they become terminated.

Intuitively, in the asynchronous scenario, a node becoming active must compute its message regarding the content of the whiteboard at this step. However, this message is actually written on the whiteboard only when the node becomes terminated, which depends on the choice of the adversary. Thus, there may be some delay between the creation of a message and the step when it is written. In particular, the order in which the messages are created and the order in which they are actually available on the whiteboard may differ. In this way, we can model real-world asynchronous systems where there are no guarantees on the time of communications.

In this paper, we combine the free/simultaneous and asynchronous/synchronous scenarios and study the four resulting models. In particular, it is easy to check that the simultaneous-asynchronous model exactly corresponds to the model studied in [3]. On the other hand, the free-synchronous model was inspired by the Multiparty Communication Protocol introduced by Chandra, Furst and Lipton [4]. We aim at deciding which kind of problems can be solved in different models. Moreover, we intend to show that these models form a hierarchy in which the computation power increases strictly.

Related work. Many variations to the CONGEST model have been proposed in order to focus on different aspects of distributed computing. In a seminal paper, Linial introduced the LOCAL model [12, 17]. In the LOCAL model, the restriction on the size of messages is removed so that every vertex is allowed to send unbounded size messages in every round. This model focuses on the issue of locality in distributed systems, and more precisely on the question “*What cannot be computed locally?*” [11]. Difficult problems like minimum vertex cover and minimum dominating set cannot be well approximated when processors can locally exchange arbitrary long messages during a bounded number of rounds [11].

In the centralized computing point of view, testing a property \mathcal{P} of a graph G may consist of determining the minimum number of elementary queries (e.g., “what is the i^{th} neighbor of some vertex v ?”) necessary to decide whether the graph satisfies it. This model was first studied by Goldreich *et al.* [7] (see [6] for a survey). In this context, probabilistic algorithms are given that always accept a graph if it satisfies the property and reject with constant probability any graph that is “far enough” from the property. For instance, [1] gives lower and upper bounds for testing the triangle-freeness in general graphs. Closer to our model, Goldreich and Ron provide efficient algorithms for testing the connectivity of bounded degree graphs when the list of neighbors of every vertex is given [8].

Other trade-offs between the size of a data structure and the complexity (in terms of the number of bits that must be checked in this structure) of algorithms for solving some problems have been provided using the

communication complexity model and the cell probe model [13, 14, 19, 20]. Testing graph properties has also been widely investigated using these frameworks (e.g., [5, 16, 18]).

2 Communication models

2.1 Protocol formulation

An interconnection network is modeled by a simple undirected connected n -node graph $G = (V, E)$. Each node $v \in V$ has a unique identifier $ID(v)$ between 1 and n . Typically, $V = \{v_1, \dots, v_n\}$, where v_i is such that $ID(v_i) = i$. Throughout the paper, a *graph* must be understood as a *labeled graph*.

At each node $v \in V$ there is an independent processing unit that knows the local knowledge of v : its own identifier, the identifier of each of its neighbors and the total number of nodes n . Moreover, any node $v \in V$ has a variable $output_v$, with arbitrary initial value and a variable $status_v \in \{awake, active, terminated\}$ initially set to *awake*.

Nodes can communicate with each other through a shared memory, called *whiteboard* and denoted by \mathcal{B} , that is initially empty. Any node can read \mathcal{B} and any active node is allowed to write exactly one $O(\log n)$ -bit message on it.

The time is divided into discrete steps. At any step, each node executes the same algorithm \mathcal{A} , or protocol, which may be divided into three sub-procedures. For any awake node v , the *activation* function takes the local knowledge of v and the current content of \mathcal{B} as input and decides to modify the $status_v$ variable to *active* or to maintain it as *awake*. For any active node v , the *message* function takes the local knowledge of v and the current content of \mathcal{B} as input and computes a message m_v of $O(\log n)$ bits. In particular, we may assume that m_v always contains the identifier of v and the number of messages present on \mathcal{B} at the step when m_v is created. It can serve as a signature with the identity of the author and the “time” of its creation. When n messages have been written on \mathcal{B} , the *decision* function takes the (final) content of \mathcal{B} as input and computes the final value of $output_v$. While some nodes are not in the *terminated* state, an active node v is chosen at the end of each time step and m_v is written on \mathcal{B} , and then $status_v$ is set to *terminated*.

Given a problem¹, our goal is to design an algorithm \mathcal{A} such that, at the end of the process (after the last step), and for any graph G , all nodes agree on the solution of the problem. That is, once all nodes have executed the decision function, all variables $output_v$, for any $v \in V$, give the correct answer for the problem. In what follows, to model the worst-case, the choice of which active node is chosen is done by an *adversary*. In this setting, we say that a problem *can be solved* in our model if there exists such an algorithm. It is important to note that, to avoid deadlock, a valid algorithm – precisely, its activation procedure – must ensure that, at any step, at least one node is active.

2.2 Scheduling of whiteboard access

We now specify four models of computation, following the general framework described above, that we study in this paper. For this purpose, we propose two additional constraints that may be satisfied or not, depending on the model.

In order to model how nodes access the whiteboard, i.e., in a synchronous way or not, we consider the following two variants: either a node must create its message as soon as it becomes active, or a node may defer creating its message until it has actually been chosen by the adversary to write it on the whiteboard. In the latter case, the node may take advantage of all the messages that have been written since it became active. More

¹Problems we are considering typically consist of a question on the graph’s topology and the output of which may be a boolean value (decision problems), the adjacency matrix of the graph, etc. However, our framework may also be used to deal with classical distributed problems like consensus, leader election, etc.

	message created when node becomes active	message created when node is chosen
all nodes initially active	SIMASYNC	SIMSYNC
no node initially active	FREEASYNC	FREESYNC

Table 1: Classification of communication models.

formally, we impose that any node executes the *message* function only once: in the *asynchronous* model, it must be executed during the same step when it becomes active, while in the *synchronous* model, it is executed when the node becomes terminated.

The second constraint we consider aims at avoiding an initial deadlock. Indeed, in some distributed contexts, it is not possible to decide whether it is better to write some message on the empty whiteboard or to wait a first message to be written. To deal with this, we consider that either the nodes may be free to decide when to become active, or they may be forced to be all active at the beginning. More formally, in the *simultaneous* model, the *activation* function is imposed to be the function that turns the status variable of each node to *active* during the first step (when the whiteboard is empty). In the *free* model, the definition of the *activation* function is part of the algorithm's design.

To satisfy or not this pair of constraints leads to four different communication models. We now detail the four communication models studied in this work. These models are summarized in Table 1.

Simultaneous Asynchronous (SIMASYNC). In this model, all nodes become active and create their messages at the first step. In other words, all nodes create their messages while the whiteboard is still empty. Hence, the message created by a node v only depends on the local knowledge of v and on n . Moreover, the ordering in which the messages are written on \mathcal{B} (i.e., in which the nodes are chosen by the adversary) is clearly not relevant. This is exactly the communication model studied in [3].

Simultaneous Synchronous (SIMSYNC). In this model, all nodes become active in the first step. However, messages are created just before being written on the whiteboard. In other words, the adversary chooses an ordering (x_1, \dots, x_n) of V such that, at step $1 \leq i \leq n$, vertex x_i computes its message according to its local knowledge, the total number of nodes n , and the $i - 1$ messages that have previously been written by x_1, \dots, x_{i-1} on \mathcal{B} . Note that the ordering is not known *a priori* by the nodes. Hence, an algorithm for solving a problem in this model must solve it for any ordering chosen by the adversary.

Free Asynchronous (FREEASYNC). In this model, the nodes decide when to turn active and create their messages at the same step that they become active. That is, a node may create a message long before being chosen by the adversary to write it down on \mathcal{B} .

Free Synchronous (FREESYNC). The nodes decide when to become active and create their messages only when they are asked by the adversary to write them on \mathcal{B} .

This paper aims at deciding what kind of problems can be solved in each of these models. For instance, [3] proves that deciding if a graph has degeneracy k , $k \geq 1$, can be solved in SIMASYNC. On the opposite, deciding whether a graph contains a triangle as a subgraph and deciding whether a graph has diameter at most 3 cannot be solved in SIMASYNC [3].

In Section 3, we focus on problems that separate the models, i.e., that can be solved in some model but not in another one. In Section 4, we focus on problems related to connectivity.

First of all, we prove the following lemma that extends a result of [3]. Let BUILD be the problem that consists in computing the adjacency matrix of an input graph G .

Lemma 1. *Let \mathcal{G} be a family of n -node labeled graphs, and $g(n)$ be the number of graphs in \mathcal{G} . In any of the four considered models, BUILD can be solved in the class \mathcal{G} only if $\log g(n) = O(n \log n)$.*

Proof. Consider any algorithm in one of the four considered models. In any model, at the end of the communication process, n messages of size $O(\log n)$ bits are written on \mathcal{B} . Notice that includes the information on the order in which the messages have been created. Hence, at the end, a total of $O(n \log n)$ bits are available on the whiteboard. For any node to distinguish two different graphs in \mathcal{G} , we must have $\log g(n) = O(n \log n)$. \square \square

For the ease of descriptions, in what follows we will not define explicitly the functions for activation, message creation and decision. Nevertheless, they always will be clear from the context.

3 A strict hierarchy

In this section, we intend to show that these models form a hierarchy in which the computation power strictly increases. Let us formalize this idea. We say that model Y is *more powerful* than model X , denoted by $X \leq Y$, if every problem that can be solved in X can also be solved in Y . Moreover, Y is *strictly more powerful* than X , denoted by $X < Y$, if $X \leq Y$ and there exists a problem \mathcal{P} that can be solved in Y but not in X .

The main result of this section is the following theorem:

Theorem 1. $\text{SIMASYNC} < \text{SIMSYNC} < \text{FREEASYNC} \leq \text{FREESYNC}$.

We start with the following weaker result:

Proposition 2. $\text{SIMASYNC} \leq \text{SIMSYNC} \leq \text{FREEASYNC} \leq \text{FREESYNC}$.

Proof. Given two models X, Y , to show that $X \leq Y$, we consider an algorithm for solving some arbitrary problem in X and show how to turn this algorithm to satisfy requirements of Y .

- $\text{SIMASYNC} \leq \text{SIMSYNC}$. In the SIMSYNC model, any node applies directly the protocol of the SIMASYNC model. Nodes create their message initially, ignoring the messages present on the whiteboard when they write their own.
- $\text{SIMSYNC} \leq \text{FREEASYNC}$. Recall that a problem is solved in the SIMSYNC model if the nodes compute the output *no matter* the order chosen by the adversary. So we can translate a SIMSYNC protocol into a FREEASYNC one if we fix an order (for instance v_1, \dots, v_n) and use this order for a sequential activation of the nodes.
- $\text{FREEASYNC} \leq \text{FREESYNC}$. It is the situation of the first inequality. It suffices to force the protocols in FREESYNC to create their messages based only on what was known at the moment when they became active.

\square

\square

3.1 SIMASYNC vs. SIMSYNC

We consider here a “rooted” version of the INCLUSION MAXIMAL INDEPENDENT SET problem. This problem, denoted by MIS, takes as input an n -node graph $G = (V, E)$ together with an identifier $ID(x)$, $x \in V$, and the desired output is any maximal (by inclusion) independent set containing x .

Proposition 3. MIS can be solved in the SIMSYNC model.

Proof. Recall that in the adversarial model, all nodes are initially active and that the adversary chooses the ordering in which the nodes write their messages. Hence, an algorithm in this model must specify the message created by a node v , according to the local knowledge of v and the messages written on the whiteboard before v is chosen by the adversary.

The protocol is trivial (it is the greedy one). When node v is chosen by the adversary, the message of v is either its own ID (meaning that v belongs to the final independent set) or v writes “no” (otherwise). The choice of the message is done as follows. The message is $ID(v)$ either if $v = x$ or if $v \notin N(x)$ and $ID(v)$ does not appear on the whiteboard for any $y \in N(v)$. Otherwise, the message of v is “no”.

Clearly, at the end, the set of vertices with their IDs on the whiteboard consists of an inclusion maximal independent set containing x . □ □

Proposition 4. MIS cannot be solved in the SIMASYNC model.

Proof. We proceed by contradiction. Let us assume that there exists a protocol \mathcal{A} for solving MIS in the SIMASYNC model. Then we show how to design an algorithm \mathcal{A}' to solve the BUILD Problem for any graph in this model, contradicting Lemma 1.

Let $G = (V, E)$ be a graph with $V = \{v_1, \dots, v_n\}$. For any $1 \leq i < j \leq n$, let $G_{i,j}^{(x)}$ be obtained from G by adding a vertex x adjacent to every vertex in V with the exception of v_i and v_j . Note that $\{x, v_i, v_j\}$ is the only inclusion maximal independent set containing x in $G_{i,j}^{(x)}$ if and only if $\{v_i, v_j\} \notin E$. Indeed, if $\{v_i, v_j\} \in E$, there are two inclusion maximal independent sets containing x : $\{x, v_i\}$ and $\{x, v_j\}$.

Recall that, in the SIMASYNC model, all nodes must create their message initially, i.e., while the whiteboard is still empty. Hence, the message created by a node only depends on its local knowledge. We denote by $\mathcal{A}(v_k, G_{i,j}^{(x)})$ the message created by node v_k following protocol \mathcal{A} when the input graph is $G_{i,j}^{(x)}$.

Notice that, for a given k , the node v_k can generate only two possible messages $\mathcal{A}(v_k, G_{i,j}^{(x)})$ depending on whether $k \in \{i, j\}$ or $k \notin \{i, j\}$. Therefore, we call m_k the message that v_k generates when $k \in \{i, j\}$ (i.e., x and v_k are not neighbors) and m'_k the message v_k generates when $k \notin \{i, j\}$ (i.e., x and v_k are neighbors).

From the previous protocol \mathcal{A} we are going to define another protocol \mathcal{A}' in the SIMASYNC model which solves the BUILD Problem for any graph. Protocol \mathcal{A}' works as follows. Every node v_k generates the pair (m_k, m'_k) of the two messages v_k would send in \mathcal{A} when it is adjacent to x and when it is not. Clearly, this consists of $O(\log n)$ bits.

Now let us prove that any node can reconstruct $G = (V, E)$ from the messages generated by \mathcal{A}' . More precisely, for any $1 \leq s < t \leq n$, any node can decide whether $\{v_s, v_t\} \in E$ or not. It is enough for any node to simulate the decision function of \mathcal{A} in $G_{s,t}^{(x)}$ by using messages m_s, m_t and $\{m'_k : k \in \{1, \dots, n\} \setminus \{s, t\}\}$. Since the output of \mathcal{A} is $\{x, v_s, v_t\}$ if and only if $\{v_s, v_t\} \notin E$, the results follows.

This means that from $O(n \log n)$ bits we can solve BUILD in the class of all graphs - a contradiction. □ □

Corollary 1. SIMASYNC < SIMSYNC.

We discuss now another problem that could possibly separate the two models. Given an $(n-1)$ -regular $2n$ -node graph G , the 2-CLIQUEs problem consists in deciding whether G is the disjoint union of two complete graphs with n vertices or not.

It is easy to show that 2-CLIQUEs can be solved in the SIMSYNC model. Indeed, a trivial protocol can partition the vertices into two cliques numbered 0 and 1 if the input consists of two cliques, or otherwise indicate that it is not the case. The first vertex f to be chosen by the adversary writes $(ID(f), 0)$ on \mathcal{B} . Then, each time a vertex v is chosen, it writes $(ID(v), 0)$ if it “believes” to be in the same clique as f , and $(ID(v), 1)$ otherwise. More precisely, let S_v be the subset of neighbors of v that have already written a message on the whiteboard. If $S_v = \emptyset$ then v writes 1. If all nodes in S_v have written that they belong to the the same clique $c \in \{0, 1\}$ then v writes c ,

and v writes “no” otherwise. Clearly, G is the disjoint union of two cliques if and only if there is no message “no” on the whiteboard at the end of the communication process.

Proving that 2-CLIQUEs cannot be solved in the SIMASYNC model is an interesting question because it would allow us to show that CONNECTIVITY (deciding whether a graph is connected or not) cannot be solved in the SIMASYNC model. Indeed, it is easy to show that an $(n-1)$ -regular $2n$ -node graph is the disjoint union of two cliques if and only if it is not connected. We leave this as an open question:

Open Problem 1. Can 2-CLIQUEs be solved in the SIMASYNC model?

3.2 SIMSYNC vs. FREEASYNC

Let SQUARE be the problem that consists in deciding whether a graph G contains a *square* (induced or not), i.e., whether $V(G)$ contains four vertices a, b, c and d such that a is adjacent to b which is adjacent to c which is adjacent to d which is adjacent to a .

In [3], it is proved that SQUARE cannot be solved in the SIMASYNC model. The proof consists of showing that if SQUARE could be solved in the SIMASYNC model, then BUILD could be solved, in this model, in the class of square-free graphs, a contradiction.

Theorem 5. [3] SQUARE cannot be solved in the SIMASYNC model.

We extend this result to the SIMSYNC model. More precisely, we show that SQUARE cannot be solved in SIMSYNC model even restricted to a specific class of graphs. We then show that, in this particular graph class, SQUARE can be solved in the FREEASYNC model. Hence, the SQUARE problem on this class of graphs separates SIMSYNC and FREEASYNC.

First, let \mathcal{C} be the class of graphs of even order $N = 2n$ ($n \geq 1$) that can be obtained as follows. $G \in \mathcal{C}$ has the vertex set $\{v_1, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$ where $G[\{v_1, \dots, v_n\}]$ induces a square-free n -node graph H , and, for any $1 \leq i \leq n$, v_{i+n} is adjacent to v_i , and finally, there is a unique additional edge between v_{n+i} and v_{n+j} for some $1 \leq i < j \leq n$ (i.e., for any $k \in \{1, \dots, n\} \setminus \{i, j\}$, v_{n+k} has degree one in G). In the following, we note $G = (H, i, j)$. Note that, since there are $\Omega(2^{n^{3/2}})$ (labeled) square-free graphs with n nodes [10], $|\mathcal{C}| = \Omega(2^{n^{3/2}}) = \Omega(2^{N^{3/2}})$.

Proposition 6. SQUARE cannot be solved in the SIMSYNC model, even when inputs are restricted to \mathcal{C} .

Proof. For purpose of contradiction, let us assume that there is a protocol \mathcal{P} for solving SQUARE in \mathcal{C} in the SIMSYNC model. We design a protocol \mathcal{P}' for solving BUILD in \mathcal{C} , contradicting Lemma 1 since $N \log N = o(\log |\mathcal{C}|)$.

In the SIMSYNC model, the nodes are asked to write their messages in some order. When it is the turn of node v_i it computes its message according to the current content of the whiteboard and to its own neighborhood.

Let $G = (H, \ell, k) \in \mathcal{C}$. The key point is that \mathcal{P} solves SQUARE in G whatever be the order in which the vertices write their messages. In particular, if the vertices in $\{v_1, \dots, v_n\}$ are interrogated first, their messages cannot bring any information on the single edge between two vertices in $\{v_{n+1}, \dots, v_{2n}\}$. Moreover, the vertices in $\{v_{n+1}, \dots, v_{2n}\}$ have degree at most two in G and so can write their full neighborhood. Using these two facts, we design \mathcal{P}' that will be used to rebuild any graph in \mathcal{C} .

The protocol \mathcal{P}' is defined as follows. For any $i \leq 2n$, the message created and written by node v_i when it is interrogated consists of

- if $n < i \leq 2n$, then v_i writes the identifiers of its at most two neighbors;
- otherwise, let O be the sequence of the vertices in $\{v_1, \dots, v_n\}$ that have been interrogated before v_i , in order. Using its local neighborhood and the messages previously written by the vertices in O , v_i writes the message it would have written following \mathcal{P} and in the ordering O .

We now show that whatever be the ordering O in which the $2n$ vertices have been interrogated, the final content of the whiteboard allows any vertex to build the adjacency matrix of G . More precisely, we show that any node can decide whether the edge $\{v_i, v_j\} \in E(G)$ for any $1 \leq i < j \leq 2n$.

Clearly, for any $1 \leq i \leq n$, the edges adjacent to v_{n+i} can be decided since they appear explicitly on the whiteboard (in particular for $i = \ell$ and $j = k$). Let O' be the restriction of O to the vertices in $\{v_1, \dots, v_n\}$. Now, for any $1 \leq i < j \leq n$, we show that, using the information on the whiteboard, any node can simulate the protocol \mathcal{P} in the graph (H, i, j) , against the ordering $O' \odot (v_{n+1}, \dots, v_{2n})$ and decide whether $\{v_i, v_j\}$ does exist in G which is the case if and only if (H, i, j) has a square.

After the execution of the protocol, the whiteboard contains the messages m_1, \dots, m_n that would have written by v_1, \dots, v_n when following protocol \mathcal{P} against the ordering O' . Using this information, any node can compute the message that v_t ($n < t \leq 2n$) would have written when executing \mathcal{P} in (H, i, j) . Therefore, it can decide whether (H, i, j) has a square, i.e., whether $\{v_i, v_j\} \in E(G)$.

Since by Lemma 1 and because of the cardinality of \mathcal{C} , no protocol can solve BUILD in \mathcal{C} and in the SIMSYNC model, we get a contradiction. \square \square

Corollary 2. SQUARE cannot be solved in the SIMSYNC model.

Proposition 7. SQUARE can be solved in \mathcal{C} in the FREEASYNC model.

Proof. The protocol is almost trivial. First, any node with identifier at least $n + 1$ becomes active and creates a message containing its neighborhood (recall that it has at most two neighbors in G). By reading the n messages that have been written by v_{n+1}, \dots, v_{2n} , all remaining nodes know the (unique) pair (i, j) such that $\{v_{i+n}, v_{j+n}\} \in E(G)$ and know the neighbor u of v_{n+i} , resp., the neighbor v of v_{j+n} , in $\{v_1, \dots, v_n\}$. Finally, all remaining vertices become active: for any $k \leq n$, v_k writes an empty message if $v_k \notin \{u, v\}$ and u and v write “yes” if they are adjacent and “no” otherwise. Clearly, the graph admits a square if and only if the whiteboard eventually contains “yes”. \square \square

Corollary 3. SIMSYNC < FREEASYNC.

Open Problem 2. Can SQUARE be solved in the FREEASYNC model, in FREESYNC?

4 Connectivity and related problems

One of the main questions arising in distributed environments concerns connectivity. For instance, one important task in wireless network consists in computing a connected spanning subgraph (e.g., a spanning tree) the links of which will be used for communications. In this section, we ask in which of our models such problems can be solved.

We consider the following three problems. By increasing level of difficulty²: the CONNECTIVITY Problem asks if an input graph is connected or not; given an input graph G and an input identifier $ID(r)$, the SPANNING-TREE Problem requires as output a spanning-tree of G rooted in r if it exists; similarly, the BFS Problem requires a BFS-tree of the input graph G rooted in some given node (if G is connected) as an output.

In [3], authors conjecture that CONNECTIVITY (and therefore SPANNING-TREE) cannot be solved in the SIMASYNC model. This is still an open question. We do not even know whether these problems can be solved in the SIMSYNC model.

Nevertheless, it is clear that SPANNING-TREE (and therefore CONNECTIVITY) can be solved in the FREEASYNC model. The protocol is the greedy one. First the root r becomes active and is chosen by the adversary and writes

²We say that a problem \mathcal{P} is more difficult than a problem \mathcal{P}' if any algorithm for solving \mathcal{P} also solves \mathcal{P}' , or equivalently, any solution for \mathcal{P} is a solution for \mathcal{P}'

its ID. Then, at each step, nodes having a neighbor already included in the spanning tree become active and choose its parent.

In the more powerful FREESYNC model it is possible not only to construct a spanning tree but a BFS tree. Table 2 summarizes the results of this section.

	SIMASYNC	SIMSYNC	FREEASYNC	FREESYNC
BFS	No	No (Prop. 9)	? Ok in Bipartite graphs (Cor. 4)	Ok (Prop. 8)
SPANNING-TREE	?	?	Ok (Remark above)	Ok
CONNECTIVITY	?	?	Ok	Ok

Table 2: Various problems related to connectivity and models where they can(not) be solved.

Proposition 8. *BFS can be solved in the FREESYNC model.*

Proof. The idea is to simulate phases at the end of which every node in the same layer, i.e., at the same distance from v_1 , knows the ID of its parent. The protocol must let the nodes of layer k , i.e., at distance $k \geq 1$ from v_1 , to know the step when *all the nodes of layer $k - 1$* have already written their messages on the whiteboard.

The algorithm for solving BFS is defined as follows. Initially (when the whiteboard is empty), only v_1 must become active and writes its ID, its degree and its layer 0. This is Phase 0.

At Phase $i > 0$, any node v in layer i becomes active and its message consists of

1. its own ID;
2. its layer i ;
3. the ID of its neighbor in layer $i - 1$ with minimum ID (this neighbor will be considered as its parent in the BFS tree);
4. the number a_v of its neighbors in layer $i - 1$;
5. the number b_v of its neighbors which are not in layer $i - 1$ ($a_v + b_v$ equals the degree of v);
6. the number c_v of its neighbors in layer i that have already written their messages on the whiteboard.

We now have to prove that every node becomes active at the right phase and that it can compute the required information from its local knowledge and what have previously been written on the whiteboard. We prove it by induction on $i \geq 1$. In particular, we prove that all nodes can decide when Phase $i - 1$ terminates and that, at this step, *the number e_{i-1} of edges between layer $i - 1$ and layer i* can be computed from the information available on the whiteboard.

Phase 0 terminates when v_1 writes its message which contains its degree, i.e., e_0 . Then, all neighbors of v_1 (the vertices of layer 1) become active. The vertices of $N(v_1)$ can easily compute the required information, no matter the ordering the adversary chooses. Note that information 6 can be obtained because we consider the FREESYNC model, i.e., any node can create its message at the step when it is chosen (and not when it becomes active). Moreover, all nodes know the number of vertices in layer 1 and so can decide when Phase 1 terminates. Finally, e_1 is exactly the sum of $b_v - 2c_v$ among the vertices v in layer 1. Hence, at the end of Phase 1, the induction hypothesis is satisfied.

Assume the induction hypothesis is satisfied at the end of Phase $i > 0$. Then, any node that has not become active yet and that has a neighbor in layer i knows that it belongs to layer $i + 1$ and becomes active. No matter the ordering the adversary chooses, the vertices in layer $i + 1$ can easily compute the required information since, by

the induction hypothesis, all nodes in layer i have written their IDs together with the corresponding layer on the whiteboard.

Moreover, the nodes can detect the end of Phase $i + 1$ since e_i exactly equals the sum of the a_v over the vertices v in layer $i + 1$. Finally, e_{i+1} is exactly the sum of $b_v - 2c_v$ among the vertices v in layer $i + 1$.

To conclude, since the vertices know n , they can detect when the communication process terminates. Moreover, any node can compute a BFS tree because every vertex has written its parent ID on the whiteboard. \square \square

Corollary 4. *BFS can be solved in the FREEASYNC model, in the class of bipartite graphs.*

Proof. In a bipartite graph there are no edges between nodes in the same layer, and therefore $c_v = 0$ for every node v . In other words, we need to apply the protocol for the general case without computing information 6. \square

Proposition 9. *BFS cannot be solved in the SIMSYNC model.*

Proof. For purpose of contradiction, let us assume that there is a protocol \mathcal{P} for solving BFS in the SIMSYNC model. We design a protocol \mathcal{P}' for solving BUILD for any n -node graph, contradicting Lemma 1.

First, let \mathcal{C} be the class of graphs of even order $N = 4n - 1$ ($n \geq 1$) that can be obtained as follows. A graph $G = (H, i) \in \mathcal{C}$ is built from any n -node graph H with vertex-set $\{v_1, \dots, v_n\}$ and any integer i , $1 \leq i \leq n$, in the following way: let us add $3n$ vertices $\{v_{n+1}, \dots, v_{4n}\} = \{r = v_{n+1}, a_1, \dots, a_n, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n, c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n\}$ such that, for any $j \leq n$, r is adjacent to a_j , for any $j \leq n$, $j \neq i$, b_j is adjacent to v_j , c_j is adjacent to a_j and to b_j , and finally, a_i is adjacent to v_i . Note that $|\mathcal{C}| = \Omega(2^{n^2})$.

Let $G = (H, k) \in \mathcal{C}$. The key point is that \mathcal{P} solves BFS in G whatever be the order in which the vertices write their messages. In particular, if the vertices in $\{v_1, \dots, v_n\}$ are interrogated first, their messages cannot bring any information on which of the a_i is adjacent to r . The neighborhood of r can easily be encoded with $O(\log n)$ bits since it is adjacent to all a_i , $i \leq n$. Moreover, the vertices in $\{v_{n+2}, \dots, v_{4n}\}$ have degree two in G and so can write their full neighborhood. Using these two facts, we design \mathcal{P}' that will be used to rebuild any graph in \mathcal{C} .

The protocol \mathcal{P}' is defined as follows. For any $i \leq 4n$, the message created and written by node v_i when it is interrogated consists of

- if $i = n + 1$, r writes its neighborhood.
- if $n + 1 < i \leq 4n$, then v_i writes the identifiers of its at most two neighbors;
- otherwise, let O be the sequence of the vertices in $\{v_1, \dots, v_n\}$ that have been interrogated before v_i , in order. Using its local neighborhood and the messages previously written by the vertices in O , v_i writes the message it would have written following \mathcal{P} and in the ordering O .

We now show that whatever be the ordering O in which the $4n$ vertices have been interrogated, the final content of the whiteboard allows any vertex to build the adjacency matrix of G . More precisely, we show that any node can decide whether the edge $\{v_i, v_j\} \in E(G)$ for any $1 \leq i < j \leq 2n$.

Clearly, for any $1 \leq i \leq 3n$, the edges adjacent to v_{n+i} can be decided since they appear explicitly on the whiteboard. Let O' be the restriction of O to the vertices in $\{v_1, \dots, v_n\}$. Now, for any $1 \leq i < j \leq n$, we show that, using the information on the whiteboard, any node can simulate the protocol \mathcal{P} in the graph (H, i) , against the ordering $O' \odot (v_{n+1}, \dots, v_{4n})$. By construction of (H, i) , if the edge $\{v_i, v_j\} \in E(G)$ it must belong to the BFS-tree computed by \mathcal{P} on (H, i) . Hence, any node can decide whether $\{v_i, v_j\}$ does exist in G .

More precisely, after the execution of the protocol, the whiteboard contains the messages m_1, \dots, m_n that would have written by v_1, \dots, v_n when following protocol \mathcal{P} against the ordering O' . Using this information, any node can compute the message that v_t ($n < t \leq 4n$) would have written when executing \mathcal{P} in (H, i) . Therefore, it can decide whether $\{v_i, v_j\}$ belongs to the BFS-tree of (H, i) , i.e., whether $\{v_i, v_j\} \in E(G)$.

Since by Lemma 1 and because of the cardinality of \mathcal{C} , no protocol can solve BUILD in \mathcal{C} and in the SIMSYNC model, we get a contradiction. \square \square

Open Problem 3. Is it true that $\text{FREEASYNC} < \text{FREESYNC}$? We conjecture that this is the case and that in fact BFS cannot be solved in the FREEASYNC model.

5 Conclusion and perspectives

We have investigated four models of distributed computing, extending the results presented in [3]. The definitions of these models are based on some intuitive conditions related to synchronicity, and we have seen that imposing them on the system has significant impact on what problems can be solved. We proved that there exists a hierarchy of non-decreasing computing power between these models. Moreover, we proved that in two cases the power strictly increases and left an open problem to check if it also holds in the third case.

We have analyzed several problems related to independence, cyclicity and connectivity. For these problems, we ask what are the conditions that a distributed computational model requires to solve them. Motivated by applications in routing [15], we paid special attention to connectivity: we analyzed the complexity of general spanning tree and BFS-tree construction. In particular, we showed that BFS-tree construction cannot be solved if nodes are not free to decide when they activate. On the other hand, the problem can be solved under the additional condition of synchronicity. The necessity of synchronicity for BFS-tree construction is left as an open problem.

We only analyzed the systems where no faults are allowed. It would be interesting to see what can be done when the system admits computation or communication errors. Another possible direction is to further the analysis of construction problems for connected spanning subgraphs that satisfy properties needed in compact routing protocols (see [15]).

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