Self-stabilizing processes in multi-wells landscape in $\mathbb{R}^d$ - Invariant probabilities[*]

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Abstract

The aim of this work is to analyse the stationary measures for a particular class of non-markovian diffusions: the self-stabilizing processes. All the trajectories of such a process attract each other. This permits to exhibit a non-uniqueness of the stationary measures in the one-dimensional case, see [HT10a]. In this paper, the extension to general multi-wells lansdcape in general dimension is provided. Moreover, the method for investigating this problem is different and needs less assumptions. The small-noise limit behavior of the invariant probabilities is also given.

Key words and phrases: Self-interacting diffusion, Free-energy, McKean-Vlasov stochastic differential equations, Stationary measures, Uniqueness problem, Granular media equation

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Introduction

We are interested in the invariant probabilities of the following non-markovian diffusion:

$$X_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t \nabla W_s(X_s) ds \qquad \text{(I)}$$

where W_s is a potential which is evolving in time. Moreover, we assume that W_s depends only on $\mathcal{L}(X_s) =: u_s$, the own law of the diffusion. In this paper,

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the potential W_s is given by

$$W_s(x) := V(x) + \int_{\mathbb{R}^d} F(x-y)u_s(dy) = \left(V + F * u_s\right)(x)\,.$$

The notation $*$ is used for denoting the convolution. V is the so-called confining potential. It corresponds to a classical drift. F is the interacting potential. Indeed, the term $\nabla F * u_s\left(X_s(\omega_0)\right)$ is equal to $\int_{\omega\in\Omega} \nabla F\left(X_s(\omega_0) - X_s(\omega)\right) d\mathbb{P}(\omega)$. We can write (I) in this way:

$$\left\{\begin{array}{l} X_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t \nabla V\left(X_s\right) ds - \int_0^t \nabla F * u_s\left(X_s\right) ds \\ u_s = \mathcal{L}\left(X_s\right) \end{array}\right. . \tag{I}$$

This equation is nonlinear in the sense of McKean, see [McK67, McK66]. We note that X_t and u_t depend on ϵ. We do not write ϵ for simplifying the reading.

The existence of the invariant probabilities of (I) and the small-noise behaviour of these measures are the subject of the article.

The diffusion X_t corresponds to the thermodynamical limit of a particle in a continuous mean-field system when the number of particles tends towards infinity. The mean-field system associated to the self-stabilizing process (I) is a random dynamical system like

$$\left\{\begin{array}{l} dX_t^1 = \sqrt{\epsilon} dB_t^1 - \nabla V\left(X_t^1\right) dt - \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_t^1 - X_t^j\right) dt \\ \vdots \\ dX_t^i = \sqrt{\epsilon} dB_t^i - \nabla V\left(X_t^i\right) dt - \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_t^i - X_t^j\right) dt \\ \vdots \\ dX_t^N = \sqrt{\epsilon} dB_t^N - \nabla V\left(X_t^N\right) dt - \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_t^N - X_t^j\right) dt \end{array}\right. \tag{II}$$

where the N brownian motions $\left(B_t^i\right)_{t\in\mathbb{R}_+}$ are independents. The link between (I) and (II) is called the propagation of chaos and is based - intuitively - on the following remark: the more N is large, the less a particle X_t^j has influence on X_t^1. Consequently, it is reasonable to consider that the particles are more and more independents and that the empirical measure $\frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$ converges towards a measure u_s which would be the own law of X_s^1. For a rigorous proof of this statement, see [Szn91, BRTV98, Mél96, BAZ99, CGM08].

The existence problem of McKean-Vlasov diffusions (I) has been investigated by two different methods. The first one consists in the application of a fixed point theorem, see [McK67, BRTV98]. The existence holds also when the confining potential is not convex, see [HIP08]. The other method consists in using the propagation of chaos ([Mél96]).

In [McK67], the author proved that the law u_t of the unique strong solution admits a $\mathcal{C}^\infty$-continuous density with respect to the Lebesgue measure for all

$t > 0$ and we will also denote it by u_t. Furthermore, this density satisfies a nonlinear partial differential equation of the following type:

$$\frac{\partial}{\partial t}u_t = \mathrm{div}\left\{\frac{\epsilon}{2}\nabla u_t + u_t\nabla W_t\right\} = \mathrm{div}\left\{\frac{\epsilon}{2}\nabla u_t + u_t\left(\nabla V + \nabla F * u_t\right)\right\}. \qquad \text{(III)}$$

This link between the granular media equation (III) and the McKean-Vlasov diffusion (I) permits to study the partial differential equation by probabilistic methods ([CGM08, Mal03, Fun84]). Reciprocally, it is a useful tool for characterizing the stationary measures and the long-time behavior, see [BRTV98, BRV98, Tam84, Tam87, Ver06].
When the confining potential V is not convex, Theorem 3.2 in [HT10a] states the thirdness of the stationary measures under natural conditions. This non-uniqueness prevents the long-time behavior to be as intuitive as in the case of a unique stationary measure.
The work in [HT10b] and [HT09] provides some estimates of the small-noise asymptotic of these three stationary measures. In particular, the convergence towards Dirac measures and its rate of convergence are investigated. In the bifurcation between the synchronized case and the asynchronized case, the rate of convergence is not linear. A note has been made on this subject: [Tug11b].

If V is identically equal to 0, the authors in [BRV98] proved the convergence in long time towards the stationary measure. Another method consists in using the propagation of chaos in order to derive the convergence of the self-stabilizing process from the one of the mean-field system, see [CGM08, Mal03] when V is convex. Nevertheless, the non-uniqueness of the stationary measures pointed out in [HT10a, Tug11a] implies that uniform propagation of chaos does not hold. The convergence in the non-convex case has been done in [Tug10b] when the dimension is equal to one.

In [HIP08], the authors investigate the exit-time of (I). For doing this, they use tacitly the stationary measures. In particular, they assume the convexity of V which ensures immediatly the existence and the uniqueness of the stationary measure. Moreover, the small-noise limit of this unique stationary measure is easy to find out. Therefore the knowledge of the stationary measure (the number and the limits) is important for the exit problem.

As noted previously, the diffusion (I) is similar to the particle X_t^1 defined in (II). However, this system is a Kolmogorov diffusion with the following potential:

$$\Upsilon^N(X_1,\cdots,X_N) := \int_{\mathbb{R}^d} V(x)\mu^N(dx) + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} F(x-y)\mu^N(dx)\mu^N(dy)$$

$$\text{with}\quad \mu^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_j}.$$

It yields $\frac{d}{dt}\mathbb{E}\left\{\Upsilon^N\left(X_t^1,\cdots,X_t^N\right)\right\}=-\mathbb{E}\left\{\left|\left|\nabla\Upsilon^N\left(X_t^1,\cdots,X_t^N\right)\right|\right|^2\right\}$ if $\epsilon=0$. The equivalent of this potential Υ^N for the flow (III) is Υ defined as

$$\Upsilon(u):=\int_{\mathbb{R}^d}V(x)u(dx)+\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}F(x-y)u(dx)u(dy)\,.$$

However, $\epsilon>0$. Consequently, we add a term which corresponds to the entropy:

$$\Upsilon_\epsilon(u):=\frac{\epsilon}{2}\int_{\mathbb{R}^d}u\log(u)+\Upsilon(u)\tag{IV}$$

for all measures u which are absolutely continuous with respect to the Lebesgue measure. As noted previously, the law u_t satisfies this hypothesis for all $t>0$. The functionnal Υ_ϵ is called the free-energy. Intuitively, we have $\Upsilon_\epsilon\left(u+\delta u\right)-\Upsilon_\epsilon(u)=\int_{\mathbb{R}^d}\left(\frac{\epsilon}{2}\log(u)+V+F*u\right)\delta u+o\left(\delta u\right)$ if δu is an infinitesimal measure such that $\int_{\mathbb{R}^d}\delta u=0$. Then, the application to the law u_t is roughly speaking:

$$\begin{aligned}\frac{d}{dt}\Upsilon_\epsilon\left(u_t\right)&=\int_{\mathbb{R}^d}\left(\frac{\epsilon}{2}\log\left(u_t\right)+W_t\right)\mathrm{div}\left\{\frac{\epsilon}{2}\nabla u_t+u_t\nabla W_t\right\}\\&=-\int_{\mathbb{R}^d}\left|\left|\frac{\epsilon}{2}\nabla u_t+u_t\nabla W_t\right|\right|^2\frac{1}{u_t}\end{aligned}$$

if an integration by parts was possible. This implies that the free-energy is non-increasing along the trajectories of the flow $(u_t)_{t\in\mathbb{R}_+}$. This statement has been proved rigorously in [CMV03]. By proceeding like in [BCCP98], it is possible to show that the family $\{u_t\,;\,t\in\mathbb{R}_+\}$ admits an adherence value which is an invariant probability of (I). The method for obtaining the existence of stationary measure consists in finding an open set $\mathcal{M}$ of measures absolutely continuous with respect to the Lebesgue measure such that $\inf_{\mu\in\partial\mathcal{M}}\Upsilon_\epsilon(\mu)>\inf_{\mu\in\mathcal{M}}\Upsilon_\epsilon(\mu)$ where $\partial\mathcal{M}$ denotes the set of the measures absolutely continuous with respect to the Lebesgue measure which are in the boundary of $\mathcal{M}$. This proceedure will permit to exhibit invariant probabilities in a much simpler way than in [HT10a] and with less assumptions. Moreover, we will obtain the convergence in the small-noise limit immediately. Then, we will provide the small-noise limit of the family of stationary measures in a more general way. The possible limits will be studied. Finally, we postpone two asymptotic and classical computationnal results in the annex.

Assumptions

We present now the properties of the confining potential V:

Assumption (V-1): *V is a polynomial function on each coordinate $x_1,\cdots,x_d$. And, the total degree of V is* $\deg(V)=:2m\geq4$.

It is possible to consider more general setting. Indeed, in the following, we only need V to be infinitely derivable in each coordinate. All the mathematical difficulties are present in the polynomial case and it permits to avoid some technical and tedious computations.

Assumption (V-2): *The equation* $\nabla V(x) = 0$ *admits a finite number of solution. We do not specify anything about the nature of these critical points. However, the wells will be denoted by* a_0*.*
The aim of this assumption is to separate the different critical points. Indeed, we aim to prove that there is an invariant probability around each wells (under an easy to verify assumption). And, the method requests that the measures δ_{a_0} and δ_{a_1} are separate if a_1 is another critical point.
Assumption (V-3): $V(x) \geq C_4 ||x||^4 - C_2 ||x||^2$ *for all* $x \in \mathbb{R}^d$ *with* $C_2, C_4 > 0$. $||\,.\,||$ *denotes the euclidian norm.*
Assumption (V-4): $\lim_{||x|| \to \pm\infty} \mathrm{Hess}\, V(x) = +\infty$ *and* $\mathrm{Hess}\, V(x) > 0$ *for all* $x \notin K$ *where* K *is a compact of* $\mathbb{R}^d$ *which contains all the critical points of* V*.*
These conditions ensure that the confining potential V confines the diffusion. It is used for proving the existence of a solution to (I), see [Tug10a].
An important constant associated to V is ϑ:

$$\vartheta := 2 \sup_{x \in \mathbb{R}^d} \sup_{z \in \mathbb{R}^d} \lim_{t \to 0} \frac{V(x) + t \langle \nabla V(x)\,;\, z \rangle - V(x + tz)}{t^2}\,.$$

In dimension one, $\vartheta = \sup_{z \in \mathbb{R}} -V''(z)$.

Let us present now the assumptions on the interaction potential F:
Assumption (F-1): *There exists an even polynomial function* G *on* $\mathbb{R}$ *such that* $F(x) = G(||x||)$*. And,* $\deg(G) =: 2n \geq 2$*.*
The choice of a polynomial function implies that W_t is a polynomial function parametrized by a finite number of parameters. Thereby, the small-noise limit of the stationary measures is tractable. Also, in [HT10a], the method used for finding the stationary measures was based on a fixed point theorem in $\mathbb{R}^{2n-1}$. Let us note that the method of this paper does not apply the fixed point theorem.
Assumption (F-2): G *and* G'' *are convex.*
In terms of mean-field systems, this means that two particles are more attracted when they are far than when they are closed. Thereby, X_t does not correspond to a spatial position.
Assumption (F-3): $G(0) = 0$.
An important constant will be used subsequently:

$$\alpha := G''(0) = \inf_{z \in \mathbb{R}_+} G''(z) \geq 0\,.$$

We present now the assumptions on the initial law u_0:
Assumption (ES): *The* $8q^2$*-th moment of the measure* u_0 *is finite with* $q := \max\{m, n\}$*.*
By Theorem 2.12 in [HIP08], we deduce that (I) admits a unique strong solution. Moreover, we have the following inequality:

$$\max_{1 \leq j \leq 8q^2} \sup_{t \in \mathbb{R}_+} \mathbb{E}\left[||X_t||^j \right] \leq M_0\,.$$

If the $2p$-th moment of u_0 is finite, the previous inequality holds with $2p$ instead of $8q^2$. We deduce immediately that the family $(u_t)_{t\in\mathbb{R}_+}$ is tight.

Definition: *Let us introduce* $\mathcal{A}_\epsilon$ *the set of all the limiting value of the family* $\{u_t\,;\,t\in\mathbb{R}_+\}$. *And, let* $\mathcal{S}_\epsilon$ *the set of all the stationary measures for* (I).

Assumption (FE): *The measure* u_0 *admits a* $\mathcal{C}^\infty$*-continuous density* u_0 *with respect to the Lebesgue measure. And, the entropy* $\int_{\mathbb{R}^d} u_0\log(u_0)$ *is finite.*

In the following, we shall use occasionnaly one of the following two additional properties concerning the two potentials V and F:

(LIN) G' is linear where G is defined in Assumption F-1. Moreover, for all $m\in\mathbb{R}^d$, the equation $\nabla V(x)+\alpha x-\alpha m=0$ admits a finite number of solutions.

(SYN) $\alpha+\vartheta>0$ and $2n=\deg(G)<\deg(V)=2m$.

For concluding the introduction, we write the statement of the main result:

Theorem: *Let a wells* a_0 *of* V *such that* $V(x)+F(x-a_0)>V(a_0)$ *for all* $x\neq a_0$. *Then, for all* $\rho>0$ *small enough, there exists* $\epsilon_0>0$ *such that for all* $\epsilon\in]0;\epsilon_0[$, *the diffusion* (I) *admits a stationary measure* u_ϵ *which satisfies* $\int_{\mathbb{R}^d}||x-a_0||^{2n}u_\epsilon(x)dx\leq\rho^{2n}$.

1 Preliminaries

We begin by providing basic results. First, we remark that we can prove (see [BCCP98] or [Tug10b] for a proof in dimension one) the following inequality:

$$\Upsilon_\epsilon(u_t)\geq -C\epsilon+\int_{\mathbb{R}^d}\left(V(x)+\frac{1}{2}F*u_t(x)-\frac{\epsilon}{4}||x||^2\right)u_t(x)dx \tag{1.1}$$

where C is a real constant. This inequality permits to eliminate the entropy term in the free-energy. The hypotheses on F and (1.1) implies

$$\Upsilon_\epsilon(u_t)\geq -C\epsilon+\int_{\mathbb{R}^d}\left(V(x)-\frac{\epsilon}{4}||x||^2\right)u_t(x)dx\,.$$

The diffusion (I) is under the influence of the potential $W_t:=V+F*u_t$. Since F is a polynomial function of the euclidian norm, it is possible to write the term $F*u_t$ as a polynomial function parametrized by the moments of the law u_t.

Lemma 1.1. *Let a measure* μ *which admits a finite moment of order* $2n$. *Then,* *the quantity* $F*\mu(x)$ *is well defined and we have the following development:*

$$F*\mu(x)=\sum_{k=1}^{n}\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1}\sum_{\sigma\in\mathcal{S}_{k-p_1-p_2}}\mathcal{C}^{\sigma}_{k,p_1,p_2}(\mu)\,||x||^{2p_1}\,\nu^{\sigma}(x) \tag{1.2}$$

$$\text{with}\quad \mathcal{C}^{\sigma}_{k,p_1,p_2}(\mu):=\frac{G^{(2k)}(0)}{(2k)!}\,\frac{k!(-2)^{k-p_1-p_2}}{p_1!p_2!(k-p_1-p_2)!}\int_{\mathbb{R}^d}||y||^{2p_2}\,\nu^{\sigma}(y)\mu(dy)$$

$$\text{and}\quad \nu^{\sigma}_{l}(x):=\prod_{i=1}^{l}x_{\sigma(i)}\quad\forall\,\sigma\in\mathcal{S}_l:=[\![1\,;\,d]\!]^{[\![1\,;\,l]\!]}\,.$$

The proof is left to the attention of the reader. Let us note that the development is much more complicate and tedious than the one in [HT10b]. We make the remark that each parameter $\mathcal{C}^{\sigma}_{k,p_1,p_2}(\mu)$ can be controlled by the quantity $\int_{\mathbb{R}^d}||y||^{2n}\,d\mu(y)$:

Remark 1.2. *For all* $k\in[\![1\,;\,n]\!]$, $p_1\in[\![0\,;\,k]\!]$, $p_2\in[\![0\,;\,k-p_1]\!]$ *and* $\sigma\in\mathcal{S}_{k-p_1-p_2}$, *we have the inequality:*

$$\left|\mathcal{C}^{\sigma}_{k,p_1,p_2}(\mu)\right|\leq\frac{\left|G^{(2k)}(0)\right|}{(2k)!}\,\frac{k!2^{k-p_1-p_2}}{p_1!p_2!(k-p_1-p_2)!}\left[\int_{\mathbb{R}^d}||y||^{2n}\,\mu(dy)\right]^{\frac{k-p_1+p_2}{2n}}\,.$$

This will be used for obtaining the convergence in the small-noise limit of the stationary measures. Indeed, the first step is the convergence of the parameters of the potential $V+F*u_\epsilon$, where u_ϵ is a stationary measure. This control implies that these parameters are bounded ; which will be the first step of the convergence.

From now, we consider only the stationary measures which admit a finite moment of order $8q^2$. Indeed, the assumption (ES) implies that the only relevants stationary measures have a finite moment of order $8q^2$. This restriction will not be specified anymore.

We present now the exponential form verified by all the invariant probabilities.

Lemma 1.3. *If there exists an invariant measure* u_ϵ, *then:*

$$u_\epsilon(x)=\frac{1}{Z_\epsilon}\exp\left[-\frac{2}{\epsilon}\Big(V(x)+F*u_\epsilon(x)\Big)\right], \tag{1.3}$$

where Z_ϵ *denotes the normalization factor:* $\int_{\mathbb{R}^d}u_\epsilon(x)dx=1$. *Conversely any measure whose density satisfies* (1.3) *is invariant for* (I) *and admits a* $8q^2$ *finite moment.*

Proof. **Step 1.** First we shall prove that any measure u_ϵ satisfying (1.3) is an invariant measure for (I). Let X_0 be a random variable with distribution u_ϵ. We consider the diffusion $(Y_t)_{t\geq0}$ solution of the classical stochastic differential equation

$$Y_t=X_0+\sqrt{\epsilon}B_t-\int_0^t\nabla W_\epsilon\left(Y_s\right)ds \tag{1.4}$$

with $W_\epsilon := V + F * u_\epsilon$. Since $(Y_t)_{t\geq 0}$ is a Kolmogorov diffusion process, it admits a unique invariant probability v_ϵ given by

$$v_\epsilon(x) := \frac{\exp\left[-\frac{2}{\epsilon}W_\epsilon(x)\right]}{\int_{\mathbb{R}^d}\exp\left[-\frac{2}{\epsilon}W_\epsilon(y)\right]dy} = u_\epsilon(x)\,.$$

Consequently the law of Y_t corresponds with u_ϵ for all $t \geq 0$. Thereby (1.4) becomes (I). Hence, u_ϵ is an invariant measure for (I). And, hypothesis (V-3) implies that the $(8q^2)$-moment of u_ε is finite.

Step 2. Let us prove now that any invariant measure u_ϵ satisfies to this exponential implicit structure. First, we note that the potential $W_t := V + F * u_t$ does not depend on t. Let a random variable X_0 with law u_ϵ. Then, for all $t > 0$, X_t has the law u_ϵ. It implies that X_t is the unique strong solution of (1.4). The law u_ϵ is invariant. Consequently, u_ϵ satisfies

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}W_\epsilon(x)\right]}{\int_{\mathbb{R}^d}\exp\left[-\frac{2}{\epsilon}W_\epsilon(y)\right]dy} = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon(x)\right)\right]}{\int_{\mathbb{R}^d}\exp\left[-\frac{2}{\epsilon}\left(V(y)+F*u_\epsilon(y)\right)\right]dy}\,.$$

This achieves the proof. ☐

Lemma 1.3 presents the essential structure of any invariant measure. The global exponential form will play a crucial role in next sections: to prove the existence of a stationary measure, it is necessary and sufficient to solve equation (1.3).

The keystone of the paper is the monotonicity of the free-energy Υ_ϵ along the orbits of (III).

Definition 1.4. *For all $t \in \mathbb{R}_+$, we introduce the functions:*

$$\xi(t) := \Upsilon_\epsilon\left(u_t\right) \quad \textit{and} \quad \eta_t(x) := \frac{\epsilon}{2}\nabla u_t + u_t\left(\nabla V + \nabla F * u_t\right)\,.$$

According to (III), we remark that if η_t is identically equal to 0 then u_t is a stationary measure for (I). Indeed, $\frac{\partial}{\partial t}u_t(x) = \operatorname{div}\eta_t(x)$.

We recall the following well-known entropy dissipation:

Proposition 1.5. *Let a probability measure u_0 which verifies (FE) and (ES). Then, for all $t, s \geq 0$, we have*

$$\xi(t+s) \leq \xi(t) \leq \xi(0) < +\infty\,.$$

Furthermore, we have:

$$\xi'(t) \leq -\int_{\mathbb{R}^d}\frac{1}{u_t}\left|\left|\eta_t\right|\right|^2\,.$$

See [CMV03] for a proof. Let us note that we can find the second point of the Lemma 1.3 by another method with this inequality. Indeed, if u_ϵ is a stationary measure, then the function $\xi'(t)$ is equal to 0 which implies directly

$\eta_t(x) = 0$ for all $x \in \mathbb{R}^d$.

Let us present two lemmas which will be important in the following. We do not write the proofs since the arguments are similar to those of Lemma 1.3 and Lemma 1.4 in [Tug10b].

Lemma 1.6. *For all $\epsilon > 0$, there exists $\Xi_\epsilon \in \mathbb{R}$ such that $\Upsilon_\epsilon(u) \geq \Xi_\epsilon$ for all the probability measure.*

Lemma 1.7. *There exists $L_0 \in \mathbb{R}$ such that $\Upsilon_\epsilon(u_t)$ converges towards L_0 as time elapses to infinity.*

2 Stationary measures

In [Tug11a], the existence of stationary measures around a wells a_0 of the potential has been proved under the conditions:

$$V(x) + F(x - a_0) > V(a_0) \quad \text{for all } \; x \neq a_0 \tag{2.1}$$

and

$$\sum_{p=0}^{2n-2} \frac{\left|F^{(p+2)}(a_0)\right|}{p!} \, |a_0|^p < F''(0) + V''(a_0)\,. \tag{2.2}$$

The inequality (2.1) is intuitive. Indeed, in the one-dimensional case, the global idea in [HT10a] consists in finding a vector $(m_1, \cdots, m_{2n-1})$ closed to $(a_0, \cdots, a_0^{2n-1})$ which verifies

$$m_j = \frac{\int_{\mathbb{R}} x^j \exp\left[-\frac{2}{\epsilon}\left(V(x) + F * u^{(m)}(x)\right)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(x) + F * u^{(m)}(x)\right)\right] dx}$$
$$\text{with} \quad \int_{\mathbb{R}} x^k u^{(m)}(x) = m_k \quad \text{for all } k \in [\![1\,;\,2n-1]\!]\,.$$

This needs a_0 to be the global minimum of $V(x) + F(x - a_0)$.

The inequality (2.2) was just used in the particular method developped in [HT10a, Tug11a] as a technical assumption when $\deg(G) \geq 4$. But, in the following, we will present a more general method which will only assume (2.1). For this, we will use the free-energy and more particularly one of its property that is to say the convergence of one of the subsequence of the family $\{u_t\,;\, t \in \mathbb{R}_+\}$ towards a stationary measure.

2.1 Subconvergence

Proposition 2.1. *There exists an element $u_\epsilon \in \mathcal{A}_\epsilon \bigcap \mathcal{S}_\epsilon$.*

Proof. **Plan:** First, we use the convergence of $\int_t^\infty \xi'(s)ds$ towards 0 when t tends to infinity and we deduce the existence of a sequence $(t_k)_k$ such that $\xi'(t_k)$ tends to 0 when k goes to infinity. Then, the tightness of the sequence

$\{u_{t_k}\,;\,k\in\mathbb{N}\}$ permits to extract a subsequence of $(t_k)_k$ - we will continue to write it $(t_k)_k$ - such that u_{t_k} converges weakly towards a limiting value of the family $\{u_t\,;\,t\in\mathbb{R}_+\}$. By using a test function and Weyl lemma, we prove that this adherence value is a stationary measure. Let us give now the details of the proof.

Step 1: Lemma 1.7 implies that the quantity $\int_t^\infty \xi'(s)ds$ collapses at infinity. According to Proposition 1.5, ξ is monotonous so we deduce the existence of an increasing sequence $(t_k)_{k\in\mathbb{N}}$ which converges towards infinity such that $\xi'(t_k)\longrightarrow 0$.

Step 2: The uniform boundedness of the first $8q^2$ moments with respect to the time allows us to use Prohorov's theorem: we can extract a subsequence (we continue to write it $(t_k)_k$ for simplifying the writting) such that u_{t_k} converges weakly towards a probability measure u_ϵ.

Step 3: We consider now a function $\varphi\in\mathcal{C}^\infty\left(\mathbb{R}^d,\mathbb{R}^d\right)\bigcap\mathcal{L}_2\left(u_\epsilon\right)$ with compact support and we estimate the following quantity:

$$\left|\int_{\mathbb{R}^d}\langle\varphi(x)\,;\,\eta_{t_k}(x)\rangle\,dx\right| = \left|\int_{\mathbb{R}^d}\left\langle\varphi(x)\sqrt{u_{t_k}(x)}\,;\,\frac{|\eta_{t_k}(x)|}{\sqrt{u_{t_k}(x)}}\right\rangle dx\right|$$
$$\leq\sqrt{\int_{\mathbb{R}^d}||\varphi(x)||^2\,u_{t_k}(x)dx}\times\sqrt{\int_{\mathbb{R}^d}\frac{1}{u_{t_k}(x)}\,||\eta_{t_k}(x)||^2\,dx}$$
$$\leq\sqrt{-\xi'(t_k)}\sqrt{\int_{\mathbb{R}^d}||\varphi(x)||^2\,u_{t_k}(x)}\longrightarrow 0$$

when k goes to infinity. Thanks to the compactness of the support of φ, we can apply an integration by parts and we obtain

$$\int_{\mathbb{R}^d}\left\langle\varphi\,;\,\frac{\epsilon}{2}\nabla u_{t_k}+u_{t_k}\left[\nabla V+\nabla F*u_{t_k}\right]\right\rangle$$
$$=\int_{\mathbb{R}^d}\langle\varphi\,;\,\nabla V+\nabla F*u_{t_k}\rangle\,u_{t_k}-\int_{\mathbb{R}^d}\frac{\epsilon}{2}\mathrm{div}\left(\varphi\right)u_{t_k}\,.$$

The weak convergence of u_{t_k} towards u_ϵ implies that the previous term tends towards $\int_{\mathbb{R}^d}\langle\varphi\,;\,\nabla V+\nabla F*u_\epsilon\rangle\,u_\epsilon-\int_{\mathbb{R}^d}\frac{\epsilon}{2}\mathrm{div}\varphi u_\epsilon$. It has been proved previously that $\int_{\mathbb{R}^d}\langle\varphi\,;\,\eta_{t_k}\rangle$ is collapsing when k goes to ∞. We get the following statement:

$$\int_{\mathbb{R}^d}\langle\varphi\,;\,\nabla V+\nabla F*u_\epsilon\rangle\,u_\epsilon-\int_{\mathbb{R}^d}\frac{\epsilon}{2}\mathrm{div}\varphi u_\epsilon=0\,. \tag{2.3}$$

This equality holds for all the function $\varphi\in\mathcal{C}^\infty\left(\mathbb{R}^d,\mathbb{R}^d\right)\bigcap\mathcal{L}_2\left(u_\epsilon\right)$ with compact support.

Step 4: This means that u_ϵ is a weak solution of the equation

$$\frac{\epsilon}{2}\nabla u+\left[\nabla V+\nabla F*u\right]u=0\,.$$

By applying Weyl lemma, we deduce that the function

$$x \mapsto \exp\left[\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon(x)\right)\right]u_\epsilon(x)$$

is smooth. Moreover, it is harmonic. Since it is bounded by below (because it is positive), Liouville theorem implies that it is a constant. This means that the measure u_ϵ satisfies the equality (1.3). Consequently, u_ϵ is an invariant probability of (I) according to Lemma 1.3. ☐

2.2 Existence

We are now able to provide the main result that is to say the existence of stationary measure around the wells of V which satisfy (2.1). First, we observe an immediate consequence of Proposition 2.1:

Corollary 2.2. *The set $\mathcal{S}_\epsilon$ is not empty that is to say that the diffusion* (I) *admits at least one stationary measure.*

It is also possible to obtain a localization result about the stationary measures. In other words, we improve Proposition 3.1 and Theorem 4.6 in [HT10a].

Theorem 2.3. *Let a_0 a point where V admits a local minimum such that*

$$V(x)+F(x-a_0)>V(a_0)\quad \text{for all}\quad x\neq a_0\,. \tag{2.4}$$

Then, for all $\rho>0$ small enough, there exists $\epsilon_0>0$ such that $\forall\epsilon\in]0;\epsilon[$, the diffusion (I) *admits a stationary measure u_ϵ satisfying*

$$\int_{\mathbb{R}^d}||x-a_0||^{2n}u_\epsilon(x)dx\leq\rho^{2n}\,.$$

Proof. **Plan.** The global idea is to prove that there exists a set $\mathcal{M}$ of measures absolutely continuous with respect to the Lebesgue measure and $\epsilon>0$ sufficiently small such that $\inf_{\mu\in\partial\mathcal{M}}\Upsilon_\epsilon(\mu)>\inf_{\mu\in\mathcal{M}}\Upsilon_\epsilon(\mu)$. Then, we exhibit an element u_0 in $\mathcal{M}$ with free-energy less than $\inf_{\mu\in\partial\mathcal{M}}\Upsilon_\epsilon(\mu)$. We take X_0 a random variable with law u_0. Theorem 2.1 tells us that there exists an adherence value of the family $\{u_t\}_{t\in\mathbb{R}_+}$ which is a stationary measure for the diffusion (I). Since, the free-energy is nonincreasing, it yields that $u_t\in\mathcal{M}$ for all $t>0$. Consequently, the set $\mathcal{M}$ contains at least one stationary measure. Moreover, we will consider a set $\mathcal{M}$ which is arbitrarily closed to the measure δ_{a_0}.

Step 1. We note $\mathcal{M}_\rho$ the set of the probability measures μ absolutely continuous with respect to the Lebesgue measure such that $\int_{\mathbb{R}^d}||x-a_0||^{2n}\mu(x)dx\leq\rho^{2n}$. In particular, for each element $\mu\in\mathcal{M}_\rho$, we have $\int_{\mathbb{R}^d}||x-a_0||^4\mu(x)dx\leq\rho^4$. We can write:

$$\begin{aligned}\Upsilon_\epsilon(\mu)\geq&\frac{\epsilon}{2}\int_{\mathbb{R}^d}\mu(x)\log[\mu(x)]\mathbb{1}_{\{\mu(x)\leq1\}}dx+\int_{\mathbb{R}^d}\left[V(x)+F(x-a_0)\right]\mu(x)dx\\&+\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left[F(x-y)-F(x-a_0)-F(y-a_0)\right]\mu(x)\mu(y)dxdy\,.\end{aligned}$$

By proceeding like in [Tug10b], we obtain the existence of a constant $C > 0$ such that

$$\frac{\epsilon}{2}\int_{\mathbb{R}^d}\mu(x)\log[\mu(x)]\mathbb{1}_{\{\mu(x)\leq 1\}}dx \geq -\frac{\epsilon}{4}\int_{\mathbb{R}^d}||x||^2\mu(x)dx + C\epsilon\,.$$

Step 2. We focus now in the second term. Since the wells a_0 satisfies (2.4), we have immediatly $V(x)+F(x-a_0)-V(a_0)\geq 0$ for all $x\in\mathbb{R}^d$.
Also, by putting $M := \mathrm{Hess}\,V(a_0)+\mathrm{Hess}\,F(0)$, for all $\kappa > 0$, there exists $\tau > 0$ sufficiently small such that

$$V(x)+F(x-a_0)-V(a_0)\geq\frac{1-\kappa}{2}\langle x-a_0\,;\,M(x-a_0)\rangle$$

for all $x\in\mathbb{R}^d$ which verifies $||x-a_0|| < \tau$. Hence, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}[V(x)+F(x-a_0)]\,\mu(x)dx\\
\geq& V(a_0)+\frac{1-\kappa}{2}\int_{||x-a_0||<\tau}\langle x-a_0\,;\,M(x-a_0)\rangle\,\mu(x)dx\\
\geq& V(a_0)+\frac{1-\kappa}{2}\int_{\mathbb{R}^d}\langle x-a_0\,;\,M(x-a_0)\rangle\,\mu(x)dx\\
&-\frac{1-\kappa}{2\tau^2}\omega\int_{\mathbb{R}^d}||x-a_0||^4\mu(x)dx
\end{aligned}$$

where $\omega := \sup_{z\in\mathbb{R}^d}\frac{\langle z\,;\,Mz\rangle}{||z||^2}$. By taking $\rho := \tau^2$, we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^d}[V(x)+F(x-a_0)]\,\mu(x)dx\\
\geq& V(a_0)+\frac{1-\kappa}{2}\int_{\mathbb{R}^d}\langle x-a_0\,;\,M(x-a_0)\rangle\,\mu(x)dx-\frac{1-\kappa}{2}\omega\rho^3\\
\geq& V(a_0)+\frac{1-\kappa}{2}\int_{\mathbb{R}^d}\langle x-a_0\,;\,M(x-a_0)\rangle\,\mu(x)dx+o\left(\rho^2\right)\,.
\end{aligned}$$

Step 3. Now, we look at the third term:

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d}[F(x-y)-F(x-a_0)-F(y-a_0)]\,\mu(x)\mu(y)dxdy\\
=&\iint_{\mathbb{R}^d\times\mathbb{R}^d}[F_0(x-y)-F_0(x-a_0)-F_0(y-a_0)]\,\mu(x)\mu(y)dxdy\\
&-G''(0)\left|\left|\int_{\mathbb{R}^d}(x-a_0)\mu(x)dx\right|\right|^2\\
\geq&-2\int_{\mathbb{R}^d}F_0(x-a_0)\mu(x)dx-G''(0)\int_{\mathbb{R}^d}||x-a_0||^2\mu(x)dx\,.
\end{aligned}$$

with $F_0(x) := F(x)-\frac{G''(0)}{2}||x||^2$. Indeed, G is convex on $\mathbb{R}_+$ so F is convex on $\mathbb{R}^d$ which implies $F_0\geq 0$. And, by definition of the set $\mathcal{M}_\rho$, for all $\mu\in\mathcal{M}_\rho$, it yields

$$\begin{aligned}\int_{\mathbb{R}^d} F_0(x-a_0)\mu(x)dx &\leq \sum_{k=2}^{n} \frac{G^{(2k)}(0)}{(2k)!} \int_{\mathbb{R}^d} ||x-a_0||^{2k}\mu(x)dx \\ &\leq C\rho^4 = o(\rho^2)\,.\end{aligned}$$

Step 4. As we have $\text{Hess}\, F(0) = G''(0)I_n$, we deduce:

$$\begin{aligned}\Upsilon_\epsilon(\mu) \geq & V(a_0) + \frac{1-\kappa}{2}\int_{\mathbb{R}^d} \langle x-a_0\,;\, M(x-a_0)\rangle\,\mu(x)dx \\ & - \frac{G''(0)}{2}\int_{\mathbb{R}^d} ||x-a_0||^2\,\mu(x)dx + C\epsilon - \frac{\epsilon}{4}\rho^2 + o\left(\rho^2\right) \\ \geq & V(a_0) + \frac{1-\kappa}{2}\int_{\mathbb{R}^d} \langle x-a_0\,;\, \text{Hess}\, V(a_0)(x-a_0)\rangle\,\mu(x)dx \\ & - \frac{\kappa}{2}G''(0)\rho^2 + C\epsilon - \frac{\epsilon}{4}\rho^2 + o\left(\rho^2\right)\,.\end{aligned}$$

Then, by taking ρ small enough then ϵ sufficiently small, we obtain

$$\Upsilon_\epsilon(\mu) \geq V(a_0) + \frac{\lambda}{2}\int_{\mathbb{R}^d} ||x-a_0||^2\mu(x)dx + o(\rho^2) \tag{2.5}$$

where λ is the smallest eigenvalue of the matrix $\text{Hess}\, V(a_0)$. Let us remark that this statement is true since κ goes to 0 with ρ.

Step 5. We prove now that $\inf_{\mu\in\partial\mathcal{M}_\rho}\Upsilon_0(\mu) > V(a_0)$. *Let us note that $\partial\mathcal{M}_\rho$ does not denote the boundary but the set of the measures absolutely continuous with respect to the Lebesgue measure and with finite entropy which are in the boundary.* We proceed a *reductio ad absurdum*. Then, we can find a sequence of measures $(\mu_k)_{k\in\mathbb{N}}$ in $\partial\mathcal{M}_\rho$ such that $\Upsilon_0(\mu_k) < V(a_0) + \frac{1}{k}$. This family is tight because its second moment is less than $2\,||a_0||^2$ for ρ sufficiently small. Moreover, $\Upsilon_0(\mu)$ depends only on the moments of the measure μ. Consequently, we can extract a subsequence which converges towards a measure $\nu\in\partial\mathcal{M}_\rho$ which would satisfy $\Upsilon_0(\nu) = V(a_0)$. However, if it is possible to prove that (2.5) holds for ν. Consequently,

$$\Upsilon_0(\nu) \geq V(a_0) + \frac{\lambda}{2}\int_{\mathbb{R}^d} ||x-a_0||^2\nu(x)dx + o(\rho^2)$$

Since ν is absolutely continuous with respect to the Lebesgue measure, the term $\int_{\mathbb{R}^d}||x-a_0||^2\nu(x)dx$ is positive. This implies that the hypothesis was wrong. Thereby, if ρ is sufficiently small, there exists $\gamma(\rho) > 0$ such that $\inf_{\mu\in\partial\mathcal{M}_\rho}\Upsilon_0(\mu) \geq V(a_0) + \gamma(\rho)$. Then, by taking ϵ sufficiently small and since the second moment is bounded by $2\,||a_0||^2$, it yields

$$\inf_{\mu\in\partial\mathcal{M}_\rho}\Upsilon_\epsilon(\mu) \geq V(a_0) + \frac{\gamma(\rho)}{2}\,.$$

Step 6. Let us consider now the measure with the density

$$v_\epsilon(x) := Z_\epsilon^{-1}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F(x-a_0)\right)\right].$$

Proposition A.1 implies the convergence of $\Upsilon_\epsilon(v_\epsilon)$ towards $V(a_0)$ when ϵ goes to 0. We assume now that ϵ is small enough such that $\Upsilon_\epsilon(v_\epsilon) < \inf_{\mu\in\partial\mathcal{M}_\rho}\Upsilon_\epsilon(\mu)$.

We consider the process (I) starting by v_ϵ. According to Theorem 2.1, there exists a sequence $(t_k)_k$ which tends to ∞ such that u_{t_k} converges towards a stationary measure u_ϵ. Since the free-energy is nonincreasing, we have furthermore: $\Upsilon_\epsilon(u_t) \leq \Upsilon_\epsilon(v_\epsilon) < \inf_{\mu\in\partial\mathcal{M}\rho}\Upsilon_\epsilon(\mu)$ for all $t\in\mathbb{R}_+$. Consequently, the measure u_ϵ is in $\mathcal{M}_\rho$. This achieves the proof. □

Let us note that this method does not hold if a_0 is not a wells of V. If it is not a wells, the inequality (2.5) would hold with a negative constant λ. Then, the measure v_ϵ has not a free-energy less than $\inf_{\mu\in\partial\mathcal{M}_\rho}\Upsilon_\epsilon(\mu)$. Reciprocally, if a_0 is a wells of V but if the function $x\mapsto V(x)+F(x-a_0)$ is not minimal in a_0, the quantity $\int_{||x-a_0||\geq\rho}(V(x)+F(x-a_0)-V(a_0))\mu(x)dx$ would not be positive. We will see subsequently that the inequality $V(x)+F(x-a_0)\geq V(a_0)$ for all $x\in\mathbb{R}^d$ is necessary.

3 Behavior in the small-noise limit of u_ϵ

In this section we shall analyze the asymptotic behavior of the invariant probabilities for (I) as $\varepsilon\to 0$. Let us consider a stationary measure u_ϵ. According to Lemma 1.3, the following exponential expression holds:

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}W_\epsilon(x)\right]}{\int_{\mathbb{R}^d}\exp\left[-\frac{2}{\epsilon}W_\epsilon(y)\right]dy} \quad\text{with}\quad W_\epsilon := V + F*u_\epsilon\,. \tag{3.1}$$

By applying Lemma 1.1 to the measure u_ϵ, we have:

$$W_\epsilon(x) = V(x) + \sum_{k=1}^{n}\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1}\sum_{\sigma\in\mathcal{S}_{k-p_1-p_2}}\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)\,||x||^{2p_1}\,\nu^\sigma(x) \tag{3.2}$$

$$\text{with}\quad \mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon) := \frac{G^{(2k)}(0)}{(2k)!}\frac{k!(-2)^{k-p_1-p_2}}{p_1!p_2!(k-p_1-p_2)!}\int_{\mathbb{R}^d}||y||^{2p_2}\,\nu^\sigma(y)u_\epsilon(y)dy$$

W_ϵ is called the *pseudo-potential*. In order to study the behavior of u_ε for small ϵ, we need to estimate precisely the pseudo-potential W_ϵ. Indeed, the convergence from u_ϵ to a measure u_0 is strongly related to an eventual convergence from the pseudo-potential.
The study will follow this plan:

- Step 1. First we will prove that, under the condition (H) that is to say the boundedness of the family $\{\int_{\mathbb{R}^d}||y||^{2n}\,u_\epsilon(y)dy,\ \varepsilon>0\}$ with $2n=\deg(G)$, we can find a sequence $(\epsilon_k)_{k\geq 0}$ satisfying $\lim_{k\to\infty}\varepsilon_k = 0$ such that W_{ϵ_k} converges uniformly on each compact of $\mathbb{R}^d$ towards a limit function W_0 associated to a measure u_0.

- Step 2. We shall describe the measure u_0: it is a discrete measure under natural assumptions. Moreover, its support and the corresponding weights satisfy particular conditions.
- Step 3. We analyze then the possible limits for sequences of invariant probabilities.
- Step 4. We prove that the assumption (H) holds if (LIN) or (SYN) are satisfied.

3.1 Weak convergence for a subsequence of invariant measures

Let $(u_\epsilon)_{\epsilon>0}$ be a family of stationary measures. We recall the main assumption in the subsequent developments:

(H) The family $\left\{\int_{\mathbb{R}^d}||y||^{2n}u_\epsilon(y)dy,\ \epsilon>0\right\}$ is bounded.

We admit the hypothesis (H). We will provide further some cases such that (H) is satisfied.
Therefore applying Bolzano-Weierstrass theorem and Remark 1.2, we obtain:

Lemma 3.1. *There exists a sequence* $(\epsilon_k)_{k\geq 0}$ *satisfying* $\lim_{k\to\infty}\varepsilon_k = 0$ *such that, for any* $k\in [\![1\,;\,n]\!]$, $p_1\in[\![0\,;\,k]\!]$, $p_2\in[\![0\,;\,k-p_1]\!]$ *and* $\sigma\in\mathcal{S}_{k-p_1-p_2}$, *the sequence* $\left\{\mathcal{C}^\sigma_{k,p_1,p_2}(u_{\epsilon_k})\,;\,k\in\mathbb{N}\right\}$ *converges towards a limit value denoted by* $\mathcal{C}^\sigma_{k,p_1,p_2}(0)$ *with* $\left|\mathcal{C}^\sigma_{k,p_1,p_2}(0)\right|<\infty$.

As presented in (3.2), the quantities $\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)$ characterize the pseudo-potential W_ϵ. We have then the convergence of the pseudo-potential. We introduce the following potential:

$$W_0(x) = V(x) + \sum_{k=1}^{n}\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1}\sum_{\sigma\in\mathcal{S}_{k-p_1-p_2}}\mathcal{C}^\sigma_{k,p_1,p_2}(0)\,||x||^{2p_1}\,\nu^\sigma(x)\,. \tag{3.3}$$

Proposition 3.2. *For all* $j_1,\cdots,j_d\in\mathbb{N}$, *the sequence* $(\frac{\partial^{j_1}}{\partial x_1^{j_1}}\cdots\frac{\partial^{j_d}}{\partial x_d^{j_d}}W_{\epsilon_k})_{k\geq 1}$ *converges towards* $\frac{\partial^{j_1}}{\partial x_1^{j_1}}\cdots\frac{\partial^{j_d}}{\partial x_d^{j_d}}W_0$, *uniformly on each compact subset of* $\mathbb{R}^d$ *- where the* limit pseudo-potential W_0 *is defined by* (3.3) *- and* $(u_{\epsilon_k})_{k\geq 1}$ *converges weakly towards a probability measure* u_0.

Proof. By definition, W_0 is a polynomial function in each coordinate $x_1,\cdots,x_d$. Consequently, ths pointwise convergence of each coefficient $\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)$ is sufficient for obtaining the uniform convergence on each compact of the sequence

$\left(\frac{\partial^{j_1}}{\partial x_1^{j_1}}\cdots\frac{\partial^{j_d}}{\partial x_d^{j_d}}W_{\epsilon_k}\right)_{k\geq 1}$ to $\frac{\partial^{j_1}}{\partial x_1^{j_1}}\cdots\frac{\partial^{j_d}}{\partial x_d^{j_d}}W_0$.
The tightness of the sequence $\{u_{\epsilon_k}\,;\,k\in\mathbb{N}\}$ - which is a consequence of (H) - and the application of Prohorov theorem permit to achieve the proof. $\square$

In [HT10b], we proved that the potential W_0 admits a finite number of critical points. The dimension one was essential. But, if $d\geq 2$, a polynomial function can have an infinite number of zeros without being identically equal to 0. Consequently, we give a weaker result concerning the number of global minima.

Lemma 3.3. *Under (LIN) or (SYN), the function W_0 admits a finite number of wells.*

Proof. Under (LIN), we have: $\nabla W_0(x)=\nabla V(x)+\alpha x-\alpha m$ with $m\in\mathbb{R}^d$. The hypothesis permits to conclude immediatly.
Under (SYN), W_0 is convex which achieves the proof. $\square$

If neither (LIN) nor (SYN) are verified, we still have results concerning the small-noise limit of u_ϵ.

Definition 3.4. *From now, we call Ω the set of all the points where W_0 reaches its global minimum. And, for all $\delta>0$, we introduce*

$$\Omega^\delta:=\left\{x\in\mathbb{R}^d\ \mid\ x=y+w\,,w\in\Omega\,,||y||\leq\delta\right\}\,.$$

Since W_0 is polynomial in each coordinate $x_1,\cdots,x_d$, $\mathcal{W}$ has empty interior. Since F is convex and $\operatorname{Hess}V(x)>0$ for $||x||\geq R$, we also deduce that Ω is bounded. So, Ω is a compact of $\mathbb{R}^d$ with empty interior.

Definition 3.5. *If $\#\Omega=r<\infty$, we define $A_1,\cdots,A_r$ by*

$$W_0(A_1)=\cdots=W_0(A_r)=\inf_{x\in\mathbb{R}^d}W_0(x)=:w_0. \tag{3.4}$$

The set Ω plays a central role in the asymptotic analysis of the measures $(u_\varepsilon)_\varepsilon$. In particular we can prove that u_0 defined in Proposition 3.2 is concentrated around these points. The following holds even if $\#\Omega=\infty$.

Proposition 3.6. *Let W_0 and $(\epsilon_k)_{k\in\mathbb{N}}$ be defined by Proposition 3.2. Then, for all $\delta>0$ sufficiently small, we have: $\lim\limits_{k\to\infty}u_{\epsilon_k}\left((\Omega^\delta)^c\right)dx=0$.*

Proof. The proof is similar to the one of Proposition 3.5 in [HT10b] so we skip the details.
Step 1. By using the hypotheses, there exists $\eta>0$ such that $W_{\epsilon_k}(x)\geq w_0+\eta$ for all $x\in\left(\Omega^\delta\right)^c$ if $k\geq k_0$.
Step 2. We take $\gamma<\delta$ such that $\sup_{z\in\overline{\Omega^\gamma}}W_0\leq w_0+\frac{\eta}{4}$. By using the compactness of $\overline{\Omega^\gamma}$ and the uniform convergence of W_{ϵ_k} towards W_0 on each compact, we obtain $W_{\epsilon_k}(x)\leq w_0+\frac{\eta}{2}$ for k large enough and for all $x\in\overline{\Omega^\gamma}$.

Step 3. Consequently, for all $x \in \left(\Omega^\delta\right)^c$, it yields $u_{\epsilon_k}(x) \leq \exp\left[-\frac{\eta}{\epsilon_k}\right]\frac{1}{\mathrm{Vol}(\Omega^\gamma)}$ which tends towards 0 as k goes to infinity.
Step 4. The tightness of the sequence $\{u_{\epsilon_k}\,;\, k \in \mathbb{N}\}$ permits to conclude. ☐

The sequence of measures $\left(u_{\epsilon_k}\right)_{k\in\mathbb{N}^*}$ converges to a measure u_0. Furthermore the open set $\left(\Omega^\delta\right)^c$ is less and less weighted by u_{ϵ_k} as k becomes large. Intuitively u_0 should be a measure whose support corresponds to the set Ω. From now, we assume that W_0 reaches its global minimum in a finite number of points which is true under (LIN) or under (SYN).

Theorem 3.7. *Let $(\varepsilon_k)_{k\geq1}$, W_0, u_0 and $A_1, \ldots, A_r$ be defined in the statement of Proposition 3.2 and in Definition 3.5. Then the sequence of measures $\left(u_{\epsilon_k}\right)_{k\geq1}$ converges weakly, as k becomes large, to the discrete probability measure $u_0 = \sum_{i=1}^r p_i\delta_{A_i}$ where*

$$p_i = \lim_{k\to+\infty}\int_{||x-A_i||\leq\delta} u_{\epsilon_k}(x)dx, \quad 1\leq i\leq r, \quad \delta>0 \text{ small enough.}$$

Moreover p_i is independent of the parameter δ.

Proof. **Step 1.** First we shall prove that the coefficients p_i are well defined. Let us fix a positive constant δ. We define $p_i(\delta)$ as the limit of $\int_{||x-A_i||\leq\delta} u_{\epsilon_k}(x)dx$ when $k\to\infty$. Of course this limit exists since, by Proposition 3.2, $(u_{\varepsilon_k})_{k\geq1}$ converges weakly. Furthermore this limit is independent of δ. Indeed let us choose $\delta' < \delta$. By definition, we obtain

$$p_i(\delta) - p_i(\delta') = \lim_{k\to\infty}\int_{\delta'<||x-A_i||\leq\delta} u_{\epsilon_k}(x)dx\,.$$

An obvious application of Proposition 3.6 implies $p_i(\delta') = p_i(\delta) =: p_i$.
Step 2. Let us prove now that u_0 is a discrete probability measure. Let f be a continuous and bounded function on $\mathbb{R}^d$. We note $U_i(\delta) := \{x \ | \ ||x-A_i|| \leq \delta\}$. The weak convergence is based on the following difference:

$$\int_{\mathbb{R}^d} f(x)u_{\epsilon_k}(x)dx - \sum_{i=1}^r p_i f(A_i) = R + \sum_{i=1}^r \Delta_i(f),$$

with $\Delta_i(f) = \int_{U_i(\delta)} f(x)u_{\epsilon_k}(x)dx - p_i f(A_i)$ and $R = \int_{(\Omega^\delta)^c} f(x)u_{\epsilon_k}(x)dx$. The boundedness of the function f and Proposition 3.6 imply that R tends to 0 as $k\to\infty$. Let us now estimate each term $\Delta_i(f)$:

$$\begin{aligned}|\Delta_i(f)| &\leq \int_{U_i(\delta)}\Big|f(x)-f(A_i)\Big|u_{\epsilon_k}(x)dx + |f(A_i)|\Big|u_{\epsilon_k}\Big(U_i(\delta)\Big) - p_i\Big| \\ &\leq \sup_{z\in U_i(\delta)}|f(z)-f(A_i)|\,u_{\epsilon_k}\Big(U_i(\delta)\Big) + |f(A_i)|\,\Big|u_{\epsilon_k}\Big(U_i(\delta)\Big) - p_i\Big|\,.\end{aligned}$$

Due to the continuity of f, $\sup_{x\in U_i(\delta)}|f(x)-f(A_i)|$ is small as soon as δ is small enough. Moreover for some fixed δ, the definition of p_i leads to the convergence of $u_{\epsilon_k}(U_i(\delta))-p_i$ towards 0 as $k\to\infty$. Combining these two arguments allows us to obtain the weak convergence of u_{ε_k} towards the discrete measure $\sum_{i=1}^r p_i\delta_{A_i}$ which can finally be identified with u_0. ☐

3.2 Description of the limit measures

We have just pointed out previously that all the limit measures are discrete probability measures in the two following cases:

- $\mathrm{Hess}\,V(z)+\alpha\geq 0$ for all $z\in\mathbb{R}^d$.
- $\deg(G)=2$ and the equation $\nabla V(x)+\alpha x=\alpha m$ has a finite number of solutions for all $m\in\mathbb{R}^d$.

Each limit measure shall be denoted in a generic way u_0 and is associated with a limit pseudo-potential W_0 defined by (3.3). Therefore we have the following expression $u_0=\sum_{i=1}^r p_i\delta_{A_i}$ where $\{A_1\,;\,\cdots\,;\,A_r\}=\Omega$ and $\sum_{i=1}^r p_i=1$, $p_i>0$. We will now refine this result by exhibiting properties of the points A_i and the weights p_i. Proposition 3.8 allows us in a suitable situation to describe precisely the set of limit measures.

Proposition 3.8. **1.** *For all* $1\leq i\leq r$ *and* $1\leq j\leq r$*, we have :*

$$\nabla V(A_i)+\sum_{l=1}^r p_l\nabla F\left(A_i-A_l\right)=0\,,\tag{3.5}$$

$$V(A_i)-V(A_j)+\sum_{l=1}^r p_l\left(F(A_i-A_l)-F(A_j-A_l)\right)=0\,,\tag{3.6}$$

$$V(z)-V(A_i)+\sum_{l=1}^r p_l\left(F(z-A_l)-F(A_j-A_l)\right)\geq 0\quad\forall\, z\in\mathbb{R}^d\tag{3.7}$$

$$\textit{and}\quad \mathrm{Hess}\,V\left(A_i\right)+\sum_{l=1}^r p_l\mathrm{Hess}\,F\left(A_i-A_l\right)\geq 0\tag{3.8}$$

2. *Under (SYN),* $\Omega=\{A_0\}$ *with* $A_0\in\mathbb{R}^d$.

Proof. **1.** We can write W_0 as follows: $W_0=V+F*u_0$. The definition of Ω implies (3.5)–(3.8).
2. Under (SYN), W_0 is convex so it is immediate. ☐

Let us remark that immediatly, if a wells $a_0\in\mathbb{R}^d$ does not satisfy $V(x)+F(x-a_0)\geq V(a_0)$ for all $x\in\mathbb{R}^d$, δ_{a_0} can not be a limit measure for stationary probabilities of the diffusion (I). However, let us note that the inequality (2.1) is more restrictive than $V(x)+F(x-a_0)\geq V(a_0)$ for all $x\in\mathbb{R}^d$.
Now, let us focus on the measures pointed out in Theorem 2.3.

Proposition 3.9. *Let* a_0 *a wells of* V *which satisfies the condition* (2.1)*. Then there exists a family of invariant measures* $(u_\varepsilon)_{\varepsilon>0}$ *which converges weakly as* $\varepsilon\to 0$ *towards the Dirac measure* δ_{a_0}.

Proof. Let us choose some sequence $(\rho_k)_{k\in\mathbb{N}^*}$ satisfying $\lim\limits_{k\to\infty}\rho_k = 0$. Using Theorem 2.3, we know that (2.1) implies the existence of a sequence of invariant measures $(u_{\varepsilon_k})_{k\in\mathbb{N}^*}$ which verifies the following asymptotic estimate

$$\int_{\mathbb{R}^d} ||x-a_0||^{2n}\, u_{\epsilon_k}(x)dx \leq \left(\frac{1}{k}\right)^{2n}. \tag{3.9}$$

Using the binomial coefficients and equation (3.9), the convergence from u_{ε_k} towards δ_{a_0} in L^{2n} is immediate. Consequently, this sequence converges weakly towards the Dirac measure δ_{a_0}. □

3.3 Assumption (H)

Let us recall that we assumed the condition (H). We will now prove that it holds under (LIN) or under (SYN).

Proposition 3.10. *Let us assume that one of the two following hypotheses is satisfied:*

- $\mathrm{Hess}\, V(z) + \alpha \geq 0$ *for all* $z \in \mathbb{R}^d$.
- $\deg(G) = 2$ *and the equation* $\nabla V(x) + \alpha x = \alpha m$ *has a finite number of solutions for all* $m \in \mathbb{R}^d$.

If $\{u_\epsilon\,;\,\epsilon > 0\}$ *is a family of stationary measures for the self-stabilizing diffusion* (I) *then it satisfies the condition (H).*

Proof. **Step 1.** By taking the previous notations, $u_\epsilon(x) = Z^{-1}\exp\left[-\frac{2}{\epsilon}W_\epsilon(x)\right]$ where

$$W_\epsilon(x) = V(x) + \sum_{k=1}^{n}\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1}\sum_{\sigma\in\mathcal{S}_{k-p_1-p_2}} \mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)\,||x||^{2p_1}\,\nu^\sigma(x)$$

$$\text{with}\quad \mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon) := \frac{G^{(2k)}(0)}{(2k)!}\frac{k!(-2)^{k-p_1-p_2}}{p_1!p_2!(k-p_1-p_2)!}\int_{\mathbb{R}^d}||y||^{2p_2}\,\nu^\sigma(y)u_\epsilon(y)dy$$

$$\text{and}\quad \nu^\sigma_l(x) := \prod_{i=1}^{l} x_{\sigma(i)} \quad \forall\,\sigma\in\mathcal{S}_l := [\![1\,;\,d]\!]^{[\![1\,;\,l]\!]}.$$

Let us introduce

$$\omega(\epsilon) := \sup\left\{\left|\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)\right|^{\frac{1}{2m+p_2-k-p_1}}\right\}$$

where the supremum is taken on the set such that $1 \leq k \leq n$, $0 \leq p_1 \leq k$, $0 \leq p_2 \leq k - p_1$ and $\sigma \in \mathcal{S}_{k-p_1-p_2}$.

Step 2. We note that $\mathcal{C}_{2n,2n,0}(u_\epsilon) = \frac{G^{(2n)}(0)}{(2n)!} > 0$. Then, $(\omega(\epsilon))_\epsilon$ is uniformly lower-bounded. Consequently, we can divide by $\omega(\epsilon)$.

Step 3. The change of variable $x := \omega(\epsilon)y$ provides

$$\frac{m_{2l}(\epsilon)}{\omega(\epsilon)^{2l}} = \frac{\int_{\mathbb{R}} ||y||^{2l} \exp\left[-\frac{2}{\widehat{\epsilon}}\widehat{W}_\epsilon(y)\right] dy}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\widehat{\epsilon}}\widehat{W}_\epsilon(y)\right] dy} \quad \forall\, l \in \mathbb{N} \quad \text{with}$$

$$\widehat{W}_\epsilon(x) := \frac{V(\omega(\epsilon)x)}{\omega(\epsilon)^{2m}} + \sum_{k=1}^{n}\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1}\sum_{\sigma\in\mathcal{S}_{k-p_1-p_2}} \frac{\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)}{\omega(\epsilon)^{2p+p_2-k-p_1}} ||x||^{2p_1} \nu^\sigma(x)$$

and $\widehat{\epsilon} := \frac{\epsilon}{\omega(\epsilon)^{2p}}$.

Step 4. The sequences $\left(\frac{\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)}{\omega(\epsilon)^{2m+p_2-k-p_1}}\right)_\epsilon$ are bounded so we can extract a subsequence (we continue to write ϵ for simplicity) such that $\frac{\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)}{\omega(\epsilon)^{2m+p_2-k-p_1}}$ converges towards $\widehat{\mathcal{C}}^\sigma_{k,p_1,p_2}$ when $\epsilon \to 0$. Also, we can extract a subsequence of ϵ such that $\frac{V(\omega(\epsilon)x)}{\omega(\epsilon)^{2m}}$ converges towards a function $\widehat{V}(x)$ uniformly on each compact. We put

$$\widehat{W}(x) := \widehat{V}(x) + \sum_{k=1}^{n}\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1}\sum_{\sigma\in\mathcal{S}_{k-p_1-p_2}} \widehat{\mathcal{C}}^\sigma_{k,p_1,p_2} ||x||^{2p_1} \nu^\sigma(x)$$

Step 5. We prove now that $\widehat{W}$ has a finite number of wells.
If $\alpha + \vartheta \geq 0$, W_ϵ is convex which implies that $\widehat{W}_\epsilon$ is also convex for all $\epsilon > 0$. Consequently, $\widehat{W}$ is convex then has a unique wells.
If (LIN) holds, we note $m_\epsilon := \int_{\mathbb{R}^d} x u_\epsilon(x)dx$. We have $\omega(\epsilon) = (\alpha\,||m_\epsilon||)^{\frac{1}{2m}}$ if $||m_\epsilon||$ goes to infinity as k tends to ∞. Then, $\widehat{W}(x) = C_{2m}\,||x||^{2m}$ admits a unique wells. If $(m_\epsilon)_{\epsilon>0}$ is bounded, there exists $C \in \mathbb{R}_+$ and $m_0 \in \mathbb{R}^d$ such that $\widehat{W}(x) = \frac{V(Cx)}{C^{2m}} + \frac{\alpha}{2}\frac{C^2||x||^2}{C^{2m}} - \alpha\left\langle \frac{Cx}{C^{2m}}\,;\, m_0\right\rangle$. We deduce immediatly that $\widehat{W}$ has a finite number of wells. We call $A_1, \cdots, A_r$ the $r \geq 1$ location(s) of the global minimum of $\widehat{W}$.
Step 6. By applying the result of Lemma A.2, we can extract a subsequence (and we continue to denote it by ϵ) such that $\frac{\int_{\mathbb{R}^d}||y||^{2l}\exp\left[-\frac{2}{\widehat{\epsilon}}\widehat{W}_\epsilon(y)\right]dy}{\int_{\mathbb{R}^d}\exp\left[-\frac{2}{\widehat{\epsilon}}\widehat{W}_\epsilon(y)\right]dy}$ converges towards $\sum_{j=1}^r p_j\,||A_j||^{2l}$ where $p_1 + \cdots + p_r = 1$ and $p_j \geq 0$; for all $l \geq 0$.
Step 7. It implies $m_{2l}(\epsilon) = O\left\{\omega(\epsilon)^{2l}\right\}$ for all $l \in \mathbb{N}^*$ then $\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon) = O\left\{\omega(\epsilon)^{k-p_1+p_2}\right\}$. Thereby:

$$\omega(\epsilon) = \sup\left\{\left|\mathcal{C}^\sigma_{k,p_1,p_2}(u_\epsilon)\right|^{\frac{1}{2m+p_2-k-p_1}}\right\} = O\left\{\omega(\epsilon)^{\frac{k-p_1+p_2}{2m-k-p_1+p_2}}\right\}.$$

Since $n < m$, we have $\frac{k-p_1+p_2}{2m-k-p_1+p_2} < 1$. It yields that $(\omega(\epsilon))_{\epsilon>0}$ is bounded then $(m_{2n}(\epsilon))_{\epsilon>0}$ is also bounded. □

A Classical asymptotic results

We shall present here some useful asymptotic results which are close to the classical Laplace method. A direct computation provides:

Proposition A.1. *By considering the probability measure with the following density:*

$$v_\epsilon(x) := Z^{-1}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F(x-a_0)\right)\right]$$

where a_0 *is a wells of* V *such that* $V(x)+F(x-a_0)>V(a_0)$ *for all* $x\neq a_0$*, we have:*

$$\lim_{\epsilon\to 0}\Upsilon_\epsilon\left(v_\epsilon\right)=V(a_0)\,.$$

We provide here a useful asymptotic result linked to the Laplace method. The proof is similar to the one of Lemma A.4 in [Tug10b]. It is sufficient to write it in the general dimension case. Consequently, the details are left to the attention of the reader.

Lemma A.2. *Let* U_k *and* $U\in\mathcal{C}^\infty\left(\mathbb{R}^d,\mathbb{R}\right)$ *such that for all* $i\in\mathbb{N}$*,* $U_k^{(i)}$ *converges uniformly on all compact subset when* k *tends to* $+\infty$*. Let a sequence* $\left(\epsilon_k\right)_k$ *which tends to* 0 *as* k *tends to* $+\infty$*. We assume that* U *has* r *global minimum locations* $A_1,\cdots,A_r$ *and that there exist* $R>0$ *and* k_c *such that* $U_k(x)>||x||^2$ *for all* $||x||>R$ *and* $k>k_c$*. Then, for* k *large enough, we get:*
1. U_k *has exactly one global minimum location* $A_j^{(k)}$ *on each open* $\mathcal{B}_j$*, where* $\mathcal{B}_j$ *represents the Voronoï cells corresponding to the central points* A_j *with* $1\leq j\leq r$*.*
2. $A_j^{(k)}$ *tends to* A_j *when* k *tends to* $+\infty$*.*

Furthermore, for all $N\in\mathbb{N}$*, there exists* $p_1,\cdots,p_r$ *which verify* $p_1+\cdots+p_r=1$ *and* $p_i\geq 0$ *for all* $1\leq i\leq r$ *such that we can extract a subsequence* $\psi(k)$ *which satisfies*

$$\lim_{k\to+\infty}\frac{\int_{\mathbb{R}^d}||x||^l\exp\left[-\frac{2}{\epsilon_{\psi(k)}}U_{\psi(k)}\right]dx}{\int_{\mathbb{R}^d}\exp\left[-\frac{2}{\epsilon_{\psi(k)}}U_{\psi(k)}\right]dx}=\sum_{j=1}^r p_j\,||A_j||^l$$

for all $1\leq l\leq N$*.*

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