Université Pierre et Marie Curie

École Doctorale de Paris Centre

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

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Calcul d'Itô étendu

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Soutenue le 30 Juin 2011 devant le jury composé de :

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Remerciements / Acknowledgements / Agradecimientos

Ante todo, quiero agradecer a mis padres Noemy y Manuel por haberme inculcado los valores necesarios para poder concluir esta nueva etapa y por haberme dado la posibilidad de poder estudiar sabiendo que su apoyo siempre fue incondicional. No me cabe la menor duda de que este trabajo es fruto de una gran labor que ellos realizan desde hace poco mas 28 años.

Je voudrais remercier très spécialement Nathalie Eisenbaum, ma directrice de thèse, pour tout ce qu'elle m'a appris depuis quatre ans, pour sa disponibilité tout au long de ces années, pour le temps passé aux nombreuses relectures de ce manuscrit et surtout pour la qualité de ses conseils.

I am very grateful to the reviewers, Andreas Kyprianou and Zhen-Qing Chen, for taking the time to read my thesis and their priceless comments. I want to thank Andreas Kyprianou for having accepted so kindly to be part of the jury.

Je tiens à remercier vivement Jean Bertoin, Francesco Russo, Peter Tankov et Marc Yor d'avoir accepté de faire partie de mon jury. J'en suis très honoré.

J'exprime mon remerciement à tous ceux qui sont intervenus d'une façon ou autre à la recherche présentée ici. Je tiens en particulier à remercier...

... À l'ensemble du Laboratoire de Probabilités et Modèles aléatoires pour m'avoir permis de travailler dans d'excellentes conditions.

... À tous les thésards que j'ai rencontrés pendant mes études et surtout à tous ceux avec qui j'ai partagé un bureau, pour l'ambiance amicale et sympathique que j'ai toujours ressenti.

... A mis amigos, compañeros y profesores de la escuela de matemática de la Universidad de Costa Rica por enseñarme las bases suficientes para poder comenzar con el pie derecho mis estudios aquí en Francia. Quiero agradecer particularmente a Eugenio Chinchilla, Alex Ramírez, Mario Villalobos y Santiago Cambronero por ayudarme también con los trámites administrativos que tuve que realizar.

... A todo el personal de la Oficina de Asuntos Internacionales y Cooperación externa de la Universidad de Costa Rica por todo su apoyo durante estos años de estudio.

... A todas las personas que me ayudaron en mis primeros días aquí en París. A Avenilde que me ayudó muchísimo cuando no hablaba ni entendía nada de francés, a Elizabeth, Claudio, Pablo y Valentina por su gran hospitalidad cuando aún no encontraba apartamento.

... A todas aquellas personas que hicieron que mi experiencia en Francia fuera excelente, mostrándome aquello que existe fuera de la universidad. En este último año tuve el agrado de poder redescubrir París junto a tres amigos costarricenses, mis agradecimientos a Samaria, Rafael y Jesús.

... A mis hermanos : Joselyn, Jonathan y Christopher porque a pesar de la distancia, me han acompañado a lo largo de todo este viaje. Yo guardo grandes recuerdos de sus visitas aquí en París, para mí, de los mejores días que pude haber pasado. Gracias también por ayudarme a tomar fuerzas durante mis vacaciones a Costa Rica. A Jimena gracias por alegrarme muchos momentos con sus fotos durante estos dos últimos años.

... À Josiane et Siegfried pour votre gentillesse et chaleureuse hospitalité pendant toutes mes visites à Montpellier. Merci aussi pour votre soutien et vos encouragements.

No hubiera vivido tantos buenos momentos aquí en París si no hubiera contado con la compañía de Ermelinde durante estos 3 últimos años. Muchas gracias por haberme soportado en los momentos de estrés, por haberme ayudado a tomar descanso y ganas de seguir adelante en momentos de tanto cansancio. Su apoyo incondicional fue un elemento esencial que me permitió finalizar este trabajo.

Calcul d'Itô étendu

Résumé : Nos différents résultats consistent principalement à établir des extensions du calcul stochastique classique. Pour $(X_t)_{t\geq 0}$ processus de Markov, il s'agissait à l'origine de donner dans les quatre cas suivants, la décomposition explicite de $F(X_t,t)$ en tant que processus de Dirichlet, sous des conditions minimum sur F fonction déterministe à valeurs réelles.
Dans le premier cas, X est un processus de Lévy réel avec composante brownienne. Dans le deuxième cas X est un processus de Lévy symétrique sans composante brownienne mais admettant des temps locaux en tant que processus de Markov. Dans le troisième cas, X est un processus de Markov symétrique général sans condition d'existence de temps locaux mais $F(x,t)$ ne dépend pas de t. Dans le quatrième cas, nous supprimons l'hypothèse de symétrie du troisième cas.
Dans chacun des trois premiers cas, on obtient une formule d'Itô à la seule condition que la fonction F admette des dérivées de Radon-Nikodym d'ordre 1 localement bornées. On rappelle que dans l'hypothèse où X est une semi-martingale, la formule d'Itô classique nécessite que F soit C^2. C'est l'hypothèse que nous devons prendre dans le quatrième cas.
Le premier cas excepté, chacune des formules d'Itô obtenues s'appuie sur la construction de nouvelles intégrales stochastiques par rapport à des processus aléatoires qui ne sont pas des semi-martingales.

Extended Itô calculus

Abstract : Our main results are extensions of the classical stochastic calculus. For a Markov process $(X_t)_{t\geq 0}$, the problem is to give the explicit decomposition as a Dirichlet process of $F(X_t,t)$ under minimal conditions on F, real-valued deterministic function. We consider the four following cases.
In the first case X is a real-valued Lévy process with a Brownian component. In the second case, X is a symmetric Lévy process without Brownian component, but admitting a local time process as a Markov process. In the third case, X is a general symmetric Markov process without condition of existence of local times, but $F(x,t)$ does not depend on t. In the fourth case, we suppress the assumption of symmetry of the third case.
In each of the first three cases, we obtain an Itô formula under the only condition that the function F admits locally bounded first order Radon-Nicodym derivatives. Note that under the assumption that X is a general semimartingale, the classical Itô formula requires C^2 functions. This is what we have to assume in the fourth case.
First case excepted, each of the obtained Itô formulas requires the construction of a new stochastic integral with respect to random processes which are not semimartingales.

Mots clés : fonctionnelle additive , décomposition de Fukushima, formule d'Itô, processus de Lévy, temps local, calcul local espace-temps, correspondance de Revuz, calcul stochastique, processus de Markov symétrique, processus symétrique stable, processus d'énergie nulle.

Keywords : additive functional, Fukushima decomposition, Itô formula, Lévy process, local time, local time-space calculus, Revuz correspondance, stochastic calculus, symmetric Markov process, symmetric stable process, zero energy process.

AMS 2010 subject classification : 31C25, 60G44, 60H05, 60J25,60J55, 60J65.

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Chapitre 1

Introduction

La formule d'Itô est un outil fondamental de la théorie des probabilités. En particulier, pour $(X_t)_{t\geq 0}$ semimartingale et F fonction déterministe de $C^{2,1}(\mathbb{R}^2)$, elle fournit le développement explicite du processus $(F(X_t,t))_{t\geq 0}$ mais également sa structure stochastique de semimartingale :

$$\begin{aligned} F(X_t,t) &= F(X_0,0)+\int_0^t \frac{\partial F}{\partial t}(X_{s-},s)ds \\ &+ \int_0^t \frac{\partial F}{\partial x}(X_{s-},s)dX_s+\frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(X_s,s)d<X^c>_s \\ &+ \sum_{0<s\leq t}\{F(X_s,s)-F(X_{s-},s)-\frac{\partial F}{\partial x}(X_{s-},s)\Delta X_s\}. \end{aligned} \quad (1.0.1)$$

De nombreux auteurs ont cherché à étendre cette formule, soit en allégeant les conditions de régularité sur F soit en considérant d'autres processus X que des semimartingales. Mais il survient toujours de nouveaux problèmes requérant l'utilisation de la formule d'Itô sous des conditions encore plus générales. C'est ce qui maintient ce sujet ouvert. A titre d'illustration, citons les articles de Peskir [39], [40]. Pour prouver un résultat d'unicité pour le problème de l'option américaine [40], il doit d'abord établir une formule d'Itô pour le mouvement brownien et des fonctions F qui sont partout $C^2(\mathbb{R}\times\mathbb{R}_+)$ sauf sur un ensemble $\{(x,t): x=b(t)\}$ avec b fonction continue.
Chacun des chapitres suivants de cette thèse va fournir un nouvel outil. Au chapitre 2, il permet de s'affranchir de façon optimale des conditions restrictives de la formule (1.0.1) pour les processus de Lévy avec composante brownienne. Puis dans chacun des chapitres 3, 4, et 5, nous construisons un nouveau calcul stochastique par rapport à des processus qui ne sont pas des semi-martingales. Il donne lieu chaque fois à une extension de la formule (1.0.1) pour successivement les processus de Lévy symétriques, les processus de Markov symétriques et les processus de Markov non nécessairement symétriques. Les processus de Markov considérés sont à valeurs dans un espace métrique. Ce qui permet d'envisager

des exemples tels que les superprocessus, les processus de branchements ou bien encore l'historique d'un processus de Markov.

Supposons qu'une fonction $F(x,t)$ possède des dérivées d'ordre 1 de Radon-Nikodym ainsi que les propriétés minimum d'intégration pour que l'expression

$$F(X_t,t) - F(X_0,0) - \int_0^t \frac{\partial F}{\partial t}(X_{s-},s)ds - \int_0^t \frac{\partial F}{\partial x}(X_{s-},s)dX_s$$

existe. Si l'on veut écrire une extension de (1.0.1) pour une telle fonction F, il faut trouver une expression alternative à

$$\frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(X_s,s)d < X^c >_s + \sum_{0<s\leq t}\{F(X_s,s) - F(X_{s-},s) - \frac{\partial F}{\partial x}(X_{s-},s)\Delta X_s\} \tag{1.0.2}$$

qui ne nécessiterait pas l'existence de dérivées d'ordre supérieur à 1. Plusieurs auteurs ont résolu cette question en utilisant la notion de temps local. Meyer [34] fut le premier à alléger les conditions sur F en introduisant une intégrale par rapport aux temps locaux, suivi par Bouleau et Yor [6] , Azéma, Jeulin, Knight et Yor [1], Eisenbaum [10], Ghomrasni et Peskir [22], Eisenbaum et Kyprianou [13]. Dans le cas où le processus X est discontinu, les expressions alternatives à (1.0.2) proposées par ces auteurs nécessitent des conditions supplémentaires sur F ou sur X du fait de la présence de l'expression :

$$\sum_{0<s\leq t}\{F(X_s,s) - F(X_{s-},s) - \frac{\partial F}{\partial x}(X_{s-},s)\Delta X_s\}$$

En ce sens, nous pouvons dire que ces formules ne sont pas optimales. Dans le cas particulier du mouvement brownien, Eisenbaum [10] a montré que (1.0.2) coincide avec

$$-\frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)d\ell_s^x$$

où $(\ell_t^x, x \in \mathbb{R}, t \geq 0)$ est le processus des temps locaux du mouvement brownien. Cette intégrale ne nécessitant pas de conditions supplémentaires sur F, la formule d'Itô ainsi obtenue peut donc être considérée comme optimale.

Au **chapitre 2**, nous présentons une formule d'Itô *optimale* pour les processus de Lévy X possédant une composante brownienne $(\sigma B_t, t \geq 0)$. Cette formule nécessite l'intégration de fonctions déterministes sur $\mathbb{R} \times \mathbb{R}^+$ par rapport au processus $(L_t^x, x \in \mathbb{R}, t \geq 0)$ des temps locaux de X en tant que processus de Markov. La condition $\sigma \neq 0$ nous permet d'exploiter directement la construction du calcul stochastique par rapport à $(\ell_t^x, x \in \mathbb{R}, t \geq 0)$ processus des temps locaux de X en tant que semimartingale, établie dans [13]. En effet ces deux processus sont reliés par

$$(L_t^x, x \in \mathbb{R}, t \geq 0) = (\frac{1}{\sigma^2}\ell_t^x, x \in \mathbb{R}, t \geq 0).$$

On obtient la décomposition explicite en tant que processus de Dirichlet de $F(X_t, t)$ sans condition supplémentaire sur F.

Au **chapitre 3**, nous traitons le cas où X est un processus de Lévy sans composante brownienne ($\sigma = 0$). Le processus $(\ell_t^x, x \in \mathbb{R}, t \geq 0)$ est alors identiquement nul. Nous devons supposer l'existence de $(L_t^x, x \in \mathbb{R}, t \geq 0)$ qui est équivalente à la condition

$$\int_{\mathbb{R}} \frac{1}{1+\psi(\xi)} d\xi < \infty$$

où ψ est l'exposant caractéristique de X.
Nous supposons de plus que X est symétrique. Selon Fukushima [19], nous savons déjà que pour toute fonction u élément de l'espace de Dirichlet de X, $u(X)$ admet la décomposition suivante :

$$u(X_t) = u(X_0) + M^u + N^u \tag{1.0.3}$$

où M^u est une martingale de carré intégrable et N^u est une fonctionnelle additive continue d'énergie quadratique nulle. De plus, pour toute fonction Φ de classe C^2, Chen, Fitzsimmons, Kuwae et Zhang [7] ont donné une décomposition de $\Phi(u(X))$ en fonction de M^u et de N^u.
Dans ce chapitre, nous écrivons une extension de (1.0.3) aux fonctions espace-temps en donnant de plus l'expression précise de chacun des termes. Nous obtenons ainsi la décomposition de $F(X_t, t)$ en processus de Dirichlet sans condition supplémentaire sur F. Cette formule d'Itô s'obtient grâce à la construction d'une integration des fonctions déterministes de $\mathbb{R} \times \mathbb{R}^+$ par rapport à $(L_t^x, x \in \mathbb{R}, t \geq 0)$. Pour cette construction, nous utilisons de nouveaux outils inspirés de la formule de Tanaka de Salminen et Yor [42]. Cette construction nous permet également de définir un temps local sur les courbes $(b(t), t \geq 0)$ pour X puis d'établir une formule d'Itô pour les fonctions espace-temps partout $C^{2,1}$ à l'extérieur d'un ensemble $\{(x,t) : x = b(t)\}$.

Au **chapitre 4**, nous considérons un processus $X = u(Z)$, où Z est un processus de Hunt associé à une forme de Dirichlet symétrique régulière $(\mathcal{E}, \mathcal{F})$ et u appartient localement à $\mathcal{F}$. Bien que X ne soit pas en général une semimartingale, Nakao [35] et Chen, Fitzsimmons, Kuwae et Zhang [7] ont montré pour $F(x,t) = F(x)$, que (1.0.1) reste valide pour un tel processus X. Cela a été montré grâce à la construction d'une intégrale stochastique par rapport à $(N_t, t \geq 0)$, la partie d'énergie nulle de X dans sa décomposition de Fukushima. Cette intégrale vient remplacer l'intégrale de Lebesgue-Stieljes par rapport à la partie à variations bornées dans (1.0.1). De même que pour la formule d'Itô classique (1.0.1), cette formule d'Itô nécessite l'utilisation de fonctions C^2.
Le problème de l'allègement des conditions de régularité de F dans la formule de Nakao et Chen, Fitzsimmons, Kuwae et Zhang, s'avère être plus complexe que dans les deux cas précédents. En effet, l'intégrale $\int_0^t F'(u(X_s))dN_s$ n'est bien

définie que lorsque $F'(u)$ est localement dans $\mathcal{F}$. Par exemple, dans le cas où Z est un mouvement brownien, cette dernière condition impose que la dérivée seconde F'' existe au moins en tant que dérivée de Radon-Nikodym.
Nous avons contourné ce problème en construisant une intégration stochastique des fonctions déterministes sur $\mathbb{R}$ par rapport à un processus $(\Gamma_t^a(u), a \in \mathbb{R})$ à t fixé, qui va jouer le rôle de temps local pour le processus $u(X)$ (cette analogie est exposée en section 4.5). Cette construction nous permet alors d'obtenir le développement explicite de $F(u(X))$ pour F admettant une dérivée de Radon-Nikodym localement bornée.

Au **chapitre 5**, nous considérons un processus de Hunt Z, associé à une forme de Dirichlet régulière $(\mathcal{E}, \mathcal{F})$. Elle n'est pas nécessairement symétrique. Les résultats de Nakao [35] et Chen et al [7] nécessitants une hypothèse de symétrie, la question d'un calcul stochastique pour Z est entière. Néanmoins, la décomposition de Fukushima (1.0.3) reste valide dans le cas général : pour tout u élément de l'espace de Dirichlet de Z, il existe une martingale de carré intégrable M^u et une fonctionnelle additive continue d'énergie quadratique nulle N^u telles que

$$u(Z_t) = u(Z_0) + M_t^u + N_t^u, t \geq 0. \tag{1.0.4}$$

Désignons par $\tilde{\mathcal{E}}$ la partie symétrique de $\mathcal{E}$. Pour construire une intégrale stochastique par rapport à N^u, nous établissons une décomposition de N^u en somme de trois processus N_1^u, N_2^u et N_3^u. Les processus N_1^u et N_2^u sont respectivement associés à la partie diffusion et à la partie saut de $\tilde{\mathcal{E}}$. Le processus N_3^u est à variations bornées. Après avoir successivement construit une intégrale stochastique par rapport à N_1^u et N_2^u, nous disposons donc d'une notion d'intégrale par rapport à $u(Z)$. Elle nous permet d'établir un développement de $F(u(Z_t))$ pour F fonction réelle C^2.
Dans le cas où Z est à valeurs dans $\mathbb{R}^d$ $(d \geq 1)$, nous optons pour une autre démarche consistant à utiliser une décomposition de Beurling-Deny de $\mathcal{E}$ due à Hu, Ma and Sun [23]. Elle nous permet d'obtenir une décomposition de Z du type de la décomposition de Itô-Lévy pour les processus de Lévy. L'intégration stochastique par rapport à Z en découle immédiatement. Nous pouvons ensuite développer $F(Z_t)$ pour F fonction réelle C^2.

Dans chacun des quatre cas traités, la formule d'Itô obtenue possède une version multidimensionnelle.

Les chapitres 2, 3 et 4 ont chacun donné lieu à une publication (voir [14], [45], et [46]).
Le chapitre 5 doit être soumis pour publication incessamment.

Chapitre 2

An optimal Itô formula for Lévy Processes

Abstract : Several Itô formulas have been already established for Lévy processes. We explain according to which criteria they are not optimal and establish an extended Itô formula that satisfies that criteria. The interest, in particular, of this formula, is to obtain the explicit decomposition of $F(X_t, t)$, for X Lévy process and F deterministic function with locally bounded first order Radon-Nikodym derivatives, as a Dirichlet process.

2.1 Introduction and main results

Let X be a general real-valued Lévy process with characteristic triplet (a, σ, ν), i.e. its characteristic exponent is equal to

$$\psi(u) = iua - \sigma^2 \frac{u^2}{2} + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy1_{\{|y| \leq 1\}})\nu(dy)$$

where a and σ are real numbers and ν is a Lévy measure. We will denote by $(\sigma B_t, t \geq 0)$ the Brownian component of X. Let F be a $C^{2,1}$ function from $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}$. The classical Itô formula gives

$$\begin{aligned} F(X_t, t) &= F(X_0, 0) + \int_0^t \frac{\partial F}{\partial t}(X_{s-}, s)ds \\ &+ \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s)dX_s + \sigma^2 \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, s)ds \qquad (2.1.1) \\ &+ \sum_{0<s\leq t} \{F(X_s, s) - F(X_{s-}, s) - \frac{\partial F}{\partial x}(X_{s-}, s)\Delta X_s\}. \end{aligned}$$

This formula can be rewritten under the following form (see [24]) : $(F(X_t, t), t \geq 0)$ is a semimartingale admitting the decomposition

$$F(X_t, t) = F(X_0, 0) + M_t + V_t \tag{2.1.2}$$

where the local martingale M and the adapted with bounded variation process V are given by

$$M_t = \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dB_s + \int_0^t \int_{\{|y|<1\}} F(X_{s-} + y, s) - F(X_{s-}, s) \tilde{\mu}_X(dy, ds) \tag{2.1.3}$$

$$V_t = \sum_{0<st} \{F(X_s, s) - F(X_{s-}, s)\} 1_{|\Delta X_s| \geq 1} + \int_0^t \mathcal{A}F(X_s, s) ds \tag{2.1.4}$$

where $\tilde{\mu}_X(dy, ds)$ denotes the compensated Poisson measure associated to the jumps of X, and $\mathcal{A}$ is the operator associated to X defined by

$$\begin{aligned} \mathcal{A}G(x,t) &= \frac{\partial G}{\partial t}(x,t) + a\frac{\partial G}{\partial x}(x,t) + \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial x^2}(x,t) \\ &+ \int_{\mathbb{R}} \{G(x+y,t) - G(x,t) - y\frac{\partial G}{\partial x}(x,t)\} 1_{(|y|<1)} \nu(dy) \end{aligned}$$

for any function G defined on $\mathbb{R} \times \mathbb{R}^+$, such that $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial t}$ and $\frac{\partial^2 G}{\partial x^2}$ exist as Radon-Nikodym derivatives with respect to the Lebesgue measure and the integral is well defined. The later condition is satisfied when $\frac{\partial^2 G}{\partial x^2}$ is locally bounded.
Note that the existence of locally bounded first order Radon-Nikodym derivatives alone guarantees the existence of

$$F(X_t, t) - F(X_0, 0) - \int_0^t \frac{\partial F}{\partial t}(X_{s-}, s) ds - \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dX_s \tag{2.1.5}$$

but then to say that this expression coincides with

$$\sigma^2 \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, s) ds + \sum_{0<s\leq t} \{F(X_s, s) - F(X_{s-}, s) - \frac{\partial F}{\partial x}(X_{s-}, s) \Delta X_s\}$$

we need to assume much more on F.
In that sense one might say that the classical Itô formula is not optimal. The interest of an optimal formula is two-fold. It allows to expand $F(X_t, t)$ under minimal conditions on F but also to know explicitly the structure of the process $F(X_t, t)$. Such an optimal formula has been established in the particular case when X is a Brownian motion [10]. Indeed in that case, under the minimal assumption

on F for the existence of (2.1.5), namely that F admits locally bounded first order Radon-Nikodym derivatives, we know that this expression coincides with

$$-\frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)dL_s^x$$

where $(L_s^x, x \in R, s \geq 0)$ is the local time process of X. Moreover the process

$$\left\{\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)dL_s^x, t \geq 0\right\}$$

has a 0-quadratic energy.
In the general case, various extensions of (2.1.1) have been established. We will quote here only the extensions exploiting the notion of local times, we send to [11] for a more exhaustive bibliography. Meyer [34] has been the first to relax the assumption on F by introducing an integral with respect to local time, followed then by Bouleau and Yor [6], Azéma et al [1], Eisenbaum [10], [11], Ghomrasni and Peskir [22], Eisenbaum and Kyprianou [13]. In the discontinuous case, none of the obtained Itô formulas is optimal because of the presence of the expression

$$\sum_{0<s\leq t}\{F(X_s,s) - F(X_{s-},s) - \frac{\partial F}{\partial x}(X_{s-},s)\Delta X_s\}$$

The Itô formula for Lévy processes presented below in Theorem 2.1.1, is available for X admitting a Brownian component. It lightens the condition on the jumps of X required by [11], and it also lightens the condition on the first order derivatives of F required by [13]. Besides it is optimal. To introduce it we need the operator I defined on the set of locally bounded measurable functions G on $\mathbb{R} \times \mathbb{R}^+$ by

$$IG(x,t) = \int_0^t G(y,t)dy.$$

We will denote the Markov local time process of X by $(L_t^x, x \in \mathbb{R}, t \geq 0)$.

Theorem 2.1.1. *Assume that $\sigma \neq 0$. Let F be a function from $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}$ such that $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial t}$ exist as Radon-Nikodym derivatives with respect to the Lebesgue measure and are locally bounded. Then the process $(F(X_t,t), t \geq 0)$ is a Dirichlet process admitting the decomposition*

$$F(X_t,t) = F(X_0,0) + M_t + V_t + Q_t$$

with M the local martingale given by (2.1.3), V is the bounded variation process

$$V_t = \sum_{0\leq s\leq t}\{F(X_s,s) - F(X_{s-},s)\}1_{\{|\Delta X_s|\geq 1\}}$$

and Q the following adapted process with 0-quadratic variation

$$Q_t = -\int_0^t \int_{\mathbb{R}} \mathcal{A}IF(x,s)dL_s^x.$$

As a simple application of Theorem 2.1.1 consider the example of the function $F(x,s) = |x|$ in the case $\int_0^1 y\nu(dy) = +\infty$. This function does not satisfy the assumption of Theorem 3 of [13] nor X does satisfy the assumption of Theorem 2.2 in [11]. But, thanks to Theorem 1.1, we immediately obtain Tanaka's formula. The proofs are presented in Section 2.

2.2 Proofs

We first remind the meaning of integration with respect to the semimartingale local time process of X denoted $(\ell_s^x, x \in \mathbb{R}, s \geq 0)$. Theorem 1.1 is expressed in terms of the Markov local time process $(L_s^x, x \in \mathbb{R}, s \geq 0)$. The two processes are connected by :

$$\{L_s^x, x \in \mathbb{R}, s \geq 0\} = \left\{\sigma^{-2}\ell_s^x, x \in \mathbb{R}, s \geq 0\right\}$$

Let σB be the Brownian component of X. Defined the norm $||.||$ of a measurable function f from $\mathbb{R} \times \mathbb{R}_+$ to $\mathbb{R}$ by

$$||f|| = 2\mathbf{E}\left(\int_0^1 f^2(X_s,s)ds\right)^{1/2} + \mathbf{E}\left(\int_0^1 |f(X_s,s)|\frac{|B_s|}{s}ds\right)$$

In [13], integration with respect to ℓ of locally bounded mesurable function f has been defined by

$$\int_0^t \int_{\mathbb{R}} f(x,s)d\ell_s^x = \sigma \int_0^t f(X_{s-},s)dB_s + \sigma \int_0^t f(\hat{X}_{s-}, 1-s)d\hat{B}_s, \; 0 \leq t \leq 1 \quad (2.2.1)$$

where $\hat{B}$ and $\hat{X}$ are the time reversal at 1 of B and X.
We have the following properties :

(i) $\mathbf{E}\left[\int_0^t \int_{\mathbb{R}} f(x,s)d\ell_s^x\right] \leq |\sigma| \, ||f||$.

(ii) If f admits a locally bounded Radon-Nikodym derivative with respect to x, then :

$$\int_0^t \int_{\mathbb{R}} f(x,s)d\ell_s^x = -\sigma^2 \int_0^t \frac{\partial f}{\partial x}(X_s,s)ds$$

(iii) The process $\left\{\int_0^t \int_{\mathbb{R}} f(x,s)d\ell_s^x, 0 \leq t \leq 1\right\}$ has 0-quadratic variation.

Proof of Theorem 2.1.1 : We start by assuming that F and $\frac{\partial F}{\partial x}$ are bounded. We set

$$F_n(x,t) = \int\int_{\mathbb{R}^2} F(x - y/n, t - s/n)f(y)h(s)dyds$$

where f and h are nonnegative C^∞ functions with compact supports such that : $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} h(x)dx = 1$. Thanks to the usual Itô formula we have :

$$\begin{aligned}
F_n(X_t,t) = \quad & F_n(0,0) + \sigma \int_0^t \frac{\partial F_n}{\partial x}(X_{s-},s)dB_s + \int_0^t \frac{\partial F_n}{\partial t}(X_s,s)ds \\
+ \quad & a\int_0^t \frac{\partial F_n}{\partial x}(X_s,s)ds + \sum_{0\le s\le t}\{F_n(X_s,s) - F_n(X_{s-},s)\}1_{\{|\Delta X_s|\ge 1\}} \\
+ \quad & \int_0^t\int_{\mathbb{R}}\{F_n(X_{s-}+y,s) - F_n(X_{s-},s)\}1_{\{|y|<1\}}\tilde{\mu}(ds,dy) \\
+ \quad & \frac{\sigma^2}{2}\int_0^t \frac{\partial^2 F_n}{\partial x^2}(X_s,s)ds \qquad (2.2.2)\\
+ \quad & \int_0^t\int_{-1}^1\{F_n(X_s+y,s) - F_n(X_s,s) - \frac{\partial F_n}{\partial x}(X_s,s)y\}\nu(dy)ds
\end{aligned}$$

With the same arguments as in the proof of Theorem 2.2 of [11], we see that as n tends to ∞, $F_n(X_t,t)$ and each of the first five terms of the RHS of (2.2.2) converges at least in probability to the corresponding expression with F replacing F_n. Besides we note that

$$\begin{aligned}
\int_0^t \frac{\partial F}{\partial t}(X_s,s)ds &= -\frac{1}{\sigma^2}\int_0^t\int_{\mathbb{R}}\left(\int_0^x \frac{\partial F}{\partial t}(y,s)dy\right)d\ell_s^x \\
&= -\int_0^t\int_{\mathbb{R}}\left(\frac{\partial}{\partial t}\int_0^x F(y,s)dy\right)dL_s^x
\end{aligned}$$

since $\frac{\partial F}{\partial t}$ is locally bounded. Hence we have :

$$\int_0^t \frac{\partial F}{\partial t}(X_s,s)ds = -\int_0^t\int_{\mathbb{R}} \frac{\partial (IF)}{\partial t}(x,s)dL_s^x. \qquad (2.2.3)$$

Since : $F(x,s) = \frac{\partial(IF)}{\partial x}(x,s)$, we immediately obtain :

$$a\int_0^t \frac{\partial F}{\partial x}(X_s,s)ds = -\int_0^t\int_{\mathbb{R}} a\frac{\partial (IF)}{\partial x}(x,s)dL_s^x \qquad (2.2.4)$$

The convergence in L^2 of the sixth term of the RHS of (2.2.2) is obtained with the same proof as in [13]. The limit is equal to

$$\int_0^t\int_{\mathbb{R}}\{F(X_{s-}+y,s) - F(X_{s-},s)\}1_{\{|y|<1\}}\tilde{\mu}(ds,dy) \qquad (2.2.5)$$

For the seventh term of the RHS of (2.2.2), we note that :

$$\sigma^2 \int_0^t \frac{\partial^2 F_n}{\partial x^2}(X,s)ds = -\frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial F_n}{\partial x}(x,s)d\ell_s^x$$

Thanks to the properties (i) and (ii) of the integration with respect to the local times, this expression converges in L^1 to

$$-\frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)(x,s)d\ell_s^x$$

We can obviously write :

$$-\frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)d\ell_s^x = -\frac{\sigma^2}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial^2 (IF)}{\partial x^2}(x,s)dL_s^x \qquad (2.2.6)$$

We now study the convergence of the last term of the RHS of (2.2.2). We have :

$$\begin{aligned} \int_0^t \int_{-1}^1 & \{F_n(X_s+y,s) - F_n(X_s,s) - \frac{\partial F_n}{\partial x}(X_s,s)y\}\nu(dy)ds \\ = & -\int_0^t \int_{\mathbb{R}} H_n(x,s)dL_s^x \qquad (2.2.7) \end{aligned}$$

where : $H_n(x,s) = \int_0^x \int_{-1}^1 \{F_n(z+y,s) - F_n(z,s) - \frac{\partial F_n}{\partial x}(z,s)y\}\nu(dy)dz$. We have :

$$\begin{aligned} |F_n(z+y,s) & - F_n(z,s) - \frac{\partial F_n}{\partial x}(z,s)y|1_{\{|y|<1\}} \\ & = \left|\int_z^{z+y} (\frac{\partial F_n}{\partial x}(v,t) - \frac{\partial F_n}{\partial x}(z,t))dv\right| 1_{\{|y|<1\}} \\ & \leq y^2 \sup|\frac{\partial^2 F_n}{\partial x^2}|1_{\{|y|<1\}}. \end{aligned}$$

Noting that : $\frac{\partial^2 F_n}{\partial x^2} = n^2 \int\int_{\mathbb{R}^2} F(x-y/n, t-s/n)f''(y)h(s)dyds$, we obtain

$$|F_n(z+y,s) - F_n(z,s) - \frac{\partial F_n}{\partial x}(z,s)y|1_{\{|y|<1\}} \leq cste\, n^2y^2 1_{\{|y|<1\}} sup|F|$$

Consequently :

$$\begin{aligned} H_n(x,s) &= \int_{-1}^1 \int_0^x \left\{F_n(z+y,s)dz - F_n(z,s) - \frac{\partial F_n}{\partial x}(z,s)y\right\} dz\nu(dy) \\ &= \int_{-1}^1 \left\{\int_0^{x+y} F_n(z,s)dz - \int_0^y F_n(z,s)dz \right. \\ & \left. - \int_0^x F_n(z,s) - yF_n(x,s) + yF_n(0,s)\right\} \nu(dy) \\ &= G_n(x,s) + \int_{-1}^1 (yF_n(0,s) - \int_0^y F_n(z,s)dz)\nu(dy) \end{aligned}$$

where $G_n(x,s) = \displaystyle\int_{-1}^{1}(IF_n(x+y,s)-IF_n(x,s)-yF_n(x,s))\nu(dy)$ .Thanks to Corollary 8 of [13], we know that

$$\int_0^t \int_{\mathbb{R}} H_n(x,s)dL_s^x = \int_0^t \int_{\mathbb{R}} G_n(x,s)dL_s^x \tag{2.2.8}$$

By dominated convergence, we have as n tends to ∞ for every (x,s)

$$IF_n(x+y,s) - IF_n(x,s) - yF_n(x,s) \to IF(x+y,s) - IF(x,s) - yF(x,s).$$

Besides, for every n :

$$|IF_n(x+y,s) - IF_n(x,s) - yF_n(x,s)| \leq y^2 1\{|y|<1\} \sup|\frac{\partial F}{\partial x}|,$$

hence for every (x,s) : $G_n(x,s)$ tends to $G(x,s)$, where

$$G(x,s) = \int_{\mathbb{R}}(IF(x+y,s) - IF(x,s) - yF(x,s))1_{\{|y|<1\}}\nu(dy).$$

By dominated convergence, $(G_n)_{n>0}$ converges for the norm $||.||$ to G. Consequently the limit of the last term of the RHS of (2.2.2) is equal by (2.2.7) and (2.2.8) to

$$-\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} (IF(x+y,s) - IF(x,s) - yF(x,s))1_{\{|y|<1\}}\nu(dy)dL_s^x. \tag{2.2.9}$$

Summing all the limits (2.2.3), (2.2.4), (2.2.5), (2.2.6) and (2.2.9), we finally obtain

$$\begin{aligned}
F(X_t,t) &= F(X_0,0) + \sigma\int_0^t \frac{\partial F}{\partial x}(X_{s-},s)dB_s \qquad (2.2.10)\\
&+ \int_0^t \int_{\mathbb{R}} \{F(X_{s-}+y,s) - F(X_{s-},s)\}1_{\{|y|<1|\}}\tilde{\mu}(ds,dy)\\
&+ \sum_{0<s\leq t}\{F(X_s,s) - F(X_{s-},s)\}1_{\{|\Delta X_s|\geq 1\}}\\
&- \int_0^t \int_{\mathbb{R}} \left\{\frac{\partial(IF)}{\partial t}(x,s) + a\frac{\partial(IF)}{\partial x}(x,s) + \frac{\sigma^2}{2}\frac{\partial^2(IF)}{\partial x^2}(x,s)\right\} dL_s^x\\
&- \int_0^t \int_{\mathbb{R}} \left\{\int_{\mathbb{R}}(IF(x+y,s) - IF(x,s) - yF(x,s))1_{\{|y|<1\}}\nu(dy)\right\} dL_s^x.
\end{aligned}$$

which summarizes in

$$\begin{aligned} F(X_t,t) = \quad & F(X_0,0) + \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-},s)dB_s \\ & + \int_0^t \int_{\mathbb{R}} \{F(X_{s-}+y,s) - F(X_{s-},s)\} 1_{\{|y|<1\}} \tilde{\mu}(ds,dy) \\ & + \sum_{0<s\le t} \{F(X_s,s) - F(X_{s-},s)\} 1_{\{|\Delta X_s| \ge 1\}} - \int_0^t \int_{\mathbb{R}} \mathcal{A}IF(x,s)dL_s^x. \end{aligned}$$

In the general case, we set :

$$\tilde{F}_n(x,s) = F(x,s)1_{[a_n,b_n]}(x) + F(a_n,s)1_{(-\infty,a_n)}(x) + F(b_n,s)1_{(b_n,\infty)}(x)$$

where $(-a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two positive real sequences increasing to ∞. We write (2.2.10) for $\tilde{F}_n$ and stop the process $(\tilde{F}_n(X_s,s), 0 \le s \le 1)$ at

$$T_m = 1 \wedge \inf\{s \ge 0 : |X_s| > m\}$$

We let n tend to ∞ and then m tend to ∞. The behavior of two terms deserves specific explanations, the other terms converging respectively to the expected expressions.
The first one is :

$$\int_0^{t\wedge T_m} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} (I\tilde{F}_n(x+y,s) - I\tilde{F}_n(x,s) - y\tilde{F}_n(x,s)) 1_{\{|y|<1\}} \nu(dy) \right\} dL_s^x$$

Thanks to the definition of the integral with respect to local time (2.2.1), it is equal to

$$\frac{1}{\sigma} \int_0^{t\wedge T_m} \tilde{H}_n(X_s,s)dB_s + \frac{1}{\sigma} \int_{1-(t\wedge T_m)}^1 \tilde{H}_n(\hat{X}_{s-},s)d\hat{B}_s \tag{2.2.11}$$

where $\tilde{H}_n(x,s) = \displaystyle\int \{I\tilde{F}_n(x+y,s) - I\tilde{F}_n(x,s) - y\tilde{F}_n(x,s)\} 1_{\{|y|<1\}} \nu(dy)$.
We set : $H(x,s) = \displaystyle\int \{IF(x+y,s) - IF(x,s) - yF(x,s)\} 1_{\{|y|<1\}} \nu(dy)$. We can choose n big enough to have $|a_n|$ and b_n bigger than $m+1$. Hence (2.2.11) is equal to

$$\frac{1}{\sigma} \int_0^{t\wedge T_m} H(X_s,s)dB_s + \frac{1}{\sigma} \int_{1-(t\wedge T_m)}^1 H(\hat{X}_{s-},s)d\hat{B}_s.$$

For every $\varepsilon > 0$

$$\begin{aligned} & \mathbf{P}\left(\sup_{0\le t\le 1} \left| \int_{1-(t\wedge T_m)}^1 H(\hat{X}_{s-},s)d\hat{B}_s - \int_{1-t}^1 H(\hat{X}_{s-},s)d\hat{B}_s \right| \ge \varepsilon \right) \\ \le \ & \mathbf{P}(T_m < 1) \\ = \ & \mathbf{P}(\sup_{0\le t\le 1} |X_t| > m) \end{aligned}$$

which shows that as m tends to ∞, $\int_{1-(t\wedge T_m)}^{1} H(\hat{X}_{s-}, s)d\hat{B}_s$ converges in probability uniformly on $[0, 1]$ to $\int_{1-(t\wedge t)}^{1} H(\hat{X}_{s-}, s)d\hat{B}_s$. Similarly $\int_0^{t\wedge T_m} H(X_{s-}, s)dB_s$ converges in probability to $\int_0^{t\wedge T} H(X_{s-}, s)dB_s$. Consequently as m tends to ∞, (2.2.11) converges to

$$\int_0^t \int_{\mathbb{R}} \left\{ \int \{IF(x+y, s) - IF(x, s) - yF(x, s)\} 1_{\{|y|<1\}} \nu(dy) \right\} dL_s^x.$$

The second term is :

$$\int_0^t \int_{\mathbb{R}} \{\tilde{F}_n(X_{s-} + y, s) - \tilde{F}_n(X_{s-}, s)\} 1_{\{s<Tm\}} 1_{\{|y|<1|\}} \tilde{\mu}(ds, dy)$$

For n big enough such that $|a_n|$, $b_n > m$, this term is equal to

$$\int_0^t \int_{\mathbb{R}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} 1_{\{s<Tm\}} 1\{|y| < 1|\} \tilde{\mu}(ds, dy)$$

As Ikeda and Watanabe [24], we then denote by

$$\left\{ \int_0^t \int_{\mathbb{R}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} 1_{\{|y|<1|\}} \tilde{\mu}(ds, dy), 0 \le t \le 1 \right\}$$

the local martingale $(Y_t, 0 \le t \le 1)$ defined by :

$$Y_{t\wedge T_m} = \int_0^t \int_{\mathbb{R}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} 1_{\{s<T_m\}} 1_{\{|y|<1|\}} \tilde{\mu}(ds, dy).$$

□

Chapitre 3

Local time-space calculus for symmetric Lévy Processes

Abstract : We construct a stochastic calculus with respect to the local time process of a symmetric Lévy process X without Brownian component. The required assumptions on the Lévy process are satisfied by the symmetric stable processes with index in $(1, 2)$. Based on this construction, the explicit decomposition of $F(X_t, t)$ is obtained for F continuous function admitting a Radon-Nikodym derivative $\frac{\partial F}{\partial t}$ and satisfying some integrability condition. This Itô formula provides, in particular, the precise expression of the martingale and the continuous additive functional present in Fukushima's decomposition.

3.1 Introduction and main results

For a given semimartingale $(X_t)_{t\geq 0}$ and any $\mathcal{C}^{2,1}$-function F on $\mathbb{R}\times\mathbb{R}^+$, Itô formula provides both an explicit expansion of $(F(X_t, t))_{t\geq 0}$ and its stochastic structure. Consider the case when X is a Lévy process with characteristic triplet (a, σ, ν) which means that for any t in $\mathbb{R}_+$ and ξ in $\mathbb{R}$: $\mathbf{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)}$,
where : $\psi(\xi) = -ia\xi + \dfrac{\sigma^2}{2}\xi^2 + \displaystyle\int_{\mathbb{R}}(1 - e^{i\xi x} + i\xi x 1_{|x|\leq 1})\nu(\mathrm{d}x)$, $a \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and ν is a measure in $\mathbb{R}$ such that : $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \frac{x^2}{1+x^2}\nu(\mathrm{d}x) < \infty$. The function ψ is called the characteristic component of X and ν the Lévy measure of X (see Bertoin [3]). Denote by σB the Brownian component of X, then Itô formula can be rewritten under the following form (see e.g., Ikeda and Watanabe [24]) :

$$F(X_t, t) = F(X_0, 0) + M_t + A_t, \tag{3.1.1}$$

where M is a local martingale and A is an adapted process of bounded variation given by

$$M_t = \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dB_s + \int_0^t \int_{\{|y|\le 1\}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} \tilde{\mu}_X(dy, ds)$$

$$A_t = \sum_{0<s\le t} \{F(X_s, s) - F(X_{s-}, s)\} 1_{\{|\Delta X_s| > 1\}} + \int_0^t \mathcal{A}F(X_s, s) ds$$

where $\tilde{\mu}_X(dy, ds)$ denotes the compensated Poisson measure associated to the jumps of X, and $\mathcal{A}$ is the operator associated to X defined by

$$\begin{aligned} \mathcal{A}G(x,t) &= \frac{\partial G}{\partial t}(x,t) + a\frac{\partial G}{\partial x}(x,s) + \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial x^2}(x,t) \\ &\quad + \int_{\mathbb{R}} \{G(x+y,t) - G(x,t) - y\frac{\partial G}{\partial x}(x,t)\} 1_{(|y|<1)} \nu(dy) \quad (3.1.2) \end{aligned}$$

for any function G defined on $\mathbb{R} \times \mathbb{R}^+$, such that $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial t}$ and $\frac{\partial^2 G}{\partial x^2}$ exist as Radon-Nikodym derivatives with respect to the Lebesgue measure and the integral is well defined.

Many authors have succeeded in relaxing the conditions on F to write extended versions of (3.1.1) (see for example Errami et al.[15], Eisenbaum [11], Eisenbaum and Kyprianou[13]). Under the assumption that X has a Brownian component (i.e. $\sigma \neq 0$), we have established in [14] an extended version of (3.1.1) that can be considered as *optimal* in the sense that it requires the sole condition of existence of locally bounded first order Radon-Nikodym derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial t}$. Under that condition, this version gives the explicit decomposition of $F(X_t, t)$ as the sum of a Dirichlet process and a bounded variation process.

Here we treat the case $\sigma = 0$. If we assume additionally that X is symmetric (i.e. $a = 0$ and ν is symmetric), then according Fukushima [21], we already know that for every continuous function u in $\mathcal{W}$, the Dirichlet space of X, i.e.

$$\mathcal{W} = \{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} (u(x+y) - u(x))^2 \mathrm{d}x \nu(\mathrm{d}y) < \infty\},$$

$u(X)$ admits the following decomposition

$$u(X_t) = u(X_0) + M_t^u + N_t^u \quad (3.1.3)$$

where M^u is a square-integrable martingale and N^u is a continuous additive functional with zero quadratic energy. Besides, for Φ in $\mathcal{C}^2(\mathbb{R})$, Chen et al. [7] give a decomposition of $\Phi(u(X))$ in terms of M^u and N^u.

In this paper we write an extension of (3.1.3) to space-time functions and give the explicit expression of the corresponding terms. In particular, the explicit expression of the processes M^u and N^u involved in (3.1.3) are obtained.

These results, precisely presented below, require two additional assumptions on X. The first one is the existence of local times for X considered as a Markov process, i.e., a jointly measurable family $\{(L_t^x)_{t\geq 0}, x \in \mathbb{R}\}$ of positive additive functionals such that for each x, the measure dL_t^x is supported by the set $\{t \geq 0 : X_t = x\}$ and satisfying for every Borel-measurable function $f : \mathbb{R} \to \mathbb{R}_+$ the occupation time formula

$$\int_0^t f(X_s)\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L_t^x \mathrm{d}x.$$

Noting that ψ is a symmetric nonnegative function, the existence of local times is equivalent to (see Bertoin [3])

(H1)

$$\int_{-\infty}^{\infty} \frac{1}{1+\psi(\xi)} d\xi < \infty.$$

Define the function β from $(0,1]$ to $\mathbb{R}^+$ by

$$\beta(t) = \left\{\int_0^{\infty} e^{-2t\psi(\xi)}\psi(\xi)\mathrm{d}\xi\right\}^{1/2}. \tag{3.1.4}$$

the second assumption is :

(H2)

$$\int_0^1 \beta(t)\mathrm{d}t < \infty.$$

Remark that if there exists q in $(0,1)$ such that $\int_{-\infty}^{\infty} \frac{1}{(1+\psi(\xi))^q} d\xi < \infty$, then X satisfies the two assumptions (H1) and (H2). In particular, the symmetric stable Lévy processes with index in $(1,2)$ satisfy the two assumptions. This is also realized when there exists α in $(1,2)$ such that $\psi(\xi)^{-1} = O(|\xi|^{-\alpha})$ as $|\xi|$ tends to ∞.
To introduce the space-time version of (3.1.3), we need the operator $\mathcal{I}$ defined on the set of locally bounded measurable functions F on $\mathbb{R} \times \mathbb{R}^+$ by

$$\mathcal{I}F(x,s) = \int_0^x F(y,s)\mathrm{d}y. \tag{3.1.5}$$

Set : $Z_t = X_t - \sum_{s\leq t} \Delta X_s 1_{\{|\Delta X_s|>1\}}$. We define the norm $\|.\|_Z$ in the space of measurable functions from $\mathbb{R} \times [0,1]$ to $\mathbb{R}$ by $\|f\|_Z^2 = \int_0^1 \mathbf{E}(f(Z_t,t))^2 \mathrm{d}t$. We denote by $(\bar{P}_t)_{0\leq t\leq 1}$ the semigroup of the Markov process $(Z_t,t)_{0\leq t\leq 1}$ i.e.

$$\bar{P}_t f(x,s) = \mathbf{E}[f(Z_t + x, s+t)1_{\{s+t\leq 1\}}].$$

We associate to $(Z_t,t)_{0\leq t\leq 1}$ the operator $\mathcal{D}$ as follows. A real valued measurable function f on $\mathbb{R} \times [0,1]$ belongs to the domain of $\mathcal{D}$ if $\|f\|_Z < \infty$ and there exists

g such that $\|g\|_Z < \infty$ and

$$\lim_{t\to 0}\left\|\frac{\bar{P}_t f - f}{t} - g\right\|_Z = 0,$$

in this case, $\mathcal{D}f = g$.

Theorem 3.1.1. *Let F be a continuous function from $\mathbb{R}\times[0,1]$ to $\mathbb{R}$ admitting a derivative with respect to the Lebesgue measure $\frac{\partial F}{\partial t}$ such that $\frac{\partial F}{\partial t}$ belongs to $L^2(\mathbb{R}\times[0,1])$ and*

$$\int_0^1 \beta(t)\int_{\mathbb{R}^2}(F(x+y,t)-F(x,t))^2 \mathrm{d}x\nu(\mathrm{d}y)\mathrm{d}t < \infty. \tag{3.1.6}$$

Then $(F(X_t,t), 0\le t\le 1)$ is a Dirichlet process admitting the following decomposition

$$F(X_t,t) = F(X_0,0) + M_t^F + N_t^F, \tag{3.1.7}$$

where M^F is a square-integrable martingale and N^F is a continuous process with 0-quadratic energy respectively defined by

$$M_t^F = \int_0^t\int_{\mathbb{R}}(F(X_{s-}+y,s)-F(X_{s-},s))\tilde{\mu}_X(\mathrm{d}y,\mathrm{d}s)$$

$$N_t^F = -\int_0^t\int_{\mathbb{R}}\mathcal{D}\mathcal{I}F(x,s)\mathrm{d}L_s^x + \int_0^t\int_{\{|y|>1\}}(F(X_s+y,s)-F(X_{s-},s))\nu(\mathrm{d}y)\mathrm{d}s.$$

Theorem 3.1.1 is based on the construction of the stochastic integration of deterministic functions on $\mathbb{R}\times[0,1]$ with respect to $(L_t^x, x\in\mathbb{R}, 0\le t\le 1)$. This construction is done in Section 3. Unlike the cases for which this notion has been already defined (for example Brownian motion [10], Lévy process with a Brownian component [14], or elliptic diffusion [2]) the considered local time process is not a semimartingale local time but a Markov local time. The classical Tanaka's formula is not available for this local time. Instead we use an alternative formulation of an identity of Salminen and Yor [42] for X. One preliminary issue, solved in Section 2, is to obtain an analogue of Tanaka's formula for the reversed process $\hat{X}$ defined by

$$\hat{X}_t = \begin{cases} X_{(1-t)-} & \text{if} \quad 0\le t<1 \\ 0 & \text{if} \quad t=1 \end{cases} \tag{3.1.8}$$

The Itô formula of Theorem 3.1.1 is established in Section 4. It appears as a consequence of the arguments developed to establish the following localized version of Theorem 3.1.1.

Theorem 3.1.2. *Let F be a continuous function from $\mathbb{R} \times [0,1]$ to $\mathbb{R}$ admitting a Radon-Nikodym derivative with respect to the Lebesgue measure $\frac{\partial F}{\partial t}$ and such that for all $k > 0$*

$$\int_0^1 \beta(t) \int_{-k}^{k} \left[\left(\frac{\partial F(x,t)}{\partial t} \right)^2 + \int_{-1}^{1} (F(x+y,t) - F(x,t))^2 \bar{\nu}(\mathrm{d}y) \right] \mathrm{d}x \mathrm{d}t < \infty \tag{3.1.9}$$

where $\bar{\nu}$ is the Levy measure defined by

$$\bar{\nu}(\mathrm{d}x) = \frac{\nu((|x|,1])}{|x|} 1_{\{|x| \le 1\}} \mathrm{d}x.$$

Then the process $(F(X_t,t), 0 \le t \le 1)$ admits the following decomposition

$$F(X_t,t) = F(0,0) + M_t + V_t + Q_t, \tag{3.1.10}$$

where M is a local martingale, V a bounded variation process and Q a continuous process with 0-quadratic energy, respectively defined by

$$\begin{aligned} M_t &= \int_0^t \int_{\{|y| \le 1\}} \{F(X_{s-}+y,s) - F(X_{s-},s)\} \tilde{\mu}_X(\mathrm{d}y,\mathrm{d}s) \\ V_t &= \sum_{0<s\le t} \{F(X_s,s) - F(X_{s-},s)\} 1_{\{|\Delta X_s| > 1\}} \end{aligned}$$

$$Q_t = -\int_0^t \int_{\mathbb{R}} \mathcal{A}\mathcal{I}F(x,s) dL_s^x,$$

with $\mathcal{A}$ the operator defined by (3.1.2).

We mention that similarly to [11], Theorems 3.1.1 and 3.1.2 both admit multi-dimensional extensions to processes $(X^1, X^2, ..., X^d)$ such that the X^i's are independent Lévy processes each component X^i being either symmetric without Brownian component, either with a nontrivial Brownian component.

As an application of the construction of integration with respect to local time, we introduce, in Section 3.5, local times on curves for the process X. This definition is then used to establish an Itô formula for space-time functions $C^{2,1}$ everywhere except on a set $\{(x,t) \in \mathbb{R} \times [0,1] : x = b(t)\}$ where $(b(t))_{0 \le t \le 1}$ is a continuous curve.

3.2 Tanaka's Formula

It is well known that $\hat{X}$ is a semimartingale (see [26], Proposition (1.3)). It has no continuous local martingale component, hence its semimartingale local time is identically equal to zero. It is not a Markov process with respect to the definition used in Blumenthal and Getoor [4] (See page 20 in [4]), but one can associate a local time process to $\hat{X}$ by setting

$$\hat{L}_t^x = L_1^x - L_{1-t}^x. \tag{3.2.1}$$

Indeed, we have the occupation time formula

$$\int_0^t f(\hat{X}_s)ds = \int_{\mathbb{R}} f(x)\hat{L}_t^x dx.$$

We use the following notation. The filtration, satisfying the usual conditions, generated by X is denoted by $\mathcal{F} = \{\mathcal{F}_t; 0 \leq t \leq 1\}$. Similarly the filtration generated by $\hat{X}$ is denoted by $\hat{\mathcal{F}} = \{\hat{\mathcal{F}}_t; 0 \leq t \leq 1\}$. Let $\mu_{\hat{X}}$ be the Poisson random measure associated to the jumps of $\hat{X}$, then $\rho = (\rho(\omega, \mathrm{d}y, \mathrm{d}t), \omega \in \Omega)$ is the $\hat{\mathcal{F}}$-compensator of $\mu_{\hat{X}}$, i.e., a predictable measure with respect to $\hat{\mathcal{F}}$ such that

$$\mathbf{E}\left[\int_{[0,t)\times\mathbb{R}} W(w,s,y)\mu_{\hat{X}}(w,\mathrm{d}y,\mathrm{d}s)\right] = \mathbf{E}\left[\int_{[0,t)\times\mathbb{R}} W(w,s,y)\rho(w,\mathrm{d}y,\mathrm{d}s)\right],$$

for every nonnegative $\hat{\mathcal{P}}\otimes\mathcal{B}(\mathbb{R})$-measurable function W, where $\hat{\mathcal{P}}$ is the predictable σ-field of $\hat{X}$, the σ-field generated by all càg $\hat{\mathcal{F}}$-adapted process. (See [27] chapter II).

Without possible confusion, we denote the measure $\nu(\mathrm{d}x)\mathrm{d}s$ on $\mathbb{R} \times [0,1]$ by $\nu(\mathrm{d}x, \mathrm{d}s)$. Here is a preliminary lemma. We denote by $\phi(t, .)$ the continuous density function of X_t with respect to the Lebesgue measure :

$$\phi(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t\psi(\xi)}\cos(x\xi)\mathrm{d}\xi.$$

Lemma 3.2.1. *The $\hat{\mathcal{F}}$-compensator of $\mu_{\hat{X}}$ is given by :*

$$\rho(\omega,\mathrm{d}y,\mathrm{d}t) = \frac{\phi(1-t, X_{1-t}+y)}{\phi(1-t, X_{1-t})}\nu(\mathrm{d}y,\mathrm{d}t).$$

As a symmetric Lévy process, X benefits from the following Tanaka's formula established by Salminen and Yor [42]

$$L_t^a = v(X_t - a) - v(a) - \int_0^t \int_{\mathbb{R}} [v(X_{s-} - a + y) - v(X_{s-} - a)]\tilde{\mu}(\mathrm{d}y,\mathrm{d}s), \tag{3.2.2}$$

where $v(x) = \frac{1}{\pi}\int_0^{\infty} \frac{1-\cos(\xi x)}{\psi(\xi)}\mathrm{d}\xi$. Unfortunately, this identity is not convenient for our purpose. The following proposition presents an alternative formulation of

(3.2.2) and an analogue Tanaka's formula for $\hat{X}$. The characteristic exponent of the Lévy Process, $(X_t - \sum_{s\le t} \Delta X_s 1_{\{|\Delta X_s|>1\}})_{t\ge 0}$ is denoted by ψ_*. Note that

$$\psi_*(\xi) = 2\int_0^1 (1-\cos(x\xi))\nu(\mathrm{d}x), \tag{3.2.3}$$

and that ψ_* satisfies also the condition (H1). Besides, we set

$$w(x) = \frac{1}{\pi}\int_0^\infty \frac{1-\cos(x\xi)}{\psi_*(\xi)}\mathrm{d}\xi.$$

Proposition 3.2.2. *For any real number a, we have :*
(i)

$$L_t^a = w(X_t - a) - w(a) - N_t^a - \sum_{s\le t}(w(X_s - a) - w(X_{s-} - a))1_{\{|\Delta X_s|>1\}},$$

where N^a is the local $\mathcal{F}$-martingale defined by

$$N_t^a = \int_0^t \int_{\{|y|\le 1\}} [w(X_{s-} - a + y) - w(X_{s-} - a)]\tilde{\mu}(\mathrm{d}y, \mathrm{d}s).$$

(ii)

$$\begin{aligned}\hat{L}_t^a &= w(X_{1-t} - a) - w(X_1 - a) - \hat{N}_{t-}^a - \hat{W}_t^a \\ &\quad + \sum_{1-t<s\le 1}(w(X_s - a) - w(X_{s-} - a))1_{\{|\Delta X_s|>1\}},\end{aligned}$$

where $\hat{N}^a$ is a local $\hat{\mathcal{F}}$-martingale and $\hat{W}^a$ is a bounded variation process, respectively defined by

$$\begin{aligned}\hat{N}_t^a &= \int_0^t \int_{\{|y|\le 1\}} [w(X_{1-s} - a + y) - w(X_{1-s} - a)](\mu_{\hat{X}} - \rho)(\mathrm{d}y, \mathrm{d}s) \\ \hat{W}_t^a &= \int_0^t \int_{\{|y|\le 1\}} [w(X_{1-s} - a + y) - w(X_{1-s} - a)](\rho - \nu)(\mathrm{d}y, \mathrm{d}s).\end{aligned}$$

The proof of Proposition 3.2.2, inspired from Yamada's work [47], relies on Lemma 3.2.1 and the following technical lemma. We denote by B the operator defined by

$$Bf(x) = \int_{\{|y|\le 1\}} [f(x+y) - f(x) - f'(x)y]\nu(\mathrm{d}y),$$

for any function such that the integral is well defined.

Lemma 3.2.3. *Let g be an infinitely differentiable function with compact support and set*

$$G(x) = \int_{-\infty}^{\infty} g(z)w(x-z)\mathrm{d}z.$$

Then we have :

$$BG(x) = g(x).$$

We need the following notation. For f in $L^1(\mathbb{R}\times[0,1])$, $\hat{f}$ denotes its x-variable Fourier transform , i.e :

$$\hat{f}(\xi,t) = \int_{-\infty}^{\infty} e^{ix\xi} f(x,t)\mathrm{d}x. \tag{3.2.4}$$

Remark 3.2.4. *If f belongs to $L^1(\mathbb{R}\times[0,1])\cap L^2(\mathbb{R}\times[0,1])$, thanks to Plancherel's Theorem we have $\int_0^1\int_{-\infty}^{\infty}(f(x,t))^2\mathrm{d}x\mathrm{d}t = \frac{1}{2\pi}\int_0^1\int_{-\infty}^{\infty}|\hat{f}(\xi,t)|^2\mathrm{d}\xi\mathrm{d}t$. One can hence extend the above transform from $L^1(\mathbb{R}\times[0,1])$ to $L^2(\mathbb{R}\times[0,1])$.*

We now successively establish Lemma 3.2.1, Lemma 3.2.3 and finally Proposition 3.2.2.

Proof of Lemma 3.2.1 : Let $\tilde{X}$ be the process defined by $\tilde{X}_t = \hat{X}_t - X_1, 0 \le t \le 1$. From the symmetry of X, $\tilde{X}$ is a Lévy process with same law as X. Obviously : $\hat{X} = \tilde{X} - \tilde{X}_1$. Let $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_t; 0 \le t \le 1\}$ be the filtration satisfying the usual conditions generated by $\tilde{X}$, then $\hat{\mathcal{F}}$ is the filtration obtained from $\tilde{\mathcal{F}}$ by an initial enlargement with the variable $\tilde{X}_1$, i.e. :

$$\hat{\mathcal{F}}_t = \bigcap_{s>t}(\tilde{\mathcal{F}}_s \vee \sigma(\tilde{X}_1)).$$

For (x,t,ω) in $[0,1)\times\mathbb{R}\times\Omega$, set $q_t^x(\omega) = \phi(1-t, x-\tilde{X}_t(\omega))$, then $q_t^x(\omega)\mathrm{d}x$ is a regular version of the conditional law of $\tilde{X}_1$ with respect to $\tilde{\mathcal{F}}_t$. According [43] we know that for every $t>0$ the set of zeros of $\phi(t,.)$ is either empty or a half line. Since the second possibility is not possible because X is symmetric, we have $q_t^x > 0$ for every (x,t) in $\mathbb{R}\times[0,1)$. We establish now the following identity :

$$q_t^x = q_0^x + \int_0^t\int_{\mathbb{R}} q_{s-}^x U^x(s,y)(\mu_{\hat{X}} - \nu)(\mathrm{d}y,\mathrm{d}s), \tag{3.2.5}$$

where U is defined by :

$$U^x(s,y) = \frac{\phi(1-s, x-\tilde{X}_{s-}-y)}{\phi(1-s, x-\tilde{X}_{s-})} - 1.$$

From (H1) and the inequality : $e^{-tx}x \leq 4\frac{e^{t-2}}{t^2(1+x)}$, $x \geq 0$, we deduce that $\frac{\partial\phi}{\partial t}$ exists and is continuous on $(0,1]\times\mathbb{R}$. For g element of $C_c^\infty(\mathbb{R})$ (the set of infinitely differentiable functions with compact support) such that $\int g(x)\mathrm{d}x = 1$, the function ϕ_n defined by : $\phi_n(t,x) = \int_{-\infty}^{\infty}\phi(t, x+z/n)g(z)\mathrm{d}z$, belongs to $C^{1,\infty}((0,1]\times\mathbb{R})$.
Set $\Phi_n(s,x) = \int_{\mathbb{R}}\left(\phi_n(s,x+y) - \phi_n(s,x) - \frac{\partial\phi_n}{\partial x}(x)y1_{\{|y|\leq 1\}}\right)\nu(\mathrm{d}y)$. Using Itô formula (3.1.1) we have for every t in $[0,1)$:

$$\begin{aligned}
& \phi_n(1-t, x-\tilde{X}_t) - \phi_n(1,x) \\
= & -\int_0^t \frac{\partial\phi_n}{\partial t}(1-s, x-\tilde{X}_s)\mathrm{d}s + \int_0^t \Phi_n(1-s, x-\tilde{X}_s)\mathrm{d}s \\
& + \int_0^t\int_{\mathbb{R}} \phi_n(1-s, x-\tilde{X}_{s-}-y) - \phi_n(1-s, x-\tilde{X}_{s-}))(\mu_{\hat{X}}-\nu)(\mathrm{d}y,\mathrm{d}s) \\
= & \int_0^t\int_{\mathbb{R}} (\phi_n(1-s, x-\tilde{X}_{s-}-y) - \phi_n(1-s, x-\tilde{X}_{s-}))(\mu_{\hat{X}}-\nu)(\mathrm{d}y,\mathrm{d}s),
\end{aligned} \tag{3.2.6}$$

since $\phi_n(1-t, x-\tilde{X}_t) = \mathbf{E}[ng(n(X_1-x))|\mathcal{F}_t]$ is a martingale.
Thanks to the continuity of ϕ, we have

$$\phi_n(1-t, x-\tilde{X}_t) - \phi_n(1,x) \underset{n\to\infty}{\longrightarrow} \phi(1-t, x-\tilde{X}_t) - \phi(1,x).$$

We show now that the martingale in (3.2.6) converges in L^2 when n tends to ∞ to the martingale :

$$\int_0^t\int_{\mathbb{R}} (\phi(1-s, x-\tilde{X}_{s-}-y) - \phi(1-s, x-\tilde{X}_{s-}))(\mu_{\hat{X}}-\nu)(\mathrm{d}y,\mathrm{d}s), \quad 0\leq t<1.$$

Define α on $(0,1]$ by $\alpha(t) = \sup_x \phi(t,x)$, i.e. :

$$\alpha(t) = \frac{1}{\pi}\int_0^\infty e^{-t\psi(\xi)}\mathrm{d}\xi. \tag{3.2.7}$$

We have

$$\begin{aligned}
& \mathbf{E}\Big[\int_0^1\int_{\mathbb{R}}\Big(\phi_n(1-s, x-\tilde{X}_s+y) - \phi_n(1-s, x-\tilde{X}_s) \\
& \qquad -\phi(1-s, x-\tilde{X}_s+y) + \phi(1-s, x-\tilde{X}_s)\Big)^2\nu(\mathrm{d}y)\mathrm{d}s\Big] \\
\leq & \int_0^1\alpha(s)\int_{\mathbb{R}^2}(\phi_n(1-s, z+y) \\
& \quad -\phi_n(1-s,z) - \phi(1-s, z+y) + \phi(1-s,z))^2\mathrm{d}z\nu(\mathrm{d}y)\mathrm{d}s.
\end{aligned}$$

Thanks to Plancherel's Theorem, this last term is equal to

$$2\pi\int_0^1 \alpha(s)\int_{\mathbb{R}^2} |(e^{-iy\xi}-1)(\hat{\phi}_n(1-s,\xi)-\hat{\phi}(1-s,\xi))|^2 \mathrm{d}\xi\nu(\mathrm{d}y)\mathrm{d}s,$$

and hence to

$$4\pi\int_0^1 \alpha(s)\int_{\mathbb{R}} \psi(\xi)e^{-2(1-s)\psi(\xi)}(\hat{g}(-\xi/n)-1)^2\mathrm{d}\xi\mathrm{d}s,$$

which converges to 0 by dominated convergence, since :

$$\int_0^1 \alpha(s)\mathrm{d}s = \int_{\mathbb{R}}(\psi(\xi))^{-1}(1-e^{-\psi(\xi)})\mathrm{d}\xi < \infty,$$

Hence we finally obtain (3.2.5). This allows, thanks to the result of Jacod (Theorem 4.1 in [25]), to claim that $(1+U^{\tilde{X}_1}(t,y))\mathrm{d}t\nu(\mathrm{d}y)$ is the $\hat{\mathcal{F}}$-compensator of $\mu_{\hat{X}}$.
□

Proof of Lemma 3.2.3 For $p>0$, set $u^{(p)}(x)=\pi^{-1}\int_0^\infty \cos(x\xi)(p+\psi_*(\xi))^{-1}\mathrm{d}\xi$ and $G_p(x)=\int_{\mathbb{R}} g(z)(u^{(p)}(0)-u^{(p)}(x-z))\mathrm{d}z$. For $p>0$, $\{u^{(p)}(z-x),(z,x)\in\mathbb{R}^2\}$ is a continuous version of the kernel of the p-potential of the Lévy process Z defined by $Z_t = X_t-\sum_{s\leq t}\Delta X_s 1_{\{|\Delta X_s|>1\}}$ i.e.,

$$U^{(p)}f(z)=\mathbf{E}_z\left(\int_0^\infty e^{-pt}f(Z_t)\mathrm{d}t\right)=\int_{\mathbb{R}} u^{(p)}(z-x)f(x)dx,$$

where $\mathbf{E}_z$ represent the mean with respect to the law of the process $(Z_t+z)_{t\in[0,1]}$. Hence we have :

$$G_p(x)=u^{(p)}(0)\int_{\mathbb{R}} g(z)\mathrm{d}z - U^p g(x).$$

On the Schwartz space of infinitely differentiable and rapidly decreasing functions, the operator pI_d-B (where I_d is the identity operator) is a one to one operator and its inverse is U^p (See Bertoin [3], p. 23), hence $BG_p(x)=-pU^pg(x)+g(x)$. From (26) of Salminen and Yor [42] we have $\lim_{p\to 0} pu^{(p)}(0)=0$, hence $\lim_{p\to 0} pU^pg(x)=0$ and we obtain :

$$\lim_{p\to 0} BG_p(x)=g(x),$$

But by dominated convergence, we also have $\lim_{p\to 0} BG_p(x)=BG(x)$. Indeed for any real x

$$\begin{aligned}|G_p(x+y)-G_p(x)-G_p'(x)y| &\leq y^2\sup_{z\in[x-1,x+1]}|G_p''(z)| \\ &\leq y^2\sup_{z\in[x-1,x+1]}\int |g''(z-\lambda)|w(\lambda)\mathrm{d}\lambda \quad (3.2.8)\end{aligned}$$

□

Proof of Proposition 3.2.2 : We only establish **(ii)**. One establishes **(i)** with similar arguments. First we will show that the processes $\hat{N}^a$ and $\hat{W}^a$ are well defined. We set :

$$w_1(x) = \pi^{-1}\int_0^1 \frac{1}{\psi_*(\xi)}(1-\cos(x\xi))\mathrm{d}\xi \text{ and } w_0(x) = \pi^{-1}\int_1^\infty \frac{1}{\psi_*(\xi)}(1-\cos(x\xi))\mathrm{d}\xi.$$

On one hand w_1 is an infinitely differentiable function, hence for any $n \in \mathbb{N}$,

$$\int_{-n}^{n}\int_{\{|y|\le 1\}}(w_1(x+y)-w_1(x))^2\nu(\mathrm{d}y)\mathrm{d}x \le 2n \sup_{z\in[n-1,n+1]}|w_1'(z)|^2\int_{\{|y|\le 1\}}y^2\nu(\mathrm{d}y) < \infty. \tag{3.2.9}$$

On the other hand, thanks to Plancherel's Theorem we have :

$$\int_{\mathbb{R}}\int_{\{|y|\le 1\}}(w_0(x+y)-w_0(x))^2\nu(\mathrm{d}y)\mathrm{d}x = \pi^{-1}\int_1^\infty \frac{1}{\psi_*(\xi)}\mathrm{d}\xi < \infty. \tag{3.2.10}$$

From (3.2.10) and (3.2.9), we obtain for every $n \in \mathbb{N}$,

$$\int_{-n}^{n}\int_{\{|y|\le 1\}}(w(x+y)-w(x))^2\nu(\mathrm{d}y)\mathrm{d}x < \infty. \tag{3.2.11}$$

For n in $\mathbb{N}$, set $\hat{T}_n = \inf\{s \ge 0 : |\hat{X}_s| > n\} \wedge t$. We have for any $n > |a|$:

$$\begin{aligned}
& \mathbf{E}\int_0^{\hat{T}_n}\int_{\{|y|\le 1\}}(w(\hat{X}_s-a+y)-w(\hat{X}_s-a))^2\rho(\mathrm{d}y,\mathrm{d}s) \\
\le\ & \mathbf{E}\int_0^1\int_{\{|y|\le 1\}}(w(X_s-a+y)-w(X_s-a))^2 1_{\{|X_s|\le n\}}\frac{\phi(s,X_s+y)}{\phi(s,X_s)}\nu(\mathrm{d}y)\mathrm{d}s \\
=\ & \int_{-n}^{n}\int_0^1\int_{\{|y|\le 1\}}(w(x-a+y)-w(x-a))^2\phi(s,x+y)\nu(\mathrm{d}y)\mathrm{d}s\mathrm{d}x \\
\le\ & \int_0^1\alpha(s)\mathrm{d}s\int_{-2n}^{2n}\int_{\{|y|\le 1\}}(w(x+y)-w(x))^2\nu(\mathrm{d}y)\mathrm{d}x,
\end{aligned} \tag{3.2.12}$$

which is finite thanks to (3.2.11) (α is defined in (3.2.7)). It follows that the process $\{\hat{N}^a_{s\wedge\hat{T}_n}; 0 \le s \le 1\}$ is a $\hat{\mathcal{F}}$-martingale, consequently $\hat{N}^a$ is a well-defined local $\hat{\mathcal{F}}$-martingale.
Now, for every real x, set

$$\hat{N}_t^x(\epsilon) = \int_0^t\int_{\{\epsilon<|y|\le 1\}}(w(X_{1-s}-x+y)-w(X_{1-s}-x))(\mu_{\hat{X}}-\rho)(\mathrm{d}y,\mathrm{d}s).$$

Similarly to (3.2.12), we have that for $|x| \leq n$,

$$\mathbf{E}[|\hat{N}^x_{\hat{T}_n} - \hat{N}^x_{\hat{T}_n}(\epsilon)|^2] \leq cste \int_{-2n}^{2n} \int_{\{|y|<\epsilon\}} (w(z+y) - w(z))^2 \nu(\mathrm{d}y)\mathrm{d}z,$$

which converges to 0 as ϵ tends to zero. Hence, if g belongs to $\mathcal{C}^\infty_c(\mathbb{R})$, $\int_{\mathbb{R}} g(x)\hat{N}^x_t(\epsilon)\mathrm{d}x$ converges in probability to $\int_{\mathbb{R}} g(x)\hat{N}^x_t \mathrm{d}x$. For any $\epsilon > 0$,

$$\int_{\mathbb{R}} g(x)\hat{N}^x_t(\epsilon)\mathrm{d}x = \int_{\{\epsilon<|y|\leq 1\}} [G(X_{1-s} + y) - G(X_{1-s})](\mu_{\hat{X}} - \rho)(\mathrm{d}y, \mathrm{d}s),$$

(where G is defined in Lemma 3.2.3 by $G(x) = \int_{\mathbb{R}} g(z)w(x-z)\mathrm{d}z$). But the right-hand side converges in probability to

$$\int_{\{|y|\leq 1\}} [G(X_{1-s} + y) - G(X_{1-s})](\mu_{\hat{X}} - \rho)(\mathrm{d}y, \mathrm{d}s).$$

This leads to **P**-a.s,

$$\int_{\mathbb{R}} g(x)\hat{N}^x_t \mathrm{d}x = \int_0^t \int_{\{|y|\leq 1\}} [G(\hat{X}_{s-} + y) - G(\hat{X}_{s-})](\mu_{\hat{X}} - \rho)(\mathrm{d}y, \mathrm{d}s), \quad (3.2.13)$$

As (3.2.12) has been show, we obtain the following inequality :

$$\begin{aligned} &\mathbf{E}\left[\int_0^{\hat{T}_n}\int_{\{|y|\leq 1\}} |w(\hat{X}_s - a + y) - w(\hat{X}_s - a)| \frac{|\phi(1-s, \hat{X}_s + y) - \phi(1-s, \hat{X}_s)|}{\phi(1-s, \hat{X}_s)} \nu(\mathrm{d}y)\mathrm{d}s\right] \\ \leq\ & (2\pi)^{-1/2} \int_0^1 \beta(s)\mathrm{d}s \left\{\int_{-2n}^{2n} \int_{\{|y|\leq 1\}} (w(x+y) - w(x))^2 \nu(\mathrm{d}y)\mathrm{d}x\right\}^{1/2}, \end{aligned}$$

which is finite thanks to condition **(H2)** and (3.2.11), then $\hat{W}^a$ is a continuous bounded variation process and by Fubini's Theorem we have :

$$\int_{\mathbb{R}} g(x)\hat{W}^x_t \mathrm{d}x = \int_0^t \int_{\{|y|\leq 1\}} (G(\hat{X}_{s-} + y) - G(\hat{X}_{s-}))(\rho - \nu)(\mathrm{d}y, \mathrm{d}s). \quad (3.2.14)$$

For any real x and any t in $[0,1]$, we define Λ^x_t by

$$\Lambda^x_t = w(X_{1-t} - x) - w(X_1 - x) - \hat{N}^x_{t-} - \hat{W}^x_t + \sum_{1-t\leq s\leq 1} (w(X_s - x) - w(X_{s-} - x))1_{\{|\Delta X_s|>1\}}.$$

We prove now that Λ satisfies the time occupation time formula for $\hat{X}$, i.e :

$$\int_0^1 g(\hat{X}_s)\mathrm{d}s \;=\; \int_{\mathbb{R}} g(x)\Lambda_t^x \mathrm{d}x. \tag{3.2.15}$$

Λ^x is a continuous function in t, thus we can rewrite it as follows

$$\Lambda_t^x = w(\hat{X}_t - x) - w(X_1 - x) - \hat{N}_t^x - \hat{W}_t^x - \sum_{s\le t}(w(\hat{X}_s - x) - w(\hat{X}_{s-} - x))1_{\{|\Delta \hat{X}_s|>1\}}.$$

Thanks to (3.2.13) and (3.2.14) we have

$$\begin{aligned}
&\int_{\mathbb{R}} g(x)\Lambda_t^x \mathrm{d}x \\
= \;& G(\hat{X}_t) - G(X_1) - \int_0^t \int_{\{|y|\le 1\}} (G(\hat{X}_{s-}+y) - G(\hat{X}_{s-}))(\mu_{\hat{X}} - \rho)(\mathrm{d}y, \mathrm{d}s) \\
& - \int_0^t \int_{\{|y|\le 1\}} (G(\hat{X}_{s-}+y) - G(\hat{X}_{s-}))(\rho - \nu)(\mathrm{d}y, \mathrm{d}s) \\
& - \sum_{s\le t}(G(\hat{X}_s) - G(\hat{X}_{s-}))1_{\{|\Delta \hat{X}_s|>1\}}.
\end{aligned}$$

By Itô formula, we know that the right-hand side of the above equality is equal to $\int_0^t \mathcal{B}G(\hat{X}_s)\mathrm{d}s$, where the operator $\mathcal{B}$ is defined in Lemma 3.2.3. Then Lemma 3.2.3 gives (3.2.15).

Consequently, we obtain $\int_{\mathbb{R}} g(x)\Lambda_t^x\mathrm{d}x = \int_{\mathbb{R}} g(x)\hat{L}_t^x\mathrm{d}x$ $\mathbf{P}$-a.s. A priori, the set of probability 1, on which the previous identity holds, depends of the function g. But we can suppress this dependency since the set of continuous function with compact support with the metric of convergence uniform is a separable topological space. We obtain $\mathbf{P}$-a.s. :

$$\int_{\mathbb{R}} g(x)\Lambda_t^x \mathrm{d}x = \int_{\mathbb{R}} g(x)\hat{L}_t^x \mathrm{d}x, \;\forall g \in \mathcal{C}_c^\infty(\mathbb{R}).$$

Hence we have for any x outside of a set of Lebesgue measure zero : $\Lambda_t^x = \hat{L}_t^x$ $\mathbf{P}$-a.s. In order to guarantee that this holds for any given a it is sufficient to show that

$$\lim_{x\to a} \Lambda_t^x = \Lambda_t^a \text{ in probability and} \tag{3.2.16}$$

$$\lim_{x\to a} \hat{L}_t^x = \hat{L}_t^a \text{ in probability.} \tag{3.2.17}$$

For n such that $|a| < n$, similarly to (3.2.12), we have for any x :

$$\begin{aligned}
&\mathbf{E}[|\hat{N}_{\hat{T}_n}^x - \hat{N}_{\hat{T}_n}^a|^2] \le \\
& cste \int_{\{|y|\le 1\}} \int_{2n}^{2n} (w(z+y) - w(z) - w(z+y+a-x) + w(z+a-x))^2 \mathrm{d}z\nu(\mathrm{d}y).
\end{aligned}$$

When $|x-a|<1$, the above integral is smaller than $4\int_{-2n-1}^{2n+1}(w(z+y)-w(z))^2 \mathrm{d}z$. Hence by dominated convergence, thanks to the continuity of w, we conclude that $\mathbf{E}[|\hat{N}^x_{\hat{T}_n} - \hat{N}^a_{\hat{T}_n}|^2]$ converges to 0 as x tends to a, thus $(\hat{N}^x_t - \hat{N}^a_t)$ converges to 0 in probability as x tends to a. Similarly, $(\hat{W}^x_t - \hat{W}^a_t)$ also converges in probability to 0 as x tends to a. These convergences and the continuity of w lead to (3.2.16).
Defining the martingale $(M^x_t)_{0\le t\le 1}$ by

$$M^x_t = \int_0^t \int_{\{|y|\le 1\}} (v(X_{s-}-x+y) - v(X_{s-}-x))\tilde{\mu}(\mathrm{d}y, \mathrm{d}s),$$

we obtain as above that $M^x_t - M^a_t$ converge to 0 in probability as x tends to a. According to Corollary 14 in Bertoin [3] p.147, one defines a distance d on $\mathbb{R}$ by setting :
$d(x,y) = v(x-y)\ \forall x,y\in\mathbb{R}$. Consequently : $|v(x)-v(y)| \le v(x-y)$ for all $x,y\in\mathbb{R}$. Defining for any real x, O^x_t by :

$$O^x_t = \int_0^t \int_{\{|y|>1\}} (v(X_s-x+y) - v(X_s-x))\nu(\mathrm{d}y)\mathrm{d}s,$$

we obtain that : $|O^x_t - O^a_t| \le 2\nu(\{|y|>1\})v(x-a)$, which converges to zero as x tends to a. Now, note that Salminen and Yor's formula for L (3.2.2) can be written as follows

$$L^a_t = v(X_t-a) - v(a) - M^a_t - \sum_{s\le t}(v(X_s-a) - v(X_{s-}-a))1_{\{|\Delta X_s|>1\}} - O^a_t,$$

which shows that L^x_t converges in probability to L^a_t as x tends to a, and (3.2.17) follows. □

3.3 Integration with respect to local time

We start by defining the stochastic integration of elementary functions from $\mathbb{R}\times[0,1]$ to $\mathbb{R}$ with respect to $(L^x_t, x\in\mathbb{R}, 0\le t\le 1)$. Let f_Δ be an elementary function i.e. there exists a finite sequence $(x_i)_{1\le i\le n}$ of real numbers, a subdivision of $[0,1]$ $(s_j)_{1\le j\le m}$ and a family of real numbers $\{f_{ij} : 1\le i\le n, 1\le j\le m\}$ such that

$$f_\Delta(x,s) = \sum_{1\le i\le n, 1\le j\le m} f_{ij} 1_{(x_i,x_{i+1}]} 1_{(s_j,s_{j+1}]}.$$

For such a function integration with respect to L is defined by

$$\int_0^t \int_{\mathbb{R}} f_\Delta(x,s)\mathrm{d}L^x_s = \sum_{1\le i\le n, 1\le j\le m} f_{ij}(L^{x_{i+1}}_{s_{j+1}\wedge t} - L^{x_i}_{s_{j+1}\wedge t} - L^{x_{i+1}}_{s_j\wedge t} + L^{x_i}_{s_j\wedge t}). \quad (3.3.1)$$

The problem is to find the space of deterministic functions to which this integration could be extended. The answer is given by the following theorem. To introduce it, we define the norm $\| \|$ by

$$\|f\|^2 = \int_0^1 \beta(t) \int_{\mathbb{R}} \left(1 + \frac{\xi^2}{\psi_*(\xi)}\right) |\hat{f}(\xi, t)|^2 \mathrm{d}\xi \mathrm{d}t.$$

where β, ψ_* and $\hat{f}$ are respectively defined by (3.1.4), (3.2.3) and Remark 3.2.4. We set

$$\Upsilon = \{f \in L^2(\mathbb{R} \times [0,1]) : \|f\| < \infty\}. \tag{3.3.2}$$

Theorem 3.3.1. *Integration with respect to L can be extended from the elementary functions to Υ. This extension satisfies :*
(i) *There exist a constant κ such that for every element f of Υ*

$$\mathbf{E}\left(\sup_{0\le t\le 1}\left|\int_0^t \int_{\mathbb{R}} f(x,s)\mathrm{d}L_s^x\right|\right) \le \kappa \|f\|.$$

(ii) *For $f \in \Upsilon$, the process $(\int_0^t \int_{\mathbb{R}} f(x,s)\mathrm{d}L_s^x, 0 \le t \le 1)$ has 0-quadratic variation.*

After proving Theorem 3.3.1, we will show that integration with respect to local time can be extended from Υ to Υ_{loc} the set of measurable functions $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $\forall k > 0$ there exists a function $f_k \in \Upsilon$ which satisfies : $f(x,s) = f_k(x,s)$ if $|x| \le k$.

The proof of Theorem 3.3.1 is based on the two analogues of Tanaka's formula established in Section 2 and on the following lemma. For a complex valued function h on $\mathbb{R} \times [0,1]$, element of $L^2(\mathbb{R} \times [0,1])$, we set

$$\check{h}(x,s) = \frac{1}{2\pi}\hat{h}(-x,s),$$

i.e., $\check{h}$ is the inverse Fourier transform in the variable x of h.

Lemma 3.3.2. **(i)** *Let f be an element of Υ. For every y in $\mathbb{R}$, set*

$$g_y(\xi,s) = \hat{f}(\xi,s)\frac{\xi}{\psi_*(\xi)}[\sin(y\xi) + i(\cos(y\xi) - 1)],$$

then g_y belongs to $L^2(\mathbb{R} \times [0,1])$.
(ii) *Let f be an element of Υ. Set $\varphi(x,y,s) = \check{g}_y(x,s)$, then the three following processes $H(f)$, $I(f)$ and $K(f)$, are well defined on $[0,1]$*

$$\begin{aligned}
H_t(f) &= \frac{1}{2}\int_0^t \int_{\{|y|\le 1\}} \varphi(X_{s-}, y, s)\tilde{\mu}_X(\mathrm{d}y, \mathrm{d}s), \\
I_t(f) &= \frac{1}{2}\int_{[1-t,1]} \int_{\{|y|\le 1\}} \varphi(X_{1-s}, y, 1-s)(\mu_{\hat{X}} - \rho)(\mathrm{d}y, \mathrm{d}s) \text{ and} \\
K_t(f) &= \frac{1}{2}\int_0^t \int_{\{|y|\le 1\}} \varphi(X_s, y, s)\left(\frac{\phi(s, X_s + y) - \phi(s, X_s)}{\phi(s, X_s)}\right)\nu(\mathrm{d}y)\mathrm{d}s.
\end{aligned}$$

(iii) *For f in Υ, set $J(f) = (H_t(f)+I_t(f)+K_t(f))_{0\leq t\leq 1}$. There exists a constant κ such that for every f in Υ*

$$\mathbf{E}(\sup_{0\leq t\leq 1}|J_t(f)|) \leq \kappa\|f\|.$$

We immediately use Lemma 3.3.2 to prove Theorem 3.3.1. Lemma 3.3.2 is established after.

Proof of Theorem 3.3.1 For a, b real numbers such that $b < a$, the function φ, defined in Lemma 3.3.2, corresponding to $f = 1_{(b,a]}$, is given by

$$\varphi(x,y,s) = w(x-b+y) - w(x-b) - w(x-a+y) + w(x-a). \qquad (3.3.3)$$

Indeed, let $h_y(x)$ be the right-hand side of the above equation. For $p > 0$, set

$$u^{(p)}(x) = \pi^{-1}\int_0^\infty \frac{\cos(x\xi)}{p+\psi_*(\xi)}\mathrm{d}\xi. \qquad (3.3.4)$$

We know that (See Lemma 1 of Salminen and Yor [42]) $u^{(p)}(0)-u^{(p)}(x)$ converges to $w(x)$ as p tends to 0. Define : $h_{y,p}(x) = -u^{(p)}(x-b+y)+u^{(p)}(x-b)+u^{(p)}(x-a+y)-u^{(p)}(x-a)$ and note that $(p+\psi_*(\xi))^{-1}$ is the Fourier's transform of $u^{(p)}$, hence

$$\hat{h}_{y,p}(\xi) = \frac{e^{ia\xi}-e^{ib\xi}}{i\xi}\frac{\xi}{p+\psi_*(\xi)}[\sin(y\xi)+i(\cos(y\xi)-1)],$$

which converges pointwise to $g_y(\xi, s)$ as p tends to 0.
For every $p > 0$, $|\hat{h}_{y,p}|^2 \leq 2|e^{ia\xi}-e^{ib\xi}|^2(\psi_*(\xi))^{-2}[1-\cos(y\xi)]$. Thus by dominated convergence, $\hat{h}_{y,p}$ converges in L^2 to g_y. It follows, thanks to Plancherel's Theorem, that $h_{y,p}$ converges in L^2 to $\varphi(.,y,.)$. But obviously $h_{y,p}$ converges pointwise to h_y as p tends to 0. Consequently $\varphi(x,y,s) = h_y(x)$.
From the definition of $\hat{L}$ (3.2.1) we have for any real x

$$L_t^x = \hat{L}_1^x - \hat{L}_{1-t}^x$$

and hence, thanks to Proposition 3.2.2.(ii) :

$$\begin{aligned} L_t^x = w(x) - w(X_t - x) \ &- \ (\hat{N}_1^x - \hat{N}_{(1-t)-}^x) - (\hat{W}_1^x - \hat{W}_{1-t}^x), \\ &+ \ \sum_{s\leq t}(w(X_s - x) - w(X_{s-} - x))1_{\{|\Delta X_s|>1\}}, \end{aligned}$$

which, comparing with Proposition 3.2.2.(i) leads to :

$$2L_t^x = -N_t^x - (\hat{N}_1^x - \hat{N}_{(1-t)-}^x) - (\hat{W}_1^x - \hat{W}_{1-t}^x). \tag{3.3.5}$$

We have obtained with the notation of Lemma 3.3.2 : $L_t^a - L_t^b = J_t(f)$. By linear combination, thanks to the definition of integration with respect to L (3.3.1), this identity immediately extends to elementary functions :

$$\int_0^t \int_{\mathbb{R}} f_\Delta(x,s)\mathrm{d}L_s^x = J_t(f_\Delta), \tag{3.3.6}$$

which leads to :

$$\mathbf{E}\left(\sup_{t\in[0,1]}\left|\int_0^t \int_{\mathbb{R}} f_\Delta(x,s)\mathrm{d}L_s^x\right|\right) \leq \kappa \|f_\Delta\|,$$

where κ is the constant introduced in Lemma 3.3.2.(iii). Thus, in order to extend the integration with respect to L to the normed space $(\Upsilon, \| \ \|)$ we must show that the set of elementary functions is dense in Υ for the topology generated by $\| \ \|$. The obtained extension will then obviously satisfy (i).
To this end, we will show the followings assertions :

(a1) The set of elementary functions is dense in $\mathcal{C}_c^{1,0}(\mathbb{R}\times[0,1])$, the set of continuous functions with compact support such that $\frac{\partial f}{\partial x}$ exists and is continuous.

(a2) $\mathcal{C}_c^{1,0}(\mathbb{R}\times[0,1])$ is dense in the set $\tilde{\Upsilon}$ of functions f such that, $\frac{\partial f}{\partial x}$ exists as Radon-Nikodym derivative and $\|f\|_\beta + \|\frac{\partial f}{\partial x}\|_\beta < \infty$, where $\| \ \|_\beta$ is the norm defined by $\|g\|_\beta^2 = \int_0^1 \beta(t)\int_{\mathbb{R}} g(x,t)^2 \mathrm{d}x\mathrm{d}t$, for g measurable function from $\mathbb{R}\times[0,1]$ to $\mathbb{R}$

(a3) $\tilde{\Upsilon}$ is dense in Υ.

(a1) : For f element of $\mathcal{C}_c^{1,0}(\mathbb{R}\times[0,1])$ and a, b real numbers such that the support of f is contained in $[a,b]\times[0,1]$, we take a family of subdivisions of $[a,b]\times[0,1]$, $\{(x(i,n), s(j,n)), 0 \leq i \leq k_n, 0 \leq j \leq m_n\}$ such that $x(0,n) = a$, $x(k_n,n) = b$, $s(0,n) = 0$, $s(m_n,n) = 1$ and
$\max\limits_{0\leq i\leq k_n-1} |x(i+1,n) - x(i,n)| \vee \max\limits_{0\leq j\leq m_n-1} |s(j+1,n) - s(j,n)| \underset{n\to\infty}{\longrightarrow} 0.$

$$f_n = \sum_{i=0}^{k_n-1}\sum_{j=0}^{m_n-1} f(x(i,n), s(j,n))1_{(x(i,n),x(i+1,n)]}1_{(s(j,n),s(j+1,n)]}.$$

By dominated convergence, we have :

$$\lim_{n\to\infty}\int_0^1 \beta(s)\int_{\mathbb{R}} |f_n(\xi,s) - f(\xi,s)|^2 \mathrm{d}\xi\mathrm{d}s = 0,$$

and equivalently :

$$\lim_{n\to\infty}\int_0^1 \beta(s)\int_{\mathbb{R}} |\hat{f}_n(\xi,s)-\hat{f}(\xi,s)|^2 \mathrm{d}\xi\mathrm{d}s = 0. \tag{3.3.7}$$

Note that for every $\delta > 0$, $\int_0^\delta x^2\nu(\mathrm{d}x) > 0$. Indeed, if $\int_0^\delta x^2\nu(\mathrm{d}x) = 0$ then $\nu((0,\delta]) = 0$ and it follows that for every $\xi > 0$, $\psi_*(\xi) = 2\int_\delta^1 (1-\cos(x\xi))\nu(\mathrm{d}x) \leq 4\nu(\delta,1]$, which contradicts $\lim_{\xi\to\infty}\psi_*(\xi) = \infty$ (see Lemma 4.2.2 of [33]). Set $k = \inf_{x\in[-1,1]}(1-\cos(x))x^{-2}$, then :

$$\psi_*(\xi) \geq 2\int_0^{\xi^{-1}\wedge 1}(1-\cos(x\xi))\nu(\mathrm{d}x) \geq 2k\xi^2\int_0^{\xi^{-1}\wedge 1} x^2\nu(\mathrm{d}x) > 0,$$

which gives $\sup_{\xi\in[-N,N]}\dfrac{\xi^2}{\psi_*(\xi)} < \infty$, $\forall N > 0$. Thanks to (3.3.7) we hence obtain :

$$\lim_{n\to\infty}\int_0^1 \beta(s)\int_{-N}^N \left(1+\frac{\xi^2}{\psi_*(\xi)}\right)|\hat{f}_n(\xi,s)-\hat{f}(\xi,s)|^2\mathrm{d}\xi\mathrm{d}s = 0,\ \forall N > 0, \tag{3.3.8}$$

Besides, a simple computation gives

$$\begin{aligned}\hat{f}_n(\xi,s) &= \frac{1}{i\xi}\sum_{i=0}^{k_n-1}\sum_{j=0}^{m_n-1} f(x(i,n),s(j,n))(e^{i\xi x(i+1,n)}-e^{i\xi x(i,n)})1_{(s(j,n),s(j+1,n)]}\\ &= \frac{1}{i\xi}\sum_{i=1}^{k_n}\sum_{j=0}^{m_n-1} e^{i\xi x(i,n)}[f(x(i-1,n),s(j,n))\\ &\qquad\qquad -f(x(i,n),s(j,n))]1_{(s(j,n),s(j+1,n)]},\end{aligned}$$

which leads to : $|\hat{f}_n(\xi,s)| \leq \dfrac{1}{|\xi|}\sup_x\left|\dfrac{\partial f}{\partial x}\right|(b-a)$ and hence to

$$\begin{aligned}&\lim_{N\to\infty}\sup_n\int_0^1\beta(s)\int_{|\xi|\geq N}\left(1+\frac{\xi^2}{\psi_*(\xi)}\right)|\hat{f}_n(\xi,s)|^2\mathrm{d}\xi\mathrm{d}s\\ \leq\ & \sup_x\left|\frac{\partial f}{\partial x}\right|(b-a)\lim_{N\to\infty}\int_0^1\beta(s)\int_{|\xi|\geq N}\left(\frac{1}{\xi^2}+\frac{1}{\psi_*(\xi)}\right)\mathrm{d}\xi\mathrm{d}s\\ =\ & 0,\end{aligned}$$

which, together with (3.3.8) gives (a1).
(a2) : For f element of $\tilde{\Upsilon}$ set $\tilde{f}(x,s) = \beta^{1/2}(s)f(x,s)$. We take $(h_n)_{n\in\mathbb{N}}$ a sequence of infinitely differentiable functions with compact support such that :

$$\|\tilde{f}-h_n\|_{L^2(\mathbb{R}\times[0,1])} + \left\|\frac{\partial\tilde{f}}{\partial x}-\frac{\partial h_n}{\partial x}\right\|_{L^2(\mathbb{R}\times[0,1])}\underset{n\to\infty}{\longrightarrow} 0.$$

We define f_n by $f_n(x,s) = h_n(x,s)\beta^{-1/2}(s)$. Since $\beta^{-1/2}$ is a continuous function f_n belongs to $C_c^{1,0}(\mathbb{R}\times[0,1])$. Indeed one shows the continuity of β on $(0,1]$ by dominated convergence and then check the continuity at $t=0$ as follows. From Fatou's Lemma we have :

$$\liminf_{t\to 0} \beta^2(t) \geq \int_0^\infty \psi(\xi) = \infty,$$

hence $\lim_{t\to 0}\beta^{-1/2}(t) = 0$.
For g measurable function from $\mathbb{R}\times[0,1]$ into $\mathbb{R}$ such that $\partial g/\partial x$ exists as Radon-Nikodym derivative, one easily shows with the identity $\widehat{\frac{\partial g}{\partial x}}(\xi,t) = -i\xi\hat{g}(\xi,t)$ that

$$\|g\| \leq c(\|g\|_\beta + \|\partial g/\partial x\|_\beta),$$

where $c^2 = 2\pi[\sup_{0\leq\xi\leq 1}(1+\xi^2\psi_*(\xi)^{-1}) \vee \sup_{\xi\geq 1}(\xi^{-2}+\psi_*(\xi)^{-1})]$. Hence we have

$$\limsup_{n\to\infty} \|f-f_n\| \leq \lim_{n\to\infty} c\left[\|f-f_n\|_\beta + \left\|\frac{\partial f}{\partial x} - \frac{\partial f_n}{\partial x}\right\|_\beta\right] = 0,$$

which proves (a2).
(a3) : Let h be an infinitely differentiable function with compact support from $\mathbb{R}$ into $\mathbb{R}$ such that $\int_{\mathbb{R}} h(x)\mathrm{d}x = 1$. For f such that $\|f\| < \infty$, set

$$f_n(x,s) = n\int_{\mathbb{R}} f(x-y,s)h(ny)\mathrm{d}y.$$

From $\hat{f}_n(\xi,s) = \hat{f}(\xi,s)\hat{h}(\xi/n)$ and $\widehat{\frac{\partial f_n}{\partial x}}(\xi,s) = n\hat{f}(\xi,s)\widehat{h'}(\xi/n)$, we obtain that f_n belongs to $\tilde{\Upsilon}$ and by dominated convergence, $\lim_{n\to\infty}\|f_n - f\| = 0$, which gives (a3).
To finish the proof of Theorem, it remains to show the point (ii). For f element of Υ, we must prove that $[J(f)]_1 = 0$, where for a stochastic process Y we denote $[Y]_1$ its quadratic variation in $[0,1]$. For every $\varepsilon > 0$ there exists an elementary function f_ε such that $\|f - f_\varepsilon\|^2 < \varepsilon/(4\kappa)$. It is obvious from the definition of the integral with respect to L for elementary functions and the relation 3.3.6 that $J(f_\epsilon)$ is a continuous bounded variation process. It hence has a 0-quadratic energy. We have, with the notation of Lemma 3.3.2

$$\begin{aligned}
\mathbf{E}([J(f)]_1) &= \mathbf{E}([J(f) - J(f_\varepsilon)]_1)\\
&= \mathbf{E}([H(f) + I(F) - H(f_\varepsilon) - I(f_\varepsilon)]_1)\\
&\leq 2\mathbf{E}([H(f) - H(f_\varepsilon)]_1) + 2\mathbf{E}([I(f) - I(f_\varepsilon)]_1)\\
&\leq 4\kappa\|f - f_\epsilon\|^2\\
&\leq \varepsilon,
\end{aligned}$$

and (ii) follows. □

Proof of Lemma 3.3.2 : In the proof of Lemma 3.2.1 we have defined the function α by $\alpha(t) = \sup_x \phi(t,x)$ (see (3.2.7)) and we have seen that α belongs to $L^1[0,1]$. Actually there exists a constant c such that

$$\alpha(t) \leq c\beta(t) \ \ \forall t \in (0,1]. \tag{3.3.9}$$

Indeed,

$$\begin{aligned}
\pi\alpha(t) &\leq 1 + \int_1^\infty e^{-t\psi(\xi)}\mathrm{d}\xi \leq 1 + \left(\int_1^\infty e^{-2t\psi(\xi)}\psi(\xi)\mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_1^\infty \frac{1}{\psi(\xi)}\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 e^{-2\psi(\xi)}\psi(\xi)\mathrm{d}\xi\right)^{-\frac{1}{2}} \left(\int_0^1 e^{-2t\psi(\xi)}\psi(\xi)\mathrm{d}\xi\right)^{\frac{1}{2}} + \left(\int_1^\infty \frac{1}{\psi(\xi)}\mathrm{d}\xi\right)^{\frac{1}{2}} \beta(t) \\
&\leq \left[\left(\int_0^1 e^{-2\psi(\xi)}\psi(\xi)\mathrm{d}\xi\right)^{-\frac{1}{2}} + \left(\int_1^\infty \frac{1}{\psi(\xi)}\mathrm{d}\xi\right)^{\frac{1}{2}}\right] \beta(t).
\end{aligned}$$

(i) According to Lemma 4.2.2 of [33], for every real y, $\sup \frac{1-\cos(y\xi)}{\psi_*(\xi)} < \infty$, hence

$$|g_y(\xi,s)|^2 = 2|\hat{f}(\xi,s)|^2 \frac{\xi^2}{\psi_*(\xi)^2}(1-\cos(y\xi)) \leq cste|\hat{f}(\xi,s)|^2 \frac{\xi^2}{\psi_*(\xi)},$$

and it follows immediately that g_y belongs to $L^2(\mathbb{R}\times[0,1])$ when $f \in \Upsilon$.

(ii) We show that the process $(H_t(f))_{0\leq t\leq 1}$ is a well-defined martingale. To this end, it is sufficient to show that

$$\mathbf{E}\int_0^1 \int_{\{|y|\leq 1\}} (\varphi(X_s,y,s))^2 \nu(\mathrm{d}y)\mathrm{d}s < \infty, \tag{3.3.10}$$

We have :

$$\begin{aligned}
\mathbf{E}\int_0^1 \int_{\{|y|\leq 1\}} (\varphi(X_s,y,s))^2 \nu(\mathrm{d}y)\mathrm{d}s &= \int_0^1 \int_{\mathbb{R}\times[-1,1]} (\varphi(z,y,s))^2 \phi(s,z)\mathrm{d}z\nu(\mathrm{d}y)\mathrm{d}s \\
&\leq \int_0^1 \alpha(s) \int_{\mathbb{R}\times[-1,1]} (\varphi(z,y,s))^2 \mathrm{d}z\nu(\mathrm{d}y)\mathrm{d}s \\
&= \frac{1}{2\pi}\int_0^1 \alpha(s) \int_{\mathbb{R}\times[-1,1]} |g_y(\xi,s)|^2 \mathrm{d}\xi\nu(\mathrm{d}y)\mathrm{d}s,
\end{aligned}$$

thanks to Plancherel's Theorem. Now, from the definition of g_y and Fubini's Theorem we have :

$$\mathbf{E}\int_0^1 \int_{\{|y|\leq 1\}} (\varphi(X_s,y,s))^2 \nu(\mathrm{d}y)\mathrm{d}s \leq \frac{1}{\pi}\int_0^1 \alpha(s) \int_{\mathbb{R}} |\hat{f}(\xi,s)|^2 \frac{\xi^2}{\psi_*(\xi)}\mathrm{d}\xi\mathrm{d}s < \infty,$$

which leads to (3.3.10). Note that thanks to Doob's inequality and (3.3.9) we have

$$\mathbf{E}(\sup_{0\le t\le 1}|H_t(f)|^2)\le 4c\pi^{-1}\int_0^1\beta(s)\int_{\mathbb{R}}|\hat{f}(\xi,s)|^2\frac{\xi^2}{\psi_*(\xi)}\mathrm{d}\xi\mathrm{d}s. \qquad (3.3.11)$$

A similar argument shows that the process

$$\tilde{I}(f)=\{\tilde{I}_t(f)=\frac{1}{2}\int_0^t\int_{\{|y|\le 1\}}\varphi(X_{1-s},y,1-s)(\mu_{\tilde{X}}-\rho)(\mathrm{d}y,\mathrm{d}s),t\in[0,1]\},$$

is a well-defined square-integrable $\hat{\mathcal{F}}$-martingale and

$$\mathbf{E}(\sup_{0\le t\le 1}|\tilde{I}_t(f)|^2)\le 4c\pi^{-1}\int_0^1\beta(s)\int_{\mathbb{R}}|\hat{f}(\xi,s)|^2\frac{\xi^2}{\psi_*(\xi)}\mathrm{d}\xi\mathrm{d}s.$$

Thus, the process $I_t(f)=\tilde{I}_1(f)-\tilde{I}_{(1-t)-}(f)$ is well defined and

$$\begin{aligned}\mathbf{E}(\sup_{0\le t\le 1}|I_t(f)|^2) &\le 2\mathbf{E}(|\tilde{I}_1(f)|^2)+2\mathbf{E}(\sup_{0\le t\le 1}|\tilde{I}_t(f)|^2)\\ &\le 16c\pi^{-1}\int_0^1\beta(s)\int_{\mathbb{R}}|\hat{f}(\xi,s)|^2\frac{\xi^2}{\psi_*(\xi)}\mathrm{d}\xi\mathrm{d}s. \qquad (3.3.12)\end{aligned}$$

Finally, we will show that the process $K(f)$ is a well-defined bounded variation process. We have :

$$\begin{aligned}&\mathbf{E}\int_0^1\int_{\{|y|\le 1\}}|\varphi(X_s,y,s)|\frac{|\phi(s,X_s+y)-\phi(s,X_s)|}{\phi(s,X_s)}\nu(\mathrm{d}y)\mathrm{d}s\\ \le&\int_0^1\left\{\int_{\mathbb{R}\times[-1,1]}(\varphi(x,y,s))^2\mathrm{d}x\nu(\mathrm{d}y)\right\}^{1/2}\left\{\int_{\mathbb{R}^2}(\phi(s,x+y)-\phi(s,x))^2\mathrm{d}x\nu(\mathrm{d}y)\right\}^{1/2}\mathrm{d}s\\ =&\frac{1}{\pi}\int_0^1\left\{\int_{\mathbb{R}}|\hat{f}(\xi,s)|^2\frac{\xi^2}{\psi_*(\xi)}\mathrm{d}\xi\right\}^{1/2}\beta(s)\mathrm{d}s\\ \le&\frac{1}{\pi}\left\{\int_0^1\beta(s)\mathrm{d}s\right\}^{1/2}\left\{\int_0^1\beta(s)\int_{\mathbb{R}}|\hat{f}(\xi,s)|^2\frac{\xi^2}{\psi_*(\xi)}\mathrm{d}\xi\mathrm{d}s\right\}^{1/2},\end{aligned}$$

where the equality is obtained thanks to Plancherel's Theorem and the last inequality follows from Cauchy-Schwarz inequality. Hence $K(f)$ is a variation bounded process and we have :

$$\begin{aligned}&\mathbf{E}(\sup_{0\le t\le 1}|K_t(f)|) \qquad (3.3.13)\\ \le&\frac{1}{\pi}\left\{\int_0^1\beta(s)\mathrm{d}s\right\}^{1/2}\left\{\int_0^1\beta(s)\int_{\mathbb{R}}|\hat{f}(\xi,s)|^2\frac{\xi^2}{\psi_*(\xi)}\mathrm{d}\xi\mathrm{d}s\right\}^{1/2}.\end{aligned}$$

(iii) We derive immediately from (3.3.11), (3.3.12) and (3.3.13) the existence of a constant κ such that

$$\begin{aligned}\mathbf{E}[\sup_{0\leq t\leq 1}|J_t(f)|] &\leq \kappa\left\{\int_0^1\beta(s)\int_{\mathbb{R}}|\hat{f}(\xi,s)|^2\frac{\xi^2}{\psi_*(\xi)}\mathrm{d}\xi\mathrm{d}s\right\}^{1/2}. \\ &\leq k\|f\|\end{aligned}$$

□

In the proof of Theorem 3.3.1, we have defined the norm $\|\|_\beta$ as follows

$$\|g\|_\beta^2=\int_0^1\beta(t)\int_{\mathbb{R}}g(x,s)^2\mathrm{d}x\mathrm{d}s, \tag{3.3.14}$$

for g measurable function from $\mathbb{R}\times[0,1]$ into $\mathbb{R}$ and we have seen that there exists a positive constant c such that for any g admitting a Radom-Nikodym derivative $\frac{\partial g}{\partial x}$:

$$\|g\|\leq c(\|g\|_\beta+\|\partial g/\partial x\|_\beta). \tag{3.3.15}$$

Lemma 3.3.3. *Let f be a measurable function from $\mathbb{R}\times[0,1]$ into $\mathbb{R}$ such that $\partial f/\partial x$ exists as Radon-Nikodym derivative and $\|f\|_\beta+\|\partial f/\partial x\|_\beta<\infty$, then the processes $\{\int_0^t\int_{\mathbb{R}}f(x,s)\mathrm{d}L_s^x;0\leq t\leq 1\}$ and $\{-\int_0^t\frac{\partial f}{\partial x}(X_s,s)\mathrm{d}s;0\leq t\leq 1\}$ are indistinguishable.*

Proof of Lemma 3.3.3 Suppose that f does not depend on t, has a compact support and a continuous first derivative with respect to x (denoted f'). Then with the arguments used to show (3.3.3) one shows that the corresponding function φ defined in Lemma 3.3.2 is given by

$$\varphi(x,y)=\int_{\mathbb{R}}f'(z)(w(x-z+y)-w(x-z))\mathrm{d}z.$$

Thanks to (3.2.13) in Proposition 3.2.2 we have :

$$\int_{\mathbb{R}}f'(x)\hat{N}_t^x\mathrm{d}x=\int_0^t\int_{\{|y|\leq 1\}}\varphi(X_{1-s},y)(\mu_{\hat{X}}-\rho)(\mathrm{d}y,\mathrm{d}s),$$

thus, with the notation of Lemma 3.3.2 we have $I_t(f)=\frac{1}{2}\int_{\mathbb{R}}f'(x)(\hat{N}_1^x-\hat{N}_{(1-t)-}^x)\mathrm{d}x$ and similarly (see (3.2.14)), $K_t(f)=\frac{1}{2}\int_{\mathbb{R}}f'(x)(\hat{W}_1^x-\hat{W}_{(1-t)}^x)\mathrm{d}x$ and $H_t(f)=\frac{1}{2}\int_{\mathbb{R}}f'(x)N_t^x\mathrm{d}x$. It follows from (3.3.5) that $J_t(f)=-\int_{\mathbb{R}}f'(x)L_t^x\mathrm{d}x$ which proves Lemma 3.3.3 in this special case thanks to the time occupation formula.
Suppose that f belongs to $\mathcal{C}_c^{1,0}(\mathbb{R}\times[0,1])$. Take a sequence of subdivisions $(\Delta_n)_{n\geq 0}$ of $[0,1]$ such that the mesh of Δ_n tends to 0 as n tends to ∞. Define $f_n(x,s)=\sum_{s_i\in\Delta_n}f(x,s_i)1_{(s_i,s_{i+1}]}$. For any $n>0$, $J_t(f_n)=-\int_0^t(\partial f_n/\partial x)(X_s,s)\mathrm{d}s$. The

right-hand side of the precedent equation converges to $\int_0^t (\partial f/\partial x)(X_s, s)\mathrm{d}s$ almost surely and $\|f - f_n\|$ converge to 0, hence $J(f_n)$ converges in L^1 to $J(f)$ and Lemma 3.3.3 is proved in this case.
For the general case, thanks to (3.3.15), $f \in \Upsilon$. Moreover, a.s, $\forall t \in [0,1]$ $Y_t = \int_0^t \frac{\partial f}{\partial x}(X_s, s)\mathrm{d}s$ is well defined as Lebesgue integral and

$$\mathbf{E}[\sup_{0\le t\le 1} |Y_t|^2] \le \mathbf{E}\int_0^1 \int_{\mathbb{R}} \frac{\partial f}{\partial z}^2 (x,s)\phi(x,s)\mathrm{d}x\mathrm{d}s \le c\|\partial f/\partial x\|_\beta^2,$$

where c is the constant involved in (3.3.9)). Let $(f_n)_{n\ge 0}$ be a sequence of $\mathcal{C}_c^{1,0}$ such that $\|f - f_n\|_\beta + \|\partial f/\partial x - \partial f_n/\partial x\|_\beta \to 0$ as $n \to \infty$. Thanks to (3.3.15) $\|f - f_n\|$ also converges to 0. Then :

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} f(x,s)\mathrm{d}L_s^x &= \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}} f_n(x,s)\mathrm{d}L_s^x \\ &= -\lim_{n\to\infty}\int_0^t \frac{\partial f_n}{\partial x}(X_s,s)\mathrm{d}s \\ &= -\int_0^t \frac{\partial f}{\partial x}(X_s,s)\mathrm{d}s,\end{aligned}$$

where the first limit is in L^1 and the second one in L^2. □

We defined Υ_{loc} as the set of measurable functions from $\mathbb{R}\times[0,1]$ into $\mathbb{R}$ such that for any positive k there exists $f_k \in \Upsilon$ such that $f_k = f$ in $[-k,k]\times[0,1]$. In the rest of this section, with a localization argument, we will construct a stochastic integral with respect to local time for the elements of Υ_{loc}. Example 3.3.8 provides a characterization of Υ_{loc} in the case when X is stable symmetric.
For n integer, set $T_n = \inf\{t : |X_t| > n\} \wedge 1$.

Definition 3.3.4. *Let f be an element of Υ_{loc}. For every integer n let f_n be a function of Υ such that $f(x,s) = f_n(x,s)$ on $\{|x| \le n+1\}$. We define the process $(J_t^n(f))_{0\le t\le 1}$ by $J_t^n(f) = \int_0^{t\wedge T_n}\int_{\mathbb{R}} f_n(x,s)\mathrm{d}L_s^x$.*

The following lemma shows that this definition makes sense and does not depend of the choice of the function f_n.

Lemma 3.3.5. *For f, g in Υ such that $f(x,s) = g(x,s)$ on $\{|x| \le n+1\}$, the processes $(\int_0^{t\wedge T_n}\int_{\mathbb{R}} f(x,s)\mathrm{d}L_s^x)_{0\le t\le 1}$ and $(\int_0^{t\wedge T_n}\int_{\mathbb{R}} g(x,s)\mathrm{d}L_s^x)_{0\le t\le 1}$ are indistinguishable.*

Proof of Lemma 3.3.5 Thanks to the continuity of the processes, it suffices to show that the first process is a modification of second one. Let h be an infinitely differentiable function with compact support included in $[0,1]$ and such that $\int_{\mathbb{R}} h(x)\mathrm{d}x = 1$. We define the sequence $(f_m)_{m\in\mathbb{N}}$ by $f_m(x,s) = \int_{\mathbb{R}} f(x - y/m, s)h(y)\mathrm{d}y$ and $g_m(x,s) = \int_{\mathbb{R}} g(x - y/m, s)h(y)\mathrm{d}y$. As for (a3), in the proof of Theorem 3.3.1, $\|f_m - f\|$ and $\|g_m - g\|$ both converge to 0 as m tends to ∞. This gives as m tends to ∞

$$\int_0^{t\wedge T_n}\int_{\mathbb{R}} f_m(x,s)\mathrm{d}L_s^x \overset{L^1(\mathbf{P})}{\longrightarrow} \int_0^{t\wedge T_n}\int_{\mathbb{R}} f(x,s)\mathrm{d}L_s^x$$

and

$$\int_0^{t\wedge T_n}\int_{\mathbb{R}} g_m(x,s)\mathrm{d}L_s^x \overset{L^1(\mathbf{P})}{\longrightarrow} \int_0^{t\wedge T_n}\int_{\mathbb{R}} g(x,s)\mathrm{d}L_s^x.$$

We show now that for every m :

$$\int_0^{t\wedge T_n}\int_{\mathbb{R}} f_m(x,s)\mathrm{d}L_s^x = \int_0^{t\wedge T_n}\int_{\mathbb{R}} g_m(x,s)\mathrm{d}L_s^x \text{ a.s.}$$

For every m, $f_m(x,s) = g_m(x,s)$ on $\{|x| \le n\}$. Hence, thanks to Lemma 3.3.3 we have :

$$\begin{aligned}\int_0^{t\wedge T_n}\int_{\mathbb{R}} f_m(x,s)\mathrm{d}L_s^x &= -\int_0^{t\wedge T_n} \frac{\partial f_m}{\partial x}(X_s,s)\mathrm{d}s \\ &= -\int_0^{t\wedge T_n} \frac{\partial g_m}{\partial x}(X_s,s)\mathrm{d}s \\ &= \int_0^{t\wedge T_n}\int_{\mathbb{R}} g_m(x,s)\mathrm{d}L_s^x.\end{aligned}$$

This finishes the proof. □

Definition 3.3.6. *For f element of Υ_{loc}, we define the process $(J_t(f)_{0\le t\le 1}$ by :*

$$J_t(f) = J_t^n(f) \text{ en } \{0 \le t \le T_n\}.$$

Thanks to Lemma 3.3.5 if $m \le n$, $J_t^n(f) = J_t^m(f)$ a.s on $\{0 \le t \le T_n\}$, which shows that this definition is consistent.

Lemma 3.3.7. (i) *For every f in Υ_{loc}, the process $J^n(f)$ converges uniformly in probability as n tends to ∞ to the process $J(f)$.*

(ii) *For f measurable function on $\mathbb{R}\times[0,1]$ such that $\frac{\partial f}{\partial x}$ exists as Radon-Nikodym derivative, $\int_0^1 \beta(s)\int_{-K}^K (\frac{\partial f}{\partial x}(x,s))^2\mathrm{d}x\mathrm{d}s < \infty\ \forall K > 0$ and $\int_0^1 \beta(s)(f(x_0,s))^2\mathrm{d}s < \infty\ \exists x_0 \in \mathbb{R}$, then f belongs to Υ_{loc} and the processes $J(f)$ and $\{-\int_0^t \frac{\partial f}{\partial x}(X_s,s)\mathrm{d}s; 0 \le t \le 1\}$ are indistinguishable.*

(iii) *For every f in Υ_{loc}, $J(f)$ is a 0-quadratic energy process.*

Proof (i) $\forall \epsilon > 0$:

$$\mathbf{P}(\sup_{0\leq t\leq 1} |J_t(f) - J_t^n(f)| > \epsilon) \leq \mathbf{P}(1 > T_n) = \mathbf{P}(\sup_{0\leq s\leq 1} |X_s| > n) \to 0$$

as $n \to \infty$ because $\sup\{|X_s| : 0 \leq s \leq 1\} < \infty$ a.s.
(ii) With these assumptions on f, one easily shows that

$$\int_0^1 \beta(s) \int_{-k}^k (f(x,s))^2 \mathrm{d}x\mathrm{d}s < \infty.$$

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of infinitely differentiable functions with compact support such that $g_n(x) = 1$ if $|x| < n+1$, then for every n

$$\|fg_n\|_\beta + \|\frac{\partial(fg_n)}{\partial x}\|_\beta < \infty.$$

Thanks to Lemma 3.3.3, fg_n belongs to Υ and thus f belongs to Υ_{loc}. Moreover :

$$\begin{aligned} J_t^n(f) &= \int_0^{t\wedge T_n} \int_{\mathbb{R}} f(x,s) g_n(x) \mathrm{d}L_s^x \\ &= -\int_0^{t\wedge T_n} \frac{\partial(g_n f)}{\partial x}(X_s, s)\mathrm{d}s \\ &= -\int_0^{t\wedge T_n} \frac{\partial f}{\partial x}(X_s, s)\mathrm{d}s. \end{aligned}$$

Consequently we obtain (ii) from (i) by letting n tend to ∞.
(iii) For every n, the quadratic variation of $J^n(f)$ is a.s. zero, hence a.s on $0 \leq t \leq T_n$, $J(f)$ is a 0-quadratic energy process, consequently it is so on $[0,1]$.□

We can now extend the stochastic integration with respect to local times from Υ to Υ_{loc} as follows. For f element of Υ_{loc}, we define the stochastic integral with respect to L by

$$\int_0^t \int_{\mathbb{R}} f(x,s)\mathrm{d}L_s^x = J_t(f).$$

Example 3.3.8. When X is a α-stable process, we have the following characterization of set Υ_{loc} : a measurable function f from $\mathbb{R} \times [0,1]$ to $\mathbb{R}$ belongs to Υ_{loc} if and only if for every positive k

$$\int_0^1 \beta(t) \int_{-k}^k [f^2(x,s) + \int_{-1}^1 (f(x+y,s) - f(x,s))^2 \varpi(\mathrm{d}y)]\mathrm{d}x < \infty, \qquad (3.3.16)$$

where ϖ is the Lévy measure of a $(2-\alpha)$-stable and symmetric Lévy process.

Indeed, set $\tau(\xi)=\int_{-1}^{1}(1-\cos(x\xi))\varpi(\mathrm{d}x)$, then there exists a constant c such that $c(1+|\xi|^{2-\alpha})\leq 1+\tau(\xi)\leq(1+|\xi|^{2-\alpha})\ \forall\xi\in\mathbb{R}$. Hence we have :

$$\begin{aligned} f\in\Upsilon \quad &\Leftrightarrow \quad \int_0^t\beta(t)\int_{\mathbb{R}}(1+\tau(\xi))|\hat{f}(\xi,t)|^2\mathrm{d}\xi\mathrm{d}t<\infty \\ &\Leftrightarrow \quad \int_0^1\beta(t)\int_{\mathbb{R}}[f^2(x,s)+\int_{-1}^{1}(f(x+y,s)-f(x,s))^2\varpi(\mathrm{d}y)\mathrm{d}x\mathrm{d}t<\infty \end{aligned}$$

thanks to Plancherel's Theorem. With this fact, one obtains the "only if" part. Conversely assume that (3.3.16) holds. Let g be an element of $\mathcal{C}_c^\infty(\mathbb{R})$ with support in $[-K,K]$, for $K>0$, then

$$\begin{aligned} &\int_0^1\beta(t)\int_{\mathbb{R}}[(fg)^2(x,s)+\int_{-1}^{1}((fg)(x+y,s)-(fg)(x,s))^2\varpi(\mathrm{d}y)]\mathrm{d}x\mathrm{d}t \\ \leq\ &(\|g\|_\infty^2+2\sup_x\int_{-1}^{1}(g(x+y)-g(x))^2\varpi(\mathrm{d}y))\int_0^1\beta(t)\int_{K-1}^{K+1}f^2(x,s)\mathrm{d}x\mathrm{d}t \\ &+2\|g\|_\infty^2\int_0^1\beta(t)\int_{K-1}^{K+1}\int_{-1}^{1}(f(x+y,s)-f(x,s))^2\varpi(\mathrm{d}y)\mathrm{d}x\mathrm{d}t \\ <\ &\infty, \end{aligned}$$

consequently

$$fg\in\Upsilon \text{ for any } g\in C_c^\infty \tag{3.3.17}$$

and the "only" part follows.
Actually, this characterization of Υ_{loc} is also available when the Lévy process is such that $\phi(\xi)\sim\xi^\alpha$ as $\xi\to\infty$ for some $1<\alpha<2$.

3.4 Extension of the Itô formula

Before proving Theorem 3.1.2, we establish some results. They show that each of the terms of (3.1.10) is well defined and they will ease their approximations. We need the following notation. The operator $\mathcal{B}$ is defined by

$$\mathcal{B}F(x,s)=\int_{\{|y|\leq 1\}}\{F(x+y,s)-F(x,s)-y\frac{\partial F}{\partial x}(x,s)\}\nu(\mathrm{d}y), \tag{3.4.1}$$

for any function F on $\mathbb{R}\times[0,1]$ such that the above integral is well defined.
To the Lévy measure $\bar{\nu}(\mathrm{d}x)=\frac{1}{|x|}\nu((|x|,1])1_{\{|x|\leq 1\}}\mathrm{d}x$, we associate the corresponding characteristic exponent $\bar{\psi}$

$$\bar{\psi}(\xi)=2\int_0^1(1-\cos(x\xi))\bar{\nu}(\mathrm{d}x)$$

and the norm $\|.\|_*$ defined on $L^2(\mathbb{R}\times[0,1])$ by

$$\|F\|_*^2 = \int_0^1 \beta(s)\int_{\mathbb{R}}(1+\bar{\psi}(\xi))|\hat{F}(\xi,s)|^2 \mathrm{d}\xi \mathrm{d}s.$$

Lemma 3.4.1. *Let $F \in L^2(\mathbb{R}\times[0,1])$ such that $\|F\|_* < \infty$, then $\mathcal{BI}F$ is a.e. well defined and belongs to Υ. Moreover there exists a constant C independent of F such that $\|\mathcal{BI}F\| \leq C\|F\|_*$.*

Proof of Lemma 3.4.1 For F in $L^2(\mathbb{R}\times[0,1])$ such that $\|F\|_* < \infty$. We first show that $\mathcal{BI}F$ is well defined. Thanks to Plancherel's Theorem, the norm $\|.\|_*$ can be written as :

$$\|F\|_*^2 = \pi\int_0^1 \beta(s)\int_{\mathbb{R}}\left\{2(F(x,s))^2 + \int_{-1}^1 [F(x+y,s)-F(x,s)]^2\bar{\nu}(\mathrm{d}y)\right\}\mathrm{d}x\mathrm{d}s.$$

Applying Fubini's Theorem and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \int_{\{|y|\leq 1\}} |\mathcal{I}F(x+y,s)-\mathcal{I}F(x,s)-yF(x,s)|\nu(\mathrm{d}y) \\
\leq & \int_{\{|y|\leq 1\}}\int_{-y^-}^{y^+} |F(x+z,s)-F(x,s)|\mathrm{d}z\nu(\mathrm{d}y) \\
= & \int_{\{|z|\leq 1\}} |F(x+z,s)-F(x,s)|\nu((|z|,1])\mathrm{d}z \\
\leq & \left\{\int_{\{|z|\leq 1\}} |F(x+z,s)-F(x,s)|^2\frac{\nu((|z|,1])}{|z|}\mathrm{d}z\right\}^{1/2}\left\{\int_{\{|z|\leq 1\}} |z|\nu((|z|,1])\mathrm{d}z\right\}^{1/2} \\
= & k\left\{\int_{\{|z|\leq 1\}} |F(x+z,s)-F(x,s)|^2\bar{\nu}(\mathrm{d}z)\right\}^{1/2}, \qquad (3.4.2)
\end{aligned}$$

where $k = (\int z^2\bar{\nu}(\mathrm{d}z))^{1/2} < \infty$. Hence $\mathcal{BI}F$ is well defined and satisfies :

$$\|\mathcal{BI}F\|_{L^2(\mathbb{R}\times[0,1])} \leq \frac{k^2}{\beta(1)}\|F\|_*^2 < \infty. \qquad (3.4.3)$$

Similarly to what has been done in the proof of Theorem 3.3.1, we can show that there exists a sequence $(F_n)_{n\in\mathbb{N}}$ in $C_c^{\infty,0}$ such that $\|F-F_n\|_* \to 0$ as $n\to\infty$ then $\mathcal{BI}F_n \to_{L^2(\mathbb{R}\times[0,1])} \mathcal{BI}F$ or equivalently $\widehat{\mathcal{BI}F_n} \to_{L^2(\mathbb{R}\times[0,1])} \widehat{\mathcal{BI}F}$. It follows, by taking a subsequence if necessary, that $\widehat{\mathcal{BI}F_n}$ converges $\mathrm{d}\xi\mathrm{d}s$-a.s. to $\widehat{\mathcal{BI}F}$. Thanks to Fatou's Lemma we have : $\|\mathcal{BI}F\| \leq \underline{\lim}_{n\to\infty}\|\mathcal{BI}F_n\|$, so we must only show that there exists a constant C which satisfies $\|\mathcal{BI}G\| \leq C\|G\|_*$ for any function $G\in C_c^{\infty,0}$.

For G in $C_c^{\infty,0}$ and g in $C_c^\infty(\mathbb{R})$, the equalities :

$$\mathcal{BI}G(x,s) = \mathcal{IB}G(x,s) + \mathcal{BI}G(0,s) \text{ and } \widehat{\mathcal{B}g} = -\psi_* \hat{g},$$

(see e.g., Bertoin [3] p.24 for the second one) imply that :

$$\widehat{\mathcal{BI}G}(\xi,s) = \frac{i}{\xi}\widehat{\mathcal{B}G}(\xi,s) = -i\frac{\psi_*(\xi)}{\xi}\hat{G}(\xi,s), \tag{3.4.4}$$

where the operator $\mathcal{I}$ is defined by (3.1.5) and consequently

$$\|\mathcal{BI}G\|^2 = \int_0^1 \beta(s) \int_{\mathbb{R}} \left(1 + \frac{\xi^2}{\psi_*(\xi)}\right) \frac{\psi_*^2(\xi)}{\xi^2} |\hat{G}(\xi,s)|^2 \mathrm{d}\xi \mathrm{d}s. \tag{3.4.5}$$

From Lemma 4.2.2 of [33], the function $\psi_*(\xi)\xi^{-2}$ is bounded, thus there exists a constant $\tilde{C}$ such that $(1 + \frac{\xi^2}{\psi_*(\xi)})\frac{\psi_*(\xi)^2}{\xi^2} \le \tilde{C}(1 + \psi_*(\xi))$ for every ξ and we obtain

$$\|\mathcal{BI}G\|^2 \le \tilde{C} \int_0^1 \beta(s) \int_{\mathbb{R}} (1 + \psi_*(\xi)) |\hat{G}(\xi,s)|^2 \mathrm{d}\xi \mathrm{d}s. \tag{3.4.6}$$

Besides there exists a constant k such that for every x : $1 - \cos x \le k \int_0^x (1 - \cos\lambda)\lambda^{-1}\mathrm{d}\lambda$, hence

$$\int_0^1 (1 - \cos(x\xi))\nu(\mathrm{d}x) \le k \int_0^1 \int_0^{x\xi} (1 - \cos\lambda)\lambda^{-1}\mathrm{d}\lambda\nu(\mathrm{d}x).$$

By Fubini's Theorem, the right-hand side of the above inequality is equal to k$\bar{\psi}(\xi)$. One obtains

$$2\psi_*(\xi) \le k\bar{\psi}(\xi), \tag{3.4.7}$$

which together with (3.4.6) gives the desired result. □

Corollary 3.4.2. *Let F be a measurable function from $\mathbb{R} \times [0,1]$ to $\mathbb{R}$ such that for every positive k,*

$$\int_0^1 \beta(t) \int_{-k}^k [(F(x,t))^2 + \int_{-1}^1 (F(x+y,t) - F(x,t))^2 \bar{\nu}(\mathrm{d}y)] \mathrm{d}x\mathrm{d}t < \infty, \tag{3.4.8}$$

then $\mathcal{BI}F$ is a.e. well defined and belongs to Υ_{loc}.

Proof of Corollary 3.4.2 $\mathcal{BI}F$ is a.e. well defined thanks to (3.4.2). For g element of $C_c^\infty(\mathbb{R})$ such that $g(x) = 1$ when $|x| \le n+1$, we show, as for (3.3.17), that $\|Fg\|_* < \infty$. Thanks to Lemma 3.4.1, $\mathcal{BI}(Fg)$ is hence well defined and belongs to Υ. Moreover $\mathcal{BI}(Fg)(x,s) = \mathcal{BI}F(x,s)$ $\forall |x| \le n$, hence $\mathcal{BI}F$ belongs to Υ_{loc}. □

Remark 3.4.3. Thanks to (3.4.4), for any F in $C_c^{\infty,0}(\mathbb{R}\times[0,1])$, and any real y, the function g_y, defined in Lemma 3.3.2, corresponding to $\mathcal{BI}F$, is given by :

$$g_y(\xi,s) = (e^{-iy\xi}-1)\hat{F}(\xi,s).$$

Thus the corresponding function φ (also defined in Lemma 3.3.2) is given by :

$$\varphi(x,y,s) = F(x+y,s) - F(x,s).$$

Consequently the stochastic integral of the function $\mathcal{BI}F$ satisfies :

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}} \mathcal{BI}F(x,s)\mathrm{d}L_s^x = \\
&\frac{1}{2}\int_0^t\int_{\{|y|\le 1\}}(F(X_{s-}+y,s)-F(X_{s-},s))\tilde{\mu}(\mathrm{d}y,\mathrm{d}s) \qquad (3.4.9)\\
&+\frac{1}{2}\int_{[1-t,1]}\int_0^t\int_{\{|y|\le 1\}}(F(X_{1-s}+y,1-s)-F(X_{1-s},1-s))(\mu_{\hat{X}}-\rho)(\mathrm{d}y,\mathrm{d}s)\\
&+\frac{1}{2}\int_0^t\int_{\{|y|\le 1\}}(F(X_{s-}+y,s)-F(X_{s-},s))\left(\frac{\phi(s,X_s+y)-\phi(s,X_s)}{\phi(x,X_s)}\right)\nu(\mathrm{d}y)\mathrm{d}s.
\end{aligned}$$

For F element of $L^2(\mathbb{R}\times[0,1])$ such that $\|F\|_*^2<\infty$, there exists a sequence of elements of $C_c^{\infty,0}(\mathbb{R}\times[0,1])$, converging to F with respect to the norm $\|\ \|_*$. This is sufficient to show that (3.4.9) holds for such a function F. Finally, with a stopping time argument, we can show that (3.4.9) holds for any measurable function $F:\mathbb{R}\times[0,1]\to\mathbb{R}$ such that every positive k

$$\int_0^1\beta(t)\int_{-k}^k\left\{(F(x,t))^2+\int_{-1}^1[F(x+y,t)-F(x,t)]^2\bar{\nu}(\mathrm{d}y)\right\}\mathrm{d}x\mathrm{d}t<\infty.$$

With the arguments used in the proof of Lemma 3.3.2, we can show the existence of a constant k_1 such that for every F in $L^2(\mathbb{R}\times[0,1])$:

$$\mathbf{E}\int_0^1\int_{\{|y|\le 1\}}(F(X_s+y,s)-F(X_s,s))^2\nu(\mathrm{d}y)\le k_1\int_0^1\beta(s)\int_{\mathbb{R}}\psi_*(\xi)|\hat{F}(\xi,s)|^2\mathrm{d}\xi\mathrm{d}s.$$

We have shown (3.4.7) that for every real ξ there exists a constant k_2 such that $\psi_*(\xi)\le k_2\bar{\psi}(\xi)$. Together with the previous inequality this leads to

$$\mathbf{E}\int_0^1\int_{\{|y|\le 1\}}(F(X_s+y,s)-F(X_s,s))^2\nu(\mathrm{d}y)\le k_1k_2\int_0^1\beta(s)\int_{\mathbb{R}}\bar{\psi}(\xi)|\hat{F}(\xi,s)|^2\mathrm{d}\xi\mathrm{d}s.$$

This remark leads to the following lemma.

Lemma 3.4.4. *Let F be a measurable function form $\mathbb{R} \times [0,1]$ to $\mathbb{R}$ such that $\|F\|_* < \infty$ (resp. (3.4.8) holds), then the process $(M_t(F))_{0\leq t\leq 1}$ defined by*

$$M_t(F) = \int_0^t \int_{\{|y|\leq 1\}} (F(X_{s-}+y,s) - F(X_{s-},s))\tilde{\mu}_X(\mathrm{d}y,\mathrm{d}s),$$

is a well-defined square-integrable martingale (resp. local martingale). Moreover there exists a constant C independent of F such that $\mathbf{E}[\sup_{0\leq s\leq 1}|M_s(F)|^2] \leq C\|F\|_*^2$.

Proof of Theorem 3.1.2 : With a stopping time argument , it is sufficient to show that the result holds for a function F such that $\|F\|_* + \|\frac{\partial F}{\partial t}\|_\beta < \infty$, where the norm $\|.\|_\beta$ is defined by (3.3.14). We already have shown that all the processes involved in (3.1.10) are well defined and they are right continuous. It is hence sufficient to prove the result for a fixed t in $[0,1]$.
Let h and g be two positive functions elements of $C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} g(\tau)\mathrm{d}\tau = \int_{\mathbb{R}} h(z)\mathrm{d}z = 1$. We assume that *supp*$[g] \subseteq \mathbb{R}^+$. For every n, m, we defined $F_{n,m}$ and F_n by :

$$\begin{aligned} F_{n,m}(x,s) &= \int_{\mathbb{R}^2} F(x+z/n, s+\tau/m)h(z)g(\tau)\mathrm{d}z\mathrm{d}\tau \\ F_n(x,s) &= \int_{\mathbb{R}} F(x+z/n,s)h(z)\mathrm{d}z, \end{aligned}$$

for every (x,s) in $\mathbb{R}\times[0,1]$. We set $F(x,s) = F(x,1)$ when $s > 1$.
First, we establish for any integer n, the following decomposition :

$$F_n(X_t,t) = F_n(0,0) + A_t^n + M_t^n + V_t^n + B_t^n \text{ a.s.} \tag{3.4.10}$$

where :

$$\begin{aligned} A_t^n &= \int_0^t \frac{\partial F_n}{\partial t}(X_s,s)\mathrm{d}s \\ M_t^n &= \int_0^t \int_{\mathbb{R}} 1_{\{|y|\leq 1\}}(F_n(X_{s-}+y,s) - F_n(X_{s-},s))\tilde{\mu}(\mathrm{d}y,\mathrm{d}s) \\ V_t^n &= \sum_{s\leq t}(F_n(X_s,s) - F_n(X_{s-},s))1_{\{|\Delta X_s|>1\}} \\ B_t^n &= \int_0^t \mathcal{B}F_n(X_s,s)\mathrm{d}s. \end{aligned}$$

Similarly to A^n, M^n, V^n and B^n, we define the processes $A^{n,m}, M^{n,m}, V^{n,m}$ and $B^{n,m}$ with $F_{n,m}$ replacing F_n. Then applying Itô formula (3.1.1), we have :

$$F_{n,m}(X_t,t) = F_{n,m}(0,0) + A_t^{n,m} + M_t^{n,m} + V_t^{n,m} + B_t^{n,m}.$$

For fixed n, we let m tend to ∞ in the above equation. Since $V_t^{n,m}$ is a finite sum, $(F_{n,m}(X_t,t) - F_{n,m}(0,0) - V_t^{n,m})$ converges a.s. to $(F_n(X_t,t) - F_n(0,0) - V_t^n)$ as $n \to \infty$. We will show now that :

$$\lim_{m\to\infty} \mathbf{E}[(M_t^{n,m} - M_t^n)^2] = 0 \tag{3.4.11}$$

$$\lim_{m\to\infty} \mathbf{E}[(B_t^{n,m} - B_t^n)^2] = 0 \tag{3.4.12}$$

$$\lim_{m\to\infty} \mathbf{E}[|A_t^{n,m} - A_t^n|] = 0. \tag{3.4.13}$$

Thanks to Lemma 3.4.4, in order to obtain (3.4.11) it suffices to show for every n :

$$\lim_{m\to\infty} \|F_{n,m} - F_n\|_* = 0.$$

From $\hat{F}_n(\xi,s) = \hat{h}(-\xi/n)\hat{F}(\xi,s)$ and $\hat{F}_{n,m}(\xi,s) = \hat{h}(-\xi/n)\int_0^\infty \hat{F}(\xi,s+\tau/m)g(\tau)\mathrm{d}\tau$, we obtain :

$$\begin{aligned}
\|F_{n,m} - F_n\|_*^2 &\leq \|g\|_{L^1}\int_0^1 \beta(s)\int_{\mathbb{R}}(1+\bar{\psi}(\xi))|\hat{h}(-\xi/n)|^2 \times \\
&\quad \int_0^\infty |\hat{F}(\xi,s+\tau/m) - \hat{F}(\xi,s)|^2 g(\tau)\mathrm{d}\tau\mathrm{d}\xi\mathrm{d}s \\
&\leq k(n)\int_0^1 \beta(s)\int_{\mathbb{R}}\int_0^\infty |\hat{F}(\xi,s+\tau/m) - \hat{F}(\xi,s)|^2 g(\tau)\mathrm{d}\tau\mathrm{d}\xi\mathrm{d}s,
\end{aligned}$$

where $k(n) = \|g\|_{L^1}\sup_\xi (1+\bar{\psi}(\xi))|\hat{h}(-\xi/n)|^2 < \infty$. Then, we have :

$$\begin{aligned}
&\|F_{n,m} - F_n\|_*^2 \\
\leq\ & 2\pi k(n)\int_0^1 \beta(s)\int_{\mathbb{R}}\int_0^\infty (F(x,s+\tau/m) - F(x,s))^2 g(\tau)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\
\leq\ & 2\pi k(n)\int_0^1 \beta(s)\int_{\mathbb{R}}\int_0^\infty \int_0^1 |\frac{\partial F}{\partial t}(x,\theta)|^2\mathrm{d}\theta\frac{\tau}{m}g(\tau)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\
=\ & \frac{1}{m}2\pi k(n)\int_0^1 \beta(s)\mathrm{d}s\int_0^\infty \tau g(\tau)\mathrm{d}\tau\int_{\mathbb{R}}\int_0^1 |\frac{\partial F}{\partial t}(x,\theta)|^2\mathrm{d}\theta\mathrm{d}x.
\end{aligned}$$

which leads to (3.4.11) since the last term converges to 0 as m tends to ∞. From (3.4.2) and (3.4.3) we know that there exists a constant k such that :

$$\mathbf{E}\int_0^t (\mathcal{B}F_{n,m}(X_s,s) - \mathcal{B}F_n(X_s,s))^2\mathrm{d}s \leq k\left\|\frac{\partial F_{n,m}}{\partial x} - \frac{\partial F_n}{\partial x}\right\|_*^2.$$

The arguments used to establish (3.4.11), are then used for the function $-nh'$ instead of the function h.and lead to (3.4.12).
To show (3.4.13), we note that

$$\begin{aligned}
&\limsup_{m\to\infty}\{\mathbf{E}(|A_t^{n,m}-A_t^n|)\}^2\\
\leq\ &\limsup_{m\to\infty}\{\mathbf{E}\int_0^1|\frac{\partial F_{n,m}}{\partial t}(X_s,s)-\frac{\partial F_n}{\partial t}(X_s,s)|\mathrm{d}s\}^2\\
\leq\ &\limsup_{m\to\infty}\int_0^1\int_{\mathbb{R}}\phi^2(x,s)\mathrm{d}x\mathrm{d}s\int_0^1\int_{\mathbb{R}}(\frac{\partial F_{n,m}}{\partial t}(x,s)-\frac{\partial F_n}{\partial t}(x,s))^2\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Since : $\int_0^1\int_{\mathbb{R}}\phi^2(x,s)\mathrm{d}x\mathrm{d}s=\frac{1}{2\pi}\int_0^1\int_{\mathbb{R}}e^{-2s\psi(\xi)}\mathrm{d}\xi\mathrm{d}s<\infty$, then the last term in the above inequality is smaller than

$$cste\lim_{m\to\infty}\int_0^1\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{\partial F_n}{\partial t}(x,s+\tau/m)-\frac{\partial F_n}{\partial t}(x,s)\right)^2 g(\tau)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s,$$

which is equal to

$$cste\int_{\mathbb{R}}\lim_{m\to\infty}\int_0^1\int_{\mathbb{R}}\left(\frac{\partial F_n}{\partial t}(x,s+\tau/m)-\frac{\partial F_n}{\partial t}(x,s)\right)^2\mathrm{d}x\mathrm{d}s g(\tau)\mathrm{d}\tau$$

and this last limit is equal to 0.
Now we let n tend to ∞ in the equation (3.4.10). We have : $(F_n(X_t,t)-F_n(0,0)-V_t^n)$ converges a.s. to $(F(X_t,t)-F(0,0)-V_t)$. To finish the proof we show now :

$$A_t^n \overset{L^1}{\longrightarrow} -\int_0^t\int_{\mathbb{R}}\frac{\partial(\mathcal{I}F)}{\partial t}(x,s)\mathrm{d}L_s^x \tag{3.4.14}$$

$$B_t^n \overset{L^1}{\longrightarrow} -\int_0^t\int_{\mathbb{R}}\mathcal{B}\mathcal{I}F(x,s)\mathrm{d}L_s^x \tag{3.4.15}$$

$$M_t^n \overset{L^1}{\longrightarrow} M_t. \tag{3.4.16}$$

With the arguments used to show (3.4.13) we can show that $\lim_{n\to\infty}\mathbf{E}|A_t^n-A_t|=0$, where $A_t=\int_0^t\frac{\partial F}{\partial t}(X_s,s)\mathrm{d}s$. Thanks to Lemma 3.3.7 and the identity : $\mathcal{I}(\frac{\partial F}{\partial t})=\frac{\partial(\mathcal{I}F)}{\partial t}$, we have :

$$A_t=-\int_0^t\int_{\mathbb{R}}\mathcal{I}(\frac{\partial F}{\partial t}(x,s))\mathrm{d}L_s^x=-\int_0^t\int_{\mathbb{R}}\frac{\partial(\mathcal{I}F)}{\partial t}(x,s)\mathrm{d}L_s^x,$$

which gives (3.4.14).

From $\hat{F}_n(\xi,s) = \hat{h}(-\xi/n)\hat{F}(\xi,s)$ and $\widehat{\frac{\partial F_n}{\partial x}}(\xi,s) = -n\widehat{h'}(-\xi/n)\hat{F}(\xi,s)$ it follows that $\|F_n\|_* + \left\|\frac{\partial F_n}{\partial x}\right\|_* < \infty$. By the Lemma 3.4.1, $\mathcal{BI}F_n$ and $\mathcal{B}F_n$ are hence well defined and belong to Υ. Moreover we have : $\mathcal{IB}F_n(x,s)+\mathcal{BI}F_n(0,s) = \mathcal{BI}F_n(x,s)$, hence thanks to Lemma 3.3.3 we obtain

$$B_t^n = -\int_0^1 \int_{\mathbb{R}} \mathcal{BI}F_n(x,s)\mathrm{d}L_s^x.$$

But we have : $\|F-F_n\|_*^2 = \int_0^1 \beta(s)\int_{\mathbb{R}}(1+\bar{\psi}(\xi))|1-\hat{h}(-\xi/n)|^2|\hat{F}(\xi,s)|^2\mathrm{d}\xi\mathrm{d}s$, which gives by dominated convergence :

$$\lim_{n\to\infty} \|F_n - F\|_*^2 = 0, \tag{3.4.17}$$

which leads to (3.4.15).
Finally, we obtain (3.4.16) thanks to (3.4.17) and Lemma 3.4.4 □

Proof of Theorem 3.1.1 We define the norm $\|.\|_+$ by

$$\|f\|_+^2 = \int_0^1 \beta(s)\int_{\mathbb{R}}(1+\psi(\xi))|\hat{f}(\xi,s)|^2\mathrm{d}s,$$

for f measurable function from $\mathbb{R}\times[0,1]$.
Note that the condition (3.1.6) holds if and only if, $\|F\|_+^2 < \infty$. From (3.4.6), we can extend the operator $\mathcal{BI}$ from $\mathcal{C}_c^{\infty,0}(\mathbb{R}\times[0,1])$ to the set of functions f such that $\|f\|_+ < \infty$. We denote this extension by $\mathcal{H}$. Moreover we obtain from (3.4.6) that if $\|F\|_+ < \infty$, then $\|\mathcal{H}F\| < \infty$. Using similar arguments as the arguments of the proof of Lemma 3.4.4, we show that M^F is a well-defined square-integrable martingale and $\mathbf{E}[(M_t^F)^2] \le cste\|F\|_+^2$. With the arguments used to prove Theorem 3.1.2, we establish the decomposition (3.1.7) with $N_t^F = -\int_0^t\int_{\mathbb{R}}(\mathcal{H}F(x,s)+\frac{\partial\mathcal{I}F}{\partial t}(x,s))\mathrm{d}L_s^x$. It remains to show that :

$$\mathcal{H}F + \partial\mathcal{I}F/\partial t = \mathcal{DI}F. \tag{3.4.18}$$

Making use of the approximations $F_{n,m}$ and F_n used in the proof of Theorem 3.1.2, one establishes the following formula for any (x,τ) in $\mathbb{R}\times[0,1]$ and $0\le t\le 1-\tau$:

$$\begin{aligned} & \mathcal{I}F(Z_t+x,\tau+t) \\ = \; & \mathcal{I}F(x,\tau) + \int_0^t \frac{\partial\mathcal{I}F}{\partial t}(Z_s+x,s+\tau)\mathrm{d}s + \int_0^t \mathcal{H}F(Z_s+x,s+\tau)\mathrm{d}s \\ & + \int_0^t\int_{\{|y|\le 1\}}(\mathcal{I}F(Z_{s-}+x+y,s+\tau) - \mathcal{I}F(Z_{s-}+x,s+\tau))\tilde{\mu}_Z(\mathrm{d}y,\mathrm{d}s), \end{aligned}$$

where $\tilde{\mu}_Z$ is the compensated Poisson measure associated to the jumps of Z. Thus we have for any positive t such that $t+\tau\le 1$:

$$
\begin{aligned}
&\bar{P}_t(\mathcal{I}F)(x,\tau) - \mathcal{I}F(x,\tau) \\
= &\int_0^t \mathbf{E}[\mathcal{H}F(Z_s+x,s+\tau)]\mathrm{d}s + \int_0^t \mathbf{E}\left[\frac{\partial \mathcal{I}F}{\partial t}(Z_s+x,s+\tau)\right]\mathrm{d}s.
\end{aligned}
$$

Dividing each member of the above equation by t, we obtain (3.4.18) by letting t to 0 and using the following assertion that one can easily checks :
If h is such that $\|h\|_Z < \infty$ then $t^{-1}\int_0^t \mathbf{E}[h(Z_s+x,s+\tau)]\mathrm{d}s 1_{(t+\tau\leq 1)}$ converges with respect to $\|.\|_Z$ to $f(x,\tau)$ as t tends to 0.

□

3.5 Local times on curves

For a semimartingale Y, the natural definition of the local time on a bounded variation curve b is the local time at 0 of the semimartingale $(Y-b)$. In the present case of a Lévy process without Brownian component, this local time is identically equal to zero. The construction done in Section 3 allows to define $(L_t^{b(.)})_{0\leq t\leq 1}$, the local time of X along any measurable curve $(b(t))_{0\leq t\leq 1}$ by setting

$$L_t^{b(.)} = \int_0^t \int_{\mathbb{R}} 1_{(-\infty,b(s))}(x)\mathrm{d}L_s^x. \tag{3.5.1}$$

To show that the definition (3.5.1) makes sense, we check the two following points.
(i) The function F defined by $F(x,s) = 1_{(-\infty,b(s))}(x)$ belongs to the space Υ_{loc} .
(ii) For every $k>0$, set $T_k = \inf\{t : |X_t| > k\}\wedge 1$, then

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^{t\wedge T_k} 1_{\{|X_s-b(s)|<\epsilon\}}\mathrm{d}s = \int_0^{t\wedge T_k}\int_{\mathbb{R}} 1_{(-\infty,b(s))}(x)\mathrm{d}L_s^x = L_{t\wedge T_k}^{b(.)} \text{ in } L^1(\mathbf{P}).$$

(i) For $k>0$, set $b_k(s) = (-k)\vee(b(s)\wedge k)$ and $F_k(x,s) = 1_{(-k,b_k(s))}(x)$. We have : $F_k(x,s) = F(x,s)$ if $|x|<k$. Note that $\hat{F}_k(\xi,s) = (i\xi)^{-1}(e^{ib_k(s)\xi} - e^{-ik\xi})$, and hence that $|\hat{F}_k(\xi,s)|^2 = 2\xi^{-2}\{1-\cos[(b_k(s)+k)\xi]\}$. According to Lemma 4.2.2 of [33] we have :

$$
\begin{aligned}
\|F_k\|^2 &\leq cste\int_0^1 \beta(s)\int_{\mathbb{R}}\frac{1-\cos[(b_k(s)+k)\xi]}{\xi^2}\mathrm{d}\xi\mathrm{d}s \\
&= cste\int_0^1 \beta(s)|b_k(s)+k|\mathrm{d}s = cste\, k\int_0^1 \beta(s)\mathrm{d}s < \infty,
\end{aligned}
$$

consequently $F_k \in \Upsilon$ and (i) is checked.

(ii) For $k, \epsilon > 0$, set

$$F_{\epsilon,k}(x,s) = \frac{1}{2\epsilon}\int_{-\infty}^{x}(1_{(-k-\epsilon,-k+\epsilon)}(y) - 1_{(b_k(s)-\epsilon,b_k(s)+\epsilon)}(y))\mathrm{d}y.$$

We have :

$$\hat{F}_{\epsilon,k}(\xi,s) = \frac{e^{ib_k(s)\xi} - e^{-ik\xi}}{i\xi}.\frac{e^{i\epsilon\xi} - e^{-i\epsilon\xi}}{2i\xi\epsilon},$$

hence by dominated convergence $\lim_{\epsilon\to 0}\|F_{\epsilon,k} - F_k\| = 0$ and consequently

$$\lim_{\epsilon\to 0}\mathbf{E}\sup_{t\in[0,1]}\left|\int_0^t\int_{\mathbb{R}}F_{\epsilon,k}(x,s)\mathrm{d}L_s^x - \int_0^t\int_{\mathbb{R}}F_k(x,s)\mathrm{d}L_s^x\right| = 0.$$

On the other hand, thanks to Lemma 3.3.3, we have

$$\int_0^t\int_{\mathbb{R}}F_{\epsilon,k}(x,s)\mathrm{d}L_s^x = -\frac{1}{2\epsilon}\int_0^t(1_{(-k-\epsilon,-k+\epsilon)}(X_s) - 1_{(b_k(s)-\epsilon,b_k(s)+\epsilon)}(X_s))\mathrm{d}s. \quad (3.5.2)$$

Note that for $\epsilon < 1/2$ and $t \in [0, T_{k-1}]$ (where $T_{k-1} = \inf\{t : |X_t| > k-1\}\wedge 1$) the integral on the right-hand side of (3.5.2) agree with $\frac{1}{2\epsilon}\int_0^t 1_{\{|X_s-b(s)|<\epsilon\}}\mathrm{d}s$. Besides by definition of integration with respect to local time for the functions in Υ_{loc},

$$\int_0^{t\wedge T_{k-1}}\int_{\mathbb{R}}F_k(x,s)\mathrm{d}L_s^x = \int_0^{t\wedge T_{k-1}}\int_{\mathbb{R}}F(x,s)\mathrm{d}L_s^x.$$

Consequently (ii) is checked since

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^{t\wedge T_{k-1}}1_{\{|X_s-b(s)|<\epsilon\}}\mathrm{d}s = \int_0^{t\wedge T_{k-1}}\int_{\mathbb{R}}1_{(-\infty,b(s))}(x)\mathrm{d}L_s^x = L_{t\wedge T_{k-1}}^{b(.)} \text{ in } L^1(\mathbf{P}).$$

Remark that $L^{b(.)}$ is an increasing continuous functional that increases only at times when X and b take the same value. The local time process of X along curves extends the definition of local time at points which represent local times along constant curves.

Lemma 3.5.1. *Let b be a continuous function from $[0,1]$ to $\mathbb{R}$. Let f be a continuous function on $\mathbb{R}\times[0,1]$ admitting a continuous derivative $\frac{\partial f}{\partial x}$. Then the function $(x,s)\to f(x,s)1_{(x<b(s))}$ belongs to Υ_{loc} and we have*

$$\int_0^t\int_{\mathbb{R}}f(x,s)1_{(x<b(s))}\mathrm{d}L_s^x = \int_0^t f(b(s),s)\mathrm{d}_sL_s^{b(.)} - \int_0^t\frac{\partial f}{\partial x}(X_s,s)1_{(X_s<b(s))}\mathrm{d}s. \quad (3.5.3)$$

Proof of Lemma 3.5.1 First, we assume that f has a compact support. In this case the function g defined by $g(x,s) = f(x,s)1_{(x<b(s))}$, belongs to Υ. Indeed, by the integration by parts formula, there exist a constant C, depending of f, such that $|\hat{g}(\xi,s)| \leq C(1_{\{|\xi|\leq 1\}} + |\xi|^{-1}1_{\{|\xi|>1\}})$, thus $\|g\| < \infty$. The identity (3.5.3) is then obtained with the same arguments as the one used in the proof of Lemma 3.1 of [11].
For the general case, for any $k > 0$ let h_k be an element of $C_c^\infty(\mathbb{R})$ such that $h_k(x) = 1$ if $|x| < k$. Set $g_k = gh_k$. Then g_k belongs to Υ and $g = g_k$ on $[-k.k] \times [0,1]$, thus g belongs to Υ_{loc}. Furthermore if $k > \sup_{s\in[0,1]} |b(s)|$, we have

$$\begin{aligned}\int_0^{t\wedge T_k}\int_{\mathbb{R}} f(x,s)1_{(x<b(s))}\mathrm{d}L_s^x &= \int_0^{t\wedge T_k}\int_{\mathbb{R}} h_{k+1}(x)f(x,s)1_{(x<b(s))}\mathrm{d}L_s^x \\ &= \int_0^{t\wedge T_k} f(b(s),s)\mathrm{d}_sL_s^b \\ &\quad -\int_0^{t\wedge T_k}\frac{\partial f}{\partial x}(X_s,s)1_{(X_s<b(s))}\mathrm{d}s,\end{aligned}$$

which leads to (3.5.3). □

Remark 3.5.2. With the assumptions of Lemma 3.5.1 we similarly have :

$$\int_0^t\int_{\mathbb{R}} f(x,s)1_{(x>b(s))}\mathrm{d}L_s^x = -\int_0^t f(b(s),s)\mathrm{d}_sL_s^{b(.)} - \int_0^t \frac{\partial f}{\partial x}(X_s,s)1_{(X_s>b(s))}\mathrm{d}s.$$

Besides thanks to Theorem 3.3.1.(i) note that

$$\int_0^t\int_{\mathbb{R}} f(x,s)1_{(x=b(s))}\mathrm{d}L_s^x = 0.$$

We present now an Itô formula inspired from Peskir's formula written for continuous semimartingales [39] . This formula concerns the continuous functions F on $\mathbb{R}\times[0,1]$ for which there exists a continuous curve $(b(t))_{0\leq t\leq 1}$ such that setting $C = \{(x,s)\in\mathbb{R}\times[0,1] : x < b(s)\}$ and $D = \{(x,s)\in\mathbb{R}\times[0,1] : x > b(s)\}$, F is $\mathcal{C}^{2,1}$ on $\bar{C}$ and $\bar{D}$. Define $F_1(x,s) = F(x\wedge b(s),s)$, $F_2(x,s) = F(x\vee b(s),s)$. For such a function F we have the following formula.

Theorem 3.5.3. *The process $(F(X_t,t))_{0\leq t\leq 1}$ is a semimartingale admitting the following decomposition*

$$\begin{aligned}F(X_t,t) &= F(0,0) + M_t + V_t + \int_0^t (\mathcal{BI}F_2(b(s),s) - \mathcal{BI}F_1(b(s),s))\mathrm{d}_sL_s^{b(.)} \\ &\quad + \int_0^t \mathcal{A}F_1(X_s,s)1_{(X_s<b(s))}\mathrm{d}s + \int_0^t \mathcal{A}F_2(X_s,s)1_{(X_s>b(s))}\mathrm{d}s,\end{aligned}$$

where M and V are the local martingale and the bounded variation process defined in Theorem 3.1.2 *and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{I}$ are the operators respectively defined by* (3.1.2), (3.4.1) *and* (3.1.5).

Proof of Theorem 3.5.3 By the usual stopping time argument, we can assume that F has a support compact. Let K be such that $supp[F] \subset [-K, K] \times [0, 1]$. For any $m, n \in \mathbb{N}$, we define the function $F_{n,m}$ as in the proof of Theorem 3.1.2 and note that, the convergences involving $\frac{\partial F}{\partial t}$ excepted, all the convergences established there as m, and then n, tends to ∞, are still available here. Indeed the present assumption on F does not guarantee the existence of this derivative.
Nevertheless (3.4.12) and (3.4.13) hold because $\lim_{m\to\infty} \|F_{n,m} - F_n\|_* = 0$. Indeed, we have seen in the proof of Theorem 3.1.2 that for any n there exists a constant $k(n)$ such that

$$\|F_{n,m} - F_n\|_* \leq k(n) \int_0^1 \beta(s) \int_{\mathbb{R}} \int_0^\infty (F(x, s + \tau/m) - F(x, s))^2 g(\tau) \mathrm{d}\tau \mathrm{d}x \mathrm{d}s. \tag{3.5.4}$$

Since F is a continuous function with compact support, we see by dominated convergence, that the right-hand side of (3.5.4) converges to 0.
Besides, for each s, the law of X_s has a density with respect to the Lebesgue measure, hence for any n, m, $\int_0^t \frac{\partial F_{n,m}}{\partial t}(X_s, s) 1_{(X_s = b(s))} \mathrm{d}s = 0$. Consequently we have, similarly as (3.4.13) :

$$\int_0^t \frac{\partial F_{n,m}}{\partial t}(X_s, s)\mathrm{d}s \xrightarrow{L^1} \int_0^t \frac{\partial F}{\partial t}(X_s, s) 1_{(X_s < b(s))} \mathrm{d}s + \int_0^t \frac{\partial F}{\partial t}(X_s, s) 1_{(X_s > b(s))} \mathrm{d}s,$$

as $m \to \infty, n \to \infty$. Regrouping all the obtained convergences, we obtain

$$\begin{aligned} F(X_t, t) &= F(0,0) + \int_0^t \frac{\partial F}{\partial t}(X_s, s) 1_{(X_s < b(s))} \mathrm{d}s + \int_0^t \frac{\partial F}{\partial t}(X_s, s) 1_{(X_s > b(s))} \mathrm{d}s \\ &\quad + M_t + V_t - \int_0^t \int_{\mathbb{R}} \mathcal{B}\mathcal{I}F(x, s) \mathrm{d}L_s^x. \end{aligned}$$

For $i = 1, 2$, F_i belongs to $\mathcal{C}^{2,1}$, hence $\mathcal{B}\mathcal{I}F_i$ admits a continuous derivative with respect to x equal to $\mathcal{B}F_i$. Thanks to Lemma 3.5.1, Remark 3.5.2 and the identity $F_1(x, s) + F_2(x, s) = F(x, s) + F(b(s), s)$, we have :

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \mathcal{BI}F(x,s)\mathrm{d}L_s^x \\
= \;& \int_0^t \int_{\mathbb{R}} \mathcal{BI}F_1(x,s)\mathrm{d}L_s^x + \int_0^t \int_{\mathbb{R}} \mathcal{BI}F_2(x,s)\mathrm{d}L_s^x \\
= \;& \int_0^t \mathcal{BI}F_1(b(s),s)\mathrm{d}_s L_s^b - \int_0^t \mathcal{B}F_1(X_s,s)1_{(X_s<b(s))}\mathrm{d}s \\
& - \int_0^t \mathcal{BI}F_2(b(s),s)\mathrm{d}_s L_s^b - \int_0^t \mathcal{B}F_2(X_s,s)1_{(X_s>b(s))}\mathrm{d}s,
\end{aligned}
$$

which leads to Theorem 3.5.3. □

Chapitre 4

Extended Itô calculus for symmetric Markov Processes

Abstract : Chen, Fitzsimmons, Kuwae and Zhang [7] have established an Itô formula consisting in the development of $F(u(X))$ for a symmetric Markov process X, a function u in the Dirichlet space of X and any $\mathcal{C}^2$-function F. We give here an extension of this formula for u locally in the Dirichlet space of X and F admitting a locally bounded Radon-Nicodym derivative. This formula has some analogies with various extended Itô formulas for semimartingales using the local time stochastic calculus. But here the part of the local time is played by a process $(\Gamma_t^a, a \in \mathbb{R}, t \geq 0)$ defined thanks to Nakao's operator [35].

4.1 Introduction and main results

For any real-valued semimartingale $Y = (Y_0 + M_t + N_t)_{t\geq 0}$ (M martingale and N bounded variation process) and any function F in $\mathcal{C}^2(\mathbb{R})$, the classical Itô formula

$$\begin{aligned} F(Y_t) &= F(Y_0) + \int_0^t F'(Y_s)\mathrm{d}M_s + \int_0^t F'(Y_s)\mathrm{d}N_s + \frac{1}{2}\int_0^t F''(Y_s)\mathrm{d}\langle M^c\rangle_s \\ &\quad + \sum_{s\leq t}\{F(Y_s) - F(Y_{s-}) - F'(Y_{s-})\Delta Y_s\} \end{aligned} \tag{4.1.1}$$

provides both an explicit expansion of $(F(Y_t))_{t\geq 0}$ and its stochastic structure of semimartingale.

Let now E be a locally compact separable metric space , m a positive Radon measure on E, and X a m-symmetric Hunt process. Under the assumption that the associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ of X is regular, Fukushima has showed that for any function u in $\mathcal{F}$, the additive functional (abbreviated as AF) $(u(X_t) - u(X_0))_{t\geq 0}$ admits the following unique decomposition :

$$u(X_t) = u(X_0) + M_t^u + N_t^u \quad \mathbf{P}_x - \text{a.e for quasi-every } x \text{ in } E, \tag{4.1.2}$$

where M^u is a martingale AF of finite energy and N^u is a continuous AF of zero energy.
Although $u(X)$ is not in general a semimartingale, Nakao [35] and Chen et al. [7] have proved that (4.1.1) is still valid with $u(X)$, M^u and N^u replacing respectively Y, M and N. This is done thanks to the construction of a new stochastic integral with respect to N^u, which takes the place of the well defined Lebesgue-Stieltjes integral for the bounded variation processes. As the classical Itô formula (4.1.1), this Itô formula for symmetric Markov processes requires the use of $\mathcal{C}^2$-functions. For the semimartingale case, there exist extended versions of (4.1.1) relaxing this regularity condition. This extensions are based on the replacement of the fourth and fifth terms of the right-hand-side of (4.1.1) by an alternative expression requiring only the existence of F' and some integrability condition on F' (see for example [12], [13], [14]). The integrability condition insures also the existence of the other terms of (4.1.1).
The question of relaxing the regularity condition on F in the formula of Nakao and Chen et al. is a more complex question. Indeed the integral $\int_0^t F'(u(X_s))\mathrm{d}N_s^u$ is well defined only when $F'(u)$ belongs to $\mathcal{F}_{loc}$, the set of functions locally in $\mathcal{F}$. As in [7], $u \in \mathcal{F}_{loc}$ means that there exists a nest of finely open Borel sets $\{G_k\}_{k\in\mathbb{N}}$ and a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that $f = f_k$ q.e on G_k. As an example, in the case X is a Brownian motion, this condition implies that the second derivative F'' exists at least as a weak derivative. Nevertheless, in the general case, we know that for any function F element of $\mathcal{C}^1(\mathbb{R})$ with bounded derivative, $F(u)$ belongs to $\mathcal{F}$ and the process $F(u(X))$ hence admits a Fukushima decomposition. We can thus hope to obtain an Itô formula for $\mathcal{C}^1$-functions F that would express each element of the decomposition of $F(u(X))$ in terms of F, u, N^u and M^u. Our purpose here is to establish such a formula. The obtained formula is actually established for the functions F with locally bounded Radon Nikodym derivative and u element of $\mathcal{F}_{loc}$.
Before introducing this extended Itô formula for symmetric Markov processes, remark that one can easily obtain an extended Itô formula in case $u(X)$ is a semimartingale. Indeed, under the assumption that X has an infinite life time, we note (see (3.4) in [7]) that $u(X)$ is then a reversible semimartingale and that one can hence make use of [12] or [15] to develop $F(u(X))$. But in general, $u(X)$ is not a semimartingale.

The extended Itô formula for symmetric Markov processes presented here is based on the construction for a fixed $t > 0$, of a stochastic integral of deterministic functions with respect to the process $(\Gamma_t^a(u))_{a\in\mathbb{R}}$, defined as follows.
For u in $\mathcal{F}$, let $M^{u,c}$ be the continuous part of M^u. For any real a and $t \geq 0$, we set

$$Z_t^a(u) = \int_0^t 1_{\{u(X_s)\leq a\}}\mathrm{d}M_s^{u,c}$$

and define Γ^a by

$$\Gamma^a(u) = (\Gamma^a_t(u))_{t\geq 0} = (\Gamma(Z^a(u))_t)_{t\geq 0} = \Gamma(Z^a(u))$$

where Γ is the operator on the space of martingale AF with finite energy constructed by Nakao [35] (its definition is recalled in Section 4.2). The process $(\Gamma^a_t(u))_{t\geq 0}$ is hence an additive functional with zero energy.
In Section 2, we will see that the definition of $\Gamma^a(u)$ can be extended to functions u in $\mathcal{F}_{loc}$. In that case, the process $M^{u,c}$ is a continuous martingale AF on $[\![0,\zeta[\![$ locally of finite energy and the process $(\Gamma^a_t(u))_{t\geq 0}$ is an Af on $[\![0,\zeta[\![$ locally with zero energy.
As shown by the Tanaka formula (4.1.4) below, the doubly-indexed process $(\Gamma^a_t(u), a\in\mathbb{R}, t\geq 0)$ plays almost the part of a local time process for $u(X)$. In Section 5, this analogy with local time will be fully clarified under some stronger assumption on u.

To introduce the obtained Itô formula, we need the objects presented by the following lemma. We denote by $(N(x,\mathrm{d}y),H)$ a Lévy system for X (See Definition A.3.7 of [21]), by ν_H the Revuz's measure of H and by ζ the life time of X.

Lemma 4.1.1. *Let $u\in\mathcal{F}$ (resp. $u\in\mathcal{F}_{loc}$).There exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers converging to 0 and such that for any locally absolutely continuous function F from $\mathbb{R}$ into $\mathbb{R}$ with a locally bounded Radon-Nikodym derivative, the following two processes are well defined.*

$$\begin{aligned} M^d_t(F,u) &= \lim_{n\to\infty}\Big\{\sum_{s\leq t}\{F(u(X_s))-F(u(X_{s-}))\}1_{\{\varepsilon_n<|u(X_s)-u(X_{s-})|<1\}}1_{\{s<\zeta\}} \\ &\quad -\int_0^t\int_{\{\varepsilon_n<|u(y)-u(X_s)|<1\}}\{F(u(y))-F(u(X_s))\}N(X_s,\mathrm{d}y)\mathrm{d}H_s\Big\} \\ A_t(F,u) &= \lim_{n\to\infty}\int_0^t\int_{\{\varepsilon_n<|u(y)-u(X_s)|<1\}}\{F(u(y))-F(u(X_s))\}N(X_s,\mathrm{d}y)\mathrm{d}H_s \end{aligned}$$

The above limits are uniform on any compact of $[0,\infty)$ (resp. $[0,\zeta)$) $\mathbf{P}_x$-a.e for q.e $x\in E$. Moreover $(M^d_t(F,u))_{t\geq 0}$ is a local martingale AF (resp. AF on $[\![0,\zeta[\![$) with locally finite energy and $(A_t(F,u))_{t\geq 0}$ is a continuous AF (resp. AF on $[\![0,\zeta[\![$) locally with 0 energy.

With the notation of Lemma 4.1.1, we have the following Itô formula.

Theorem 4.1.2. *Let $u\in\mathcal{F}$ (resp. $u\in\mathcal{F}_{loc}$). For any locally absolutely continuous function F from $\mathbb{R}$ into $\mathbb{R}$ with a locally bounded Radon-Nikodym derivative F' such that $F(0)=0$, the process $(F(u(X_t)), t\in[0,\infty))$ (resp. $t\in[0,\zeta)$) admits the following decomposition $\mathbf{P}_x$-a.e for q.e $x\in E$*

$$F(u(X_t)) = F(u(X_0)) + M_t(F,u) + Q_t(F,u) + V_t(F,u) \tag{4.1.3}$$

where $M(F,u)$ is a local martingale AF (resp. AF on $[\![0,\zeta[\![$) locally of finite energy, $Q(F,u)$ is an AF (resp. AF on $[\![0,\zeta[\![$) locally of zero energy, and $V(F,u)$ is a bounded variation process respectively given by :

$$\begin{aligned} M_t(F,u) &= M_t^d(F,u) + \int_0^t F'(u(X_s))\mathrm{d}M_s^{u,c} \\ Q_t(F,u) &= \int_{\mathbb{R}} F'(z)\mathrm{d}_z\Gamma_t^z(u) + A_t(F,u) \\ V_t(F,u) &= \sum_{s\le t}\{F(u(X_s)) - F(u(X_{s-}))\}1_{\{|u(X_s)-u(X_{s-})|\ge 1\}}1_{\{s<\xi\}} \\ &\quad -F(u(X_{\xi-}))1_{\{t\ge\xi\}} \end{aligned}$$

Note that for u element of $\mathcal{F}$ and F in $\mathcal{C}^2(\mathbb{R})$, (4.1.3) provides the Itô formula of Chen et al. [7] together with the identity connecting integration with respect to $(N_t^u)_{t\ge 0}$ and integration with respect to $(\Gamma_t^a(u))_{a\in\mathbb{R}}$ for smooth enough functions.

As a consequence of Theorem 4.1.2, we obtain the following Tanaka formula for Γ_t^a :

$$\begin{aligned} \Gamma_t^a(u) &= (u(X_0)-a)^- - (u(X_t)-a)^- + \int_0^t 1_{\{u(X_{s-})\le a\}}\mathrm{d}M_s^{u,c} \\ &\quad + \lim_{n\to\infty}\sum_{s\le t}\{(u(X_s)-a)^- - (u(X_{s-})-a)^-\}1_{\{|u(X_s)-u(X_{s-})|>\varepsilon_n\}} \end{aligned} \tag{4.1.4}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is the sequence of Lemma 4.1.1 and the limit is uniform on any compact $\mathbf{P}_x$-a.e for q.e $x \in E$. Using Tanaka's formula for semimartingales (see [41]), we obtain that when $u(X)$ is a martingale, $-2\Gamma^a(u)$ is the local time process of $u(X)$ at level a. This is the case when $u(x) = x$ and X is a symmetric Lévy process.

Formula (4.1.3) is hence reminiscent of various extensions of Itô formula involving stochastic integrals with respect to local time, as for example the extensions given in [6] for some martingales, [10] for the Brownian Motion, [11] and [14] for Lévy processes with Brownian component and [45] for Lévy processes without Brownian component. Note that in case the martingale part of $u(X)$ has no continuous component, the process $\Gamma^a(u)$ is identically equal to 0. But (4.1.3) still represents an improvement of Fukushima's decomposition since (4.1.3) requires only u in $\mathcal{F}_{loc}$ and F with a locally bounded Radon-Nikodym derivative.

Integration with respect to $(\Gamma_t^a(u))_{a\in\mathbb{R}}$ is constructed in Section 3 and the Itô formula (4.1.3) is established in Section 4.

In Section 4.5, we will show that, when $\Gamma(M^{u,c})$ is of bounded variation, $u(X)$ admits a local time process $(L_t^a, a \in \mathbb{R}, t < \zeta)$ satisfying an occupation time formula of the same type as the occupation time formula for the semimartingales and in this case, the process of locally zero energy $Q(F,u)$ can be rewritten as :

$$Q_t(F,u) = -\frac{1}{2}\int_{\mathbb{R}} F'(z)\mathrm{d}_z L_t^z + \int_0^t F'(u(X_s))\mathrm{d}\Gamma(M^{u,c})_s + A_t(F,u), t < \zeta$$

Finally in Section 4.6 we give a multidimensional version of Theorem 4.1.2.

4.2 Preliminaries on m-symmetric Hunt processes

Let E be a locally compact separable metric space, m a positive Radon measure on E such that $Supp[m] = E$, Δ be a point outside E and $E_\Delta = E \cup \Delta$. Let $X = \{\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \theta_t, \zeta, \mathbf{P}_x, x \in E_\Delta, t \geq 0\}$ be a m-symmetric Hunt Processes such that its associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ is regular on $L^2(E; m)$. We may take as Ω the space $D([0,\infty[\to E_\Delta)$ of càdlàg functions from $[0,\infty[$ to E_Δ, for which Δ is a cemetery (i.e. if $\omega(t) = \Delta$, then $\omega(s) = \Delta$ for any $s > t$) and denote by θ_t the operator $\omega(s) \to \theta_t\omega(s) := \omega(t+s)$. Every element u of $\mathcal{F}$ admits a quasi-continuous m-version. In the sequel, the functions in $\mathcal{F}$ are always represented by their quasi-continuous m-versions. We use the term "quasi everywhere" or "q.e" to mean "except on an exceptional set".

We say that a subset Ξ of Ω is a defining set of a process $A = (A_t)_{t\geq 0}$ with values in $[-\infty, \infty]$, if for any $\omega \in \Xi, t, s \geq 0 : \theta_t\Xi \subset \Xi$, $A_0(\omega) = 0$, $A_.(\omega)$ is càdlàg and finite on $[0, \zeta[$,

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$$

and $A_t(\omega_\Delta) = 0$, where ω_Δ is the constant path equal to Δ. A $(\mathcal{F}_t)$-adapted process is an additive functional if it has a defining set $\Xi \in \mathcal{F}_\infty$ admitting an exceptional set, i.e : $\mathbf{P}_x(\Xi) = 1$ for q.e $x \in E$.

An $(\mathcal{F}_t)$-adapted process is an additive functional on $[\![0, \zeta[\![$ or a local additive functional if it satisfies all the conditions to be an additive functional except that the additive property $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$ is required only for $t+s < \zeta(\omega)$.

Let $\mathcal{F}_\infty^m$ (resp. $\mathcal{F}_t^m$) be the $\mathbf{P}_m$-completion of $\sigma\{X_s, 0 \leq s < \infty\}$ (resp. $\sigma\{X_s, 0 \leq s \leq t\}$). An $(\mathcal{F}_t)$-adapted process is an additive functional admitting m-null set if it has a defining set $\Xi \in \mathcal{F}_\infty^m$ such that $\mathbf{P}_x(\Xi) = 1$ for m-a.e $x \in E$.

The abbreviations AF, PAF, CAF, PCAF and MAF stand respectively for "additive functional", "positive additive functional", "continuous additive functional", "positive continuous additive functional" and "martingale additive functional", respectively. Let $\mathcal{M}$ and $\mathcal{N}_c$ denote, respectively, the space of MAF's of finite energy and the space of continuous additive functionals of zero energy N such that

$\mathbf{E}_x(|N_t|) < \infty$ q.e. for each $t > 0$. Moreover $\mathcal{M}^c$ denotes the subset of continuous elements of $\mathcal{M}$ and $\mathcal{M}^d$ denotes the subset of purely discontinuous elements of $\mathcal{M}$.

For $u \in \mathcal{F}$, the elements M^u and N^u of the Fukushima's decomposition (4.1.2) are elements of respectively $\mathcal{M}$ and $\mathcal{N}_c$. We denote by $M^{u,c}$, $M^{u,j}$ and $M^{u,\kappa}$ respectively the continuous part, the jump part and the killing part of M^u (see Section 5.3 of [21]). This three martingales are elements of $\mathcal{M}$.

Let Γ the linear operator from $\mathcal{M}$ to $\mathcal{N}_c$ constructed by Nakao [35] in the following way. It is shown in [35] that for every $Z \in \mathcal{M}$, there is a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}(w,v) + (w,v)_m = \frac{1}{2}\mu_{\langle M^v + M^{v,\kappa}, Z\rangle}(E) \text{ for every } v \in \mathcal{F},$$

where $(w,v)_m = \int_E w(x)v(x)m(\mathrm{d}x)$ and $\mu_{\langle M^v + M^{v,\kappa}, Z\rangle}$ is the smooth signed measure corresponding to $\langle M^v + M^{v,\kappa}, Z\rangle$ by the Revuz correspondence. The process $\Gamma(Z)$ is then defined by :

$$\Gamma_t(Z) = N_t^w - \int_0^t w(X_s)\mathrm{d}s$$

This operator satisfies : $\Gamma(M^u) = N^u$ for $u \in \mathcal{F}$. Thus N^u admits the decomposition :

$$N^u \;=\; {}^cN^u \;+\; {}^jN^u \;+\; {}^\kappa N^u, \tag{4.2.1}$$

where for $p \in \{c, j, \kappa\}$: ${}^pN^u = \Gamma(M^{u,p})$.

For a Borel subset B of $E \cup \{\Delta\}$, it is known that $\tau_B = \inf\{t > 0 : X_t \notin B\}$ and $\sigma_B = \inf\{t > 0 : X_t \in B\}$ are $(\mathcal{F}_t)$-stopping times.

An increasing sequence of Borel sets $\{G_k\}$ in E is called a nest if

$$\mathbf{P}_x\left(\lim_{k\to\infty} \tau_{G_k} = \zeta\right) = 1 \text{ for q.e } x \in E$$

Let $\mathcal{D}$ be a class of AF's. We say that an AF (resp. AF on $[\![0,\zeta[\![$) is locally in $\mathcal{D}$ and write $A \in \mathcal{D}_{loc}$ (resp. $A \in \mathcal{D}_{f\text{-}loc}$) if there exists a sequence $\{A^n\}$ in $\mathcal{D}$ and an increasing sequence of stopping times T_n with $T_n \to \infty$ (resp. a nest $\{G_n\}$ of finely open Borel sets) such that $\mathbf{P}_x$-a.e for q.e $x \in E$, $A_t = A_t^n$ for $t < T_n$ (resp. $t < \tau_{G_n}$).

Let $\{A^n\}$ be a sequence in $\mathcal{D}$ such that for $k > n$, $\mathbf{P}_x$-a.e for q.e $x \in E$, $A_t^k = A_t^n$ for $t < \tau_{G_n}$, then it is clear that the process

$$A_t := \begin{cases} A_t^n & \text{for } t < \tau_{G_n} \\ 0 & \text{for } t \geq \zeta \end{cases}$$

is a well-defined element of $\mathcal{D}_{f\text{-}loc}$. A Borel function f from E into $\mathbb{R}$ is said to be locally in $\mathcal{F}$ (and denoted as $f \in \mathcal{F}_{loc}$), if there is a nest of finely open Borel sets $\{G_k\}$ and a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that $f = f_k$ q.e on G_k. This is equivalent to (see Lemma 3.1.(ii) in [7]) there is a nest of closed sets $\{D_k\}$ and a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_b$ such that $f = f_k$ q.e on D_k. For a such f,

$$M_t^{f,c} := \begin{cases} M_t^{f_k,c} & \text{for} \quad t < \sigma_{E\setminus G_k} \\ 0 & \text{for} \quad t \geq \lim_{k\to\infty} \sigma_{E\setminus G_k} \end{cases}$$

is well defined and belongs to $\mathcal{M}_{f\text{-}loc}$ because, for $n > k$, $M_t^{f_n,c} = M_t^{f_k,c}\ \forall t \leq \sigma_{E\setminus G_k}$ $\mathbf{P}_x$-a.e for q.e $x \in E$. Indeed, the last property is shown in Lemma 5.3.1 in [21] for τ_{G_k} instead of $\sigma_{E\setminus G_k}$, we conclude with the following observation :
For a CAF A, and a Borel set $G \subset E$, $\mathbf{P}_x$-a.e for q.e $x \in E$:

$$A_t = 0 \text{ for } t < \tau_G \Leftrightarrow A_t = 0 \text{ for } t < \sigma_{E\setminus G} \tag{4.2.2}$$

Every $f \in \mathcal{F}_{loc}$ admits a quasi-continuous m-version, so we may assume that all $f \in \mathcal{F}_{loc}$ are quasi-continuous and we set $f(\Delta) = 0$.
We use the following notation for a locally bounded measurable function f and a $(\mathcal{F}_t)_{t\geq 0}$-semimartingale M :

$$(f * M)_t \ = \int_0^t f(X_{s-})dM_s$$

We will use repeatedly the following fact (see Theorem 5.6.2 in [21]) :
For any F in $\mathcal{C}^1(\mathbb{R}^d)$ (d is a positive integer) and $u_1, ..., u_d$ in $\mathcal{F}_b$, the composite function $Fu = F(u_1, ..., u_d)$ belongs to $\mathcal{F}_{loc}$ and

$$M^{Fu,c} = \sum_{i=1}^{d} F_{x_i}(u) * M^{u_i,c} \tag{4.2.3}$$

Chen et al. [7] have extended Nakao's definition of the operator Γ to the set of locally square-integrable MAF. We keep using the letter Γ for this extension without possible confusion since thanks to Theorem 3.6 of [7] on the set $\mathcal{M}$, both definitions given in [7] and [35] agree $\mathbf{P}_m$-a.e. on $[\![0,\zeta[\![$. For a continuous locally square-integrable MAF M, $\Gamma(M)$ is defined to be the following CAF admitting m-null set on $[\![0,\zeta[\![$:

$$\Gamma_t(M) = -\frac{1}{2}(M_t + M_t \circ r_t) \text{ for } t \in [0,\zeta[\tag{4.2.4}$$

where the operator r_t is defined by

$$r_t(\omega)(s) = \omega((t-s)-)1_{\{0\leq s\leq t\}} + \omega(0)1_{\{s>t\}} \text{ for a path } \omega \in \{t < \zeta\}$$

and $r_t(\omega) := \omega_\Delta$ for a path $\omega \in \{t \geq \zeta\}$.
The continuity of $\Gamma(M)$ $\mathbf{P}_m$-a.e on $[0,\zeta[$ is a consequence of Theorem 2.18 in [7].
For f a bounded element of $\mathcal{F}$ and M in $\mathcal{M}$, Nakao has defined the stochastic integral of $f(X)$ with respect to $\Gamma(M)$. We use here the extension of this definition

set by Chen et al. [7] for f in $\mathcal{F}_{loc}$ and M continuous locally square-integrable MAF as follows :

$$f * \Gamma(M)_t \;=\; \int_0^t f(X_{s-})\mathrm{d}\Gamma_s(M) \;:=\; \Gamma_t(f * M) - \frac{1}{2}\langle M^{f,c}, M\rangle_t \quad (4.2.5)$$

It is a CAF admitting m-null set on $[\![0,\zeta[\![$.
When $M \in \mathcal{M}$ and $f \in \mathcal{F}_{loc}$ the integral $f * \Gamma(M)_t$ can be well defined $\mathbf{P}_x$-a.e. for q.e. $x \in E$. In particular the process $(f * \Gamma(M)_t)_{t\geq 0}$ is a local CAF of X (Lemma 4.6 of [7]).
The argument developed by Chen et al. to write "q.e $x \in E$" instead of "m-a.e $x \in E$" in the proof of their Lemma 4.6 in [7], is sufficient to establish Lemma 4.2.1 below.

Lemma 4.2.1. *Let A be an AF of X (resp. AF on $[\![0,\zeta[\![$). Let G be a measurable subset of E_Δ (resp. $G \subset E$) and $\Xi := \{\omega \in \Omega : A_t \geq 0, \forall t < \tau_G\}$, then $\mathbf{P}_x(\Xi) = 1$ for m-a.e. $x \in E$ if and only if $\mathbf{P}_x(\Xi) = 1$ for q.e $x \in E$.*

Lemma 4.2.2. *Let $\{D_n\}$ be a nest of closed sets and $\sigma := \lim_{n\to\infty} \sigma_{E\setminus D_n}$. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{M}^c$ such that for $n < k$, $\mathbf{P}_x$-a.e for q.e $x \in E$, $M^n_t = M^k_t$ if $t < \sigma_{E\setminus D_n}$. Define a continuous locally square-integrable MAF M by :*

$$M_t = \begin{cases} M^n_t & on \quad t < \sigma_{E\setminus D_n} \\ 0 & on \quad t \geq \sigma \end{cases}$$

Then $\Gamma_t(M)$ can be well defined for all t in $[0,\infty)$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$, by setting

$$\Gamma_t(M) = \begin{cases} \Gamma_t(M^n) & on \quad t < \sigma_{E\setminus D_n} \\ 0 & on \quad t \geq \sigma \end{cases} \quad (4.2.6)$$

Moreover $\Gamma(M)$ belongs to $\mathcal{N}_{c,f\text{-}loc}$.

For f element of $\mathcal{F}_{loc}$, (4.2.5) shows then that $f * \Gamma(M)$ is a well-defined CAF on $[\![0,\zeta[\![$.

Proof of Lemma 4.2.2. A consequence of the m-symmetry assumption on X is that the measure $\mathbf{P}_m$, when restricted to $\{t < \zeta\}$ is invariant under r_t, so we have $\mathbf{P}_m$-a.e on $t < \zeta$:
$M_t \circ r_t = M^n_t \circ r_t$ if $t \leq \tau_{D_n} \circ r_t$, but since D_n is closed, for any $\omega \in \Omega$ and $t < \zeta(\omega)$: $t \leq \tau_{D_n}(\omega) \Leftrightarrow t \leq \tau_{D_n}(r_t\omega)$. Hence it follows from (4.2.4) that (4.2.6) hold $\mathbf{P}_m$-a.e on $[\![0,\tau_{D_n}[\![$. This show also, with Lemma 4.2.1 that if $l > n$, $\mathbf{P}_x$-a.e for q.e $x \in E$: $\Gamma_t(M^n) = \Gamma_t(M^l)$ for $t \leq \tau_{D_n}$ (and consequently for $t \leq \sigma_{E\setminus D_n}$ by (4.2.2)). Hence, the right-hand side of (4.2.6) is well defined as a CAF belongs to $\mathcal{N}_{c,f\text{-}loc}$. □

Remark 4.2.3. Lemma 4.2.2 shows that for any $u \in \mathcal{F}_{loc}$, ${}^cN^u := \Gamma(M^{u,c})$ is an element of $\mathcal{N}_{c,f\text{-}loc}$.

The above Lemma 4.2.1 and Theorem 4.1 of [7] lead to the following lemma.

Lemma 4.2.4. *Let M be an element of $\mathcal{M}$ such that $\Gamma(M)$ is of bounded variation on each compact interval of $[0,\zeta[$. Then for every element f of $\mathcal{F}_{loc}$, $\mathbf{P}_x$-a.e. q.e for $x \in E$, on $t < \zeta$, $\int_0^t f(X_s)\mathrm{d}\Gamma_s(M)$ coincides with the Lebesgue-Stieljes integral of $f(X)$ with respect to $\Gamma(M)$.*

For the reader's convenience, we recall the following result which is Theorem 5.2.1 of [21] and Theorem 3.2 of [35], the last statement can be seen directly from their proofs. By $e(M)$ we denote the energy of M.

Theorem 4.2.5. *Let $\{M^n : n \in \mathbb{N}\}$ be a e-Cauchy sequence of $\mathcal{M}$. There exists a unique element M of $\mathcal{M}$ such that $e(M^n - M)$ converges to zero. The subsequence n_k such that there exists $C \in \mathbb{R}_+$ such that for every k in $\mathbb{N}$: $e(M - M^{n_k}) < C2^{-4k}$, satisfies : $\mathbf{P}_x$-a.e for q.e $x \in E$, $M_t^{n_k}$ and $\Gamma_t(M^{n_k})$ converge uniformly on any finite interval of t to M_t and $\Gamma_t(M)$ respectively.*

4.3 Integration with respect to Γ^z

We fix a function u of $\mathcal{F}_{loc}$. Let $\{D_k\}_{k\in\mathbb{N}}$ be a nest of closed sets and $(u_k)_{k\in\mathbb{N}}$ be a sequence of bounded elements of $\mathcal{F}$ associated to u such that $u = u_k$ q.e on D_k. Let $\sigma := \lim_{n\to\infty} \sigma_{E\setminus D_n}$. For any real number a, define $Z^a = Z^a(u)$ by

$$Z_t^a = \begin{cases} \int_0^t 1_{\{u_k(X_{s-})\le a\}} \mathrm{d}M_s^{u_k,c} & \text{for} \quad t \le \sigma_{E\setminus D_k} \\ 0 & \text{for} \quad t \ge \sigma \end{cases}$$

Z^a is a MAF on $[\![0,\zeta[\![$ locally of finite energy. In particular, when u belongs to $\mathcal{F}$, Z^a is in $\mathcal{M}^c$ for any real a . By Lemma 4.2.2, $\Gamma(Z^a)$ is well defined and belongs to $\mathcal{N}_{c,f\text{-}loc}$.

Remark 4.3.1. *For u element of $\mathcal{F}$, we can choose D_k such that*

$$\sigma = \lim_{k\to\infty} \sigma_{E\setminus D_k} = \infty \ \mathbf{P}_x\text{-}a.e \text{ for q.e. } x \in E \tag{4.3.1}$$

Indeed, in this case, take $u_k := (-k) \vee u \wedge k$ and $G_k := \{x : |u(x)| < k\}$, then it follows from the strict continuity of u that $\lim_{k\to\infty} \sigma_{E\setminus G_k} = \infty$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$. Therefore, the nest of closed sets $\{F_k\}_{k\in\mathbb{N}}$ built in the proof of Lemma 3.1.(ii) in [7] satisfies the property (4.3.1) and $u = u_k$ q.e. on F_k. Choose then, $\{D_k\} = \{F_k\}$

Definition 4.3.2. *The process $(\Gamma_t^a, a \in \mathbb{R}, t \ge 0)$ is defined by $\Gamma_t^a = \Gamma_t^a(u) = \Gamma_t(Z^a)$.*

Consider an elementary function f, i.e. there exists two finite sequences $(z_i)_{0\leq i\leq n}$ and $(f_i)_{0\leq i\leq n-1}$ of real numbers such that :

$$f(z)=\sum_{i=0}^{n-1} f_i 1_{(z_i,z_{i+1}]}(z)$$

For such a function integration with respect to $\Gamma_t=\{\Gamma_t^z; z\in\mathbb{R}\}$ is defined to be the following CAF on $[\![0,\zeta[\![$:

$$\int_{\mathbb{R}} f(z)\mathrm{d}_z\Gamma_t^z \;=\; \sum_{i=0}^{n-1} f_i(\Gamma_t^{z_{i+1}}-\Gamma_t^{z_i}) \tag{4.3.2}$$

Thanks to the linearity property of the operator Γ we have for any elementary function f :

$$\int_{\mathbb{R}} f(z)\mathrm{d}_z\Gamma_t^z=\Gamma_t\left(\int_0^{\cdot} f(u(X_s))\mathrm{d}M_s^{u,c}\right)$$

For any $k\in\mathbb{N}$ we define the norm $\|.\|_k$ on the set of measurable functions f from $\mathbb{R}$ into $\mathbb{R}$ by

$$\|f\|_k=\left(\int_E f^2(u_k(x))\mu_{\langle M^{u_k,c}\rangle}(\mathrm{d}x)\right)^{1/2} \tag{4.3.3}$$

Let $\mathcal{I}_k$ be the set of measurable functions from $\mathbb{R}$ into $\mathbb{R}$ such that $\|f\|_k<\infty$. On $\mathcal{I}=\bigcap_{k\in\mathbb{N}}\mathcal{I}_k$, we define a distance d by setting :

$$d(f,g)=[f-g]$$

where

$$[f]=\sum_{k=1}^{\infty}2^{-k}(1\wedge\|f\|_k). \tag{4.3.4}$$

Note that $\mathcal{I}$ contains the measurable locally bounded functions and that the set of elementary functions is dense in $(\mathcal{I},d)$. Indeed, by a monotone class argument, we can show that if f is bounded, for any $n\in\mathbb{N}$, there exists f_n elementary such that $\sup_{k\leq n}\|f-f_n\|_k\leq 2^{-n}$. Hence

$$\sum_{n=1}^{\infty}[f-f_n]\leq\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}2^{-k}(1\wedge\|f-f_n\|_k)+2^{-n}\right)<2.$$

Consequently it is sufficient to show that the set of bounded functions is dense in $\mathcal{I}$. By dominated convergence, $\lim_{n\to\infty}[f-(-n)\vee f\wedge n]=0$ for any $f\in\mathcal{I}$.
Let f be an element of $\mathcal{I}$. The MAF W^k defined by : $W_t^k=\int_0^t f(u_k(X_s))\mathrm{d}M_s^{u^k,c}$, has finite energy since : $e(W^k)=\frac{1}{2}\|f\|_k^2$. Hence :

$$fu*M_s^{u,c}:=\begin{cases} fu_k*M_s^{u_k,c} & \text{for} \quad t<\sigma_{E\setminus D_k}\\ 0 & \text{for} \quad t\geq\sigma\end{cases}$$

belongs to $\mathcal{M}^c_{f\text{-}loc}$ ($\mathcal{M}^c_{loc}$ if $u \in \mathcal{F}$) and by Lemma 4.2.2, $\Gamma(fu * M^{u,c})$ is well defined and is an element of $\mathcal{N}_{c,f\text{-}loc}$ ($\mathcal{N}_{c,loc}$ if $u \in \mathcal{F}$).

Theorem 4.3.3. *The application defined by (4.3.2) on the set of elementary functions can be extended to the set $\mathcal{I}$. This extension, denoted by $\int f(z)\mathrm{d}_z\Gamma^z$, for f in $\mathcal{I}$, satisfies :*

(i) $\int f(z)\mathrm{d}_z\Gamma^z_t = \Gamma_t(fu * M^{u,c})$ $\forall t \geq 0$, $\mathbf{P}_x$*-a.e for q.e* $x \in E$.

(ii) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence $\mathcal{I}$. Assume that : $[f_n - f] \to 0$. Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $(\int f_{n_k}(z)\mathrm{d}_z\Gamma^z_t)_{k\in\mathbb{N}}$ converges uniformly on any compact of $[0,\zeta)$ ($[0,\infty)$ if $u \in \mathcal{F}$) to $\int f(z)\mathrm{d}_z\Gamma^z_t$ $\mathbf{P}_x$-a.e for q.e $x \in E$*

Proof. Elementary functions are dense in $\mathcal{I}$ and (i) holds for elementary functions. It is sufficient to prove that that if $[f_n - f]$ converge to zero, there exists a subsequence n_k such that for any $p \in \mathbb{N}$, $\Gamma(f_{n_k}u * M^{u,c})$ converges to $\Gamma(fu * M^{u,c})$ uniformly on any compact of $[0, \sigma_{E\setminus D_p}[$. Let n_k be such that $[f_{n_k} - f] < 2^{-4k}$ and $p \in \mathbb{N}$, hence $\|f - f_{n_k}\|_p \leq 2^p 2^{-4k}$ for any $k > p/4$ and it follows from Theorem 4.2.5 that $\Gamma(f_{n_k}u_p * M^{u_p,c})$ converges uniformly on any compact to $\Gamma(fu_p * M^{u_p,c})$ $\mathbf{P}_x$-a.e for q.e $x \in E$. But thanks to (4.2.6), $\Gamma(f_{n_k}u_p * M^{u_p,c})$ and $\Gamma(fu_p * M^{u_p,c})$ agrees on $t < \sigma_{E\setminus D_p}$ with $\Gamma(f_{n_k}u * M^{u,c})$ and $\Gamma(fu * M^{u,c})$ respectively $\mathbf{P}_x$-a.e for q.e $x \in E$. □

We finish this section with a characterization of the set $\mathcal{I}$ when u belongs to $\mathcal{F}$. Let $\mathcal{E}^{(c)}$ be the local part in the Beurling-Deny decomposition for $\mathcal{E}$ (See Theorem 3.2.1 of [21].). $\mathcal{E}^{(c)}$ has the local property, hence with the same argument used to proof Theorems 5.2.1 and 5.2.3 of [5], there exists a function U in $L^1(\mathbb{R}, \mathrm{d}x)$ such that for any function F in $\mathcal{C}^1$ with bounded derivative f :

$$\mathcal{E}^{(c)}(F(u), F(u)) = \frac{1}{2}\int_{\mathbb{R}} f^2(x)U(x)\mathrm{d}x.$$

Then thanks to (4.2.3) and Lemma 3.2.3 of [21],

$$\int_E f^2(u(x))\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x) = \int_{\mathbb{R}} f^2(x)U(x)\mathrm{d}x.$$

hence it follows by a monotone class argument that for any measurable positive function f we have :

$$\int_E f(u(x))\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x) = \int_{\mathbb{R}} f(x)U(x)\mathrm{d}x. \tag{4.3.5}$$

Lemma 4.3.4. *For u element of $\mathcal{F}$, the set $\mathcal{I}$ coincides with the set $L^1_{loc}(\mathbb{R}, U(x)\mathrm{d}x)$, where the function U is defined by (4.3.5).*

Proof. For k integer, the function u_k is defined be $(-k) \vee u \wedge k$. Associate U_k to u_k as U is associated to u. We have then : $\|f\|_k^2 = \int_{\mathbb{R}} f^2(x)U_k(x)\mathrm{d}x$ for any measurable function f. In order to proof Lemma 4.3.4, it is sufficient to prove that : $U_k(x) = 1_{[-k,k]}U(x)$ for a.e. x in $\mathbb{R}$.
Let f be a continuous function with support in $[-k, k]$ and set $F(x) := \int_0^x f(z)\mathrm{d}z$. We have hence : $F(u(x)) = F(u_k(x))$ for any x in E and therefore $f(u_k) * M^{u_k,c} = f(u) * M^{u,c}$, indeed thanks to (4.2.3) both martingales coincides with $M^{Fu_k,c}$ $(= M^{Fu,c})$.
We have therefore : $\int_E f^2(u_k(x))\mu_{\langle M^{u_k,c}\rangle}(\mathrm{d}x) = \int_E f^2(u(x))\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x)$. This shows that

$$\int_{\mathbb{R}} f^2(x)U_k(x)\mathrm{d}x = \int_{\mathbb{R}} f^2(x)U(x)\mathrm{d}x$$

for any function f continuous with compact support in $[-k.k]$, hence $U_k(x) = U(x)$ for a.e. x in $[-k, k]$.
Now if g is a continuous positive function with support in $\mathbb{R} \setminus [-k, k]$ then :

$$\int_{\mathbb{R}} g(x)U_k(x)\mathrm{d}x = \int_E g(u_k(x))\mu_{\langle M^{u_k,c}\rangle}(\mathrm{d}x) = 0$$

therefore $U_k(x) = 0$ for a.e. x in $\mathbb{R} \setminus [-k, k]$. This finishes the proof. □

4.4 Itô Formula

In this section, we first prove Lemma 4.1.1 and then Theorem 4.1.2.

Proof of Lemma 4.1.1. Let u be an element of $\mathcal{F}_{loc}$, thanks to the proof of Lemme 3.1 of [7], there exists a nest of finely open Borel sets $\{\mathcal{G}_k\}_{k\in\mathbb{N}}$ and a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{F}$ such that $u(x) = u_k(x)$ for q.e. $x \in \mathcal{G}_k$ and $\|u_k\|_\infty < k$. Let $\phi \in L^1(E; m)$ such that $0 < \phi \le 1$ and for any k let

$$h_k(x) := \mathbf{E}_x\left(\int_0^{\sigma_{E\setminus\mathcal{G}_k}} e^{-t}\phi(X_t)\mathrm{d}t\right)$$

$G_k := \{x \in E : h_k(x) > k^{-1}\}$ and $g_k(x) := 1 \wedge (kh_k(x))$. For any k, $G_k \subset \mathcal{G}_k$, thus $u(x) = u_k(x)$ for q.e. $x \in G_k$. Moreover, by the proof of Lemme 3.8 of [30], $\{G_k\}_{k\in\mathbb{N}}$ is a nest and we have : $0 \le g_k \le 1$, $g_k(x) = 1$ q.e. on G_k, $g_k(x) = 0$ on $E\setminus\mathcal{G}_k$. Since h_k is quasi-continuous we can suppose that each G_k is finely open (Theorem 4.6.1 of [21]). For any $k \in \mathbb{N}$ we have :

$$\begin{aligned}
&\int_{G_k}\int_{\{|u(x)-u(y)|<1\}} |u(x)-u(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\\
= &\int_{G_k} |g_k(x)|^2 \int_{\{|u(x)-u(y)|<1\}} |u(x)-u(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\\
\leq\ & 2\int_{G_k}\int_{\{|u(x)-u(y)|<1\}} |g_k(x)-g_k(y)|^2|u(x)-u(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\\
&+2\int_{\mathcal{G}_k\times\mathcal{G}_k\cap\{|u(x)-u(y)|<1\}} |g_k(y)|^2|u(x)-u(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\\
\leq\ & 2\int_{E\times E} |g_k(x)-g_k(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\\
&+2\int_{E\times E} |u_k(x)-u_k(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\\
\leq\ & 4\mathcal{E}(g_k,g_k)+4\mathcal{E}(u_k,u_k)<\infty
\end{aligned}$$

Therefore, if for any $\varepsilon>0$, we set :

$$S_\varepsilon=\sum_{k=1}^{\infty}2^{-k}\left(1\wedge\int_{G_k}\int_{\{|u(x)-u(y)|<\varepsilon\}}|u(x)-u(y)|^2N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\right)$$

We have then $\lim_{\varepsilon\to 0}S_\varepsilon=0$. We choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $S_{\varepsilon_n}<2^{-4n}$. Let F be a locally absolutely continuous function with a locally bounded Radon-Nikodym derivative f. For k in $\mathbb{N}$, define (F_k) by

$$F_k(x)=F(x)1_{[-k-1,k+1]}(x)+F(k+1)1_{[k+1,\infty)}(x)+F(-k-1)1_{(-\infty,-k-1]}(x).$$

Note that F_k has a bounded Radon-Nikodym derivative : $f_k=f1_{[-k-1,k+1]}$.
For a function $\beta:E^2\to\mathbb{R}$, define :

$$A_t(\beta,n):=\int_0^t\int_{\{\varepsilon_n<|u(y)-u(X_s)|<1\}}\beta(y,X_s)N(X_s,\mathrm{d}y)\mathrm{d}H_s \text{ and}$$

$$M^d(\beta,n)=\sum_{s\leq t}\beta(X_s,X_{s-})1_{\{\varepsilon_n<|u(X_{s-})-u(X_s)|<1\}}1_{\{s<\xi\}}-A_t(\beta,n)$$

Denote by $M^d(F,u,n)$ (resp $M^d(F,u,n,k)$) the process $M^d(\beta,n)$ for $\beta(y,x)=F(u(y))-F(u(x))$ (resp. $\beta(y,x)=[F(u(y))-F(u(x))]1_{G_k}(x)$). Similarly, define $A^d(F,u,n)$ and $A(F,u,n,k)$.
We just have to prove that $\mathbf{P}_x$-a.e for q.e $x\in E$, the limits $\lim_{n\to\infty}M^d(F,u,n)$ and $\lim_{n\to\infty}A(F,u,n)$ exist uniformly on any compact of $[0,\sigma_{E\setminus G_k}[$. We have : $M^d_t(F,u,n)=M^d_t(F_k,u,n,k)$ and $A_t(F,u,n)=A_t(F_k,u,n,k)$ on $[0,\sigma_{E\setminus G_k}[$.

For every k, the process $M^d(F_k, u, n, k)$ belongs to $\mathcal{M}$ and for $4n > k$, we have

$$e(M^d(F_k, u, n+1, k) - M^d(F_k, u, n, k)) \leq c_k 2^k 2^{-4n}$$

where $c_k = \|f_k\|_\infty$. Indeed, from the definition of ε_n :

$$\begin{aligned}
& e(M^d(F_k, u, n+1, k) - M^d(F_k, u, n, k)) \\
= \; & \frac{1}{2}\int_{G_k \times E} (F_k(u(x)) - F_k(u(y)))^2 1_{\{\varepsilon_{n+1} \leq |u(x)-u(y)| < \varepsilon_n\}} N(x, \mathrm{d}y)\nu_H(\mathrm{d}x) \\
\leq \; & c_k \int_{G_k \times E} |u(x) - u(y)|^2 1_{\{|u(x)-u(y)| < \varepsilon_n\}} N(x, \mathrm{d}y)\nu_H(\mathrm{d}x) \\
\leq \; & c_k 2^k 2^{-4n}
\end{aligned}$$

thus, the convergence of $M^d(F, u, n)$ follows from Theorem 4.2.5. Still thanks to Theorem 4.2.5, the convergence of $A(F, u, n)$ can be seen as a consequence of :

$$\Gamma(M^d_t(F_k, u, n, k)) = A_t(F_k, u, n, k) \; \mathbf{P}_x - \text{a.e for q.e}\, x \in E \tag{4.4.1}$$

To prove (4.4.1), we note that $(A_t(F_k, u, n, k))_{t\geq 0}$ is of bounded variation, so $A_t(F_k, u, n, k) \circ r_t = A_t(F_k, u, n, k)$ $\mathbf{P}_m$-a.e on $t < \zeta$ (Theorem 2.1 of [16]). Hence making use of the operator Λ defined in [7], instead of Γ, we first obtain :

$$\Lambda(M^d_t(F_k, u, n, k)) = A_t(F_k, u, n, k) \; \mathbf{P}_m - \text{a.e for q.e}\, x \in E \text{ on } [\![0, \zeta[\![$$

Finally by Theorem 3.6 in [7] and Lemma 4.2.1, (4.4.1) holds, $\mathbf{P}_x$-a.e for q.e $x \in E$ on $[\![0, \zeta[\![$, and therefore on $[\![0, \infty[\![$ thanks to the continuity of $\Gamma(M^d_t(F_k, u, n, k))$ and $A_t(F_k, u, n, k)$.
It is clear that $M^d(F, u) \in \mathcal{M}_{f\text{-}loc}$ and $A(F, u) \in \mathcal{N}_{c,f\text{-}loc}$. Moreover for u element of $\mathcal{F}$, we can take $G_n = \{x : |u(x)| < n\}$ for any n. In this case, from the strict continuity of u we have, $\mathbf{P}_x(\lim_{n\to\infty} \sigma_{E\setminus G_n} = \infty) = 1$ for q.e. $x \in E$, thus the convergence of $M^d(F, u, n)$ and $A(F, u, n)$ are uniformly on any compact of $[0, \infty)$. Thus we obtain : $M^d(F, u) \in \mathcal{M}_{loc}$ and $A(F, u) \in \mathcal{N}_{c,loc}$.

□

Remark 4.4.1. (i) If $u \in \mathcal{F}$ and f is bounded, then $M^d(F, u) \in \mathcal{M}$ and $\Gamma(M^d(F, u)) = A(F, u)$.
(ii) With the notation of the proof of Lemma 4.1.1, it holds that if $u_k = u$ q.e. on G_k, $\mathbf{P}_x$-a.e for q.e $x \in E$:

$$M^d_t(F, u) + A_t(F, u) = M^d_t(F_k, u_k) + A_t(F_k, u_k) \;\text{ for}\, t \in [0, \sigma_{E\setminus G_k}[$$

Proof of Theorem 4.1.2. We use the notation of the proof of Lemma 4.1.1. Thus, if $u \in \mathcal{F}$, we take $G_n := \{x : |u(x)| < n\}$, $n \in \mathbb{N}$. Let F be a locally absolutely continuous function F with a locally bounded Radon-Nikodym derivative f.

Let I_t be the difference of the left-hand side and the right-hand side of (4.1.3). For any k, we define I_t^k as I_t with u_k and f_k replacing u and f respectively. Hence : $I_t = I_t^k$ for $t < \sigma_{E\setminus G_k}$, $\mathbf{P}_x$-a.e for q.e $x \in E$. Since $\sigma_{E\setminus G_n} \wedge \zeta$ converges to ζ if $u \in \mathcal{F}_{loc}$ and $\sigma_{E\setminus G_n}$ converges to ∞ if $u \in \mathcal{F}$, it is sufficient to prove (4.1.3) on $[0, \sigma_{E\setminus G_k}[$ for any $k \in \mathbb{N}$. Consequently we can assume (and we do) that u is an element of $\mathcal{F}_b$ and f is bounded.

If f is continuous, thanks to (4.2.3), $F(u) \in \mathcal{F}$ and $M^{Fu,c} = fu * M^{u,c}$ and we have the Fukushima decomposition :

$$F(u(X_t)) = F(u(X_0)) + fu * M_t^{u,c} + \Gamma(fu * M^{u,c})_t + M_t^{u,d} + \Gamma(M^{u,d})_t$$

We obtain (4.1.3) from Lemma 4.3.3 (i) and Remark 4.4.1 (i).

If f is not necessarily continuous, let g be in $L^1(\mathbb{R})$ be a strictly positive function on $\mathbb{R}$ such that g and $1/g$ are locally bounded . Define the norms $\|.\|$ and $\|.\|_*$ on the Borel measurable functions as follows :

$$\|h\|_* = \left(\int_E h^2(u(x))\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x)\right)^{1/2}$$

$$\begin{aligned} \|h\| &= \|h\|_* + \int |h(x)|g(x)\mathrm{d}x \\ &+ \left(\int_{E\times E-\delta} |u(x)-u(y)| \int_{u(x)\wedge u(y)}^{u(x)\vee u(y)} h(z)^2 \mathrm{d}z N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\right)^{\frac{1}{2}} \end{aligned}$$

Since u is in $\mathcal{F}$, we have : $\|f\| < \infty$. By a monotone class argument, one shows that there exists a sequence of bounded continuous functions $(f_n)_{n\in\mathbb{N}}$ with compact support such that $\|f_n - f\|$ converges to 0 as n tends to infinity. We set : $F_n(x) = \int_0^x f_n(z)\mathrm{d}z$.

In order to show (4.1.3), we will show that there exists a subsequence n_k such that the terms in the expansion (4.1.3) for F_{n_k} converge as $k \to \infty$ to the corresponding expression with f replacing f_{n_k}. The convergence of $F_n(u(X_t)) - F_n(u(X_0)) - V_t(F_n, u)$ to $F(u(X_t)) - F(u(X_0)) - V_t(F, u)$ is a consequence of the pointwise convergence of F_n to F, indeed, for any $x \in \mathbb{R}$,

$$|F_n(x) - F(x)| \le \int_{-x^-}^{x^+} |f_n(z) - f(z)|\mathrm{d}z \le \sup_{|\lambda|\le|x|} \frac{1}{g(\lambda)} \int_{-\infty}^{\infty} |f_n(z) - f(z)|g(z)\mathrm{d}z \to 0$$

The existence of a subsequence $\{n_k\}$ such that $\int_0^t f_{n_k}(u(X_s))\mathrm{d}M_s^{u,c}$ and $\int_{\mathbb{R}} f_{n_k}(z)\mathrm{d}_z\Gamma_t^z(u)$ converge to $\int_0^t f(u(X_s))\mathrm{d}M_s^{u,c}$ and $\int_{\mathbb{R}} f(z)\mathrm{d}_z\Gamma_t^z(u)$ respectively is a consequence of the fact that $e(fu * M^{u,c} - f_n u * M^{u,c}) = \frac{1}{2}\|f - f_n\|_* \to 0$ as $n \to \infty$, and Theorem 4.2.5. Thanks to Theorem 4.2.5 and Remark 4.4.1 (i), it is then sufficient to show that $e(M(F_n, u) - M(F, u))$ converges to zero as $n \to \infty$. But

$$\begin{aligned} e(M - M^n) &\leq \frac{1}{2}\int_{E\times E-\delta} (F(u(x)) - F_n(u(x)) - F(u(y)) \\ &\qquad + F_n(u(y)))^2 N(x, \mathrm{d}y)\nu_H(\mathrm{d}x) \\ &\leq \frac{1}{2}\|f - f_n\|_*^2 \to 0 \text{ as } n \to \infty. \end{aligned}$$

□

As an example, for $F(z) = z$ and u in $\mathcal{F}_{loc}$, one obtains a Fukushima decomposition for the process $u(X)$. This case can be seen as a refinement of Lemma 2.2 in [9].

4.5 Local Time

We fix an element u of $\mathcal{F}_{loc}$. The associated process ${}^cN^u$ has been defined in (4.2.1) by ${}^cN^u = \Gamma(M^{u,c})$. By Remark 4.2.3, ${}^cN^u$ is a CAF locally of zero energy or merely a CAF of zero energy when u belongs to F. We suppose that u satisfies the additional assumption that ${}^cN^u$ is of bounded variation on $[0, \zeta)$, i.e. there exists two PCAF's $A^{(1)}$ and $A^{(2)}$ such that $\mathbf{P}_x$-a.e for q.e $x \in E$:

$${}^cN_t^u = A_t^{(1)} - A_t^{(2)}, \;\; \forall t \in [0, \zeta) \tag{4.5.1}$$

We remind that a measure ν on E is a smooth signed measure on E if there exists a nest $\{F_k\}$ such that for each k, $1_{F_k}.\nu$ is a finite signed Borel measure charging no set of zero capacity. Such nest is said to be associated to ν. For a closed set $F \subset E$ we set :

$$\mathcal{F}_{b,F} = \{u \in \mathcal{F}_b : u = 0 \text{ q.e. on } E \setminus F\}.$$

We also need the following definition :

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_m.$$

Lemma 4.5.1. *The process ${}^cN^u$ is of bounded variation if and only if there exists a smooth signed measure ν on E with associated nest $\{F_k\}$ such that*

$$\mathcal{E}^{(c)}(u, v) = < v, \nu >, \quad \forall v \in \bigcup_{k=1}^{\infty} \mathcal{F}_{b,F_k}.$$

Proof. From theorem 5.2.4 of [21], ${}^cN^u$ is the only AF of zero energy such that for any $h \in \mathcal{F}$,

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbf{E}_{h.m}[{}^cN_t^u] = -e(M^{u,c}, M^{h,c}) = -\mathcal{E}^{(c)}(u,h)$$

On the other hand, since : $|\mathcal{E}^{(c)}(u,h)| \leq (\mathcal{E}^{(c)}(u,u))^{1/2}(\mathcal{E}_1(h,h))^{1/2}$, there exists a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}^{(c)}(u,h) = \mathcal{E}_1(w,h) \text{ for any } h \in \mathcal{F}.$$

Hence : $\lim_{t\downarrow 0} \frac{1}{t}\mathbf{E}_{h.m}[N_t^w - \int_0^t w(X_s)\mathrm{d}s] = -\mathcal{E}^{(c)}(u,h)$ for all $h \in \mathcal{F}$. This implies that the AF $N^w - \int_0^\cdot w(X_s)\mathrm{d}s$ is equivalent to ${}^cN^u$. Consequently ${}^cN^u$ is of bounded variation if and only if N^w is of bounded variation. But thanks to Theorem 5.4.2 of [21], this last condition is equivalent to the existence of a smooth signed measure ν with an associated nest $\{F_k\}$ such that

$$\mathcal{E}_1(w,v) =< v, \nu >, \ \forall v \in \bigcup_{k=1}^{\infty} \mathcal{F}_{b,F_k}.$$

□

4.5.1 Definition of local time

Definition 4.5.2. *The local time at a of $u(X)$, denoted by $L_t^a = L_t^a(u)$ is the following CAF on $[\![0,\zeta[\![$:*

$$\frac{1}{2}L_t^a = -\Gamma(Z^a)_t + \int_0^t 1_{\{u(X_{s-})\leq a\}}\mathrm{d}^cN_s^u \quad \textit{for } t \in [0,\zeta)$$

The name “local time” is justified by Proposition 4.5.3 and Corollary 4.5.4 below.

Proposition 4.5.3. *There exists a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty^m$-measurable version of the local time process $\{\tilde{L}_t^a; a \in \mathbb{R}, t \geq 0\}$ such that $\mathbf{P}_m$-a.e we have the occupation time density formula :*

$$\int_{\mathbb{R}} f(x)\tilde{L}_t^x \mathrm{d}x = \int_0^t f(u(X_s))\mathrm{d}\langle M^{u,c}\rangle_s \quad \textit{for any } f \textit{ Borel bounded and } t < \zeta$$

Proof. We start with the case when u is a bounded element of $\mathcal{F}$. From (4.2.4) we have : $\mathbf{P}_m$-a.e. on $[\![0,\zeta[\![$: $L_t^a = Z_t^a + Z_t^a \circ r_t + 2\int_0^t 1_{\{u(X_{s-})\leq a\}}\mathrm{d}N_s^{u,c}$. Moreover, thanks to Theorem 63 chapter IV of [41], there exists a function $\tilde{Z}(a,t,\omega)$ in $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty^m$, such that for each $a \in \mathbb{R}$, $\tilde{Z}(a,t,w)$ is a continuous $(\mathcal{F}_t^m)$-adapted version of the stochastic integral Z^a, and thanks to Lemma 2.10 and

Theorem 2.18 of [7], $\tilde{Z}(a,t,w)\circ r_t(\omega) \in \mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}^m_\infty$ is a continuous $(\mathcal{F}^m_t)$-adapted version of $Z^a_t \circ r_t$ for each $a \in \mathbb{R}$. Besides we can take $\int_0^t 1_{\{u(X_{s-})\leq a\}} \mathrm{d}N^{u,c}_s$ jointly continuous in t and right continuous in a, $\mathbf{P}_m$-a.e on $[\![0,\zeta[\![\times\mathbb{R}$. Thus, we have constructed a version $\{\tilde{L}^a_t, a\in\mathbb{R}, t\in[0,\zeta[\}$ of $\{L^a_t, a\in\mathbb{R}, t\in[0,\zeta)\}$ which is $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}^m_\infty$-measurable.
Let f be a continuous positive element of $L^1(\mathbb{R})$. Using the proof presented in [41] of Fubini's Theorem for stochastic integrals (Theorem 64, Chapter IV of [41]), we know that $\int_{\mathbb{R}} \tilde{Z}(z,t,\omega)f(z)\mathrm{d}z$ is a well-defined Lebesgue integral since $\mathbf{P}_m$-a.e :

$$\int_{\mathbb{R}} |\tilde{Z}(z,t,\omega)| f(z)\mathrm{d}z < \infty \text{ for all } t.$$

Moreover still thanks to this theorem, $\int_{\mathbb{R}} \tilde{Z}(z,t,\omega)f(z)\mathrm{d}z$ is a continuous $\mathbf{P}_m$-version of $\int_0^t F(u(X_s))\mathrm{d}M^{u,c}_s$, where $F(z)=\int_z^\infty f(\lambda)\mathrm{d}\lambda$. Consequently, for $t>0$, $\mathbf{P}_m$-a.e on $\{t<\zeta\}$, $\int_{\mathbb{R}} |\tilde{Z}(z,t,r_t(\omega))|f(z)\mathrm{d}z<\infty$ and $\int_{\mathbb{R}} \tilde{Z}(z,t,r_t(\omega))f(z)\mathrm{d}z$ is a continuous $\mathbf{P}_m$-version of $\int_0^t F(u(X_s))\mathrm{d}M^{u,c}_s\circ r_t$.
Since $(\int_0^t 1_{\{u(X_s-)\leq a\}}\mathrm{d}N^{u,c}_s)_{a\in\mathbb{R}}$ is of bounded variation on $\{t<\zeta\}$, we obtain $\mathbf{P}_m$-a.e on $\{t<\zeta\}$: $\int_{\mathbb{R}} f(z)|\tilde{L}^z_t|\mathrm{d}z<\infty$ and

$$\int_{\mathbb{R}} f(z)\tilde{L}^z_t\mathrm{d}z = \int_0^t F(u(X_s))\mathrm{d}M^{u,c}_s + \int_0^t F(u(X_s))\mathrm{d}M^{u,c}_s\circ r_t + 2\int_0^t F(u(X_s))\mathrm{d}N^{u,c}_s$$

which leads to

$$\int_{\mathbb{R}} f(z)\tilde{L}^z_t\mathrm{d}z = -2\Gamma(Fu * M^{u,c})_t + 2\int_0^t F(u(X_s))\mathrm{d}N^{u,c}_s. \tag{4.5.2}$$

Now thanks to (4.2.3), Fu belongs to $\mathcal{F}_{loc}$ and $M^{Fu,c}_t = -\int_0^t f(u(X_s))\mathrm{d}M^{u,c}_s$. Thus

$$\langle M^{Fu,c}, M^{u,c}\rangle_t = -\int_0^t f(u(X_s))\mathrm{d}\langle M^{u,c}\rangle_s.$$

Thanks to Lemma 4.2.4 we have $\mathbf{P}_m$-a.e on $\{t<\zeta\}$:

$$\int_0^t F(u(X_s))\mathrm{d}^c N^u_s = \int_0^t F(u(X_s))\mathrm{d}\Gamma(M^{u,c})_s.$$

On the other hand the definition of the integral with respect to $\Gamma(M^{u,c})$ (Chen et al. [7]) gives :

$$\int_0^t F(u(X_s))\mathrm{d}\Gamma(M^{u,c})_s = \Gamma(Fu * M^{u,c})_t + \frac{1}{2}\int_0^t f(u(X_s))\mathrm{d}\langle M^{u,c}\rangle_s$$

which together with (4.5.2) lead to

$$\int_{\mathbb{R}} f(z)\tilde{L}_t^z \mathrm{d}z = \int_0^t f(u(X_s))\mathrm{d}\langle M^{u,c}\rangle_s \ \mathbf{P}_m\text{-a.e on } \{t < \zeta\} \tag{4.5.3}$$

Actually, the set of null $\mathbf{P}_m$-measure on which (4.5.3) could fail can be chosen independently of f. Indeed, the set of continuous functions with compact support, is a separable topological space for the metric of uniform convergence.
We show now that the set of null $\mathbf{P}_m$-measure on which (4.5.3) could fail does not depend on t either. We have thanks to (4.5.3)

$$\mathbf{P}_m\text{-a.e on } \{t < \zeta\},\ \tilde{L}_t^z \geq 0 \text{ for } \mathrm{d}z\text{-a.e } z \tag{4.5.4}$$

hence by a monotone class argument, (4.5.3) holds $\mathbf{P}_m$-a.e on $\{t < \zeta\}$ for any f Borel bounded. It remains to show that (4.5.3) holds $\mathbf{P}_m$-a.e on $[\![0,\zeta[\![$. To do so it is sufficient to show that the left-hand side of (4.5.3) is continuous in t.
It follows from Theorem 2.18 in [7] that for any z, $\tilde{Z}(z,t,r_t(\omega))$ is continuous and has the additivity property $\mathbf{P}_m$-a.e for on $[\![0,\zeta[\![$. Hence thanks to (4.5.4) for $\mathrm{d}z$-a.e z, $\tilde{L}_t^z$ is increasing. One shows then by monotone convergence that for any positive Borel function f, $t \to \int_{\mathbb{R}} f(z)\tilde{L}_t^z\mathrm{d}z$ is continuous $\mathbf{P}_m$-a.e on $[\![0,\zeta[\![$.
For a function u in $\mathcal{F}_{loc}$, take an nest of closed sets $\{D_k\}$ and a sequence $(u_k)_{n\in\mathbb{N}}$ of bounded elements of $\mathcal{F}$ such that $u = u_k$ for q.e $x \in E$. For any $k \in \mathbb{N}$, let $\tilde{L}_t^z(u_k)$ be the version $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}_\infty^m$-measurable of local time obtained above. Then $\tilde{L}_t^z := \tilde{L}_t^z(u_k)$ on $t < \tau_{D_k}$ is a $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}_\infty^m$-measurable version of L_t^z and satisfies the occupation time density formula on $[0,\tau_{D_k}[$, for any $k \in \mathbb{N}$, so it satisfies it on $[0,\zeta[$. □

Corollary 4.5.4. *For any real a, L^a is a PCAF and $\mathbf{P}_x$ -a.e. for q.e $x \in E$, the measure in t, $\mathrm{d}_t L_t^a$ is carried by the set $\{s : u(X_{s-}) = u(X_s) = a\}$.*

Proof. We use u_k and $\{D_k\}$ defined as in the end of the proof of Proposition 4.5.3. Since we need to show the assertion of Cororally 4.5.3 only on $[0,\tau_{D_k}[$, we can assume that u is a bounded element of $\mathcal{F}$. It follows from the occupation time density formula and the $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes F_\infty^m$-measurability of $\tilde{L}$, that there exists a subset R of $\mathbb{R}$ of Lebesgue's measure zero, such that for any a outside of R : $\mathbf{P}_m$-a.e $\tilde{L}_t^a \geq 0$ on $[\![0,\zeta[\![$. Consequently L^a has the same property. This property holds for any $a \in \mathbb{R}$. Indeed for any real a, take a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}\setminus R$ such that $a_n \downarrow a$. We have : $e(Z^{a_n} - Z^a) = \int 1_{\{a<u(x)\leq a_n\}}\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x)$, which converges to 0 as n tends to ∞ by dominated convergence. Thus, thanks to Theorem 4.2.5 (taking a subsequence if necessary) $\Gamma(Z^{a_n})$ converges to $\Gamma(Z^a)$ uniformly on any finite interval of t, $\mathbf{P}_m$-a.e. On the other hand, for $\mathbf{P}_m$-a.e $w \in \Omega$, $\int_0^t 1_{\{u(X_s)\leq a_n\}}\mathrm{d}N_s^{u,c}(\omega)$ converges to $\int_0^t 1_{\{u(X_s)\leq a\}}\mathrm{d}N_s^{u,c}(\omega)$ for any $t < \zeta(\omega)$. Conssequently, we obtain for $\mathbf{P}_m$-a.e $\omega \in \Omega$, $L_t^a(\omega) \geq 0$ for any $t < \zeta(\omega)$.
It follows from Lemma 4.2.1 that for any real a, L^a is a PCAF on $[\![0,\zeta[\![$. By Remark 2.2 in [7], it can be extended to a PCAF.

Now defining : $f(x) = (x-a)^4$ and $h(x) = (x-a)^4 1_{\{x \le a\}}$, it follows from (4.2.3) that fu and hu belong to $\mathcal{F}_{loc}$. Moreover we have :

$$M_t^{fu,c} = 4\int_0^t (u(X_s)-a)^3 \mathrm{d}M_s^{u,c} \quad \text{and} \quad M_t^{hu,c} = 4\int_0^t (u(X_s)-a)^3 1_{\{u(X_s)\le a\}} \mathrm{d}M_s^{u,c}$$

thus, $\langle M^{fu,c}, Z^a\rangle = \langle M^{hu,c}, M^{u,c}\rangle$, and from the definition of the stochastic integral (4.2.5) we have that $\mathbf{P}_m$-a.e on $\{t < \zeta\}$

$$\int_0^t (u(X_s)-a)^4 \mathrm{d}\Gamma(Z^a)_s = \int_0^t (u(X_s)-a)^4 1_{\{u(X_s)\le a\}} \mathrm{d}\Gamma(M^{u,c})_s.$$

By Lemmas 4.2.1 and 4.2.4, we finally obtain : $\int_0^t (u(X_s)-a)^4 \mathrm{d}L_s^a = 0$ $\mathbf{P}_x$-a.e for q.e $x \in E$. □

4.5.2 Integration with respect to local time

We fix u an element of $\mathcal{F}$ satisfying (4.5.1) and set : $l_t^a = \int_0^t 1_{\{u(X_{s-})\le a\}} \mathrm{d}N_s^{u,c}$. Hence the local time at a of $u(X)$ satisfies :

$$L^a = -2\Gamma^a + 2l^a.$$

For any $\omega \in \Omega$ and $t < \zeta(\omega)$, the function $z \to l_t^z(\omega)$ is of bounded variation. The application defined for the elementary functions by

$$f \to \sum_{i=0}^{n-1} f_i (l_t^{z_{i+1}} - l_t^{z_i}),\ t < \zeta$$

can hence be extended to the set of locally bounded Borel measurable functions f from $\mathbb{R}$ into $\mathbb{R}$ as a Lebesgue-Stieljes integral and we have :

$$\int_{\mathbb{R}} f(z) \mathrm{d}_z l_t^z = \int_0^t f(u(X_s)) \mathrm{d}N_s^{u,c} \quad t < \zeta.$$

Using the stochastic integral with respect to Γ, the application defined for the elementary functions by

$$f \to \sum_{i=0}^{n-1} f_i (L_t^{z_{i+1}} - L_t^{z_i}),\ t < \zeta$$

can hence be extended to the set of locally bounded Borel measurable functions f from $\mathbb{R}$ into $\mathbb{R}$ and we have :

$$-\frac{1}{2}\int_{\mathbb{R}} f(z) \mathrm{d}_z L_t^z = \int_{\mathbb{R}} f(z) \mathrm{d}_z \Gamma_t^z - \int_0^t f(u(X_s)) \mathrm{d}N_s^{u,c}, \quad t < \zeta.$$

4.6 Multidimensional case

In this section we need the following notation. For $d \in \mathbb{N}$, $x = (x^1, ..., x^d), y = (y^1, ..., y^d) \in \mathbb{R}^d$, we set $x \leq y$ (resp. $x < y$) if and only of $x^i \leq y^i$ (resp. $x^i < y^i$) for each $i = 1, ..., d$ and $]x, y] = \{z \in \mathbb{R}^d : x < z \leq y\}$. The vector $\hat{x}$ is obtained from x by elimination of its coordinate x^d, i.e. $\hat{x} = (x^1, ..., x^{d-1})$, $\widehat{]x, y]} = \{z \in \mathbb{R}^{d-1} : \hat{x} < z \leq \hat{y}\}$.
Let φ be a measurable function from $\mathbb{R}^d$ into $\mathbb{R}$. We define integration of simple functions with respect to φ as follows. For f a simple function, i.e. there exists $x, y \in \mathbb{R}^d$ such that $f(z) = 1_{]x,y]}(z)$ for all $z \in \mathbb{R}^d$:

$$\text{if } d = 1 : \qquad \int_{\mathbb{R}} f(z) \mathrm{d}\varphi(z) = \varphi(y) - \varphi(x)$$
$$\text{if } d > 1 : \qquad \int_{\mathbb{R}^d} f(z) \mathrm{d}\varphi(z) = \int_{\mathbb{R}^{d-1}} 1_{\widehat{]x,y]}}(z) \mathrm{d}\varphi(z, y^d) - \int_{\mathbb{R}^{d-1}} 1_{\widehat{]x,y]}}(z) \mathrm{d}\varphi(z, x^d)$$

As an example, if there exist functions h_i, $1 \leq i \leq d$ such that $\varphi(z) = \prod_{i=1}^d h(z_i)$, then $\int_{\mathbb{R}^d} f(z) \mathrm{d}\varphi(z) = \prod_{i=1}^d (h_i(y^i) - h_i(x^i))$.
We extend this integration to the elementary functions $f : \mathbb{R}^d \to \mathbb{R}$ (i.e. $f(z) = \sum_{i=1}^n a_i f_i(z)$ where f_i , $1 \leq i \leq n$ are simple functions and a_i $1 \leq i \leq n$ are real numbers) by setting

$$\int_{\mathbb{R}^d} f(z) \mathrm{d}\varphi(z) = \sum_{i=1}^d a_i \int_{\mathbb{R}^d} f_i(z) \mathrm{d}\varphi(z).$$

A elementary function has many representations as linear combination of simple functions, but as in the Riemann integration theory, the integral does not depend on the choice of its representation.

Let u be in $\mathcal{F}_{loc}^d$ where $\mathcal{F}_{loc}^d = \{(u^1, u^2, ..., u^d) : u^i \in \mathcal{F}_{loc}, 1 \leq i \leq d\}$. Let $\{D_k\}_{k \in \mathbb{N}}$ be a nest of closed set, $\sigma := \lim_{k \to \infty} \sigma_{E \setminus D_k}$ and $(u_k)_{k \in \mathbb{N}}$ a sequence of bounded elements of $\mathcal{F}^d$ such that $u = u_k$ q.e. on D_k.
For any a in $\mathbb{R}^d$ and i in $\{1, 2, ..., d\}$, we define $Z^a(u^i)$ and $\Gamma^a(u^i)$ respectively in $\mathcal{M}_{f\text{-}loc}^c$ and $\mathcal{N}_{c,f\text{-}loc}$ by

$$\begin{aligned} Z_t^a(u^i) &= \begin{cases} \int_0^t 1_{\{u_k(X_{s-}) \leq a\}} \mathrm{d}M_s^{u_k^i, c} & \text{for} \quad t \leq \sigma_{E \setminus D_k} \\ 0 & \text{for} \quad t \geq \sigma \end{cases} \\ \Gamma^a(u^i) &= \Gamma(Z^a(u^i)) \end{aligned}$$

Thanks to the linearity property of Γ, we have for any elementary function f :

$$\int_{\mathbb{R}^d} f(z) \mathrm{d}_z \Gamma_t^z(u^i) = \Gamma_t \left(\int_0^t f(u(X_s)) \mathrm{d}M_s^{u^i, c} \right).$$

We extend (4.3.3) of Section 4.3 from $d = 1$ to $d \geq 1$, by defining for $k \in \mathbb{N}$, the norm $\|.\|_k$ on the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$

$$\|f\|_k := \sum_{i=1}^{d} \left(\int_E f^2(u_k(x)) \mu_{\langle M^{u^i_k,c} \rangle}(\mathrm{d}x) \right)^{1/2}.$$

and we define the set $\mathcal{I}$ with the metric $[.,.]$ as in (4.3.4) of Section 4.3. The set of elementary functions is dense in $\mathcal{I}$. We have the following version of Lemma 4.3.3.

Lemma 4.6.1. *The applications $f \to \int_{\mathbb{R}^d} f(z)\mathrm{d}_z\Gamma^z_t(u^i)$ $(1 \leq i \leq d)$ defined on the set of elementary functions, can be extended to the set $\mathcal{I}$. This extensions, denoted by $\int_{\mathbb{R}^d} \mathrm{d}_z\Gamma^z(u^i)$, satisfy :*

(i) $\int_{\mathbb{R}^d} f(z)\mathrm{d}_z\Gamma^z_t(u^i) = \Gamma(fu * M^{u^i,c})_t$ $\forall t \geq 0$, *$\mathbf{P}_x$-a.e for q.e $x \in E$.*

(ii) *For $(f_n)_{n\in\mathbb{N}}$ sequence of $\mathcal{I}$ such that $[f_n - f] \to 0$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $\int f_{n_k}(z)\mathrm{d}_z\Gamma^z_t(u^i)$ converges uniformly on any compact of $[0,\zeta)$ ($[0,\infty)$ if $u \in \mathcal{F}^d$) to $\int f(z)\mathrm{d}_z\Gamma^z_t(u^i)$ for every $1 \leq i \leq d$ $\mathbf{P}_x$-a.e for q.e $x \in E$*

With can prove a multidimensional version of Lemma 4.1.1 with the same arguments used in its proof. We have the following multidimensional Itô Formula.

Proposition 4.6.2. *Let u be an element of $\mathcal{F}^d$ (resp. $\mathcal{F}^d_{loc}$) and $F : \mathbb{R}^d \to \mathbb{R}$ a continuous function admitting locally bounded Radon-Nikodym derivatives $f_i = \partial F/\partial x_i$, $1 \leq i \leq d$, satisfying the following condition for any $1 \leq i \leq d$ and $k \in \mathbb{N}$*

$$\lim_{h\to 0} \int_E \{f_i(u_k(x) + h) - f_i(u_k(x))\}^2 \mu_{\langle M^{u^i_k,c} \rangle}(\mathrm{d}x) = 0. \tag{4.6.1}$$

Then, $\mathbf{P}_x$-a.e for q.e $x \in E$, the process $F(u(X_t)), t \in [0,\infty)$ (resp. $[0,\zeta)$) admits the decomposition

$$F(u(X_t)) = F(u(X_0)) + M_t(F,u) + Q_t(F,u) + V_t(F,u) \tag{4.6.2}$$

where $M(F,u) \in \mathcal{M}_{loc}$, (resp. $\mathcal{M}_{f\text{-}loc}$) $Q(F,u) \in \mathcal{N}_{c,loc}$ (resp. $\mathcal{N}_{c,f\text{-}loc}$) and $V(F,u)$ is a bounded variation process given by :

$$\begin{aligned} M_t(F,u) &= M^d_t(F,u) + \sum_{i=1}^{d} \int_0^t f_i(u(X_s))\mathrm{d}M^{u^i,c}_s \\ Q_t(F,u) &= \sum_{i=1}^{d} \int_{\mathbb{R}} f_i(z)\mathrm{d}_z\Gamma^z_t(u_i) + A_t(F,u) \\ V_t(F,u) &= \sum_{s\leq t}\{F(u(X_s)) - F(u(X_{s-}))\}1_{\{|u(X_s)-u(X_{s-})|\geq 1\}}1_{\{s<\xi\}} \\ &\quad -F(u(X_{\xi-}))1_{\{t\geq\xi\}} \end{aligned}$$

Proof. As in the proof of Theorem 4.1.2, we can assume that u is a bounded element of $\mathcal{F}$ and each f_i is bounded. For $\phi : \mathbb{R}^d \to \mathbb{R}$ an infinitely differentiable function with compact support, the function F_n defined by $F_n(z) := \int_{\mathbb{R}^d} F(z + y/n)\phi(y)\mathrm{d}y$ converges pointwise to $F(z)$. Setting : $f_{n,i} = \partial F_n/\partial x_i$ we obtain thanks to (4.6.1) :

$$\lim_{n\to\infty} \int_E [f_{n,i}(u(x)) - f_i(u(x))]^2 \mu_{\langle M^{u^i,c}\rangle}(\mathrm{d}x) = 0$$

The rest of the proof follows step by step the proof of Theorem 4.1.2.

□

In the case where $E = \mathbb{R}^d$ and $\mathcal{E}^{(c)}$ is given by

$$\mathcal{E}^{(c)} = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}(x)\mathrm{d}x$$

where for every (i, j), a_{ij} is a bounded measurable function. The coordinates functions $\pi_i(x) = x_i, 1 \leq i \leq d$, belong to $\mathcal{F}_{loc}$ and $M = (M^{\pi_1,c}, ..., M^{\pi_d,c})$ is a martingale additive functional with quadratic co-variation $\langle M^i, M^j\rangle_s = \int_0^t a_{ij}(X_s)\mathrm{d}s$, hence :

$\mu_{\langle M^{i,c}\rangle}(\mathrm{d}x) = a_{ii}(x)\mathrm{d}x$, and the condition (4.6.1) holds for any locally bounded measurable function.

Chapitre 5

Stochastic calculus for non necessarily symmetric Markov processes

Abstract : We consider a Markov process X associated to a non-necessarily symmetric Dirichlet form $\mathcal{E}$ and establish a representation theorem for the class of its local additive functionals locally of zero energy. We define a stochastic integral with respect to the elements of this class and can then obtain an Itô formula for the process $u(X)$, when u is in the domain of $\mathcal{E}$. In case the state space is $\mathbb{R}^d$, we establish a decomposition of X similar to the Lévy-Itô decomposition for Lévy process and obtain an Itô formula for X.

5.1 Introduction

The semimartingale theory has produced a fundamental tool based on stochastic integration and Itô's formula : the stochastic calculus. Since Markov processes are not in general semimartingales, Fukushima [20] developed another stochastic calculus in the framework of symmetric Dirichlet spaces. For a symmetric Markov process X with a regular Dirichlet form $\mathcal{E}$, and for any element u of the domain $\mathcal{F}$ of $\mathcal{E}$, the process $(u(X_t) - u(X_0), t \geq 0)$ admits the decomposition

$$u(X_t) - u(X_0) = M_t^u + N_t^u \tag{5.1.1}$$

where M^u is a martingale additive functional of finite energy and N^u is a continuous additive functional of zero energy. This decomposition is called Fukushima's decomposition and it can be seen as a substitute of the Doob-Meyer decomposition of super-martingales and Itô's formula for semimartingales. The part of the class of bounded variation processes in the semimartingale theory is played by $\mathcal{N}$, the class of additive functionals of zero energy. In general an additive functional is not of bounded variation and therefore the Lebesgue-Stieltjes integrals can not be defined. Nevertheless, Nakao [35] introduced a stochastic integral $\int_0^t f(X_s)\mathrm{d}N_s$ for f bounded function element of $\mathcal{F}$ and N element of the following subclass of $\mathcal{N}$:

$$\tilde{\mathcal{N}_c} \ = \ \{N^u_\cdot - \int_0^\cdot u(X_s)\mathrm{d}s : u \in \mathcal{F}\}.$$

In his Itô formula expending $u(X)$ [35], this integral replaces the Lebesgue-Stieltjes integral in the Itô formula for semimartingales. Besides, this integral is used by Fitzsimmons and Kuwae [17], to study the lower order perturbation of diffusion processes.
The conditions of existence of Nakao's integral being too restrictive, this notion could not be used to study the lower order perturbation of symmetric Markov processes that are not diffusions. Therefore Chen et al. [7] have extended Nakao's integral to a larger class of integrators as well as integrands. Using time reversal they have defined an integral $\int_0^t f(X_s)\mathrm{d}C_s$ for f locally in $\mathcal{F}$ and C in a class of processes containing $\tilde{\mathcal{N}_c}$. The process C is not in general of zero energy but of zero quadratic variation and the integral is not an additive functional or a local additive functional but a local additive functional admitting null set. Kuwae [31] gives a refinement of Chen et al. work, redefining the stochastic integral without using time reversal but restricting the class of integrands.

Our aim in this paper, is to construct an integral $\int_0^t f(X_s)\mathrm{d}C_s$ for a Markov process X associated to a non necessarily symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, f locally in $\mathcal{F}$ and C local continuous additive functional with zero quadratic variation. To do so, one can not extend the construction of Chen et al. neither Kuwae's construction because they both heavily rely on the symmetry of the Markov process.
On one hand, it is legitimate to solve this question since many results for symmetric Dirichlet forms hold for non-symmetric Dirichlet forms, see e.g., [28], [29], [30], [32] and [37]. In particular, Fukushima's decomposition (5.1.1) holds for non-symmetric regular Dirichlet forms, but also the correspondence between Markov processes and (non-necessarily symmetric) Dirichlet forms, Revuz correspondence and other relations between probabilistic notions for a Markov process X and analytic notions for $\mathcal{E}$.
On the other hand, the interest of constructing such an integral is that it leads to an Itô formula for $u(X)$ when u is element of $\mathcal{F}$. Moreover when the state space of X is $\mathbb{R}^d$, we obtain an Itô formula for X. With this paper we would like to offer new tools to study Markov processes.

The paper, based on Fukushima's decomposition (5.1.1) for X non necessarely symmetric, is organized as follows. In Section 2, we present some preliminaries. For u element of $\mathcal{F}$, we give in Section 3, a necessary and sufficient condition for N^u to be of bounded variations. In Section 4, we show that a continuous additive functional with zero quadratic variation is locally in $\tilde{\mathcal{N}_c}$. Then in Section 5, we construct a stochastic integration with respect to N^u. To do so we first establish in Section 5.1, a decomposition of N^u as the sum of three processes N_1^u, N_2^u and N_3^u such that N_1^u and N_2^u are respectively associated to the diffusion part and the jumping part of the symmetric part of $\mathcal{E}$, and N_3^u is of bounded variations. In Sections 5.2 and 5.3, we present respectively stochastic integration with respect to N_1^u and N_2^u. These results lead to an integral with respect to N^u which is used in Section 5.4 to establish an Itô formula for the process $u(X)$, u in $\mathcal{F}$, in which this new integral takes the place of the Lebesgue-Stieltjes integral in the classical Itô formula for semimartingales. Still in Section 5.4, we show, thanks to Section 3, that when N^u is of bounded variations the obtained stochastic integral coincides with the Lebesgue-Stieljes integral with respect to N^u. We also show that when the Dirichlet form is symmetric, the obtained stochastic integral with respect to N^u coincides with the integral defined by Chen et al [7]. In Section 5.6, we work with Markov processes in $\mathbb{R}^d$. We establish, using a Berling-Deny formula for $\mathcal{E}$ shown by Hu et al. [23], a decomposition of the coordinate process X^i for $i = 1, \ldots, d$, similar to the Lévy-Itô decomposition for Lévy processes. The drift part process is replaced by a local continuous additive functional. Under the assumption of symmetry, similar decompositions have been established in the previous chapter (Chapter 4 - Theorem 4.1.2) and in Chen et al. [9]. Thanks to this decomposition, we establish an Itô formula for the process X.

5.2 Preliminaries

This section presents mostly notation and vocabulary from the book of Fukushima et al. [21] still available in the non necessarily symmetric case. It contains also some immediate consequences of existing results that will be useful for the other sections.

Throughout this paper, we assume that E is a locally compact separable metric space and m is a σ-finite Borel measure on E such that $Supp[m] = E$. $L^2(E;m)$ denotes the real L^2-space with inner product

$$(f,g)_m := \int_E f(x)g(x)m(\mathrm{d}x), \text{ for any } f,g \in L^2(E;m)$$

We adjoin to E a point Δ and endow $E_\Delta := E \cup \{\Delta\}$ with the topology of one point compactification. If E is already compact, Δ is an isolated point. A real function f on E is extended to a function on E_Δ by setting $f(\Delta) = 0$.

We fix a (non necessarily symmetric) regular Dirichlet form $\mathcal{E}$ on $L^2(E;m)$ with domain $\mathcal{F}$. Many examples of Dirichlet forms on $L^2(E;m)$ with E distinct from $\mathbb{R}^d$ are given in the book of Ma and Röckner as well as examples with $E = \mathbb{R}^d$ (see [32], chapter 2).

We set $\mathcal{E}_1(u,v) := \mathcal{E}(u,v) + (u,v)_m$, $\hat{\mathcal{E}}(u,v) := \mathcal{E}(v,u)$,

$\tilde{\mathcal{E}}(u,v) := 1/2\{\mathcal{E}(u,v)+\hat{\mathcal{E}}(u,v)\}$, $\tilde{\mathcal{E}}_1(u,v) := \tilde{\mathcal{E}}(u,v)+(u,v)_m$ and $\check{\mathcal{E}}(u,v) := 1/2\{\mathcal{E}(u,v)-\hat{\mathcal{E}}(u,v)\}$ for any $u,v \in \mathcal{F}$.
For any Borel set B let $\mathcal{F}_B := \{u \in \mathcal{F} : u = 0$ m-a.e on $E\backslash B\}$. An increasing sequence of closed sets $\{F_k\}$ is called a nest if and only if $\cup_{n\geq 1}\mathcal{F}_{F_n}$ is $\tilde{\mathcal{E}}_1$-dense in $\mathcal{F}$. A subset $N \subset E$ is called exceptional if $N \subset \cap_{k\geq 1}(E\backslash F_k)$ for some nest $\{F_k\}$. We say that a property of points in E holds quasi-everywhere (abbreviated q.e), if the property holds outside some exceptional set. An q.e. defined function f on E is called quasi-continuous if there exists an nest $\{F_k\}$ such that $f_{|F_k}$ is continuous on F_k for any k.
Let $\{T_t\}_{t\geq 0}$ and $\{G_\alpha\}_{\alpha>0}$ (resp. $\{\hat{T}_t\}_{t\geq 0}$ and $\{\hat{G}_\alpha\}_{\alpha>0}$) be respectively the semi-group and the resolvent associated to $(\mathcal{E},\mathcal{F})$ (resp. $(\hat{\mathcal{E}},\mathcal{F})$).
By Chapter V, section 2 of [32], every element u of $\mathcal{F}$ admits a quasi-continuous m-version. Moreover, there exists a Hunt processes

$$\mathbf{M} = (\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{\mathbf{P}_z\}_{z\in E_\Delta})$$

with space E and life time ζ which is properly associated to $(\mathcal{E},\mathcal{F})$ i.e., for any $t,\alpha > 0$ and $f \in L^2(E,m)$, $p_t f$ and $R_\alpha f$ are respectively quasi-continuous m-versions of $T_t f$ and $G_\alpha f$, where for any $x \in E$:

$$p_t f(x) = \mathbf{E}_x[f(X_t)] \text{ and } R_\alpha f(x) = \int_0^\infty e^{-\alpha t} p_t f(x) \mathrm{d}t$$

Therefore for $u \in L^2(E;m)$ we have

$$\begin{aligned} u \in \mathcal{F} \quad &\Leftrightarrow \quad \sup_{\alpha>0} \alpha(u - R_\alpha u, u)_m < \infty \text{ and in this case :} \\ \mathcal{E}(u,v) \quad &= \quad \lim_{\alpha\to\infty} \alpha(u-\alpha R_\alpha u, v)_m, \text{ for all } v \in \mathcal{F} \end{aligned} \tag{5.2.1}$$

Recall that a subset B of E is said to be nearly Borel measurable if for any probability measure μ in E_Δ there exist Borel sets $B_1, B_2 (\subset E_\Delta)$ such that $B_1 \subset B \subset B_2$ and $\mathbf{P}_\mu(\exists t \geq 0, X_t \in B_2 - B_1) = 0$. Here for any $A \subset \mathcal{F}$ we set $\mathbf{P}_\mu(A) = \int_{E_\Delta} \mathbf{P}_x(A)\mu(\mathrm{d}x)$. We denote by $\mathcal{B}^n$ the set of nearly Borel measurable subsets of E.
For a nearly Borel set $B(\subset E_\Delta)$, σ_B and τ_B represent the first hitting time to B and the first exit time from B respectively, i.e :

$$\begin{aligned} \sigma_B &:= \inf\{t > 0 : X_t \in B\} \\ \tau_B &:= \inf\{t > 0 : X_t \notin B\} \end{aligned}$$

It is well known that for a nearly Borel set B, σ_B and τ_B are $(\mathcal{F}_t)-$stopping times.
A set $N \subset E$ is said $\mathbf{M}$-exceptional if there exists $\tilde{N} \in \mathcal{B}(E)$ such that $N \subset \tilde{N}$ and $\mathbf{P}_m(\sigma_{\tilde{N}} < \infty) = 0$. We say that a set $N \subset E$ is properly exceptional if $N \in \mathcal{B}^n$, $m(N) = 0$ and for any $x \in E \setminus N$, $\mathbf{P}_x(X_t \in E_\Delta \setminus N, X_{t-} \in E_\Delta \setminus N, \forall t \geq 0) = 1$.

For any set $N \subset E$, N is exceptional if and only if N is $\mathbf{M}$-exceptional. Moreover any properly exceptional set is exceptional and for any exceptional set N, there exists a properly Borel exceptional set B containing N. (see [30], Lemma 4.5.(ii) and Lemma 4.6.(iii)).
By proposition V.2.28 of [32], we have that a function u is quasi-continuous if and only if is finely continuous q.e. i.e., if and only if there exists a properly exceptional set N such that u is nearly Borel measurable on $E \backslash N$ and for $x \notin N$:

$$\mathbf{P}_x(t \to u(X_t) \text{ is right continuous}) = 1$$

If additionally $u \in \mathcal{F}$, the above property is equivalent to :

$$\mathbf{P}_x(u(X_t) \text{ is right continuous and } \lim_{s \uparrow t} u(X_s) = u(X_{t-}) \ \forall t \geq 0) = 1 \tag{5.2.2}$$

For a nearly Borel set B, define $\mathcal{F}_{E\backslash B} := \{u \in \mathcal{F} : u = 0 \text{ q.e. on } B\}$. This is a closed subspace of $(\mathcal{F}, \tilde{\mathcal{E}}_1)$. It follows from Corollary 2.1 in [30] that for any u in $\mathcal{F}$ there exists a unique v in $\mathcal{F}_{E\backslash B}$ such that $\mathcal{E}_1(u - v, w) = 0$ for all $w \in \mathcal{F}_{E\backslash B}$. This unique v is called the 1-projection of u on $\mathcal{F}_{E\backslash B}$ and denoted by $\Pi^1_{\mathcal{F}_{B^c}}(u)$.
For any f Borel measurable function such that $\mathbf{E}_x[e^{-\sigma_B}|f(X_{\sigma_B})|]$ is finite q.e. we define :

$$H^1_B f(x) := \mathbf{E}_x[e^{-\sigma_B} f(X_{\sigma_B})] \tag{5.2.3}$$

Then for any $u \in \mathcal{F}$, $H^1_B|u|(x)$ is finite q.e. and we can show with the same arguments used in the proof of Theorem 4.3.1 of [21] that $H^1_B u$ is a quasi-continuous m-version of $u - \Pi^1_{\mathcal{F}_{B^c}} u$.
A subset G of E is said to be quasi-open if there exists a nest $\{F_k\}$ of closed sets such that $G \cap F_k$ is open with respect to the relative topology on F_k for each $k \in \mathbb{N}$. In the same way one defines quasi-closed subsets.
For two subsets A, B of E we say that $A \subset B$ q.e. if $A \backslash B$ is exceptional and we say that A is q.e. equivalent to B and write $A = B$ q.e. if the symmetric difference $A \Delta B$ is exceptional. A set G is called q.e. finely open if it is q.e. equivalent to a nearly Borel finely open set. In the same way one defines a q.e. finely closed set.
A function f is quasi-continuous if and only if $f^{-1}(I)$ is quasi-open for any I open set of $\mathbb{R}$. (see [21] pp. 68). It follows from Proposition 4.1.(ii) in [30] that a subset of E is quasi-open if and only if is q.e. finely open.
Set : $\mathcal{O} := \{G \subset E : \ G \text{ is nearly Borel and finely open}\}$ and for a subset A of E, set : $\mathcal{O}_A := \{G \in \mathcal{O} : G \subset A\}$. For $G \in \mathcal{O}$ define :

$$\Xi_G := \left\{ \{G_n\} \subset \mathcal{O}_G : G_n \subset G_{n+1} \ \forall n \text{ and } G = \bigcup_{n=1}^{\infty} G_n \text{ q.e.} \right\}$$

We denote Ξ_E by Ξ.

Remark 5.2.1. *For $B \subset E$, denote by B^r the set of regular points for B. Let $\{F_n\}$ be an increasing sequence of closed sets and for any $n \in \mathbb{N}$, let G_n be the fine interior of F_n. Then $\{F_n\}$ is a nest if and only if $\{G_n\} \in \Xi$. Indeed, note that $G_n = F_n \backslash (E \backslash F_n)^r$, thus it is nearly Borel measurable (see Theorem A.2.5 of [21] and Proposition 10.6 in [44]). Moreover, by Lemma 3.3 of [29], $\{F_n\}$ is a nest if and only if $\bigcup_{n=1}^{\infty} G_n = E$ q.e.*

Let $(\hat{R}_\alpha)_{\alpha>0}$ be the resolvent associated to $\hat{\mathcal{E}}$. Define :

$$\mathcal{K}(E) := \{g \in L^1(E,m) : 0 < g(x) \leq 1 \ \forall x \in E\}.$$

We fix an element $\varphi \in \mathcal{K}(E)$ and let $h = R_1\varphi$ and $g = \hat{R}_1\varphi$. Define the capacity Cap as the capacity $\mathrm{Cap}_{h,g}$ defined in [32], Defintion III.2.4. Then a sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed sets is an nest if and only if $Cap(E \setminus F_n)$ converges to zero as n tends to infty.

The above lemma is actually Lemma 4.6. in [30] that we recall for the reader's convenience. We add point (iv). Indeed the equivalence (i)⇔(iv) is shown in the proof of Lemma 3.6. in [30].

Lemma 5.2.2. *For an element G in $\mathcal{O}$ and an increasing sequence $\{G_n\}$ of nearly Borel finely open subsets of G, the following are then equivalent :*

(i) $\{G_n\} \in \Xi_G$.

(ii) $\mathbf{P}_x(\lim_{n\to\infty} \tau_{G_n} = \tau_G) = 1$ *for m-a.e.* $x \in E$

(iii) $\mathbf{P}_x(\lim_{n\to\infty} \tau_{G_n} = \tau_G) = 1$ *for q.e.* $x \in E$

(iv) $\lim_{n\to\infty} Cap(G \backslash G_n) = 0$

Lemma 5.2.2 is used to establish the following lemma.

Lemma 5.2.3. *Let $\{G_n\}_{n\in\mathbb{N}}$ be an element of Ξ and for each n, let $\{G_{n,k}\}_{k,\in\mathbb{N}}$ be in Ξ_{G_n}. Then there exists $\{\mathcal{G}_n\}_{n\in\mathbb{N}} \in \Xi$ such that for each $n \in \mathbb{N}$, there exists $j, k \in \mathbb{N}$ such that $\mathcal{G}_n \subset G_{j,k}$.*

Proof. Let φ be in $\mathcal{K}(E)$. For any G element of $\mathcal{O}$, set : $R_1^G\varphi(x) = \mathbf{E}_x[\int_0^{\tau_G} e^{-s}\varphi(X_s)\mathrm{d}s]$. For any $n \in \mathbb{N}$ let $H_n := \{x \in E : R_1^{G_n}\varphi > n^{-1}\}$ and $\bar{H}_n := \{x \in E : R_1^{G_n}\varphi \geq n^{-1}\}$. It is known that : $E \setminus \bar{H}_n \in \mathcal{O}$. From the proof of Lemma 3.8. in [30] : $\{H_n\} \in \Xi$, $H_n \subset G_n$ q.e. and $H_n \subset \bar{H}_n \subset H_{n+1}$ q.e.
For each n, we apply the above procedure to $\{G_{n,k} \cap H_n\}_{k\in\mathbb{N}}$ which is in Ξ_{H_n}, in order to get $\{H_{n,k}\}_{k\in\mathbb{N}}$ in Ξ_{H_n} such that : $H_{n,k} \subset G_{n,k}$ q.e., $H_{n,k} \subset \bar{H}_{n,k} \subset H_{n,k+1}$ q.e. and $E \setminus \bar{H}_{n,k} \in \mathcal{O}$, for each k. In view of Lemma 5.2.2 we assume, by taking subsequences if necessary, that $\mathrm{Cap}(E \setminus H_n) \vee \mathrm{Cap}(H_n \setminus H_{n,n}) < 2^{-n}$. Set : $\bar{\mathcal{G}}_n := \bigcap_{k\geq n} \bar{H}_{k,k}$, then $\mathrm{Cap}(E \setminus \bar{\mathcal{G}}_n) < 2^{-n}$.
Since each $\bar{\mathcal{G}}_n$ is quasi closed, there exists a common nest of closed sets $\{F_k\}$ such that $F_k \cap \bar{\mathcal{G}}_n$ is closed for any k, n. Then we have : $\mathrm{Cap}(E \setminus (F_n \cap \bar{\mathcal{G}}_n)) \leq 2^{-n} + \mathrm{Cap}(E \setminus F_n)$, which converges to 0 as n tends to infinity. Let $\mathcal{G}_n$ be the fine interior of $\bar{\mathcal{G}}_n \cap F_n$. Then $\mathcal{G}_n \subset G_{n,n+1}$ and $\{\mathcal{G}_n\} \in \Xi$ (see Remark 5.2.1). □

Definition 5.2.4. *$\mathcal{F}_{loc}$ is the set of the real-valued measurable functions u on E such that there exists $\{G_k\}$ in Ξ and a sequence $\{u_k\}$ of $\mathcal{F}$ such that for any k, $u(x) = u_k(x)$ for q.e. $x \in G_k$.*

Evidently any u in $\mathcal{F}_{loc}$ admits a quasi-continuous m-version. From now on, we always assume that functions in $\mathcal{F}_{loc}$ are represented by their quasi-continuous m-versions. For $u, \{G_k\}$ and $\{u_k\}$ connected as in the above definition, set$H_k := G_k \cap \{x : |u(x)| < k\}$. Then each H_k is q.e. equivalent to a nearly Borel finely open set $\tilde{H}_k$. Hence $\{\tilde{H}_k\}$ belongs to Ξ and : $u(x) = (-k) \vee u_k \wedge k$ q.e. on $\tilde{H}_k$. Therefore in Definition 5.2.4 one can always assume that each u_k satisfies $\|u_k\|_\infty \leq k$.

An $(\mathcal{F}_t)$-adapted process A is an additive functional (AF in abbreviation) if there exists Λ in $\mathcal{F}_\infty$ and a properly exceptional set N such that : $\mathbf{P}_x(\Lambda) = 1$ for $x \in E \setminus N$, $\theta_t \Lambda \subset \Lambda$ for all $t \geq 0$ and for all $\omega \in \Lambda : t \to A_t(\omega)$ is finite càdlàg on $[0, \zeta(\omega))$, $A_0(\omega) = 0$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$ and $A(\omega)$ has the additive property : $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega), s, t \geq 0$.

A local AF or AF on $[\![0, \zeta[\![$ is a process that satisfies all requirements to be an AF except that the additive property is required only for $t, s \geq 0$ with $t + s < \zeta(\omega)$. In the sequel, we say that Λ is a defining set for A or a defining set admitting a properly exceptional set N or that N is a properly exceptional set for A.
An AF A is said to be continuous (resp. càdlàg, resp. finite) if it has a defining set Λ such that $A_.(\omega)$ is continuous (resp. càdlàg, resp. finite) in $[0, \infty)$ for any $\omega \in \Lambda$. The abbreviations CAF and PCAF stand for, "continuous additive functional" and "positive continuous additive functional" respectively.
We say that a local AF A is continuous or a local CAF if there exists a defining set Λ such that $A_.(\omega)$ is continuous in $[0, \zeta(\omega)[$ for any $\omega \in \Lambda$.
We denote by $\mathbf{A}_c^+$ the set of PCAF. A Borel measure μ on E is called smooth if it does not charge exceptional sets and there exists a nest $\{F_n\}$ such that $\mu(F_n) < \infty$ for all n. We denote by S the family of all smooth measures. There exists a correspondence between the elements of S and $\mathbf{A}_c^+$ called the Revuz correspondence characterized as follows (see e.g. [32]) : μ in S is the Revuz measure of A if for any Borel measurable positive f

$$\mu(f) = \lim_{t \downarrow 0} \frac{1}{t} E_m \left(\int_0^t f(X_s) \mathrm{d}A_s \right) = \sup_{t > 0} \frac{1}{t} E_m \left(\int_0^t f(X_s) \mathrm{d}A_s \right)$$

Definition 5.2.5. *For any A in $\mathbf{A}_c^+$ we denote its Revuz measure by μ_A. A sequence $\{G_n\}$ element of Ξ is said to be associated to μ in S if $\mu(G_n) < \infty$ for each n. $\{G_n\}$ is said to be associated to $A \in \mathbf{A}_c^+$ if it is associated to μ_A.*

Let μ be a measure charging no exceptional set of E. In view of Remark 5.2.1, μ has an associated nest $\{G_n\} \in \Xi$ iff $\mu \in S$. Indeed the direct way is a consequence of Remark 2.1, and the converse can be established exactly as in Lemma 2.2 in [31].

A positive Radon measure μ on E is said to be of finite energy integral (relatively to $\mathcal{E}_1$) if there exists $C > 0$ such that :

$$\int_E |v(x)|\mu(\mathrm{d}x) \le C\sqrt{\mathcal{E}_1(v,v)}, \quad v \in \mathcal{F} \cap C_0(E).$$

In this case, $\mathcal{F} \subset L^1(E,\mu)$ and the above equation holds for any $h \in \mathcal{F}$ (see Theorem 2.2.2 on [21]). We denote by S_0 the set of finite energy integral measures. A measure μ belongs to S_0 if and only if there exists an unique element in $\mathcal{F}$ denoted $U_1\mu$ such that $\mathcal{E}_1(U_1\mu, h) = \int_E h(x)\mu(\mathrm{d}x)$ for all $h \in \mathcal{F}$ or equivalently if there exists $\hat{U}_1\mu \in \mathcal{F}$ such that $\mathcal{E}_1(h, \hat{U}_1\mu) = \int_E h(x)\mu(\mathrm{d}x)$ for all $h \in \mathcal{F}$. Moreover $S_0 \subset S$ and a set $N \subset E$ is exceptional if and only if $\mu(N) = 0$ for all $\mu \in S_0$ if and only if $\mu(N) = 0$ for all $\mu \in \hat{S}_{00} := \{\mu \in S_0 : \mu(E) = 1, \|\hat{U}_1\mu\|_\infty < \infty\}$ (see [37], section 2.3).
According to Lemma 4.3 of [28], for any $A \in \mathbf{A}_c^+$, $\nu \in \hat{S}_{00}$,

$$\mathbf{E}_\nu[A_t] \le e^t \|\hat{U}_1\nu\|_\infty \mu_A(E) \tag{5.2.4}$$

Lemma 5.2.6. *Let A^n be a sequence of $\mathbf{A}_c^+$. Suppose that $\mu_{A_n}(E)$ converges to zero as $n \to \infty$. Then there exists a subsequence (n') satisfying the condition that for q.e. $x \in E$,*

$$\mathbf{P}_x\left(A_t^{n'} \text{ converges to zero uniformly on any compact}\right) = 1 \tag{5.2.5}$$

This can be proved using (5.2.4) and the arguments used in the proof of Lemma 5.1.2 of [21].

To simplify the formulation of the results, we define the energy of an AF A by

$$e(A) := \limsup_{\alpha\to\infty} \frac{\alpha^2}{2}\mathbf{E}_m\left[\int_0^\infty e^{-\alpha t}A_t^2 \mathrm{d}t\right]$$

For two AF A, B, their mutual energy is defined by

$$e(A,B) := \frac{1}{2}[e(A+B) - e(A) - e(B)]$$

An AF M is called a martingale additive functional (abbreviated as MAF) if it is finite, càdlàg and for q.e x in E : $\mathbf{E}_x[M_t^2] < \infty$ and $\mathbf{E}_x[M_t] = 0$ for any $t \ge 0$. Denote by $\mathcal{M}$ the set of MAF, by $\mathring{\mathcal{M}}$ the set of MAF's of finite energy and

$$\mathcal{N}_c := \left\{ N : \begin{array}{c} N \text{ is a finite continuous AF, } e(A) = 0, \\ \mathbf{E}_x(|N_t|) < \infty \text{ q.e for each } t > 0 \end{array} \right\}.$$

Lemma 5.2.7. *Let M be an AF admitting a properly exceptional set N such that for $x \in E \setminus N$ and $t \ge 0$, $\mathbf{E}_x[M_t^2] < \infty$ and $\mathbf{E}_x[M_t] = 0$. Then M is a MAF.*

Proof. The only point that we have to show is that there exists a defining set on which M is finitely càdlàg. Evidently M is finite. Let Λ be a defining set of M admitting a properly exceptional set N such that $M_{.}(\omega)$ is finite on $[0,\infty)$ for any $\omega \in \Lambda$. For any $x \in E\backslash N$, M is a $(\mathbf{P}_x, (\mathcal{F}_t))$-martingale. Thus for $x \in E\backslash N$, M has a $\mathbf{P}_x$-modification càdlàg denoted by M^x. For $\omega \in \Omega$ such that $\zeta(\omega) > 0$ define :

$$\begin{aligned} M^s_\zeta(\omega) &= \inf_{s<\zeta, s\in\mathbb{Q}} \sup\{M_r(\omega) : s \le r < \zeta(\omega), r \in \mathbb{Q}\} \\ M^i_\zeta(\omega) &= \sup_{s<\zeta, s\in\mathbb{Q}} \inf\{M_r(\omega) : s \le r < \zeta(\omega), r \in \mathbb{Q}\} \end{aligned}$$

and $M^s_\zeta(\omega) = M^i_\zeta(\omega) = 0$ if $\zeta(\omega) = 0$. Define :

$$\tilde{\Lambda} = \{\omega \in \Lambda : 0 < \zeta(\omega) < \infty \text{ and } M^s_\zeta(\omega) = M^i_\zeta(\omega)\} \cup \{\omega \in \Lambda : \zeta(\omega) \in \{0,\infty\}\} \in \mathcal{F}_\infty$$

The fact that for $\omega \in \Lambda$, $M(\omega)$ is càdlàg on $[0, \zeta(\omega))$, leads to $M^s_\zeta(\omega) = \limsup_{t\uparrow\zeta(\omega)} M_t(\omega)$ and $M^i_\zeta(\omega) = \liminf_{t\uparrow\zeta(\omega)} M_t(\omega)$. Therefore, $M(\omega)$ is finite càdlàg on $[0,\infty)$ for $\omega \in \tilde{\Lambda}$.
We shall prove that $\tilde{\Lambda}$ is a defining set for M. We can check easily that $\theta_t\tilde{\Lambda} \subset \tilde{\Lambda}$ for all $t \ge 0$. We must prove that $\mathbf{P}_x(\tilde{\Lambda}) = 1$ for all $x \in E\backslash N$. For x in $E\backslash N$, set $\Lambda^x := \Lambda \cap \{w \in \Omega : M^x_{.} \text{ is càdlàg}\} \cap \{w \in \Omega : M_t(\omega) = M^x_t(\omega) \forall t \in \mathbb{Q}^+\}$. We have then : $\mathbf{P}_x(\Lambda^x) = 1$. For $\omega \in \Lambda^x$ such that $0 < \zeta(\omega) < \infty$, one obtains : $M^s_\zeta(\omega) = \limsup_{t\uparrow\zeta(\omega)} M^x_t(\omega) = \liminf_{t\uparrow\zeta(\omega)} M^x_t(\omega) = M^i_\zeta(\omega)$. Consequently : $\Lambda^x \subset \tilde{\Lambda}$ and $\mathbf{P}_x(\tilde{\Lambda}) = 1$. □

For any $u \in \mathcal{F}$, M^u and N^u denote the elements of $\mathcal{M}$ and $\mathcal{N}_c$ respectively, that are present in Fukushima decomposition of $u(X_t) - u(X_0), t \ge 0$, i.e. :

$$u(X_t) - u(X_0) = M^u_t + N^u_t \text{ for } t \ge 0,\ \mathbf{P}_x\text{-a.e for q.e } x \in E.$$

The following lemma can be proved in the same way as Lemma 5.1.2 and Corollary 5.2.1 of [21].

Lemma 5.2.8. *Let (u_n) be a sequence of quasi continuous functions in $\mathcal{F}$ and $\tilde{\mathcal{E}}_1$-convergent to u. Then there exists a subsequence $\{u_{n_k}\}$ such that for q.e $x \in E$,*

$$\mathbf{P}_x(u_{n_k}(X_t) \text{ converges uniformly to } u(X_t) \text{ on each compact interval of } [0,\infty)) = 1$$

and the same holds for $N^{u_{n_k}}$ and N^u, and for $M^{u_{n_k}}$ and M^u, replacing $u_{n_k}(X)$ and $u(X)$ respectively.

We close this section by mentioning that in the literature the integral of H with respect to K is denoted by $\int H_s dK_s$ or by $H * K$.

5.3 CAF of bounded variation

In this section we adopt the following convention : $\infty - \infty = 0$. Thus for two PCAFs of X, B and A, the process $A - B := \{A_t - B_t : t \geq 0\}$ is always a local CAF. Set :

$$\mathbf{A}_c := \{A - B : A, B \in \mathbf{A}_c^+\}$$

We recall the following definition given in [21] : μ is called a smooth signed measure if there exists a nest $\{F_k\}$ of closed sets such that each $1_{F_k}.\mu$ is the difference of two finite smooth measures, thus $\mu = \mu^{(1)} - \mu^{(2)}$ for some smooth measures $\mu^{(1)}$ and $\mu^{(2)}$ finite on $\{F_k\}$.

By the Revuz correspondence between S and $\mathbf{A}_c^+$, there is a bijection between smooth signed measures and $\mathbf{A}_c$. In the sequel we refer to this bijection also as the Revuz correspondence. For any $A \in \mathbf{A}_c$, we denote by μ_A its Revuz signed measure. it is clear that for $A \in \mathbf{A}_c$, $|\mu_A|$ is the smooth measure associated to V, where V_t represents the total variation of A on $[0, t]$.

Definition 5.3.1. *We denote by S-S the set of smooth signed measures. We say that a smooth signed measure μ belongs to S_0-S_0 if and only if is the difference of two measures in S_0.*

It is clear that a smooth measure signed μ belongs to S_0-S_0 if and only if $\mu^{(1)}, \mu^{(2)}$ belongs to S_0, where $\mu = \mu^{(1)} - \mu^{(2)}$ denote its Jordan decomposition, and in this case there exists a unique element of $\mathcal{F}$ denoted by $U_1\mu$ such that,

$$h \in L^1(E; |\mu|) \text{ and } \mathcal{E}_1(U_1\mu, h) = \int_E h(x)\mu(\mathrm{d}x), \text{ for any } h \in \mathcal{F}$$

Definition 5.3.2. *For μ in S-S and G in $\mathcal{O}$, $\{G_k\}$ element of Ξ_G is said to be associated to μ if for each n, $1_{G_n}.\mu$ is the difference of two finite measures. We say that a nest is associated to $A \in \mathbf{A}_c$ if it is associated to μ_A.*

It is clear from Remark 5.2.1 that for any smooth signed measure $\mu \in S$-S and $G \in \mathcal{O}$, there is a $\{G_n\} \in \Xi_G$ associated to μ.
For $G \subset E$ we have defined $\mathcal{F}_G$ as the set $\{u \in \mathcal{F} : u(x) = 0 \text{ for q.e. } x \notin G\}$. Set $\mathcal{F}_{b,G} := \mathcal{F}_G \cap \mathcal{F}_b$, and denote $\mathcal{F}_{b,E}$ by $\mathcal{F}_b$ ($\mathcal{F}_b$ is the set of bounded elements of $\mathcal{F}$).

The following Theorem is an extension of Theorem 5.4.2 of [21] to the non symmetric case. We omit its proof which consists in the replacement of Theorem 5.2.4. and Lemmas 5.4.1, 5.4.2 and 5.4.3 of [21] by respectively Lemmas 5.3.6, 5.3.7, 5.3.8 and 5.3.9.

Theorem 5.3.3. *For $u \in \mathcal{F}$ and $G \in \mathcal{O}$, N^u is of bounded variation on $[0, \tau_G)$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$ if and only if there exists $\mu \in S$-S such that*

$$\mathcal{E}(u,v) =< \mu, v >, \ \forall v \in \bigcup_{n=1}^{\infty} \mathcal{F}_{b,G_n}$$

where $\{G_n\} \in \Xi_G$ associated with μ. In this case,

$$\mathbf{P}_x(N_t^u = -A_t \text{ for } t < \tau_G) = 1 \text{ for q.e } x \in E \tag{5.3.1}$$

where A is an element of $\mathbf{A}_c$ with Revuz signed measure μ.

Definition 5.3.4. *We define $\mathcal{N}_c^0$ as the set of local CAF's C such that, there exists u in $\mathcal{F}$ and A in $\mathbf{A}_c$ satisfying :*

$$\mathcal{F}_b \subset L^1(E, |\mu_A|)$$

and

$$\mathbf{P}_x(C_t = N_t^u + A_t \text{ for } t < \zeta) = 1 \text{ for q.e } x \in E$$

In this case, we define the linear functional $\Theta(C)$ on $\mathcal{F}_b$ by

$$< \Theta(C), h >:= -\mathcal{E}(u,h)+ < \mu_A, h >, \ h \in \mathcal{F}_b$$

It follows from Theorem 5.3.3 that the definition of $\Theta(C)$ for $C \in \mathcal{N}_c^0$ is consistent in the sense that it does not depend of the elements which represent C.
The following Lemma is an immediate consequence of Theorem 5.3.3 :

Lemma 5.3.5. *Let $C^{(1)}$ and $C^{(2)}$ be elements of $\mathcal{N}_c^0$ and G in $\mathcal{O}$. Then $C^{(1)} = C^{(2)}$ on $t < \tau_G$ $\mathbf{P}_x$-a.e for q.e $x \in E$ if and only if*

$$< \Theta(C^{(1)}), h >=< \Theta(C^{(2)}), h > \text{ for all } h \in \mathcal{F}_{b,G}.$$

Lemmas 5.3.6, 5.3.7, 5.3.8 and 5.3.9 below are versions for the non-symmetric case of Theorem 5.2.4. and Lemmas 5.4.1, 5.4.2 and 5.4.3 in [21] respectively. This results are proved in [21] using the approximation of $\mathcal{E}$, $\mathcal{E}^{(t)}(u,v) := \frac{1}{t}(u - p_t u, v)_m$ for $u, v \in \mathcal{F}$, available for symmetric Dirichlet forms. The proof for the non-symmetric case can be done following the same arguments but using instead the approximation of $\mathcal{E}$ given by (5.2.1). We omit their proof.

Lemma 5.3.6. *Let an AF A be an AF element of $\mathcal{N}_c$ and u a function in $\mathcal{F}$. Then : $A = N^u$ if and only if :*

$$\lim_{\alpha \to \infty} \alpha^2 \mathbf{E}_{v.m} \left[\int_0^\infty e^{-\alpha t} A_t \mathrm{d}t \right] = -\mathcal{E}(u,v) \ \forall v \in \mathcal{F} \tag{5.3.2}$$

For an AF A, set : $g(t) := \{\mathbf{E}_m[A_t^2]\}^{1/2}$. Then we have : $g(t+s) \leq g(t) + g(s)$ for all $t, s \geq 0$, thus $\lim_{t\to\infty} g(t)/t$ exists in $\mathbb{R}$. Now suppose that $a := \lim_{t\to 0} \frac{1}{t}\mathbf{E}_m[A_t^2]$ exists also in $\mathbb{R}$, then there exists a constant $p > 0$ such that $\frac{1}{t}\mathbf{E}_m[A_t^2] \leq p(1+t)$ for all t, therefore by dominated convergence, $\lim_{\alpha\to\infty} \alpha \int_0^\infty e^{-t}\mathbf{E}_m[A_{t/\alpha}^2]\mathrm{d}t = a$, thus $e(A) = a$. This is the case when $A_t = \int_0^t g(X_s)\mathrm{d}s$ for $g \in L^2(E,m)$ and also when A in $\mathbf{A}_c$ is such that μ_A belongs to S_0-S_0. (see pp. 201 of [21]). The following lemma is then an immediate consequence of Lemma 5.3.6.

Lemma 5.3.7. *Let $\mu \in S_0$-S_0, $w := (U_1\mu)$ and $A \in \mathbf{A}_c$ the CAF of bounded variation associated to μ, then ;*

$$A_t = -N_t^w + \int_0^t w(X_s)\mathrm{d}s \ t \geq 0 \ \mathbf{P}_x\text{-}a.e \text{ for } q.e\ x \in X$$

For $u \in \mathcal{F}$ and $G \in \mathcal{O}$, $H^1_{E\setminus G}u$ was defined in (5.2.3), this is a quasi-continuous m-version of $u - \Pi^1_{\mathcal{F}_G}u$.

Lemma 5.3.8. *Let $G \in \mathcal{O}$, then for any $u \in \mathcal{F}$,*

$$\mathbf{P}_x\left(N_t^{H^1_{E\setminus G}u} = \int_0^t H^1_{E\setminus G}u(X_s)\mathrm{d}s,\ \forall t < \sigma_{E\setminus G}\right) = 1 \ q.e\ x \in E$$

For $G \in \mathcal{O}$, let $\mathcal{E}^G$ be the restriction of $\mathcal{E}$ to $\mathcal{F}_G \times \mathcal{F}_G$, and for $u, v \in \mathcal{F}_G$, $\hat{\mathcal{E}}^G(u,v) = \mathcal{E}^G(v,u)$. $\mathcal{E}^G$ and $\hat{\mathcal{E}}^G$ are Dirichlet forms in $L^2(G,m)$. We denote by R^G_α and $\hat{R}^G_\alpha$ the resolvent associated to $\mathcal{E}^G$ and $\hat{\mathcal{E}}^G$ respectively.

Lemma 5.3.9. *For any $N \in \mathcal{N}_c$ and relatively compact set $G \in \mathcal{O}$:*

$$\lim_{\alpha\to\infty} \alpha^2 \mathbf{E}_{h.m}\left[\int_{\tau_G}^\infty e^{-\alpha t} N_t \mathrm{d}t\right] = 0$$

for $h = \hat{R}^G_1 f$ with f a bounded Borel function.

5.4 A representation for local CAF's of zero quadratic variation

Let $\mathcal{D}$ be a class of local AF's. Following [7], we say that a $(\mathcal{F}_t)$-adapted process A is locally in $\mathcal{D}$, and write : $A \in \mathcal{D}_{f\text{-}loc}$, if there exists a sequence A^n in $\mathcal{D}$ and $\{G_n\}$ in Ξ such that $A_t = A_t^n$ for $t < \tau_{G_n}$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$. In view of Lemma 5.2.2, A is hence a local AF. Thanks to Lemma 5.2.3, note that : $(\mathcal{D}_{f\text{-}loc})_{f\text{-}loc} = \mathcal{D}_{f\text{-}loc}$.

Definition 5.4.1. *We denote by $\mathfrak{E}$ and $\mathcal{N}$ the set of AF's of finite energy and the set of AF's of zero energy respectively.*

Recall that $\mathcal{K}(E) := \{g \in L^1(E,m) : 0 < g(x) \leq 1\ \forall x \in E\}$.

Definition 5.4.2. *We say that a process* $(V_t, t \geq 0)$ *is of zero quadratic variation (on* $[\![0,\zeta[\![$*) if for any* $T > 0$*, for all* η *positive, some (and hence for all)* g *in* $\mathcal{K}(E)$*,* $\mathbf{P}_{g.m}(Q_T^n > \eta, T < \zeta)$ *converges to zero as* n *tends to* ∞ *where*

$$Q_T^n := \sum_{i=0}^{n-1} \int_0^T (V_{t(i+1)/n} - V_{ti/n})^2 \mathrm{d}t.$$

The main result of this section is the following representation theorem for additive functionals of zero quadratic variation. It will be proved at the end of this section.

Theorem 5.4.3. *Let* A *be a local CAF element of* $\mathfrak{E}_{f\text{-}loc}$ *such that* A *is of zero quadratic variation. Then there exists* $\{G_n\} \in \Xi$ *and* $(u_n) \in \mathcal{F}$ *such that* $\mathbf{P}_x$*-a.e.for q.e.* $x \in E$ *:*

$$A_t = N_t^{u_n} - \int_0^t u_n(X_s)\mathrm{d}s \text{ for all } t < \tau_{G_n} \tag{5.4.1}$$

Remark 5.4.4. *Every element* N *of* $\mathcal{N}$ *is of zero quadratic variation. Indeed, for any* $T \geq 0$ *:*

$$\begin{aligned}
\mathbf{E}_m[Q_T^n] &= \sum_{i=0}^{K-1} \int_0^T \mathbf{E}_m[N_{t/K}^2 \circ \theta_{ti/K}]\mathrm{d}t \\
&\leq K \int_0^T \mathbf{E}_m[N_{t/K}^2]\mathrm{d}t \\
&\leq e^T K^2 \int_0^\infty e^{-tK}\mathbf{E}_m[N_t^2]\mathrm{d}t \to 0 \text{ as } K \to \infty
\end{aligned}$$

In view of Lemma 5.2.2, any N *in* $\mathcal{N}_{f\text{-}loc}$ *is also of zero quadratic variation and evidently if* N *is of zero quadratic variation then also* $(N_{t\wedge\tau}, t \geq 0)$ *for any* $(\mathcal{F}_t)$*-stopping time* τ*.*

Hence a local CAF of locally finite energy is of zero quadratic variation if and only if it belongs to $\mathcal{N}^0_{c\,f\text{-}loc}$ (Definition 5.3.4). This set plays an important part in the construction of Nakao's stochastic integral [35]. Before giving the proof of Theorem 5.4.3, we will establish a series of lemmas and remarks that will help us in the demonstration.

Lemma 5.4.5. *Let* V *be a* $(\mathcal{F}_t)$*-predictable process, finite* $\mathbf{P}_m$*-a.e. and of zero quadratic variation and* W *be a* $(\mathcal{F}_t, \mathbf{P}_{g.m})$*-semimartingale, where* g *is a element of* $\mathcal{K}(E)$*. Define* Y *by* $Y_t = W_tV_t - \int_0^t V_s\mathrm{d}W_s, t \geq 0$*, then* Y *is also of zero quadratic variation.*

Proof.

$$
\begin{aligned}
& \sum_{i=0}^{K-1}(Y_{t(i+1)/K}-Y_{ti/K})^2 \\
= \;& \sum_{i=0}^{K-1}\{(WV)_{t(i+1)/K}-(WV)_{ti/K}\}^2 \\
& +\sum_{i=0}^{K-1}\left(\int_{ti/k}^{t(i+1)/K} V_s \mathrm{d}W_s\right)^2 \\
& -2\sum_{i=0}^{K-1}\left(\int_{ti/k}^{t(i+1)/K} V_s \mathrm{d}W_s\right)\{(WV)_{t(i+1)/K}-(WV)_{ti/K}\} \\
=: \;& S_{K,1}(t)+S_{K,2}(t)-2S_{K,2}(t)
\end{aligned}
$$

it is not hard to show that for $i=1,2,3$, $\int_0^T S_{K,i}(t)\mathrm{d}t$ converges in $\mathbf{P}_{g.m}(.,T<\zeta)$-measure to $\int_0^T\int_0^t N_s^2\mathrm{d}[W]_s\mathrm{d}t$, where $[W]$ denotes the quadratic variation of W. □

The following result is a little modification of II.4.14 of [4] and wil be used to build sequences of Ξ.

Lemma 5.4.6. *Let Y be $\mathcal{F}_\infty$-measurable and N be a properly exceptional set such that for any $x\in E\backslash N$, $Y\circ\theta_t\to Y$ as $t\to 0$ $\mathbf{P}_x$-a.e and there exists $\delta>0$ such that*

$$
\sup_{x\in E\backslash N}\mathbf{E}_x[\sup_{t<\delta}|Y\circ\theta_t|]<\infty \tag{5.4.2}
$$

Then for any open set $I\subset\mathbb{R}$: $\{x:\mathbf{E}_x[Y]\in I\}\backslash N\in\mathcal{O}$.

Proof. We have : $E\backslash N\in\mathcal{O}$, and $\mathbf{P}_x(\sigma_N=\infty)=1$ for all $x\in E\backslash N$. The function $f(x):=\mathbf{E}_x[Y]$ is universally measurable (see theorem I.5.8 in [4]) hence $1_{E\backslash N}(x)f(x)$ is nearly measurable, indeed :

$$
1_{E\backslash N}(x)f(x)=\lim_{\alpha\to\infty}1_{E\backslash N}(x)\alpha R_\alpha f(x)
$$

where in the last equality we have use (5.4.2) and the fact that $Y\circ\theta_t\to Y$ as $t\to 0$ $\mathbf{P}_x$-a.e. For a real a, set : $A:=\{x:\mathbf{E}_x[Y]<a\}\backslash N$, then A is nearly Borel. We shall prove that A is finely open. For x in A and $\epsilon>0$ such that $\mathbf{E}_x[Y]<a-\epsilon$, set $B(x)=B_\epsilon(x)\cup N$ where $B_\epsilon(x):=\{y:\mathbf{E}_y[Y]\geq a-\epsilon/2\}$. Then : $B(x)\in\mathcal{B}^n$, and $E\backslash A\subset B(x)$. On the other hand :

$$\begin{aligned}
\mathbf{P}_x(\sigma_{B(x)}=0) &\leq \mathbf{P}_x(\limsup_{t\to 0}\mathbf{E}_{X_t}[Y]\geq a-\epsilon/2)\\
&= \mathbf{P}_x(\limsup_{t\to 0}\mathbf{E}_x[Y\circ\theta_t|\mathcal{F}_t]\geq a-\epsilon/2)\\
&\leq \mathbf{P}_x(\limsup_{t\to 0}\mathbf{E}_x[|Y\circ\theta_t - Y||\mathcal{F}_t]\geq \epsilon/2)\\
&\leq \lim_{t\to 0}\mathbf{P}_x(\sup_{s\leq t}\mathbf{E}_x[\sup_{r\leq t}|Y\circ\theta_r - Y||\mathcal{F}_s]\geq \epsilon/2)\\
&\leq \frac{2}{\epsilon}\lim_{t\to 0}\mathbf{E}_x[\sup_{r\leq t}|Y\circ\theta_r - Y|]\\
&= \frac{2}{\epsilon}\mathbf{E}_x[\limsup_{t\to 0}|Y\circ\theta_t - Y|]=0 \qquad (5.4.3)
\end{aligned}$$

This shows that A belongs to $\mathcal{O}$. With the same arguments we can show that for any $a\in\mathbb{R}$, $\{x:\mathbf{E}_x[Y]>a\}\backslash N\in\mathcal{O}$. □

The following fact will be used in the proof of the next lemma :

$$\mathcal{F}_\infty=\mathcal{F}_\zeta := \{A\in\mathcal{F}_\infty : A\cap\{\zeta\leq t\}\in\mathcal{F}_t\ \forall t\geq 0\}.$$

Indeed, obviously $\mathcal{F}_\zeta\subset\mathcal{F}_\infty$. For any $s\geq 0$ and $A\in\mathcal{B}(E)$, $\{X_s\in A\}=\{X_s\in A\}\cap\{s<\zeta\}\in\mathcal{F}_\zeta$, thus $\mathcal{F}^0_\infty\subset\mathcal{F}_\zeta$. Therefore, $\mathcal{F}_\infty=\bigcap_{\mu\in\mathcal{P}(E)}\mathcal{F}^{0,\mu}_\infty\subset\bigcap_{\mu\in\mathcal{P}(E)}\mathcal{F}^\mu_\zeta=\mathcal{F}_\zeta$ (see (6.20) in [44]).

Lemma 5.4.7. *Let A be a local AF with defining set Λ. Then A can be extended to an AF $\tilde{A}$ with defining set Λ such that for $\omega\in\Lambda$ satisfying $\zeta(\omega)<\infty$, the function $t\to\bar{A}_t(\omega)$ is continuous at $t=\zeta(\omega)$, where $\bar{A}_t=\sup_{s\leq t}|\tilde{A}_s|$ ($\leq\infty$).*

Proof. We will use te same argument as in [7], Remark 2.2. For $w\in\Omega$ and $s<t$ let $A^*_{s,t}(\omega):=\sup\{A_r(\omega): s\leq r<t, r\in\mathbb{Q}\}$ and $A^*_\zeta(\omega):=\inf\{A^*_{s,\zeta(\omega)}(\omega): s<\zeta(\omega), s\in\mathbb{Q}\}$ if $0<\zeta(\omega)$ and $A^*_\zeta(\omega)=0$ if $\zeta(\omega)=0$. For any $t\geq 0$, set :

$$\tilde{A}_t(\omega):=\begin{cases} A_t(\omega) & \text{if } t<\zeta(\omega)\\ A^*_\zeta(\omega) & \text{if } t\geq\zeta(\omega)\end{cases} \qquad (5.4.4)$$

First, we shall prove the $(\mathcal{F}_t)$-adaptedness of $\tilde{A}$. Let $I\subset\mathcal{B}(\mathbb{R})$ and $t\geq 0$. It is clear that $A^*_\zeta\in\mathcal{F}_\infty=\mathcal{F}_\zeta$ then $\{\tilde{A}_t\in I\}\cap\{\zeta\leq t\}=\{A^*_\zeta\in I\}\cap\{\zeta\leq t\}\in\mathcal{F}_t$. Since $\{\tilde{A}_t\in I\}\cap\{t<\zeta\}=\{A_t\in I\}\cap\{t<\zeta\}\in\mathcal{F}_t$ we obtain that $\{\tilde{A}_t\in I\}\in\mathcal{F}_t$ which gives the $(\mathcal{F}_t)$-adaptedness of $\tilde{A}$.
Now we shall proof the additivity of $\tilde{A}(\omega)$ for ω in Λ. We will prove only the case, $t<\zeta(\omega)\leq t+s$, for the other cases, the additivity is evident.Thanks to the right continuity of $A_s(\omega)$ for $s<\zeta(\omega)$ we have that $A^*_\zeta(\omega)=\limsup_{s\uparrow\zeta(\omega)}A_s(\omega)$. Since $\zeta(\theta_t\omega)=\zeta(\omega)-t>0$, $\tilde{A}_\zeta(\theta_t\omega)=\limsup_{s\uparrow(\zeta(t)-t)}A_s(\theta_t\omega)=\limsup_{s\uparrow\zeta}A_s(\omega)-A_t(\omega)=\tilde{A}_\zeta(\omega)-\tilde{A}_t(\omega)$. Finally, $\tilde{A}_{t+s}(\omega)=\tilde{A}_\zeta(\omega)=\tilde{A}_t(\omega)+\tilde{A}_\zeta(\theta_t\omega)=\tilde{A}_t(\omega)+\tilde{A}_s(\theta_t\omega)$. □

From now on, for any local AF A, $\bar{A}$ denotes the process defined in Lemma 5.4.7. For any $G \in \mathcal{O}$ we define X^G by :

$$X_t^G = \begin{cases} X_t & \text{if} \quad t < \tau_G \\ \Delta & \text{if} \quad t \geq \tau_G \end{cases}$$

Remark 5.4.8. *Let A be an AF admitting a properly exceptional set N and G be an element of $\mathcal{O}$. Suppose that N contains $(E\backslash G)\backslash(E\backslash G)^r$. With the arguments used in the proof of Lemma 2.1. in [38], we show that $(A_{s\wedge\tau_G}, s \geq 0)$ is an AF of X^G admitting N as properly exceptional set.*

Lemma 5.4.9. *Let A be a local CAF. There exists $\{G_n\}$ in Ξ such that for any $t \geq 0, p \geq 1$ and $n \in \mathbb{N}$, $\sup_{x\in G_n} \mathbf{E}_x[\bar{A}^p_{t\wedge\tau_n}] < \infty$, where $\tau_n := \tau_{G_n}$.*

Proof. For $n \in \mathbb{N}$ set $\mathcal{G}_n := \{x \in E : \mathbf{P}_x(n^{-1} < \zeta) > 0\}$, then $\{\mathcal{G}_n\}$ is in Ξ. Indeed, if $f_n(x) := \mathbf{P}_x(n^{-1} < \zeta)$, then $p_t f_n(x) = \mathbf{P}_x(n^{-1} + t < \zeta) \uparrow f_n(x)$ as $t \downarrow 0$, i.e., f_n is 0-excessive. Consequently f_n is nearly Borel and finely continuous, then $\mathcal{G}_n$ belongs to $\mathcal{O}$.
For n in $\mathbb{N}$, set $\psi_n(x) := \mathbf{E}_x[\exp(-\bar{A}_{n^{-1}})]$. Let N be a properly exceptional set for A, then by Lemma 5.4.6, for any $n \in \mathbb{N}$, $G_n := \{x \in \mathcal{G}_n \setminus N : \psi_n(x) > n^{-1}\} \in \mathcal{O}$. it is clear that for any n, $G_n \subset G_{n+1}$. Moreover, if $x \in E \setminus N$, there exists $k, n \in \mathbb{N}$ such that $f_n(x) > 0$ and $\psi_n(x) > k^{-1}$, then $x \in G_{k\vee n}$. Hence $\{G_n\}$ belongs to Ξ.
In order to finish the proof we have to prove that

$$\sup_{x\in G_n} \mathbf{E}_x[\bar{A}^p_{t\wedge\tau_n}] < \infty \ \forall t \geq 0 \tag{5.4.5}$$

The following argument is used in the proof of Theorem 5.5.6 of [21]. Set $t := n^{-1}$ and take $\lambda > 0$ such that $\beta := 1 - t + e^{-\lambda} < 1$. For any x in G_n, we have : $t < \psi_n(x) \leq 1 - \mathbf{P}_x(\bar{A}_t \geq \lambda) + e^{-\lambda}$, thus $\mathbf{P}_x(\bar{A}_t \geq \lambda) \leq \beta$. Set $\eta_k := \inf\{s > 0 : \bar{A}_s = k\lambda\}$. it is clear that $\eta_{k+1} \geq \eta_k + \eta_1 \circ \theta_{\eta_k}$ when $\eta_k < \infty$. We have :

$$\begin{aligned}
\mathbf{P}_x(\bar{A}_{t\wedge\tau_n} \geq (k+1)\lambda) &= \mathbf{P}_x(\eta_{k+1} \leq t \wedge \tau_n) \\
&\leq \mathbf{P}_x(\eta_1 \circ \theta_{\eta_k} \leq t, \eta_k \leq t \wedge \tau_n) \\
&= \mathbf{E}_x(\mathbf{P}_{X_{\eta_k}}(\bar{A}_t \geq \lambda), \eta_k \leq t \wedge \tau_n) \\
&\leq \beta \mathbf{P}_x(\eta_k \leq t \wedge \tau_n) \leq \beta^{k+1}
\end{aligned}$$

which leads, for every $p \geq 1$, to

$$\mathbf{E}_x(\bar{A}^p_{t\wedge\tau_n}) \leq \sum_{k=0}^{\infty} \{\lambda(k+1)\}^p \beta^k < \infty.$$

We assume that N contains $(E\backslash G_n)\backslash(E\backslash G_n)^r$ for all $n \in \mathbb{N}$ (if it is not the case one can always expand N). Hence $(A_{s\wedge\tau_n}, s \geq 0)$ is a AF of X^{G_n} admitting N as a properly exceptional set (see remark 5.4.8).

We show now by induction that $\sup_{x\in G}\mathbf{E}_x[\bar{A}^p_{(kt)\wedge\tau_n}] < \infty$ for any $k\in\mathbb{N}$. For bounded f, set $p^G_s f(x) = \mathbf{E}_x[f(X_t)1_{\{t<\tau\}}]$,

$$\begin{aligned}
\sup_{x\in G}\mathbf{E}_x[\bar{A}^p_{[(k+1)t]\wedge\tau}] &= \sup_{x\in G}\mathbf{E}_x[\bar{A}^p_{(kt)\wedge\tau}\vee\sup_{s\le t}|A_{(s+kt)\wedge\tau}|^p] \\
&\le (1+2^p)\sup_{x\in G}\mathbf{E}_x[\bar{A}^p_{(kt)\wedge\tau}] + 2^p\sup_{x\in G}p^G_{kt}(\mathbf{E}_{(.)}[\bar{A}^p_{t\wedge\tau}])(x) \\
&\le (1+2^p)\sup_{x\in G}\mathbf{E}_x[\bar{A}^p_{(kt)\wedge\tau}] + 2^p\sup_{x\in G}\mathbf{E}_x[\bar{A}^p_{t\wedge\tau}]
\end{aligned} \tag{5.4.6}$$

which is finite if $\sup_{x\in G}\mathbf{E}_x[\bar{A}^p_{(kt)\wedge\tau}] < \infty$. This finishes the proof of the lemma. □

We define $\mathcal{M}^c$ as the set of CAF in $\mathcal{M}$ and $\mathcal{M}^d$ its orthogonal complement, i.e. the set of purely discontinuous MAF of finite energy. We set :

$$\begin{aligned}
\mathcal{M}^j &:= \{M\in\mathcal{M}^d : \mathbf{P}_x\text{-a.e. for q.e. } x\in E:\ M_\zeta = M_{\zeta-} \text{ if } \zeta<\infty\} \\
\mathcal{M}^k &:= \{M\in\mathcal{M}^d : e(M,L)=0\ \forall L\in\mathcal{M}^j\}
\end{aligned}$$

The set $\mathcal{M}^j$ is a closed linear subspace of $\mathcal{M}^d$. This is a consequence of the fact that if (M^n) converges to M in $(\mathcal{M},e)$, then there exists a subsequence (n_k) such that (M^{n_k}) converges uniformly on any compact of $[0,\infty)$, $\mathbf{P}_x$-a.e. for q.e. $x\in E$. Hence $\mathcal{M}$ admits the following decomposition :

$$\mathcal{M} = \mathcal{M}^c\oplus\mathcal{M}^j\oplus\mathcal{M}^k.$$

For $M\in\mathcal{M}$ we denote by M^p the part of M in $\mathcal{M}^p$, with $p=c,j,k$. For a MAF M, $[M]_t$ denotes its quadratic variation on $[0,t]$, i.e.,

$$[M]_t = \langle M^c\rangle_t + \sum_{s\le t}(\Delta M_s)^2$$

Lemma 5.4.10. *We have :*

(i) *If $M\in\mathcal{M}^k$, there exists A in $\mathbf{A}_c$ such that $M_t = A_t$ for $t<\zeta$ $\mathbf{P}_x$-a.e. for q.e. $x\in E$. Moreover, μ_A belongs to S_0-S_0.*

(ii) *For $M\in\mathcal{M}$, $M\in\mathcal{M}^k$ if and only if for all t, $[M]_t = 0$ $\mathbf{P}_m$-a.e. on $\{t<\zeta\}$.*

Proof. (i). Note that $\{f*M^{u,k} : f,u\in\mathcal{F}\}$ is dense in $\mathcal{M}^k$. Indeed, if $L\in\mathcal{M}^k$ is orthogonal to $f*M^{u,k}$ for any $u,f\in\mathcal{F}$, then $0=e(L,f*M^{u,k}) = e(L,f*M^u)$, thus $L=0$ (see Lemma 5.6.3 in [21]).
Now for M in $\mathcal{M}^k$, there exists $(f_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\tilde{n}\mathbb{N}}$ sequences in $\mathcal{F}$ such that $e(f_n*M^{u_n,k}-M)$ converges to 0. Denote by (N,H) a Lévy system for X and k the killing measure of $\mathcal{E}$, i.e. $k(\mathrm{d}x) = N(x,\Delta)\mu_H(\mathrm{d}x)$. For $n,m\in\mathbb{N}$, $\int_E(f_n(x)u_n(x)-$

$f_m(x)u_m(x))^2k(\mathrm{d}x) = e(f_n * M^{u_n,k} - f_m * M^{u_m,k})$ converges to zero as n, m tend to ∞. Hence there exists $g \in L^2(E,k)$ such that $f_n u_n$ converges to g in $L^2(E,k)$. For any $h \in \mathcal{F}$, it follows from Beurling-Deny formula (Theorem 2.15 in [28]) that

$$\int_E h(x)|g(x)|k(\mathrm{d}x) \leq (\mathcal{E}(h,h))^{1/2}(\int_E g^2(x)k(\mathrm{d}x))^{1/2}$$

which implies that : $g.k \in S_0$-S_0. Let A be the element of $\mathbf{A}_c$ associated to $g.k$ by the Revuz correspondance. Similarly, note that : $g_n.k \in S_0$-S_0, where $g_n := f_n u_n$ and

$$\mathcal{E}_1(U_1(g.k) - U_1(g_n.k), U_1(g.k) - U_1(g_n.k)) \leq \int_E (g(x) - g_n(x))^2 k(\mathrm{d}x) \to 0.$$

Let A^n be the element of $\mathbf{A}_c$ associated to $g_n.k$: $A^n_t = \int_0^t g_n(X_s)N(X_s,\Delta)\mathrm{d}H_s$. By Lemmas 5.2.8 and 5.3.7, $\mathbf{P}_x$-a.e. for q.e. $x \in E$ and by taking a subsequence if necessary, A^n converges uniformly on any compact to A. But : $A^n_t = f_n * M^{u_n,k}$ on $t < \zeta$. This shows that $M_t = A_t$ for $t < \zeta$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$.
(ii). The necessacity is a consequence of (i). For the sufficiency, suppose that for any t, $[M]_t = 0$ $\mathbf{P}_m$-a.e. on $t < \zeta$, then $\mathbf{P}_m$-a.e., $[M] = 0$ on $[\![0,\zeta[\![$. Then $[M^j] = 0$ on $[\![0,\zeta[\![$ $\mathbf{P}_m$-a.e. but : $\Delta[M^j]_\zeta = (\Delta M^j_\zeta)^2 = 0$, hence : $[M^j] = 0$ on $[\![0,\infty[\![$. $\mathbf{P}_m$-a.e., thus $e(M^j) = 0$ and therefore, $M^j = 0$. In the same way we can show that $M^c = 0$. □

Let G be an element of $\mathcal{O}$ and denote by $\mathcal{E}^G$ the restriction of $\mathcal{E}$ to $\mathcal{F}_G \times \mathcal{F}_G$. Then $\mathcal{E}^G$ is also a Dirichlet form and the process X^G_t is properly associated to $\mathcal{E}^G$ (see Theorem 4.3 in [30]). When G is not open, $\mathcal{E}^G$ is not necessarily regular, nevertheless, all results of regular Dirichlet forms used in this paper are valid for $\mathcal{E}^G$, in fact, thanks to a regularization method, these results hold for any quasi-regular Dirichlet form, see chapter V of [32] for more details. When we introduce a class of AF's associated to $\mathcal{E}^G$, we add the symbol $(\mathcal{E}^G)$ in order to differentiate it from the same class associated with $\mathcal{E}$. For example for u element of $\mathcal{F}_G$, $N^u(\mathcal{E}^G)$ denotes the CAF of zero energy associated to X^G obtained from Fukushima decomposition for $u(X^G_t) - u(X^G_0)$. $\mathcal{M}(\mathcal{E}^G)$ denotes for example the set of MAF's of X^G of finite energy.

Lemma 5.4.11. *Let G be a element of $\mathcal{O}$ and u in $\mathcal{F}_G$. Then :*

$$N^u_{t\wedge\tau_G} = N^u_t(\mathcal{E}^G) \textit{ for } t \geq 0 \ \mathbf{P}_x\textit{-a.e for q.e } x \in E$$

Proof. First we shall prove the lemma for $u = R^G_1 f$ with $f \in L^2(G,m)$. In this case, $N^u_t(\mathcal{E}^G) = \int_0^{t\wedge\tau_G}(u(X_s) - f(X_s))\mathrm{d}s$. On the other hand, for any $w \in \mathcal{F}_G$, $\mathcal{E}(u,w) = \mathcal{E}^G(u,w) = (f-u,w)_m$. Then it follows by Theorem 5.3.3 that for

$\mathbf{P}_x$-a.e for q.e $x \in E$,

$$N^u_{t\wedge\tau_G} = \int_0^{t\wedge\tau_G} (u(X_s) - f(X_s))\mathrm{d}s = N^u_t(\mathcal{E}^G)$$

For the general case, let $u \in \mathcal{F}^G$ and $(f_n) \subset L^2(G, m)$ such that $u_n := R^G_1 f_n$ converges to u with respect to $\tilde{\mathcal{E}}^G_1$ and hence, with respect to $\tilde{\mathcal{E}}_1$. Then $\mathbf{P}_x$-a.e. for q.e $x \in E$, $N^{u_n}_t(\mathcal{E}^G)$ and $N^{u_n}_t$ converge uniformly on any compact to $N^u_t(\mathcal{E}^G)$ and N^u_t respectively. $\square$

The following lemma can be found in Nakao [36] under the assumption that $\mathcal{E}$ is symmetric. We relax this assumption.

Lemma 5.4.12. *Let $(c_t)_{t>0}$ be a function such that for every $t > 0$, c_t belongs to $L^2(E, m)$ and*

$$c_{t+s} = c_t + p_t c_s, \; t, s > 0$$

and $\lim_{t\to 0} \|c_t\| = 0$. Then there exists a unique u in $L^2(E, m)$ such that $c_t = p_t u - u - S_t u$ where $S_t = \int_0^t p_s \mathrm{d}s$.

Proof. Since : $\|c_{t+s}\| \le \|c_t\| + \|c_s\|$, then : $\lim_{t\to\infty} \|c_t\|/t$ exists in $\mathbb{R}_+$. Set : $u = -\int_0^\infty e^{-t} c_t \mathrm{d}t$ and $C_\alpha = \int_0^\infty e^{-\alpha t} c_t \mathrm{d}t$, $\alpha > 0$. Then u and C_α are in $L^2(E, m)$. Straightforward computations show that for any $\alpha > 0$:

$$\alpha C_\alpha = (\alpha - 1) R_\alpha u - u \tag{5.4.7}$$

One also has : $\alpha \int_0^\infty e^{-\alpha t}(p_t u - u - S_t u)\mathrm{d}t = (\alpha - 1) R_\alpha u - u$. Hence by the right continuity of (c_t) and $(p_t u - u - S_t u)$ and the uniqueness of the Laplace transform we have that : $c_t = p_t u - u - S_t u$. Let v be another function satisfying : $c_t = p_t v - v - S_t u$. Thanks to (5.4.7) we have for any $\alpha > 0$: $u - v = (\alpha - 1) R_\alpha(u - v)$. In particular, for $\alpha = 1$ we obtain : $u - v = 0$. $\square$

Proof of Theorem 5.4.3. In view of the commentary following the proof of Lemma 5.4.10, when G belongs to $\mathcal{O}$, we can apply to $\mathcal{E}^G$ all the results so far used and established for the form $\mathcal{E}$.
Let $(A^n)_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{C}$ and $\{G_n\}$ in Ξ such that : $A = A^n$ on $[\![0, \tau_n[\![$ $\mathbf{P}_x$-a.e for q.e. $x \in E$, where $\tau_n := \tau_{G_n}$. Thanks to Lemma 5.4.9 we can assume that for any $t \ge 0, p \ge 1$, $\sup_{x\in G_n} \mathbf{E}_x[\bar{A}^p_{t\wedge\tau_n}] < \infty$.
For any local AF B of X, set $B^{G_n}_t := B_{t\wedge\tau_n}$. By Remark 5.4.8, B^{G_n} is an AF of X^{G_n}.
Let v_n be the following element of $L^2(G_n, m)$:

$$v_n(x) = -\int_0^\infty e^{-t} \mathbf{E}_x[A^{G_n}_t]\mathrm{d}t.$$

By Lemma 5.4.12 :

$$\mathbf{E}_x[A_t^{G_n}] = p_t^{G_n} v_n(x) - v_n(x) - \int_0^t p_s^{G_n} v(x)\mathrm{d}s,\ \text{m-a.e. for } x \in E, t \geq 0$$

where for any $f \in L^2(G_n, m)$, $p_s^{G_n} f(x) = \mathbf{E}_x[f(X_s^{G_n})]$.
Since $A_t^{G_n}$ satisfies the condition of Lemma 5.4.6 (with $\theta_{s\wedge\tau_n}$ instead of θ_s), the function h_n defined by $h_n(x) := \mathbf{E}_x[A_t^{G_n}]$ is quasi continuous. We can hence assume that v_n is quasi continuous and therefore, for q.e. $x \in E$: (Proposition IV.5.30 in [32])

$$\mathbf{P}_x(t \to v_n(X_t) \text{ be càdlàg on } [0, \zeta)) = 1$$

i.e., v_n is q.e. finely continuous. Thanks to Lemma 4.1.6 in [21], we can also assume that v_n is Borel measurable. For $n \in \mathbb{N}$, define $\tilde{C}^n$ the following local CAF of X^{G_n} :

$$\tilde{C}_t^n := A_t^{G_n} + \int_0^t v_n(X_r^{G_n})\mathrm{d}r, t \geq 0$$

and M^n the following element of $\mathcal{M}(\mathcal{E}^{G_n})$:

$$M_t^n = v_n(X_t^{G_n}) - v_n(X_0^{G_n}) - \tilde{C}_t^n, t \geq 0$$

(use the fact that v_n is bounded and Lemma 5.2.7).
Set : $C_t^n := v_n(X_{t-}^{G_n}) - v_n(X_0^{G_n}) - M_{t-}^n$, then obviously $C^n = \tilde{C}^n$ on $[\![0, \zeta[\![$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$, hence C^n is $(\mathcal{F}_t)$-predictable and of zero quadratic variation.
For $n \in \mathbb{N}$ and $i \geq n$, let $\mu_{n,i}$ be the Revuz measure of $\langle M^i \rangle^{G_n}$. For any $n \in \mathbb{N}$ there exists a $\mathcal{E}^{G_n}$-nest $(F_{n,j})_{j\in\mathbb{N}}$ of compact sets of G_n such that $\mu_{n,i}(F_{n,j}) < \infty$ for any $j, i \in \mathbb{N}$, $i \geq n$. (see e.g. Lemma 3.2 of [29]). For any $n, j \in \mathbb{N}$ let $\tilde{H}_{n,j}$ be the fine-interior of $F_{n,j}$. It belongs to $\mathcal{O}$ and $\bigcup_{j=1}^{\infty} \tilde{H}_{n,j} = G_n$ q.e. (see remark 5.2.1). Thus, if for any $j \in \mathbb{N}$ we define $H_j = \bigcup_{n=1}^{j} \tilde{H}_{n,j}$, then : $\{H_j\} \in \Xi$. Indeed :

$$\bigcup_{j=1}^{\infty} H_j = \bigcup_{j=1}^{\infty}\bigcup_{n=1}^{j} \tilde{H}_{n,j} = \bigcup_{n=1}^{\infty}\bigcup_{j=n}^{\infty} \tilde{H}_{n,j} = \bigcup_{n=1}^{\infty} G_n = E \text{ q.e}$$

Evidently $H_n \subset G_n$ q.e. and $\mu_{n,k}(H_n) < \infty$ for any $n \leq k$. Hence for any $n \leq k$: $1_{H_n} * M^k \in \mathcal{M}(\mathcal{E}^{G_k})$.
For φ in $L^1(E, m)$ such that $0 < \varphi \leq 1$ and for any $n \in \mathbb{N}$, set $h_n(x) = R_1^{H_n}\varphi(x)$ and $\mathcal{G}_n = \{x \in G : h_n > n^{-1}\}$. We can see from the proof of Lemma 3.8 of [30] that $(\mathcal{G}_k)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest of quasi open sets and for any n, $\mathcal{G}_n \subset H_n$ q.e.
For any $n \in \mathbb{N}$ set $\mathcal{L}_n := \{w \in \mathcal{F}_{H_n} : w \geq 1 \text{ q.e on } \mathcal{G}_n\}$. $\mathcal{L}_n$ is not empty because it contains nh_n. There exists g_n in $\mathcal{L}_n$ such that $\mathcal{E}_1^{H_n}(g_n, w - g_n) \geq 0$ for any $w \in \mathcal{L}_n$. (see e.g. Corollary 2.1 of [30]). For all $w \in \mathcal{F}_{H_n}$ such that $w \geq 0$, $\mathcal{E}_1^{H_n}(g_n, w) = \mathcal{E}_1^{H_n}(g_n, w + g_n - g_n) \geq 0$. Hence g_n is $\mathcal{E}^{H_n}$-1-excessive. Since $g_n \wedge 1$ element of $\mathcal{L}_n$, is also $\mathcal{E}^{H_n}$-1-excessive, by Proposition III.1.5 of [32] we have : $g_n = g_n \wedge 1$ q.e. It follows that $g_n = 1$ q.e on $\mathcal{G}_n$ and $g_n = 0$ q.e on $E \backslash H_n$

On the other hand, since g_n is $\mathcal{E}^{H_n}$-1-excessive, by Proposition VI.2.1 of [32], there exists a signed measure $\nu_n \in S_0$-S_0 such that $\mathcal{E}^{H_n}(g_n, w) = \int w \mathrm{d}\nu_n$ for any $w \in \mathcal{F}_{H_n}$. Let D^n be the PCAF of X^{H_n} associated to ν_n by the Revuz correspondence. By Lemma 5.3.7 we have :

$$N_s^{g_n}(\mathcal{E}^{H_n}) = -D_s^n + \int_0^s g_n(X_r^{H_n})\mathrm{d}r, r \geq 0 \tag{5.4.8}$$

Thus we have :

$$(g_n(X_t^{H_n}), t \geq 0) \text{ is a } \mathbf{P}_x\text{-semimartingale for q.e } x \in E. \tag{5.4.9}$$

For any integer n, set $w_n := g_n v_n$. For a strictly positive function f in $L^1(H_n, m)$ such that $\int_{H_n} f(x)m(\mathrm{d}x) = 1$, $\mathbf{P}_{f.m}$-a.e. we have for $t < \tau_{H_n}$:

$$\begin{aligned} & w_n(X_t^{H_n}) - w_n(X_0^{H_n}) \\ = & \int_0^t g_n(X_{s-}^{H_n})\mathrm{d}M_s^n + \int_0^t v_n(X_{s-}^{H_n})\mathrm{d}M_s^{g_n}(\mathcal{E}^{H_n}) \\ & + \int_0^t v_n(X_s^{H_n})\mathrm{d}N_s^{g_n}(\mathcal{E}^{H_n}) + g_n(X_t^{H_n})C_t^n - \int_0^t C_s^n \mathrm{d}g_n(X_s^{H_n}) \end{aligned} \tag{5.4.10}$$

Indeed, the above equation can be proved by elementary arguments after performing the following integration by parts

$$g_n(X_t^{H_n})M_{t\wedge\tau_{H_n}}^n = \int_0^t g_n(X_{s-}^{H_n})\mathrm{d}M_{s\wedge\tau_{H_n}}^n + \int_0^t M_{s\wedge\tau_{H_n}-}^n \mathrm{d}g_n(X_s^{H_n}).$$

Now we shall prove that w_n belongs to $\mathcal{F}$.

$$\begin{aligned} \alpha(w_n - \alpha R_\alpha w_n, w_n)_m &= \frac{\alpha^2}{2}\mathbf{E}_m\left[\int_0^\infty e^{-\alpha s}(w_n(X_s) - w_n(X_0))^2 \mathrm{d}s\right] \\ &\quad + \frac{\alpha}{2}\int_E w_n^2(x)(1 - \alpha \hat{R}_\alpha 1(x))m(\mathrm{d}x) \\ &= I_\alpha + J_\alpha \end{aligned}$$

Note that : $M_s^n g_n(X_s) = [1_{H_n} * M_s^n]g_n(X_s), s \geq 0$ for q.e $x \in E$, therefore :

$$\begin{aligned} I_\alpha &\leq \|v_n\|_\infty^2 \alpha^2 \mathbf{E}_m\left[\int_0^\infty e^{-\alpha s}(g_n(X_s)-g_n(X_0))^2 \mathrm{d}s\right] \\ &\quad +2\alpha^2 \mathbf{E}_m\left[\int_0^\infty e^{-\alpha s}(M_s^n)^2 g_n^2(X_s)\mathrm{d}s\right] \\ &\quad +2\alpha^2 \mathbf{E}_m\left[\int_0^\infty e^{-\alpha s}(\tilde{C}_s^n)^2 g_n^2(X_s)\mathrm{d}s\right] \\ &\leq \|v_n\|_\infty^2 \alpha^2 \mathbf{E}_m\left[\int_0^\infty e^{-\alpha s}(g_n(X_s)-g_n(X_0))^2 \mathrm{d}s\right] \\ &\quad +2\alpha^2 \mathbf{E}_m\left[\int_0^\infty e^{-\alpha s}(1_{H_n} * M_s^n)^2 \mathrm{d}s\right] \\ &\quad +2\alpha^2 \mathbf{E}_m\left[\int_0^\infty e^{-\alpha s}(A_s^n + \textstyle\int_0^s v_n(X_r)\mathrm{d}r)^2 \mathrm{d}s\right] \end{aligned}$$

Consequently :

$$\limsup_{\alpha\to\infty} I_\alpha \leq \|v\|_\infty^2 \left\{2\mathcal{E}(g_n,g_n) - \int_E g_n^2(x)\hat{k}(\mathrm{d}x)\right\} + 4e(1_{H_n} * M^n) + 4e(A^n) < \infty$$

On the other hand :

$$J_\alpha \leq \|v_n^2\|_\infty \frac{\alpha}{2}\int_E g_n^2(x)(1-\alpha\hat{R}1(x))m(\mathrm{d}x) \to \|v_n\|_\infty^2 \frac{1}{2} < g_n^2, \hat{k} > \text{ as } \alpha\to\infty$$

We finally obtain : $\sup_{\alpha>0}\alpha(w_n-\alpha R_\alpha w_n, w_n)_m < \infty$, which implies that w_n belongs to $\mathcal{F}$.
Since : $g_n = 0$ on q.e $E\backslash H_n$, then w_n belongs to $\mathcal{F}_{H_n}$. Fukushima's decomposition gives :

$$w_n(X_t^{H_n}) - w_n(X_0^{H_n}) = M_t^{w_n}(\mathcal{E}^{H_n}) + N_t^{w_n}(\mathcal{E}^{H_n}) \tag{5.4.11}$$

Comparing (5.4.10) with (5.4.11), one obtains :

$$W_t^n = V_t^n, t < \tau_{H_n}\ \mathbf{P}_{f.m}\text{-a.e.}$$

where W^n and V^n are given by :

$$\begin{aligned} W_t^n &= \int_0^t g_n(X_{s-}^{H_n})\mathrm{d}M_{s\wedge\tau_{H_n}}^n + \int_0^t v_n(X_{s-}^{H_n})\mathrm{d}M_s^{g_n}(\mathcal{E}^{H_n}) - M_t^{w_n}(\mathcal{E}^{H_n}) \\ V_t^n &= N_t^{w_n}(\mathcal{E}^{H_n}) - \int_0^t v_n(X_s^{H_n})\mathrm{d}N_s^{g_n}(\mathcal{E}^{H_n}) - g_n(X_t^{H_n})C_{t\wedge\tau_{H_n}}^n \\ &\quad + \int_0^t C_{s\wedge\tau_{H_n}}^n \mathrm{d}g_n(X_s^{H_n}) \end{aligned}$$

Since W^n is a $\mathbf{P}_{f.m}$-martingale, for any $T \geq 0$, $\int_0^T \sum_{i=0}^{K-1}(W^n_{t(i+1)/K} - W^n_{ti/K})^2 \mathrm{d}t$ converges to $\int_0^T [W^n]_s \mathrm{d}s$ in $\mathbf{P}_{f.m}$-probability.
Besides, since $(\int_0^t v_n(X_s^{H_n})\mathrm{d}N_s^{g_n}(\mathcal{E}^{H_n}), t \geq 0)$ is a continuous process of bounded variation, it is of zero quadratic variation. It follows from Lemma 5.4.5 and Remark 5.4.4 that $1_{\{T<\tau_{H_n}\}} \int_0^T \sum_{i=0}^{K-1}(V^n_{t(i+1)/K} - V^n_{ti/K})^2 \mathrm{d}t$ converges to 0 in $\mathbf{P}_{f.m}$-probability. We conclude that for any t : $[W^n]_t = 0$ $\mathbf{P}_{f.m}$-a.e. on $\{t < \tau_{H_n}\}$. Note that W^n belongs to $\mathcal{M}(\mathcal{E}^{H_n})$, hence thanks to Lemma 5.4.10 W^n is in $\mathcal{M}^k(\mathcal{E}^{H_n})$ and there exists a CAF B^n of X^{H_n} of bounded variation on $[0, \tau_{H_n})$ such that $W_n = B^n$ on $[0, \tau_{H_n})$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$. Moreover the Revuz smooth signed measure μ_{B^n} of B^n belongs to S_0-S_0. Consequently, thanks to Lemma 5.3.7,

$$W_t^n = -N_t^{\gamma_n}(\mathcal{E}^{H_n}) + \int_0^t \gamma_n(X_s^{H_n})\mathrm{d}s, t < \tau_{H_n}$$

where γ_n denotes the 1-potential of μ_{B^n} (with respect to $\mathcal{E}^{H_n}$).
Since $g_n = 1$ q.e on $\mathcal{G}_n$, it follows by (5.4.8) that $\mathbf{P}_m$-a.e. on $[\![0, \tau_{\mathcal{G}_n}[\![$:

$$\begin{aligned} A_t &= N_t^{w_n}(\mathcal{E}^{H_n}) - \int_0^t w_n(X_s)\mathrm{d}s + \int_0^t v_n(X_s)\mathrm{d}D_s^n - \int_0^t v_n(X_s)\mathrm{d}s \\ &\quad + N_t^{\gamma_n}(\mathcal{E}^{H_n}) - \int_0^t \gamma_n(X_s^{H_n})\mathrm{d}s \end{aligned} \tag{5.4.12}$$

The Revuz signed measure associated to $\int_0^t v_n(X_s)\mathrm{d}(D_s^n - s)$ is $v_n.(\nu_n - m)$ which belongs to S_0-S_0 since v_n is a bounded element of $L^2(E, m)$. Let δ_n be the 1-potential of $v_n.(\nu_n - m)$ with respect to $\mathcal{E}^{H_n}$. Set $u_n := w_n + \gamma_n - \delta_n$. Note that u_n belongs to $\mathcal{F}_{H_n}$. It follows by Lemma 5.3.7 that :

$$A_t = N_t^{u_n}(\mathcal{E}^{H_n}) - \int_0^t u_n(X_s^{H_n})\mathrm{d}s, \text{ for } t < \tau_{\mathcal{G}_n} \ \mathbf{P}_m\text{-a.e}$$

Thanks to Lemma 5.4.11, we hence have $\mathbf{P}_x$-a.e. for m-a.e x :

$$A_t = N_t^{u_n} - \int_0^t u_n(X_s)\mathrm{d}s, \text{ for } t < \tau_{\mathcal{G}_n}, n \in \mathbb{N} \tag{5.4.13}$$

In order to show (5.4.13) for q.e $x \in E$ we use an argument presented in the proof of Lemma 4.6 of [7]. Let Ξ_0 be an defining set admitting an exceptional set for all the CAF taking part in (5.4.13). Set : $\Xi = \{\omega \in \Omega : (5.4.13) \text{ holds}\}$. Then $\mathbf{P}_x(\Xi^c) = 0$ for m-a.e. $x \in E$. For any $k \in \mathbb{N}$, set : $\Xi_k = \theta_{k^{-1}}^{-1}(\Xi)$. Then $\mathbf{P}_x(\Xi_k^c) = p_{k^{-1}}(\mathbf{P}_.(\Xi^c))(x) = 0$ for q.e. $x \in E$. Finally set $\Lambda = \bigcap_{k=0}^{\infty} \Xi_k$. Then $\mathbf{P}_x(\Lambda^c) = 0$ for q.e. $x \in E$. We shall prove that (5.4.13) holds for any $\omega \in \Lambda$. For ω in Λ, $n \in \mathbb{N}$ and $t < \tau_{\mathcal{G}_n}(\omega)$, take k such that : $t + k^{-1} < \tau_{\mathcal{G}_n}(\omega)$. One has : $t < \tau_{\mathcal{G}_n}(\theta_{k^{-1}}\omega)$, and hence : $A_t(\theta_{k^{-1}}\omega) = N_t^{w_n}(\theta_{k^{-1}}\omega) - \int_0^t w_n(X_{s+k^{-1}}(\omega))\mathrm{d}s$. One let then k tend to ∞ to obtain (5.4.13). □

5.5 Stochastic integration

Consider an element u of $\mathcal{F}$ and a finite smooth signed measure μ such that $\mathcal{E}(u,h) =< \mu, h >$ for any element h of $\mathcal{F}_b$. Thanks to Theorem 5.3.3, we know that N^u is of bounded variation. The integral $(f * N^u)_t := \int_0^t f(X_s)\mathrm{d}N^u_s$ is hence well defined as a Lebesgue-Stieltjes integral, moreover, if f belongs to $\mathcal{F}_b$, $f * N^u$ belongs to $\mathcal{N}_c^0$ (see Definition 5.3.4) and for any h in $\mathcal{F}_b$ we have :

$$< \Theta(f * N^u), h >=< \Theta(N^u), fh > \tag{5.5.1}$$

Thanks to Lemma 5.3.5, the above equation characterizes the local CAF $f * N^u$. In order to define the integral of f with respect to a process N^u which is not necessarily of bounded variation, it is hence natural to construct a local CAF still denoted by $f * N^u$ satisfying the equation (5.5.1). This has been done by Nakao [35] for the symmetric case and the aim of this section is to do it for the non-necessarily symmetric case.
The construction of $f*N^u$ is based on a decomposition of N^u in three components (see Lemma 5.5.10 below). The first component is associated to the diffusion part of $\tilde{\mathcal{E}}$, the symmetric component of $\mathcal{E}$. The second one is associated to the jump part of $\tilde{\mathcal{E}}$ and the third one is a local CAF of bounded variation. Once this decomposition done, the construction of $f * N^u$ will be close to Nakao's construction in the symmetric case.
Thanks to a localization argument, we will construct the integral $f * C$ for any $f \in \mathcal{F}_{loc}$ and $C \in \tilde{\mathcal{N}}_{c,f\text{-}loc}$. We always consider $\mathcal{F}$ to be equipped with the norm $\tilde{\mathcal{E}}_1$. We will use repeatedly the following facts :

(1) If a PCAF A satisfy $\mu_A(E) < \infty$ then A is finite continuous. Indeed, it is consequence of (5.2.4). This is the case when $A = \langle M \rangle$ for $M \in \mathcal{M}$. If for an element $A \in \mathbf{A}_c$, μ_A is the difference of two finite smooth measures, then A is a CAF of bounded variation in $[0,\infty)$. This is the case when $A = \langle M, L \rangle$ with M and L in $\mathcal{M}$.
(2) For two CAF, A, B and $G \in \mathcal{O}$ we have for q.e. $x \in E$, $\mathbf{P}_x(A{=}B \text{ on } [\![0,\tau_G[\![) = 1$ if and only if for q.e. $x \in E$, $\mathbf{P}_x(A = B \text{ on } [\![0,\sigma_{E\setminus G}[\![) = 1$.
(3) If $J : \mathcal{F} \to \mathbb{R}$ is a continuous linear functional, there exists a unique $w \in \mathcal{F}$ such that $J(h) = \mathcal{E}_1(w,h)$ for any $h \in \mathcal{F}$. (See Theorem I.2.6. in [32]).

5.5.1 A decomposition of N^u

Denote by $\tilde{\mathcal{E}}^{(c)}$ the diffusion part of $\tilde{\mathcal{E}}$. For u in $\mathcal{F}$, the application $h \to \tilde{\mathcal{E}}^{(c)}(u,h)$ is continuous. This leads to the following lemma.

Lemma 5.5.1. *For u in $\mathcal{F}$, there exists a unique w in $\mathcal{F}$ such that*
$\mathcal{E}_1(w,h) = \tilde{\mathcal{E}}^{(c)}(u,h)$ *for any* $h \in \mathcal{F}$.

Definition 5.5.2. *For any $u \in \mathcal{F}$, set :* ${}^c\tilde{N}_t^u := N_t^w - \int_0^t w(X_s)\mathrm{d}s$, *where w is the element of $\mathcal{F}$ given by Lemma 5.5.1.*

It is clear that ${}^c\tilde{N}^u$ belongs to $\mathcal{N}_c^0$ and

$$< \Theta({}^c\tilde{N}^u), h >= -\tilde{\mathcal{E}}^{(c)}(u,h) \text{ for all } h \in \mathcal{F}_b. \tag{5.5.2}$$

Lemma 5.5.3. **(i)** *For u, v in $\mathcal{F}$, G in $\mathcal{O}$ such that $\tilde{\mathcal{E}}^{(c)}(u,h) = \tilde{\mathcal{E}}^{(c)}(v,h)$ for every $h \in \mathcal{F}_G$, we have*

$$\mathbf{P}_x({}^c\tilde{N}^u = {}^c\tilde{N}^v \textit{ on } [\![0, \sigma_{E\setminus G}[\![) = 1 \textit{ for q.e } x \in E \tag{5.5.3}$$

In particular, if $u(x) = v(x)$ for q.e $x \in G$, (5.5.3) holds.

(ii) If (u_n) converges to u, there exists a subsequence (n_k) such that for q.e $x \in E$:

$$\mathbf{P}_x({}^c\tilde{N}^{u_{n_k}} \text{ converges to } {}^c\tilde{N}^u \text{ uniformly on any compact}) = 1$$

Proof. (i) The first assertion is consequence of (5.5.2) and Lemma 5.3.5. The second assertion is consequence of local property of $\tilde{\mathcal{E}}^{(c)}$, indeed if $u = v$ q.e on G, $\tilde{\mathcal{E}}^{(c)}(u,h) = \tilde{\mathcal{E}}^{(c)}(v,h)$ for any $h \in \mathcal{F}_G$.
(ii) One can assume that $u = 0$. Let w_n the function associated to u_n by Lemma 5.5.1. it is clear that $\mathcal{E}_1(w_n, w_u) \leq \mathcal{E}_1(u_n, u_n)$, hence (w_n) converges to 0. We conclude thanks to Lemma 5.2.8. □

Definition 5.5.4. *For u element of $\mathcal{F}_{loc}$, one extends Definition 5.5.2 as follows. Let $\{G_k\}_{k\in\mathbb{N}} \in \Xi$, $\sigma = \lim_{n\to\infty} \sigma_{E\setminus G_k}$ and $\{u_k\}$ be a sequence of $\mathcal{F}$ such that $u_k(x) = u(x)$ for q.e $x \in G_k$. Then :*

$${}^c\tilde{N}_t^u := \begin{cases} {}^c\tilde{N}_t^{u_k} & \textit{for} \quad t < \sigma_{E\setminus G_k} \\ 0 & \textit{for} \quad t \geq \sigma \end{cases}$$

Remark 5.5.5. *By Lemma 5.5.3 and (iii) of Lemma 5.2.2, for any u in $\mathcal{F}_{loc}$, ${}^c\tilde{N}^u$ is well defined and it is a local CAF of X. Moreover, the definition of ${}^c\tilde{N}^u$ does not depend of the choice of $\{G_k\}$ nor $\{u_k\}$.*

Denote by (N, H) the Lévy system of X. Let $\hat{X}$ be the Markov process associated to the Dirichlet form $\hat{\mathcal{E}}$ and $(\hat{N}, H_{\hat{X}})$ its Lévy system. Let $\nu_{\hat{H}}$ be the smooth measure associated to $H_{\hat{X}}$ and $\hat{H}$ be the PCAF of X associated to $\nu_{\hat{H}}$ by the Revuz correspondence. Let J, $\hat{J}$ and $\tilde{J}$ denote respectively the jumping measure of $\mathcal{E}$, $\hat{\mathcal{E}}$ and $\tilde{\mathcal{E}}$, that is, $J(\mathrm{d}y, \mathrm{d}x) = \frac{1}{2}N(x, \mathrm{d}y)\nu(\mathrm{d}x)$, $\hat{J}(\mathrm{d}y, \mathrm{d}x) = \frac{1}{2}\hat{N}(x, \mathrm{d}y)\nu_{\hat{H}}(\mathrm{d}x)$ and $\tilde{J}(\mathrm{d}x, \mathrm{d}y) = \frac{1}{2}[J(\mathrm{d}x, \mathrm{d}y) + \hat{J}(\mathrm{d}x, \mathrm{d}y)]$. It is known that $\hat{J}(\mathrm{d}y, \mathrm{d}x) = J(\mathrm{d}x, \mathrm{d}y)$. We will use the following notations :

$$\mathbf{N}(\mathrm{d}y, \mathrm{d}s) := N(X_s, \mathrm{d}y)\mathrm{d}H_s$$

$$\hat{\mathbf{N}}(\mathrm{d}y, \mathrm{d}s) := \hat{N}(X_s, \mathrm{d}y)\mathrm{d}\hat{H}_s$$

$$\tilde{\mathbf{N}}(\mathrm{d}y, \mathrm{d}s) := \frac{1}{2}(\mathbf{N}(\mathrm{d}y, \mathrm{d}s) + \hat{\mathbf{N}}(\mathrm{d}y, \mathrm{d}s))$$

From now on, we fix a metric ρ on E compatible with the given topology. We set $\delta := \{(x,x) : x \in E\}$. $\tilde{\mathcal{E}}^j$ denote the jumping part of $\tilde{\mathcal{E}}$, that is, for any $u, h \in \mathcal{F}$:

$$\tilde{\mathcal{E}}^{(j)}(u,h) = \int_{E\times E\setminus\delta} (u(x) - u(y))(h(x) - h(y))J(\mathrm{d}x, \mathrm{d}y) \tag{5.5.4}$$

Lemma 5.5.6. *For any $u \in \mathcal{F}$, $r > 0$ we have $\mathbf{P}_x$-a.e. for q.e. $x \in E$:*

$$\int_0^t \int_{\{\rho(x,X_s)>r\}} |u(x) - u(X_s)|\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s) < \infty \ \forall t < \zeta. \tag{5.5.5}$$

Let $\tilde{D}^u$ be the local CAF defined by

$$\tilde{D}^u_t := \int_0^t \int_{\{\rho(x,X_s)>r\}} (u(x) - u(X_s))\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s),\ t < \zeta$$

and $\tilde{D}^u_t := 0$ for $t \geq \zeta$. Then there exists w in $\mathcal{F}$ such that $\tilde{D}^u_t = N^w_t - \int_0^t w(X_s)\mathrm{d}s$ for all $t < \zeta$, $\mathbf{P}_x$-a.e. for q.e. $x \in E$. Moreover : $\mathcal{E}_1(w,w) \leq \mathcal{E}_1(u,u)$.

Proof. Let $\tilde{\mathcal{E}}^{(j,r)}(u,h)$ be the right-hand-side of (5.5.4) with $E \times E\setminus\delta$ replaced by $\{(x,y) \in E \times E : r < \rho(x,y)\}$. It is clear that $h \to \tilde{\mathcal{E}}^{(j,r)}(u,h)$ is continuous, hence there exists $w \in \mathcal{F}$ such that $\tilde{\mathcal{E}}^{(j,r)}(u,h) = \mathcal{E}_1(w,h)$ for all $h \in \mathcal{F}$. In particular, if we take $h = w$ we obtain : $\mathcal{E}_1(w,w) \leq \tilde{\mathcal{E}}^{(j,r)}(u,u) \leq \mathcal{E}_1(u,u)$. Let $\{G_k\}$ be in Ξ such that each G_k is relatively compact. For any $k \in \mathbb{N}$ let h_k be in $\mathcal{C}_0(E) \cap \mathcal{F}$, positive such that $h_k(x) = 1$ for $x \in G_k$. Set $D_k := Supp[h_k]$. We have,

$$\begin{aligned}
& \int_{\{r<\rho(x,y)\}} h_k(y)|(u(y) - u(x))|\tilde{J}(\mathrm{d}x, \mathrm{d}y) \\
= & \int_{D_k\times E\cap\{r<\rho(x,y)\}} h_k(y)|(u(y) - u(x))|\tilde{J}(\mathrm{d}x, \mathrm{d}y) \\
\leq & \int_{E\times E\setminus\delta} |(h_k(y) - h_k(x))(u(y) - u(x))|\tilde{J}(\mathrm{d}x, \mathrm{d}y) \\
& + \int_{D_k\times D_k\cap\{r<\rho(x,y)\}} h_k(x)|(u(y) - u(x))|\tilde{J}(\mathrm{d}x, \mathrm{d}y) \\
\leq & (\mathcal{E}(h_k,h_k))^{1/2}(\mathcal{E}(u,u))^{1/2} \\
& + \|h_k\|_\infty(\mathcal{E}(u,u))^{1/2}\left[\tilde{J}(D_k \times D_k \cap \{r < \rho(x,y)\})\right]^{1/2} \\
< & \infty
\end{aligned} \tag{5.5.6}$$

consequently : $\int_0^t \int_{\{r<\rho(x,X_s)\}} |u(x) - u(X_s)|\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s) < \infty$ for all $t < \tau_{G_k}$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$. Thus (5.5.5) follows from Lemma 5.2.2.(iii).
Now let μ be the Revuz signed measure of $\tilde{D}^u$. For any $h \in \bigcup_{k=1}^{\infty} \mathcal{F}_{b,G_k}$,

$$\begin{aligned}
\int_E h(x)\mathrm{d}\mu &= 2\int_{\{r<\rho(x,y)\}} h(y)[u(x)-u(y)]\tilde{J}(\mathrm{d}x,\mathrm{d}y) \\
&= -\int_{\{r<\rho(x,y)\}} [h(x)-h(y)][u(x)-u(y)]\tilde{J}(\mathrm{d}x,\mathrm{d}y) \qquad (5.5.7)\\
&= -\mathcal{E}_1(w,h)
\end{aligned}$$

which leads, thanks to Theorem 5.3.3 to : $\tilde{D}_t^u = N_t^w - \int_0^t w(X_s)\mathrm{d}s, t < \zeta$, $\mathbf{P}_x$-a.e. for q.e. $x \in E$ □

Lemma 5.5.7. *For every u in $\mathcal{F}$, there exists a unique w in $\mathcal{F}$ such that for any sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0, there exists a subsequence (n_k) satisfying :*

$$\lim_{k\to\infty} \int_0^t \int_{\{\epsilon_{n_k}<\rho(x,X_s)\}} [u(x) - u(X_s)]\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s) = N_t^w - \int_0^t w(X_s)\mathrm{d}s \qquad (5.5.8)$$

uniformly on compacts of $[0,\zeta)$ $\mathbf{P}_x$-a.e for q.e $x \in E$.

Proof. Since $h \to \tilde{\mathcal{E}}^{(j)}(u,h)$ is continuous, there exists w in $\mathcal{F}$ such that $\tilde{\mathcal{E}}^{(j)}(u,h) = \mathcal{E}_1(w,h)$ for any $h \in \mathcal{F}$. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence converging to 0. For any $n \in \mathbb{N}$, let w_n be the element of $\mathcal{F}$ given by Lemma 5.5.6 for $r = \epsilon_n$.
Then we have for any $h \in \mathcal{F}$:

$$\begin{aligned}
(\mathcal{E}_1(w_n - w, h))^2 &= [\tilde{\mathcal{E}}^{(j)}(u,h) - \tilde{\mathcal{E}}^{(j,\epsilon_n)}(u,h)]^2 \\
&= \left(\int_{E\times E\setminus\delta} (u(y)-u(x))(h(y)-h(x))1_{\{\rho(x,y)\le\epsilon_n\}} J(\mathrm{d}x,\mathrm{d}y)\right)^2 \\
&\le \int_{E\times E\setminus\delta} (u(y)-u(x))^2 1_{\{\rho(x,y)\le\epsilon_n\}} J(\mathrm{d}x,\mathrm{d}y)\tilde{\mathcal{E}}_1(h,h)
\end{aligned}$$

In particular choosing : $h = w_n - w$, we obtain :

$$\mathcal{E}_1(w_n - w, w_n - w) \le \int_{E\times E\setminus\delta} (u(y)-u(x))^2 1_{\{\rho(x,y)\le\epsilon_n\}} J(\mathrm{d}x,\mathrm{d}y)$$

which converges to 0 as n tends to ∞. It follows from Lemma 5.2.8 that there exists a subsequence (n_k) such that

$$\lim_{k\to\infty}\left(N_t^{w_{n_k}} - \int_0^t w_{n_k}(X_s)\mathrm{d}s\right) = N_t^w - \int_0^t w(X_s)\mathrm{d}s$$

uniformly on compacts $\mathbf{P}_x$-a.e for q.e $x \in E$.

□

Definition 5.5.8. *For every u in $\mathcal{F}$, define :* ${}^j\tilde{N}_t^u := N_t^w - \int_0^t w(X_s)\mathrm{d}s$*, where w is the element of $\mathcal{F}$ given by Lemma 5.5.7.*

Note that ${}^j\tilde{N}^u$ belongs to $\mathcal{N}_c^0$ and that for every h in $\mathcal{F}_b$

$$< \Theta({}^j\tilde{N}^u), h >= -\tilde{\mathcal{E}}^{(j)}(u,h). \tag{5.5.9}$$

With the same argument as the one used to prove Lemma 5.5.3.(ii), one proves the following lemma.

Lemma 5.5.9. *Let (u_n) be $\tilde{\mathcal{E}}_1$-converging sequence to u, there exists a subsequence (n_k) such that for q.e $x \in E$:*

$$\mathbf{P}_x({}^j\tilde{N}^{u_{n_k}} \textit{ converges to } {}^j\tilde{N}^u \textit{ uniformly on any compact}) = 1$$

For u in $\mathcal{F}$, the application $h \to \mathcal{E}_1(u,h)$ is continuous. Hence there exists a unique u^* in $\mathcal{F}$ such that

$$\mathcal{E}_1(u,h) = \tilde{\mathcal{E}}_1(u^*,h), h \in \mathcal{F}. \tag{5.5.10}$$

Moreover we have :

$$\mathcal{E}_1(u^*,u^*) \leq K^2\mathcal{E}_1(u,u) \tag{5.5.11}$$

where K is a continuity constant of $\mathcal{E}$, which means that $\mathcal{E}$ satisfies the sector condition :

$$|\mathcal{E}_1(v,w)| \leq K(\mathcal{E}_1(v,v))^{1/2}(\mathcal{E}_1(w,w))^{1/2} \text{ for all } v,w \in \mathcal{F}$$

Lemma 5.5.10. *For u in $\mathcal{F}$, let u^* be given by (5.5.10). Denote by $\tilde{k}$ the killing measure of $\tilde{\mathcal{E}}$ and by $\tilde{H}$ the PCAF associated to $\tilde{k}(\mathrm{d}x)$ by the Revuz correspondence. Then we have $\mathbf{P}_x$-a.e for q.e $x \in E$ for any $t > 0$*

$$N_t^u = {}^c\tilde{N}^{u^*} + {}^j\tilde{N}^{u^*} - \int_0^t u^*(X_s)\mathrm{d}\tilde{H}_s + \int_0^t (u-u^*)(X_s)\mathrm{d}s \tag{5.5.12}$$

Proof. From the Beurling-Deny decomposition of $\tilde{\mathcal{E}}$, we have that for any $h \in \mathcal{F}$,

$$\int_E |h(x)u^*(x)|\tilde{k}(\mathrm{d}x) \leq [\mathcal{E}_1(h,h)]^{1/2}[\mathcal{E}_1(u^*,u^*)]^{1/2}$$

thus $A \in \mathbf{A}_c^*$, where A denote the third element in the right-hand side of (5.5.12). Therefore the right-hand side of (5.5.12) belongs to $\mathcal{I}$. Denote this element by C. The killing part $\tilde{\mathcal{E}}^{(k)}$ of $\tilde{\mathcal{E}}$ satisfies

$$\tilde{\mathcal{E}}^{(k)}(u^*,h) =< \mu_A, h > \text{ for any } h \in \mathcal{F}$$

It follows from (5.5.2) and (5.5.9) that for all $h \in \mathcal{F}$:

$$\begin{aligned} < \Theta(C), h > &= -\tilde{\mathcal{E}}(u^*,h) + (u-u^*,h)_m \\ &= -\mathcal{E}(u,h) \end{aligned}$$

Then (5.5.12) follows from Lemma 5.3.5. □

5.5.2 Stochastic integration with respect to ${}^c\tilde{N}^u$

Lemma 5.5.11. *For every u in $\mathcal{F}$ and f in $\mathcal{F}_b$, there exists a unique w in $\mathcal{F}$, such that :*

$$e(f * M^{u,c}, M^h) = \mathcal{E}_1(w,h),\ \forall h \in \mathcal{F}.$$

Proof. For $h \in \mathcal{F}$, $[e(f * M^{u,c}, M^h)]^2 \leq e(f * M^{u,c})e(M^{h,c}) \leq e(f * M^{u,c})\tilde{\mathcal{E}}_1(h,h)$. Since $e(f * M^{u,c}) < \infty$, the functional $h \to e(f * M^{u,c}, M^h)$ is continuous. □

Definition 5.5.12. *For every u in $\mathcal{F}$ and f in $\mathcal{F}_b$, the stochastic integral of f with respect to ${}^c\tilde{N}^u$ denoted by $\int_0^{\cdot} f(X_s)\mathrm{d}^c\tilde{N}^u_s$ or by $f * {}^c\tilde{N}^u$ is defined by :*

$$\int_0^t f(X_s)\mathrm{d}^c\tilde{N}^u_s := N^w_t - \int_0^t w(X_s)\mathrm{d}s - \frac{1}{2}\langle M^{f,c}, M^{u,c}\rangle_t, t \geq 0$$

where w is the element of $\mathcal{F}$ associated to (u,f) by Lemma 5.5.11.

For any $u, v \in \mathcal{F}$, let $\mu^c_{<u,v>}$ be the Revuz measure associated to $\langle M^{u,c}, M^{v,c}\rangle$. We have : $\frac{1}{2}\mu^c_{<u,v>}(E) = \tilde{\mathcal{E}}^{(c)}(u,v)$. For f, h in $\mathcal{F}_b$ we have (Lemma 3.2.5 of [21]) :

$$\mathrm{d}\mu^c_{<u,hf>} = f\mathrm{d}\mu^c_{<u,h>} + h\mathrm{d}\mu^c_{<u,f>} \tag{5.5.13}$$

Lemma 5.5.13. **(i)** *For u in $\mathcal{F}$ and f in $\mathcal{F}_b$, we have*

$$f * {}^c\tilde{N}^u \in \mathcal{N}^0_c$$

and

$$< \Theta(f * {}^c\tilde{N}^u), h >= -\tilde{\mathcal{E}}^{(c)}(u, hf) \text{ for all } h \in \mathcal{F}_b \tag{5.5.14}$$

In particular the integral is well defined in the following sense :
If $u, v \in \mathcal{F}$ are such that ${}^c\tilde{N}^u = {}^c\tilde{N}^v$, then for any $f \in \mathcal{F}_b$, $f * {}^c\tilde{N}^u = f * {}^c\tilde{N}^v$.

(ii) *For $(u_n), (f_n)$ two sequences of $\mathcal{F}$ converging to u and f respectively, and such that $\sup_n \|f_n\|_\infty < \infty$, there exists a subsequence (n_k) such that for q.e $x \in E$:*

$$\mathbf{P}_x(f_n * {}^c\tilde{N}^{u_n} \text{ converges to } f * {}^c\tilde{N}^u \text{ uniformly on any compact}) = 1$$

Proof. (i) $f * {}^c\tilde{N}^u \in \mathcal{N}^0_c$ because $|\mu^c_{<u,f>}|(E) < \infty$. Besides for any $h \in \mathcal{F}_b$,

$$\begin{aligned} < \Theta(f * {}^c\tilde{N}^u), h > &= -e(f * M^{u,c}, M^h) - \frac{1}{2}\int_E h(x)\mathrm{d}\mu^c_{<f,u>} \\ &= -\frac{1}{2}\int_E f(x)\mathrm{d}\mu^c_{<h,u>} - \frac{1}{2}\int_E h(x)\mathrm{d}\mu^c_{<f,u>} \end{aligned}$$

Then (5.5.14) is consequence of (5.5.13). For the second statement note that if ${}^c\tilde{N}^u = {}^c\tilde{N}^v$ then by (5.5.2), $\tilde{\mathcal{E}}^{(c)}(u, hf) = \tilde{\mathcal{E}}^{(c)}(v, hf)$ for any $h \in \mathcal{F}_b$. We conclude thanks to (5.5.14) and Lemma 5.3.5.

(ii) First we shall prove the following statements (a), (b) and (c).

(a) If (u_n) and (f_n) converge to 0 and $\sup_n \|f_n\| < \infty$, there exists a subsequence (n_k) such that for q.e $x \in E$:

$$\mathbf{P}_x(f_n * {}^c\tilde{N}^{u_{n_k}} \text{ converges to 0 uniformly on any compact}) = 1$$

(b) If (u_n) converges to 0 and $f \in \mathcal{F}_b$, there exists a subsequence (n_k) such that for q.e $x \in E$:

$$\mathbf{P}_x(f * {}^c\tilde{N}^{u_{n_k}} \text{ converges to 0 uniformly on any compact}) = 1$$

(c) If $u \in \mathcal{F}$, (f_n) converges to 0 and $\sup_n \|f_n\| < \infty$, there exists a subsequence (n_k) such that for q.e $x \in E$:

$$\mathbf{P}_x(f_n * {}^c\tilde{N}^{u} \text{ converges to 0 uniformly on any compact}) = 1$$

Proof of (a) : For each n, let w_n be the function associated to (f_n, u_n) by Lemma 5.5.11. Then for any $h \in \mathcal{F}$ we have : $\mathcal{E}_1(w_n, h)^2 \leq \|f_n^2\|_\infty \mathcal{E}_1(h, h)\mathcal{E}_1(u_n, u_n)$. In particular, choosing $h = w_n$, one obtains :

$$\mathcal{E}_1(w_n, w_n) \leq \|f_n^2\|_\infty \mathcal{E}_1(u_n, u_n) \to 0 \text{ as } n \to \infty$$

It follows from Lemma 5.2.8 that there exists a subsequence (n') such that $\mathbf{P}_x$-a.e for q.e $x \in E$, $N_t^{w_{n'}} - \int_0^t w_{n'}(X_s)\mathrm{d}s$ converges to 0 uniformly on compacts.
Besides : $\mu^c_{<u_n>}(E) + \mu^c_{<f_n>}(E) = \tilde{\mathcal{E}}^{(c)}(u_n, u_n) + \tilde{\mathcal{E}}^{(c)}(f_n, f_n)$, which converges to 0. Hence by Lemma 5.2.6 there exists a subsequence $(\bar{n})$ such that

$$|\langle M^{u_{\bar{n}},c}, M^{f_{\bar{n}},c}\rangle| \leq \langle M^{u_{\bar{n}},c}\rangle^{1/2}\langle M^{f_{\bar{n}},c}\rangle^{1/2}$$

which converges to 0 on compacts $\mathbf{P}_x$-a.e for q.e $x \in E$.
One proves (b) simillarly as (a).
(c) For each n, let w_n be the function associated to (f_n, u) by Lemma 5.5.11. Since (f_n) converges to zero, there exists a subsequence (f_{n_k}) converging q.e. to 0 and therefore converging to 0 $\mathrm{d}\mu^c_{<u>}$-a.e. Thus by dominated convergence, $\int_E f_{n_k}^2 \mathrm{d}\mu^c_{<u>}$ converges to 0. For any $h \in \mathcal{F}$, $|\tilde{\mathcal{E}}_1(w_n, h)|^2 \leq \mathcal{E}_1(h, h) \int_E f_n^2 \mathrm{d}\mu^c_{<u>}$. In particular choosing $h = w_{n_k}$ we obtain for any k :

$$\mathcal{E}_1(w_{n_k}, w_{n_k}) \leq \int_E f_{n_k}^2 \mathrm{d}\mu^c_{<u>} \to 0$$

and one finishes then the proof is as in (a).

Now (iii) is consequence of (a), (b) and (c) below because

$$f * {}^c\tilde{N}^u - f_n * {}^c\tilde{N}^{u_n} = (f_n - f) * {}^c\tilde{N}^{u-u_n} + (f - f_n) * {}^c\tilde{N}^u + f * {}^c\tilde{N}^{u-u_n}$$

□

5.5.3 Stochastic integration with respect to ${}^j\tilde{N}^u$

Lemma 5.5.14. *Let $u \in \mathcal{F}$ and $f \in \mathcal{F}_b$, there exists a unique w in $\mathcal{F}$ such that for any sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0, there exists a subsequence (n_k) which satisfies :*

$$\begin{aligned} & \lim_{k\to\infty} \int_0^t \int_{\{\epsilon_{n_k} < \rho(x,X_s)\}} f(X_s)[u(x) - u(X_s)]\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s) \\ = \;& N_t^w - \int_0^t w(X_s)\mathrm{d}s - \frac{1}{2}\int_0^t \int_E [f(x) - f(X_s)][u(x) - u(X_s)]\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s) \end{aligned} \tag{5.5.15}$$

uniformly on compacts of $[0, \zeta)$ $\mathbf{P}_x$-a.e for q.e $x \in E$. Moreover, we have :

$$\mathcal{E}_1(w, h) = \int_{E\times E\backslash\delta} [h(x) - h(y)][u(x) - u(y)]f(y)\tilde{J}(\mathrm{d}x, \mathrm{d}y) \tag{5.5.16}$$

Proof. Let $u \in \mathcal{F}$, $f \in \mathcal{F}_b$ and $\{\epsilon_n\}_{n\in\mathbb{N}}$ a sequence converging to 0. For any $h \in \mathcal{F}$ define $\lambda(h) = \lambda_{u,f}(h)$ as the right-hand side of (5.5.16) and $\lambda_n(h)$ as $\lambda(h)$ with $\{\epsilon_n < \rho(x, y)\}$ instead of $E \times E\backslash\delta$. Using the fact that f is bounded, one proves with the arguments used in the proof of Lemma 5.5.7, that there exists $(w_n)_{n\in\mathbb{N}}$ and w in $\mathcal{F}$ such that for any $n \in \mathbb{N}$, $\lambda_n(h) = \mathcal{E}_1(w_n, h)$ and $\lambda(h) = \mathcal{E}_1(w, h)$ for any $h \in \mathcal{F}$, and $\mathbf{P}_x$-a.e for q.e $x \in E$:

$$\lim_{k\to\infty} \left(N_t^{w_n} - \int_0^t w_n(X_s)\mathrm{d}s \right) = N_t^w - \int_0^t w(X_s)\mathrm{d}s$$

uniformly on compacts.
Set $\Gamma(x, y) = \frac{1}{2}(u(x) - u(y))(f(x) - f(y))$ and $\Gamma_n(x, y) = 1_{\{\epsilon_n<\rho(x,y)\}}\Gamma(x, y)$. Set : $B_t = \int_0^t \int \Gamma(x, X_s)\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s)$, $B_t^n = \int_0^t \int \Gamma_n(x, X_s)\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s)$ and $C_t^n = \int_0^t \int |\Gamma(x, X_s) - \Gamma_n(x, X_s)|\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s)$.

$$\mu_{C^n}(E) \leq 2 \int_{\{\rho(x,y)\leq\epsilon_n\}} |[u(x) - u(y)][f(x) - f(y)]|\tilde{J}(\mathrm{d}x, \mathrm{d}y)$$

which converges to zero as n tends to ∞. It follows from Lemma 5.2.6 that there exists a subsequence (n_k) such that $\mathbf{P}_x$-a.e for q.e $x \in E$, $B_t^{n_k}$ converges to B_t

uniformly on any compacts. In order to prove (5.5.15), we must prove that for any n,

$$\begin{aligned}&\int_{\{\epsilon_n<\rho(x,X_s)\}} f(X_s)[u(x)-u(X_s)]\tilde{\mathbf{N}}(\mathrm{d}x,\mathrm{d}s)\\ =\ & N_t^{w_n} - \int_0^t w_n(X_s)\mathrm{d}s - \frac{1}{2}\int_0^t\int_{\{\epsilon_n<(\rho(x,X_s)\}}[f(x)-f(X_s)][u(x)-u(X_s)]\tilde{\mathbf{N}}(\mathrm{d}x,\mathrm{d}s)\end{aligned}$$

which can be proved with the arguments used to prove (5.5.7). □

Definition 5.5.15. *For every u in $\mathcal{F}$ and f in $\mathcal{F}_b$, the stochastic integral of f with respect to ${}^j\tilde{N}^u$ denoted by $\int_0^\cdot f(X_s)\mathrm{d}^j\tilde{N}_s^u$ or by $f * {}^j\tilde{N}^u$ is defined as the right-hand side of (5.5.15).*

Lemma 5.5.16. (i) *For u in $\mathcal{F}$ and f in $\mathcal{F}_b$, $f * {}^j\tilde{N}^u$ belongs to $\mathcal{N}_c^0$ and for every h in $\mathcal{F}_b$:*

$$\begin{aligned}&<\Theta(f*{}^j\tilde{N}^u),h>\\ =\ & -\int_{E\times E\setminus\delta}[h(x)f(x)-h(y)f(y)][u(x)-u(y)]J(\mathrm{d}x,\mathrm{d}y)\end{aligned} \quad (5.5.17)$$

In particular the integral is well defined in the following sense :
If u, v in $\mathcal{F}$ are such that ${}^j\tilde{N}^u = {}^j\tilde{N}^v$, then for any f in $\mathcal{F}_b$: $f*{}^j\tilde{N}^u = f*{}^j\tilde{N}^v$.

(ii) *If (u_n) and (f_n) are converging to u and f respectively and $\sup_n \|f_n\|_\infty < \infty$, there exists a subsequence (n_k) such that for q.e $x \in E$:*

$$\mathbf{P}_x(f_n * {}^j\tilde{N}^{u_n} \text{ converges to } f * {}^j\tilde{N}^u \text{ uniformly on any compact}) = 1$$

Proof. The proof of (ii) is similar to the proof of (ii) of Lemma 5.5.13. We prove (i). Set

$$A_t = \frac{1}{2}\int_0^t\int_E (f(x)-f(X_s))(u(x)-u(X_s))\tilde{\mathbf{N}}(\mathrm{d}x,\mathrm{d}s)$$

Since : $|\mu_A|(E) \le \int_{E\times E\setminus\delta}|(f(x)-f(y))(u(x)-u(y))|\tilde{J}(\mathrm{d}x,\mathrm{d}s)$
$\le (\mathcal{E}(f,f))^{1/2}(\mathcal{E}(u,u))^{1/2}$, $f * {}^j\tilde{N}^u$ belongs to $\mathcal{N}_c^0$ and by (5.5.16), for any $h \in \mathcal{F}_b$:

$$\begin{aligned}&<\Theta(f*{}^j\tilde{N}^u,h)>\\ =\ & -\int_{E\times E\setminus\delta}[f(y)\{h(x)-h(y)\}+h(y)\{f(x)-f(y)\}][u(x)-u(y)]\tilde{J}(\mathrm{d}x,\mathrm{d}y)\end{aligned}$$

Using the symmetry of $\tilde{J}$ and the fact that $J(\mathrm{d}x,\mathrm{d}y)+J(\mathrm{d}y,\mathrm{d}x) = 2\tilde{J}(\mathrm{d}x,\mathrm{d}y)$, one proves that the right-hand side of the above equation coincides with the right-hand side of (5.5.17). The second statement can be shown in the same way that its analogous in Lemma 5.5.13.(i).

□

5.5.4 Stochastic integration with respect to N^u

To introduce integration with respect to N^u, one still needs some preliminary results and remarks.
In Section 5.5 we have pointed that a local CAF A in $\mathbf{A}_c$, such that μ_A is the difference of two finite measures, is a CAF of bounded variation on $[0,\infty)$. This is true also when $\mu_A \in S_0$-S_0. Indeed, let μ be a smooth measure in S_0, associated to a PCAF A by the Revuz correspondence. Then $x \to \mathbf{E}_x[\int_0^\infty e^{-t}\mathrm{d}A_t]$ is a quasi-continuous m-version of the 1-potential $U_1\mu$. In particular $\mathbf{E}_x[A_t] \leq e^t U_1\mu(x) < \infty$ for q.e. $x \in E$, therefore $\mathbf{P}_x(A_t < \infty, \forall t \geq 0) = 1$ for q.e. $x \in E$.
We denote by A_c^* the set of CAF A of bounded variation on $[0,\infty)$ such that μ_A belongs to S_0-S_0. For a Borel function f and a CAF A in $\mathbf{A}_c^*$, $f * A$ denotes the Lebesgue-Stieltjes integral of $f(X)$ with respect to A if the integral is well defined and 0 if not i.e., If V denotes the total variation of A on $[0,t)$ then :

$$f * A_t := \begin{cases} \int_0^t f(X_s)\mathrm{d}A_s & \text{if} \quad \int_0^t |f(X_s)|\mathrm{d}V_s < \infty \\ 0 & \text{otherwise} \end{cases}$$

Thanks to (5.2.2), for any $f \in \mathcal{F}$ and $A \in \mathbf{A}_c^*$, $f * A$ is a CAF element of $\mathbf{A}_c^*$.

Lemma 5.5.17. *Let $u, v \in \mathcal{F}$ and $A \in \mathbf{A}_c^*$ such that*

$$\mathbf{P}_x({}^c\tilde{N}^u + {}^j\tilde{N}^v + A = 0 \text{ on } [\![0,\infty[\![) = 1 \text{ for q.e } x \in E \tag{5.5.18}$$

then for any $f \in \mathcal{F}_b$:

$$\mathbf{P}_x(f * {}^c\tilde{N}^u + f * {}^j\tilde{N}^v + f * A = 0 \text{ on } [\![0,\infty[\![) = 1 \text{ for q.e } x \in E$$

Proof. Set : $C = {}^c\tilde{N}^u + {}^j\tilde{N}^v + A$ and $C^f = f * {}^c\tilde{N}^u + f * {}^j\tilde{N}^v + f * A$. It follows from Lemmas 5.5.13.(i) and 5.5.16.(i) that $C, C^f \in \mathcal{N}_c^0$. On the other hand, thanks to (5.5.2), (5.5.14), (5.5.9) and (5.5.17) :

$$< \Theta(C^f), h > = < \Theta(C), fh > \text{ for all } h \in \mathcal{F}_b \tag{5.5.19}$$

But by (5.5.18) and Lemma 5.3.5, $< \Theta(C), h > = 0$ for any $h \in \mathcal{F}_b$, thus we have that $< \Theta(C^f), h > = 0$ for any $h \in \mathcal{F}_b$. We conclude thanks to Lemma 5.3.5. □

Definition 5.5.18. *Denote by $\mathcal{I}$ the set of CAF C of X such that there exists $u, v \in \mathcal{F}$ and $A \in \mathbf{A}_c^*$ such that :*

$$\mathbf{P}_x(C_t = {}^c\tilde{N}^u + {}^j\tilde{N}^v + A \text{ on } [\![0,\infty[\![) = 1 \text{ for q.e } x \in E$$

In this case for any $f \in \mathcal{F}_b$, the stochastic integral of f with respect to C denoted by $\int_0^t f(X_s)\mathrm{d}C_s$ or by $(f * C)_t$ is defined by

$$f * {}^c\tilde{N}^u + f * {}^j\tilde{N}^u + f * A \text{ on } [\![0,\infty[\![.$$

It follows from Lemma 5.5.17, that for $C \in \mathcal{I}$, the stochastic integral $f * C_t$ is well defined in the sense that it is not depending of the elements which represent C. In view of the definitions of ${}^c\tilde{N}^u$, ${}^j\tilde{N}^u$ for $u \in \mathcal{F}$, (5.5.12) and Lemma 5.3.7 we have the following identity :

$$\mathcal{I} = \left\{ N^u_\cdot - \int_0^\cdot u(X_s)\mathrm{d}s : u \in \mathcal{F} \right\} \tag{5.5.20}$$

For u, v in $\mathcal{F}$ and A, B in $\mathbf{A}^*_c$, set : $C^{(1)}_t := N^u_t + A_t, t \geq 0$ and $C^{(2)}_t := N^v_t + B_t, t \geq 0$. Suppose that for some G in $\mathcal{O}$, $C^{(1)}_t = C^{(2)}_t$ for $t < \sigma_{E \setminus G}$, $\mathbf{P}_x$-a.e. for q.e. $x \in E$. Then thanks to Lemma 5.3.5 for any $h \in \mathcal{F}_G$, $< \Theta(C^{(1)}), h >=< \Theta(C^{(2)}), h >$, thus if $f, g \in \mathcal{F}_b$ coincide q.e. on G then thanks to (5.5.19) we have for any $h \in \mathcal{F}_G$:

$$\begin{aligned} < \Theta(f * C^{(1)}), h > &= < \Theta(C^{(1)}), fh > \\ &= < \Theta(g * C^{(2)}), h > \end{aligned}$$

Finally thanks to Lemma 5.3.5, we have : $\mathbf{P}_x(f * C^{(1)}_t = g * C^{(2)}_t, \text{ for } t < \sigma_{E\setminus G}) = 1$, $\mathbf{P}_x$-a.e. for q.e. $x \in E$. We can now define the stochastic integral of $f \in \mathcal{F}_{loc}$ with respect to $C \in \mathcal{I}_{f\text{-}loc}$ as follows.

Definition 5.5.19. *Let $C \in \mathcal{I}_{f-loc}$ and $f \in \mathcal{F}_{loc}$. Let $\{G_n\} \in \Xi$, $\{C^{(n)}\} \subset \mathcal{I}$ and $\{f_n\} \subset \mathcal{F}_b$ such that $\mathbf{P}_x(C_t = C^{(n)}_t$ for $t < \sigma_{E\setminus G_n}) = 1$ for q.e. $x \in E$ and $f_n(x) = f(x)$ for q.e. $x \in G_n$ and $n \in \mathbb{N}$. Then if $\sigma := \lim_{n\to\infty} \sigma_{E\setminus G_n}$, we define the stochastic integral of f with respect to C and denoted by $f * C_t, t \geq 0$ or by $\int_0^t f(X_s)\mathrm{d}C_s, t \geq 0$ as the following local CAF :*

$$f * C_t := \begin{cases} f_n * C^{(n)}_t & \textit{for} \quad t < \sigma_{E\setminus G_n} \\ 0 & \textit{for} \quad t \geq \sigma \end{cases} \tag{5.5.21}$$

Note that the above definition does not depend of the sequences $C^{(n)}$, (f_n) nor G_n. It follows from Theorem 5.4.3 that the stochastic integral $f * C$ for $f \in \mathcal{F}_{loc}$ and C a local CAF locally of zero energy is well defined. Moreover, if $C \in \mathcal{I}_{f\text{-}loc}$ and $f \in \mathcal{F}_{loc}$ then $f * C$ belongs to $\mathcal{I}_{f\text{-}loc}$.

Consequently, thanks to Lemma 5.5.10 and (5.5.20), we have the following lemma.

Lemma 5.5.20. *For any u in $\mathcal{F}$, N^u belongs to $\mathcal{I}$ thus the stochastic integral $\int_0^t f(X_s)\mathrm{d}N^u_s$ is well defined for any $f \in \mathcal{F}_{loc}$.*

Remark 5.5.21. *If $\varphi : \mathbb{R} \to \mathbb{R}$ is a function admitting a continuous derivative, we know that $\varphi(u)$ belongs to $\mathcal{F}_{loc}$, then the integral $\int_0^t \varphi(u(X_s))\mathrm{d}N^u_s$ is well defined and it is a local CAF. In fact using the arguments in Remark 4.3.1 we can show that $\int_0^t \varphi(u(X_s))\mathrm{d}N^u_s$ is defined in $[0, \infty)$ and is in fact a CAF.*

We will use the following result in the proof of the Itô formula for X.

Lemma 5.5.22. *For (u_n) a sequence of $\mathcal{F}$ and u element of $\mathcal{F}$, (f_n) a sequence of $\mathcal{F}_b$ and f an element of $\mathcal{F}_b$ such that (u_n) and (f_n) converge to u and f respectively with respect to $\tilde{\mathcal{E}}_1$. Moreover suppose that $\sup_n \|f_n\|_\infty < \infty$. Then there exists a subsequence (n') such that for q.e $x \in E$:*

$$\mathbf{P}_x(f_{n'} * N^{u_{n'}} \text{ converge to } f * N^u \text{ uniformly on any compact}) = 1$$

Proof. Let $\tilde{H}$ the PCAF defined in Lemma (5.5.10). Then :

$$|f_n u_n^* * \tilde{H}_t - f u^* * \tilde{H}_t| \leq |(f_n - f) u_n^* * \tilde{H}_t| + |f(u_n^* - u^*) * \tilde{H}_t| \tag{5.5.22}$$

Thanks to (5.5.11) and Lemma 5.2.8 there exists a subsequence $\bar{n}$ such that $\mathbf{P}_x$-a.e. for q.e. $x \in E$, $u_{\bar{n}}(X)$ and $f_{n'}(X)$ converge uniformly on any compact to $u^*(X)$ and $f(X)$. By dominated convergence, the first term in the right-hand side of (5.5.22) converges uniformly on compacts to 0, with $\bar{n}$ replacing n. Let w_n and w be respectively the 1-potential of $f(x)u_n^*(x)\tilde{k}(\mathrm{d}x)$ and $f(x)u^*(x)\tilde{k}(\mathrm{d}x)$. One shows that $\{\mathcal{E}_1(w - w_n, w - w_n)\}^{1/2} \leq \|f\|_\infty \{\mathcal{E}_1(u_n^* - u^*, u_n^* - u^*)\}^{1/2}$, which converges to 0. Thanks to Lemmas 5.2.8 and 5.3.7, there exists a subsequence $(n")$ such that the second term in the right-hand side of (5.5.22) (with $n"$ replacing n) converges uniformly on any compact to zero, $\mathbf{P}_x$-a.e. for q.e. $x \in E$. We conclude thanks to (5.5.12) and Lemmas 5.2.8, 5.5.13.(ii) and 5.5.16.(ii) . □

Theorem 5.5.23. *For every Φ in $\mathcal{C}^2(\mathbb{R}^d)$ and every $u = (u_1, ..., u_d)$ in $\mathcal{F}^d$, for q.e $x \in E$, $\mathbf{P}_x$-a.e for all $t \in [0, \infty)$ we have :*

$$\begin{aligned}
&\Phi(u(X_t)) - \Phi(u(X_0)) \\
= \ &\sum_{i=1}^d \int_0^t \frac{\partial \Phi}{\partial x_i}(u(X_{s-}))\mathrm{d}M_s^{u_i} + \sum_{i=1}^d \int_0^t \frac{\partial \Phi}{\partial x_i}(u(X_s))\mathrm{d}N_s^{u_i} \\
&+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(u(X_s))\mathrm{d}\langle M^{u_i,c}, M^{u_j,c}\rangle_s \\
&+ \sum_{s \leq t} \left(\Phi(u(X_s)) - \Phi(u(X_{s-})) - \sum_{i=1}^d \frac{\partial \Phi}{\partial x_i}(u(X_{s-}))(u_i(X_s) - u_i(X_{s-})) \right)
\end{aligned} \tag{5.5.23}$$

Proof. Let I_t be the difference of the left-hand side and the right-hand side of (5.5.23). First suppose that

$$\eta := \sup_{k=1,\ldots,d} \|u_k\|_\infty < \infty$$

therefore we can suppose that Φ is of compact support. For any $i = 1, ..., d$, $n \in \mathbb{N}$ set $u_{n,i} := nR_n u_i$ and $u_n := (u_{n,1}, ..., u_{n,d})$. Define $I_t(n)$ as I_t with u_n replacing u. We fix a smooth measure ν such that $\nu(E) = 1$. First we prove :

$$I_t(u_n) \text{ in } \mathbf{P}_\nu\text{-probability to } I_t(u) \quad (\star)$$

The statement $(\star)$ will be an immediate consequence of the following statements (i), (ii), (iii), (iv) and (v). All convergences are in the sense of convergence in $\mathbf{P}_\nu$-probability if not specified.

(i) $\Phi(u_n(X_t)) - \Phi(u_n(X_0))$ converges to $\Phi(u(X_t)) - \Phi(u(X_0))$.
(ii)For $i = 1, ..., d$, $M_t^n(i) := \int_0^t \frac{\partial \Phi}{\partial x_i}(u_n(X_{s-}))\mathrm{d}M_s^{u_{n,i}}$ converges to $M_t(i) := \int_0^t \frac{\partial \Phi}{\partial x_i}(u(X_{s-}))\mathrm{d}M_s^{u_i}$.
(iii) For $i = 1, ..., d$, $\int_0^t \frac{\partial \Phi}{\partial x_i}(u_n(X_{s-}))\mathrm{d}N_s^{u_{n,i}}$ converges to $\int_0^t \frac{\partial \Phi}{\partial x_i}(u(X_{s-}))\mathrm{d}N_s^{u_i}$.
(iv) For $i, j \in \{1, ..., d\}$ and $g = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}$, $A_t^n = \int_0^t g(u_n(X_s))\mathrm{d}\langle M^{u_{n,i},c}, M^{u_{n,j},c}\rangle_s$ converges to $A_t = \int_0^t g(u(X_s))\mathrm{d}\langle M^{u_i,c}, M^{u_j,c}\rangle_s$.
(v) Let V_t be the last term in the right-hand side of (5.5.23) and V_t^n defined similarly but with u_n replacing u. Then V_t^n converges to V_t.

Proof :
(i) Indeed, this is consequence of Lemma 5.2.8 and the continuity of Φ.
(ii) In order to prove this, it is sufficient to show that $e(M^n(i) - M(i))$ converges to 0. But

$$\begin{aligned} e(M^n(i) - M(i)) \quad &\leq \quad 2 \sup \left\| \frac{\partial \Phi}{\partial x_i} \right\|_\infty^2 e(M^{u_{i,n} - u_i}) \\ &\quad + \int_E \left(\frac{\partial \Phi}{\partial x_i}(u_n(x)) - \frac{\partial \Phi}{\partial x_i}(u(x)) \right)^2 \mathrm{d}\mu_{<M^{u_i}>} \end{aligned}$$

it is known that the first term in the right-hand side of the above equation converges to 0. By taking a subsequence if necessary, $u_{n,i}$ converges q.e to u_i, and therefore, $\mathrm{d}\mu_{<M^{u_i}>}$-a.e. Thus by dominated convergence, $e(M^n(i) - M(i)) \to 0$.
(iii) follows from Lemma 5.5.22.
(iv) Indeed, straightforward computations using a Kunita-Watanabe inequality show that

$$\begin{aligned} |A_t - A_t^n| \quad &\leq \quad \sup_{s \leq t} |g(u_n(X_s)) - g(u(X_s))| \times \\ &\quad \frac{1}{4}\left(\langle M^{u_{n,i}+u_{n,j},c}\rangle_t + \langle M^{u_{n,i}-u_{n,j},c}\rangle_t\right) \\ &\quad + \|g^2\|_\infty \langle M^{u_{n,i},c}\rangle_t^{1/2} \langle M^{u_{n,j}-u_j,c}\rangle_t^{1/2} \\ &\quad + \|g^2\|_\infty \langle M^{u_j,c}\rangle_t^{1/2} \langle M^{u_{n,i}-u_i,c}\rangle_t^{1/2} \end{aligned}$$

thus $|A_t - A_t^n|$ converges to 0 by Lemmas 5.2.8 and 5.2.6.

(v)Indeed, set $a := \sup_{i,j} \|\partial\Phi/\partial x_i \partial x_j\|_\infty$, and $h_n = u - u_n$

$$|V^n_t - V_t| \leq a \sum_{s \leq t} |h_n(X_s) - h_n(X_{s-})|^2 =: aS^n_t$$

Set $B^n_t := \int_0^t \int_E |h_n(x) - h_n(X_s)|^2 \mathbf{N}(\mathrm{d}x, \mathrm{d}s)$ and $L^n_t := S^n_t - B^n_t$. L^n is a MAF and $e(L^n) = \int_{E\times E - d} |h_n(x) - h_n(y)|^4 J(\mathrm{d}x, \mathrm{d}y) \leq 16.\eta.d.\mathcal{E}(h_n, h_n)$, which converges to 0. Hence L^n_t converges to zero. Since $\mu_{B^n}(E) \leq \mathcal{E}(h_n, h_n)$, it follows by Lemma 5.2.6 that B^n_t converges to zero. Therefore, $S^n_t = L^n_t + B^n_t$ converges to 0.

This finishes the proof of ($\star$). Since for each n, $u_n(X)$ is a $\mathbf{P}_\nu$-semimartingale, by the classical Itô formula, $\mathbf{P}_\nu(I_t(n) = 0) = 1$, thus $\mathbf{P}_\nu(I_t = 0) = 1$. Therefore it follows from Theorem 2.2.3 of [21] that $\mathbf{P}_x(I_t = 0) = 1$ for q.e $x \in E$ and we conclude (5.5.23) because a.e. all its terms are right continuous in $[0, \infty)$.
Now suppose that u is not bounded. For each n, let $G_n := \{x \in E : |u_i(x)| < n \; \forall i \leq d\}$, $u^n_i := (-n) \vee u_i \wedge n$ and $u_n := (u^n_1, \dots u^n_d)$. Since each u_i is quasi-continuous we assume that $\{G_n\} \in \Xi$. Let I^n_t be defined as I_t with u^n instead of u. Then $I^n_t = 0$ for all t, $\mathbf{P}_x$-a-e. for q.e. $x \in E$. But $I_t = I^n_t$ for $t < \sigma_{E\setminus G_n}$ and thanks to (5.2.2), $\lim_{n\to\infty} \sigma_{E\setminus G_n} = \infty$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$. This finishes the proof of Theorem 5.5.23 □

Define $\mathcal{F}^{\mathcal{S}}$ as the set of functions f such that the process $(f(X_t), t \geq 0)$ is a semimartingale on $[\![0, \zeta[\![$, i.e. if N^f is of bounded variation on $[\![0, \zeta[\![$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$.

Lemma 5.5.24. *Let C be an element of $\mathcal{I}_{f\text{-}loc}$. Then the two following convergences exist in the sense that a sequence of processes (A^n) converges to a process A if for any $T \geq 0$, $\int_0^T |A^n_t - A_t| \mathrm{d}t$ converges to 0 in measure with respect to $\mathbf{P}_{g.m}$ on $\{T < \zeta\}$ for every $g \in L^1(E; m)$ with $0 < g \leq 1$ m-a.e.*

(i) *For f in $\mathcal{F}^{\mathcal{S}}$, we have : $(f * C)_t = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(X_{tk/n})(C_{t(k+1)/n} - C_{tk/n})$.*

(ii) *Let (f_n) be a sequence of $\mathcal{F}_b$ converging with respect to the $\tilde{\mathcal{E}}_1$-norm to f such that $\sup_n \|f_n\|_\infty < \infty$. Then, we have : $(f * C)_t = \lim_{n\to\infty} (f_n * C)_t$.*

Proof. In view of (5.5.20) and Lemma 5.2.2.(ii) we can assume that : $C = N^u - \int_0^\cdot u(X_s)\mathrm{d}s$, for some $u \in \mathcal{F}$. Since (i) and (ii) are obvious when $C = \int_0^\cdot u(X_s)\mathrm{d}s$, one has just to show (i) and (ii) in the case $C = N^u$ for some $u \in \mathcal{F}$. In this case (ii) is aconsequence of Lemma 5.5.22. We shall prove (i).

For f in $\mathcal{F}^{\mathcal{S}}$, we have :

$$\begin{aligned}
&\sum_{k=0}^{n-1} f(X_{tk/n})(N^u_{t(k+1)/n} - N^u_{tk/n}) \\
= \;& f(X_t)u(X_t) - f(X_0)u(X_0) - \sum_{k=0}^{n-1} u(X_{tk/n})(f(X_{t(k+1)/n}) - f(X_{tk/n})) \\
& - \sum_{k=0}^{n-1} f(X_{tk/n})(M^u_{t(k+1)/n} - M_{tk/n}) \\
& - \sum_{k=0}^{n-1} (f(X_{t(k+1)/n}) - f(X_{tk/n}))(u(X_{t(k+1)/n}) - u(X_{tk/n}))
\end{aligned}$$

which converges to $\int_0^t f(X_s)\mathrm{d}N^u_s$, thanks to Theorem 5.5.23 and Remark 5.4.4. □

Example 5.5.25. In this exemple we show that the stochastic integral constructed by Chen et al. [7] for symmetric Dirichlet forms can be defined in the sense of Definition 5.5.19. Moreover both definitions coincide $\mathbf{P}_m$-a.e. for q.e. $x \in E$. We use the notations and definitions of [7], thus Λ is a linear operator that maps some class of local MAF's on $[\![0, \zeta[\![$ into even CAF's on $[\![0, \zeta[\![$ admitting m-null set. Let M be a locally square-integrable MAF on $[\![0, \zeta[\![$ that belongs to the domain of Λ. We see from the proof of [[7], Theorem 3.7 and Lemma 3.2] that there exists a nest $\{F_k\}$ of closed sets such that $\mathbf{P}_m$-a.e. on $[\![0, \tau_{F_k}[\![$:

$$\Lambda(M) = \Lambda(M^k) + A^k_t + L^k_t \tag{5.5.24}$$

where $M^k \in \mathcal{M}$, $A^k \in \mathbf{A}_c$ and $L^k \in (\mathcal{M}_{loc})^{[\![0,\zeta[\![}$. With a refinement argument used in the proof of [[7], Lemma 4.6], one checks that $\Lambda(M)$ is a local CAF of X. Recall that $\mathfrak{E}$ denotes the set of CAF of X of finite energy. In view of [[7], Proposition 2.8] the right-hand side of (5.5.24) belongs to $\mathfrak{E}_{f\text{-}loc}$, hence $\Lambda(M)$ belongs to $(\mathfrak{E}_{f\text{-}loc})_{f\text{-}loc} = \mathfrak{E}_{f\text{-}loc}$.
By [[7], Theorem 3.7], $\Lambda(M)$ is of zero quadratic variation on the sets of Definition 5.4.2. Therefore thanks to Theorem 5.4.3, $\Lambda(M)$ belongs to $\mathcal{I}_{f\text{-}loc}$ and the integral $f * \Lambda(M)$ is well defined for any $f \in \mathcal{F}_{loc}$.
Thanks to [[7], Theorem 4.4] and the way that the stochastic integral was defined in [7], it satisfies (i) and (ii) of Lemma 5.5.24 where the convergence is in measure with respect to $\mathbf{P}_{gm}$ on $\{t < \zeta\}$ for every $g \in L^1(E, m)$ with $0 < g \leq 1$ m-a.e. Consequently the integrals $f * \Lambda(M)$ given by [7] and Definition 5.5.19 both coincide $\mathbf{P}_m$-a.e. on $[\![0, \zeta[\![$ for any $f \in \mathcal{F}_b$ and therefore for any $f \in \mathcal{F}_{loc}$.

5.6 Markov Processes on $\mathbb{R}^d$

Throughout this section we assume that $E = \mathbb{R}^d$, for d positive integer and $(\mathcal{E}, \mathcal{F})$ is a regular non necessarily symmetric Dirichlet form on $L^2(\mathbb{R}^d; m)$ satisfying : $C_0^\infty(\mathbb{R}^d) \subset \mathcal{F}$. Denote by $X = (X^1, ..., X^d)$ its properly associated Hunt process. Note that in this case, if $u_1, ..., u_d$ belong to $C_0^\infty(\mathbb{R})$ then for any F in $C_0^1(\mathbb{R}^d)$, $F(u_1, ..., u_d)$ belongs to $\mathcal{F}$. Therefore : $\mathcal{C}_0^1(\mathbb{R}^d) \subset \mathcal{F}$.
For a class $\mathcal{D}$ of additive functionals, the set $\mathcal{D}_{f\text{-}loc}$ that has been defined at the beginning of section 5.4. We will also make use of the random measures, $\mathbf{N}(\mathrm{d}y, \mathrm{d}s)$, $\hat{\mathbf{N}}(\mathrm{d}y, \mathrm{d}s)$ and $\tilde{\mathbf{N}}(\mathrm{d}y, \mathrm{d}s)$ introduced at the beginning of section 5.5.3. We define :

$$\check{\mathbf{N}}(\mathrm{d}y, \mathrm{d}s) := \frac{1}{2}(\mathbf{N}(\mathrm{d}y, \mathrm{d}s) - \hat{\mathbf{N}}(\mathrm{d}y, \mathrm{d}s))$$

The aim of this section is to prove the following two theorems :

Theorem 5.6.1. *For u in $C^2(\mathbb{R}^d)$, the process $u(X)$ admits the following decomposition $\mathbf{P}_x$-a.e for q.e $x \in \mathbb{R}^d$:*

$$u(X_t) = u(X_0) + V_t^u + W_t^u + C_t^u, \ t < \zeta \tag{5.6.1}$$

where $W^u \in \mathcal{M}_{f\text{-}loc}$, $C^u \in \mathcal{N}_{f\text{-}loc}$ and V^u is the AF of bounded variation given by :

$$V_t^u = \sum_{s \le t} (u(X_s) - u(X_{s-})) 1_{\{|u(X_s) - u(X_{s-})| > 1\}} \tag{5.6.2}$$

Moreover, the jumps of W^u are bounded by 1.

In particular, if we take u the coordinate function $\pi_i : x \to x_i$, $i = 1, ..., d$, the above result can be seen as a generalization of the Itô-Lêvy decomposition for Lévy processes.
Set : $V^i = V^{\pi_i}$, $W^i = W^{\pi_i}$ and $C^i = C^{\pi_i}$ and define the stochastic integral of f element of $\mathcal{F}_{loc}$ with respect to X^i by :

$$\int_0^t f(X_{s-}) \mathrm{d}X_s^i := \int_0^t f(X_{s-}) \mathrm{d}W_s^i + \int_0^t f(X_s) \mathrm{d}C_s^i + \int_0^t f(X_{s-}) \mathrm{d}V_s^i$$

where the third term in the right-hand side is given by Definition 5.5.19. We shall prove the following Itô formula.

Theorem 5.6.2. *For Φ in $\mathcal{C}^2(\mathbb{R}^d)$, for q.e $x \in \mathbb{R}^d$, $\mathbf{P}_x$-a.e for all $t \in [0, \zeta)$ we have :*

$$\begin{aligned} \Phi(X_t) &= \Phi(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial \Phi}{\partial x_i}(X_{s-}) \mathrm{d}X_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(X_s) \mathrm{d}\langle W^{i,c}, W^{j,c} \rangle_s \\ &\quad + \sum_{s \le t} \left(\Phi(X_s) - \Phi(X_{s-}) - \sum_{i=1}^d \frac{\partial \Phi}{\partial x_i}(X_{s-})(X_s^i - X_{s-}^i) \right) \end{aligned} \tag{5.6.3}$$

The proof of Theorem 5.6.1 will be based on the decomposition of $\mathcal{E}$ given in (5.6.4) below which is just another way to write the decomposition given by Hu et al., Theorem 4.8 of [23]. Moreover we can see directly from its proof that the hypothesis $u, v \in C_0^\infty(\mathbb{R}^d)$ can be replace by $u \in \mathcal{C}_0^2(\mathbb{R}^d)$ and $v \in C_0(\mathbb{R}^d) \cap \mathcal{F}$.

Lemma 5.6.3. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $\mathbb{R}^d$ satisfying $C_0^\infty(\mathbb{R}^d) \subset \mathcal{F}$. Then for u in $C_0^2(\mathbb{R}^d)$ and v in $C_0(\mathbb{R}^d) \cap \mathcal{F}$, $\mathcal{E}(u,v)$ has the following decomposition :*

$$\begin{aligned}\mathcal{E}(u,v) &= \tilde{\mathcal{E}}^{(c)}(u,v) + \check{\mathcal{E}}^{c'}(u,v) \\ &\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d-d} (u(y)-u(x))(v(y)-v(x))1_{\{|x-y|\le 1\}} J(\mathrm{d}x,\mathrm{d}y) \qquad (5.6.4) \\ &\quad -2\int_{\mathbb{R}^d\times\mathbb{R}^d-d} \{u(x)-u(y)-\sum_{i=1}^d (x_i-y_i)\frac{\partial u}{\partial y_i}(y)\} v(y) 1_{\{|x-y|\le 1\}} \check{J}(\mathrm{d}x,\mathrm{d}y) \\ &\quad -2\int_{\mathbb{R}^d\times\mathbb{R}^d-d} (u(x)-u(y))v(y)1_{\{|x-y|>1\}} J(\mathrm{d}x,\mathrm{d}y) + \int_{\mathbb{R}^d} u(x)v(x)k(\mathrm{d}x)\end{aligned}$$

where $v \to \check{\mathcal{E}}^{c'}(u,v)$ is a linear functional that equals 0 when $Supp[u] \cap Supp[v] = \emptyset$.

Before proving Theorem 5.6.1 and 5.6.2, we are going to built some local CAF locally of zero energy and some local MAF of locally finite energy, that will take part in the decomposition of X.

Lemma 5.6.4. **(i)** *For u in $\mathcal{C}^2(\mathbb{R}^d)$, the process*

$$\check{B}_t^u := 1_{\{t<\zeta\}} \int_0^t \int_{\mathbb{R}^d} \left(u(x) - u(X_s) - \sum_{i=1}^d (x_i - X_s^i) \frac{\partial u}{\partial x_i}(X_s) \right) 1_{\{|X_s - x|\le 1\}} \check{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s)$$

is well defined on $[0,\infty)$. Moreover $\check{B}^u$ is a local CAF element of $\mathbf{A}_c$ and for any relatively compact $G \subset \mathbb{R}^d$, $|\mu_{\check{B}^u}|(G) < \infty$.

(ii) *For u in $\mathcal{F}$, the process*

$$D_t^u := 1_{\{t<\zeta\}} \int_0^t \int_{\mathbb{R}^d} (u(x) - u(X_s))\, 1_{\{|X_s-x|>1\}} \mathbf{N}(\mathrm{d}x, \mathrm{d}s)$$

is well defined on $[0,\infty)$. Moreover D^u is a local CAF element of $\mathbf{A}_c$ and for any relatively compact $G \subset \mathbb{R}^d$, $|\mu_{D^u}|(G) < \infty$. The same holds for $\check{D}^u$ defined as D^u with $\check{N}$ replacing N.

Proof. (i) Let $\Gamma(x,y) := \left(u(x) - u(y) - \sum_{i=1}^d (x_i - y_i)\frac{\partial u}{\partial x_i}(y)\right) 1_{\{|y-x|\le 1\}}$ and G be a relatively compact set.

$$\begin{aligned}&\int_{\{|x-y|\le 1\}} 1_G(y)\, |\Gamma(x,y)|\, J(\mathrm{d}x,\mathrm{d}y) \\ \le\ & c(G) \int_{G\times\mathbb{R}^d} |x-y|^2 1_{\{|x-y|\le 1\}} J(\mathrm{d}x,\mathrm{d}y) \qquad (5.6.5)\end{aligned}$$

where $c(G) = \sup\left\{\sum_{i,j=1}^{d}\left|\frac{\partial^2 u(z)}{\partial x_i \partial x_j}\right| : |z-y| \leq 1 \exists y \in G\right\} < \infty$

The right-hand side of (5.6.5) is finite thanks to Lemma 4.1 of [23], hence $\int_0^t 1_G(X_s)\int_{\mathbb{R}^d}\Gamma(x,X_s)\mathbf{N}(\mathrm{d}x,\mathrm{d}s)$ is well defined on $[0,\infty)$. We can prove in the same way that $\int_0^t 1_G(X_s)\int_{\mathbb{R}^d}\Gamma(x,X_s)\hat{\mathbf{N}}(\mathrm{d}x,\mathrm{d}s)$ is well defined on $[0,\infty)$, and hence $\check{B}^u$ is well defined on $[0,\sigma_{\mathbb{R}^d\setminus G})$ for any relatively compact G and therefore well defined on $[0,\zeta)$. Since by definition $\check{B}^u_t$ equals 0 for $t \geq \zeta$, $\check{B}$ is well defined on $[\![0,\infty[\![$ We have shown also that $|\mu|(G) < \infty$.

(ii) This can be proved as (5.5.6). □

For u in $\mathcal{C}_0^2(\mathbb{R}^d)$, thanks to Lemma 5.6.4 for any relatively compact open set G and $h \in \mathcal{F}_G$, we have :

$$\begin{aligned}
& < \mu_{\check{B}^u}, h > + < \mu_{\check{D}^u}, h > \\
= & \int_{\mathbb{R}^d\times\mathbb{R}^d - d} 2h(y)\left((u(x)-u(y)-\sum_{i=1}^{d}(x_i-y_i)\frac{\partial u}{\partial x_i}(y)1_{\{|y-x|\leq 1\}}\right)\check{J}(\mathrm{d}x,\mathrm{d}y)
\end{aligned}$$

where $\check{J}(\mathrm{d}x,\mathrm{d}y) := \frac{1}{2}\{J(\mathrm{d}x,\mathrm{d}y) - J(\mathrm{d}y,\mathrm{d}x)\}$. But from Lemma 4.5 of [23] (which is also valid for $u \in \mathcal{C}_0^2(\mathbb{R})$), there exists $z \in \mathcal{F}$ such that the right-hand side of the above equation is equal to : $\check{\mathcal{E}}^{c'}(u,h) - \tilde{\mathcal{E}}_1(z,h)$, $\forall h \in \mathcal{C}_0(\mathbb{R}^d)\cap\mathcal{F}$. By defining w such that $\mathcal{E}_1(w,h) = \tilde{\mathcal{E}}_1(z,h) =$, $\forall h \in \mathcal{F}$, we obtain the following lemma.

Lemma 5.6.5. *For u in $\mathcal{C}_0^2(\mathbb{R}^d)$, there exists w in $\mathcal{F}$ such that*

$$\mathcal{E}_1(w,h) = \check{\mathcal{E}}^{c'}(u,h) - < \mu_{\check{B}^u}, h > - < \mu_{\check{D}^u}, h >$$

for all $h \in \mathcal{F}\cap\mathcal{C}_0(\mathbb{R}^d)$.

Definition 5.6.6. *For u in $\mathcal{C}_0^2(\mathbb{R}^d)$, define :*

$$^c\check{N}^u_t := N^w_t - \int_0^t w(X_s)\mathrm{d}s - \check{B}^u_t - \check{D}^u_t$$

where w is the element of $\mathcal{F}$ given by Lemma 5.6.5.

Definition 5.6.7. *Let $\mathcal{O}_c$ be the set of relatively compact open sets of $\mathbb{R}^d$. For G in $\mathcal{O}_c$, we define $\mathcal{N}^0_{c,G}$ as the set of local CAF's C such that, there exists u in$\mathcal{F}$ and A in $\mathbf{A}_c$ satisfying :*

$$\mathcal{F}_{b,G} \subset L^1(\mathbb{R}^d, |\mu_A|)$$

and

$$\mathbf{P}_x(C_t = N^u_t + A_t \text{ for } t < \zeta) = 1 \text{ for q.e } x \in \mathbb{R}^d$$

For C element of $\mathcal{N}^0_{c,G}$, we define the linear functional $\Theta(C,G)$ on $\mathcal{F}_{b,G}$ by

$$< \Theta(C,G), h > := -\mathcal{E}(u,h) + < \mu_A, h >, \; h \in \mathcal{F}_{b,G}$$

It follows from Theorem 5.3.3 that the definition of $\Theta(C,G)$ for $C \in \mathcal{N}^0_{c,G}$ is consistent in the sense that it does not depend of the elements which represents C.
The following lemma is an immediate consequence of Theorem 5.3.3 :

Lemma 5.6.8. *For G in $\mathcal{O}_c$, $C^{(1)}$ and $C^{(2)}$ elements of $\mathcal{N}^0_{c,G}$, we have :*
$C^{(1)} = C^{(2)}$ on $t < \tau_G$ $\mathbf{P}_x$-a.e for q.e $x \in \mathbb{R}^d$ if and only if

$$< \Theta(C^{(1)}, G), h >=< \Theta(C^{(2)}, G), h > \text{ for all } h \in \mathcal{F}_{b,G}.$$

Let $u \in \mathcal{F}$ and $G \in \mathcal{O}_c$, since $|\mu_{\check{B}^u}|(G) + |\mu_{\check{D}^u}|(G) < \infty$ by Lemma 5.6.4, ${}^c\check{N}^u$ belongs to $\mathcal{N}^0_{c,G}$ and :

$$< \Theta({}^c\check{N}^u, G), h >= -\check{\mathcal{E}}^{c'}(u,h) \text{ for any } h \in \mathcal{F}_{b,G} \cap C_0(\mathbb{R}^d) \tag{5.6.6}$$

Lemma 5.6.9. *For u, v in $\mathcal{C}^2_0$ and G in $\mathcal{O}_c$ such that $u = v$ on G, we have :*
${}^c\check{N}^u = {}^c\check{N}^v$ on $[\![0, \tau_G[\![$ $\mathbf{P}_x$-a.e for q.e $x \in \mathbb{R}^d$.

Proof. Since $Supp[u - v] \subset \mathbb{R}^d \backslash G$, $\check{\mathcal{E}}^{c'}(u - v, h) = 0$ for any $h \in \mathcal{F}_G$. We conclude thanks to (5.6.6) and Lemma 5.6.8. □

Thanks to Lemma 5.6.9 we can extend the definition of ${}^c\check{N}^u$ to every u in $C^2(\mathbb{R}^d)$.

Definition 5.6.10. *For u in $C^2(\mathbb{R}^d)$, define ${}^c\check{N}^u$ as follows. For any $G \in \mathcal{O}_c$,*

$${}^c\check{N}^u_t := \begin{cases} {}^c\check{N}^v_t & \text{for} \quad t < \tau_G \\ 0 & \text{for} \quad t \geq \zeta \end{cases}$$

where v is any element of $\mathcal{C}^2_0(\mathbb{R}^d)$ such that : $u = v$ on G.

Note that ${}^c\check{N}^u$ belongs to $\mathcal{N}^0_{c,G}$ for any G in $\mathcal{O}_c$.
For $r > 0$, set $\beta_r := \{x \in \mathbb{R}^d : |x| < r\}$. For u in $\mathcal{F}$, define : ${}^j\tilde{N}^{u,b} = {}^j\tilde{N}^u - \tilde{D}^u$, where ${}^j\tilde{N}^u$ is given by Definition 5.5.8 and

$$\tilde{D}^u_t = \int_0^t \int_{\{|X_s - x| > 1\}} (u(x) - u(X_s))\tilde{\mathbf{N}}(\mathrm{d}x, \mathrm{d}s),\ t < \zeta$$

which is well defined thanks to Lemma 5.5.6.
From (5.5.8) we see that for u, v in $C^1_0(\mathbb{R}^d)$ such that $u = v$ on β_{r+1}, we have :

$$\mathbf{P}_x({}^j\tilde{N}^{u,b}_t = {}^j\tilde{N}^{v,b}_t \text{ on } t < \tau_{\beta_r}) = 1 \text{ for q.e } x \in \mathbb{R}^d$$

Therefore the following definition makes sense.

Definition 5.6.11. *For u in $C^1(\mathbb{R}^d)$, define ${}^j\tilde{N}^u$ as follows. For any $r > 0$,*

$${}^j\tilde{N}^u_t := \begin{cases} {}^j\tilde{N}^v_t & \text{for} \quad t < \tau_{\beta_r} \\ 0 & \text{for} \quad t \geq \zeta \end{cases}$$

where v is any element of $\mathcal{C}^1_0(\mathbb{R}^d)$ such that : $u = v$ on β_{r+1}.

Thanks to Lemma 5.5.6, ${}^j\tilde{N}^{u,b}$ belongs to $\mathcal{N}_c^0$.
We also need the following process.

Definition 5.6.12. *For u in $C^1(\mathbb{R})$, ${}^kN^u$ is the local CAF of bounded variation defined by :*

$$ {}^kN_t^u := 1_{\{t<\zeta\}} \int_0^t u(X_s) 1_{\{|u(X_s)|\leq 1\}} N(X_s, \{\Delta\}) \mathrm{d}H_s $$

Now we are going to define the elements of $\mathcal{M}_{f\text{-}loc}$ that will be in the decomposition of X. It is known that for u, v in $\mathcal{F}$ and G in $\mathcal{O}$ such that $u = v$ q.e. on G, then $M_t^{u,c} = M_t^{v,c}$ on $[0, \tau_G)$, $\mathbf{P}_x$-a.e for q.e. $x \in \mathbb{R}^d$. Therefore it follows from (iii) of Lemma 5.2.2 that for u in $\mathcal{F}_{loc}$, $\{G_k\}$ in Ξ and $\{u_k\} \subset \mathcal{F}$ are such that $u = u_k$ q.e. on G_k, then by setting

$$ M_t^{u,c} := \begin{cases} M_t^{u_k,c} & \text{for} \quad t < \tau_{G_k} \\ 0 & \text{for} \quad t \geq \zeta \end{cases}, $$

one defines an AF element of $\mathcal{M}_{f\text{-}loc}$.
For u in $\mathcal{F}$ and $\epsilon > 0$, define the MAF $M^{u,b,\epsilon}$ on $\mathcal{M}$ by :

$$ \begin{aligned} M_t^{u,b,\epsilon} \quad := \quad & \sum_{s\leq t} (u(X_s) - u(X_{s-})) 1_{\{\epsilon<|u(X_s)-u(X_{s-})|\leq 1\}} \\ & - \int_0^t \int_{\{\epsilon<|u(x)-u(X_{s-})|\leq 1\}} (u(x) - u(X_s)) \mathbf{N}(\mathrm{d}x, \mathrm{d}s) \\ & + \int_0^t u(X_s) 1_{\{\epsilon<|u(X_s)|\leq 1\}} N(X_s, \{\Delta\}) \mathrm{d}H_s \end{aligned} \tag{5.6.7} $$

If $(\epsilon_n)_{n\in\mathbb{N}}$ converges to zero, (M^{u,b,ϵ_n}) is a Cauchy sequence in the real Banach space $(\mathcal{M}, e)$. Hence there exists an element of $\mathcal{M}$ denoted by $M^{u,b}$ and a subsequence (ϵ_{n_k}) such that $\mathbf{P}_x$-a.e for q.e $x \in \mathbb{R}^d$, $(M^{u,b,\epsilon_{n_k}})$ converges uniformly on compacts to $M^{u,b}$.
The following Lemma is a immediate consequence of (5.6.7) and the definition of $M^{u,b}$ for u in $\mathcal{F}$.

Lemma 5.6.13. **(i)** *For u, v in $\mathcal{F}$ such that $u = v$ q.e on β_{r+1}, $r > 0$, we have :*

$$ \mathbf{P}_x(M_t^{u,b} = M_t^{v,b} \textit{ for all } t < \tau_{\beta_r}) = 1 \textit{ for q.e. } x \in \mathbb{R}^d $$

(ii) *For u in $\mathcal{F}$, $\mathbf{P}_x$-a.e for q.e $x \in \mathbb{R}^d$ we have for any $t \geq 0$:*

$$ M_t^{u,b} = M_t^{u,d} - V_t^u + D_t^u - \int_0^t u(X_s) 1_{\{|u(X_s)|>1\}} N(X_s, \{\Delta\}) \mathrm{d}H_s \tag{5.6.8} $$

where $M^{u,d}$ denotes the discontinuous part of M^u and V^u is given by (5.6.2).

It follows from Lemma 5.6.13 that the following definition makes sense.

Definition 5.6.14. *For u in $C^1(\mathbb{R}^d)$, define $M^{u,b}$ as follows. For $r>0$,*

$$M_t^{u,b} := \begin{cases} M_t^{v,b} & \textit{for} \quad t<\tau_{\beta_r} \\ 0 & \textit{for} \quad t\geq\zeta \end{cases}$$

where v is any element of $\mathcal{C}_0^1(\mathbb{R}^d)$ such that : $u=v$ on β_{r+1}.

It is clear that : $M^{u,b}\in\mathcal{M}_{f\text{-}loc}, \forall u\in C^1(\mathbb{R}^d)$

Proof of Theorem 5.6.1. First we shall prove that for u in $\mathcal{C}_0^2(\mathbb{R}^d)$, $\mathbf{P}_x$-a.e for q.e $x\in\mathbb{R}^d$,

$$N^u = {}^c\tilde{N}^u + {}^c\check{N}^u + {}^j\tilde{N}^{u,b} + \tilde{B}^u + D^u - P^u \tag{5.6.9}$$

where $P_t^u := \int_0^t u(X_s)N(X_s,\{\Delta\})\mathrm{d}H_s$.
Denote by C^u the right-hand of (5.6.9). From the definition of all the terms of this sum, we can see that C^u belongs to $\mathcal{N}_{c,G}^0$ for any F in $\mathcal{O}_c$. Moreover thanks to (5.6.4), for any $h\in C_0\cap\mathcal{F}_G$:

$$<\Theta(N^u,G),h>=<\Theta(C^u,G),h> .$$

Thanks to the regularity of $\mathcal{E}$, the above identity can be extended to any $h\in\mathcal{F}_G$. One obtains then (5.6.9) thanks to Theorem 5.3.3.
From (5.6.8) and (5.6.9) we get (5.6.1) with

$$W^u = M^{u,c} + M^{u,b} \text{ and}$$

$$C^u = {}^c\tilde{N}^u + {}^c\check{N}^u + {}^j\tilde{N}^{u,b} + \tilde{B}^u - {}^kN^u$$

Now for u in $C^2(\mathbb{R}^d)$, define C^u and W^u as above. For any $k\in\mathbb{N}$ let $u_k\in C_0^2(\mathbb{R}^d)$ be such that $u=u_k$ on β_{k+1}, then

$$u(X_t)-u(X_0)-W_t^u-C_t^u = u_k(X_t)-u_k(X_0)-W_t^{u_k}-C_t^{u_k}=0 \text{ for } t<\tau_{\beta_k}$$

$\mathbf{P}_x$-a.e for q.e. $x\in\mathbb{R}^d$. We finally obtain (5.6.1) thanks to (iii) of Lemma 5.2.2. Note that $W^u\in\mathcal{M}_{f\text{-}loc}$ and $C^u\in\mathcal{N}_{f\text{-}loc}$. This finishes the proof. □

Proof of Theorem 5.6.2. This theorem can be proved with the arguments used in the proof of Theorem 5.6.1, but using the Itô formula of Theorem 5.5.23 instead of the Fukushima decomposition. □

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