

An Interpretation of Belief Functions by means of a Probabilistic Multi-modal Logic

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Abstract

While belief functions may be seen formally as a generalization of probabilistic distributions, the question of the interactions between belief functions and probability is still an issue in practice. This question is difficult, since the contexts of use of these theory are notably different and the semantics behind these theories are not exactly the same. A prominent issue is increasingly regarded by the community, that is the management of the conflicting information. Recent works have introduced new rules for handling the conflict redistribution while combining belief functions. The notion of conflict, or its cancellation by an hypothesis of open world, seems by itself to prevent a direct interpretation of belief function in a probabilistic framework. This paper addresses the question of a probabilistic interpretation of belief functions. It first introduces and implements a theoretically grounded rule, which is in essence an adaptive conjunctive rule. It is shown, how this rule is derived from a logical interpretation of the belief functions by means of a probabilistic multimodal logic; in addition, a concept of source independence is introduced, based on a principle of entropy maximization.

Keywords: Belief Functions, Modal Logic, Probability, Entropy maximization

1 Introduction

In [2, 3], A. Dempster introduced an elegant modelling of imprecise probabilistic beliefs, which formally implies a generalisation of probability, and a process for inferring combinations of these beliefs. This seminal work completed by G. Shafer [4] initiated the theory of belief functions for manipulating uncertain and imprecise information. However, while belief functions may be seen formally as a generalization of probabilistic distributions, the question of the interactions between belief functions and probability is still an issue in practice. Indeed, the contexts of use of these theory are notably different implying slightly different semantics in background. On the one hand, belief functions may deal with non technical sources of information, and as a consequence are also confronted inherently to conflictual information. On the other hand, the contexts of use of probability and bayesian reasoning are more technical and controlled, and prevent, for the most part, the issues of conflictual

information. It is often necessary to map the belief information to the probabilistic information for actual applications. As will be shown subsequently, some issue for a probabilistic interpretation of belief function is implied by the notion of conflict between the sources, which is a product of the conjunctive combination rule for fusing belief function. The conflict and the management of the conflict in the combination rules are topics of increasing interest for the community. Recent works have proposed evolutions of the historical rule of Dempster and Shafer (DS) for managing the conflict; Lefevre *and al* explicitly introduced this issue [9], and many rule constructions followed [5, 8, 10, 11, 13, 14]. Especially, Florea, Jousselme and al [8] proposed a family of rules which is adaptive with the conflict level. In this case, there is an important idea: the redistribution policy is now changing automatically as a function of the conflict. But this concern for the conflict is not new, and are for example ingredients of the famous rules of Dubois&Prade [6, 7] or Yager [17]. The notion of conflict may be cancelled by an hypothesis of open world [15], but this open world hypothesis does not map to the common formalisms of probability, and have to be interpreted. By the way, the open world hypothesis does not prevent any practical consideration of the conflict [16] implied by the interpretation of the belief put on the empty set.

This paper addresses the question of a probabilistic interpretation of belief functions. It is the continuation of reflexions done in [19, 20], which introduced and implemented a theoretically grounded rule, obtained by applying a principle of sources independence based entropy maximization. This rule is in essence an adaptive conjunctive rule, where the conflict redistribution is automatically derived from the considered independence principle. It is shown that this rule identifies to the conjunctive rule in cases without conflict. This rule is an introduction to the definition and study of a multimodal logic for modelling beliefs, since it is indeed derived from this logic by applying the independence principle. We point out by means of the axiomatizations how the conflict may be interpreted from the logical level.

This paper is divided in two parts. The first part introduces a methodology for computing combination rules, on the basis of an entropic principle for defining the independence between the sources. It is started by a general discussion about belief functions in section 2. Section 3 settles a new methodology for deriving the combination rule, based on an entropic notion of sensor independence. Then, section 4 discusses about the properties of the new rule and the implementations. A small example is considered as an illustration, but since the purpose of this paper is mainly theoretical, these experimental aspects are not developed. The second part exposes the logical fundamentals of our methodology. Section 7 provides some logical backgrounds and introduces the definition of the multimodal logic. In section 8, it is shown how a logical combination rule is derivable from the logic.

In section 9, a non trivial model is defined, and properties concerning the probabilistic extension of the logic are deduced. Belief combination rules are then derived theoretically from the logical rule, by applying a maximization of the entropic information. Section 10 concludes.

Part I

A combination rule based on the entropy maximization principle

2 Overview

2.1 Frame of discernment

In order to manipulate the information, it is necessary to have an algebraic representation of the events of the system. Various structures [21, 23] have been introduced recently, and in the most general case, distributive lattices may be used for this algebraic representation [23]. In this paper, however, we will restrict our reflexion to Boolean algebras, and especially to power sets, which are the historical and prominent structures used by the belief function community.

Definition 1 (frame of discernment). *It is given a finite Boolean algebra $\mathbf{B} = (G, \cap, \cup, \sim, \emptyset, \Omega)$ and a subset $\Theta \subset G$ of propositions, called frame of discernment, which generates G .¹*

The frame of discernment Θ is the set of basical known events of our model, and the Boolean algebra \mathbf{B} describes the logical interaction between the basical event of Θ . Subsequently, the notations $\mathbf{B}^\Theta \triangleq \mathbf{B}$ and $G^\Theta \triangleq G$ are used, in order to mark the structures with their frames of discernment.

While this paper addresses the general case of Boolean structures, it is known that power sets are sufficiently versatile structures, which may be equivalent representations of any finite Boolean algebra, and may as well include as substructures some lattice structures, like hyperpower sets [21]. Thus, a fundamental example of Boolean structure \mathbf{B}^Θ is the example of power sets.

Example 2 (power set). • *The frame of discernment identifies to singletons of a set Ω , so that*

$$\Theta = \{\{\omega\}/\omega \in \Omega\},$$

- *The Boolean algebra \mathbf{B}^Θ identifies to the algebra of the subsets of Ω , so that $G^\Theta = 2^\Omega$, and $\cap, \cup, \sim, \emptyset, \Omega$ are respectively the set intersection, set union, set complement, the empty set and the entire set Ω .*

¹Which means that G is the set of all propositions derived from \emptyset, Ω , the elements of Θ and the operators \cap, \cup, \sim .

2.2 Belief

2.2.1 Basic belief assignment

The information from different sources or expert is not reducible to the only logical representation. Information of real life are typically uncertain and imprecise and come with an uncertainty degree, or belief. In belief function theory, the basic belief assignment (bba) is a belief density, describing the information intrinsic to each proposition. This intrinsic information is the fraction of information of the proposition which is not inherited from connex propositions. There are two definitions of bba related respectively to an *open world* or a *closed world* hypothesis.

Definition 3 (basic belief assignment). *Being given a finite Boolean algebra $\mathbf{B}^\Theta = (G^\Theta, \cap, \cup, \sim, \emptyset, \Omega)$, a basic belief assignment m to G^Θ is a real valued function defined over G^Θ such that:*

$$\sum_{X \in G^\Theta} m(X) = 1 \quad \text{and} \quad m \geq 0,$$

and:

Open world hypothesis: *No other constraint,*

Closed world hypothesis: $m(\emptyset) = 0$.

The theory of Transferable Belief Model, as defined by Smets [15], implements typically the open world hypothesis. While the bba describes the information intrinsic to each proposition, the *full* belief of a proposition is potentially the compilation of the basic beliefs connex to this proposition. There is a pessimistic bound and an optimistic bound for the belief of a proposition, which are the credibility and the plausibility.

Definition 4 (credibility function). *The credibility Bel related to a bba m is defined for any $X \in G^\Theta$ by:*

$$\text{Bel}(X) = \sum_{Y \in G^\Theta: \emptyset \subsetneq Y \subset X} m(Y) = \sum_{Y \in G^\Theta: Y \subset X} m(Y) - m(\emptyset). \quad (1)$$

The credibility is a pessimistic bound, only talking into account the belief of the propositions known with certainty.

Definition 5 (plausibility function). *The plausibility is defined by $\text{Pl}(X) = 1 - \text{Bel}(\sim X)$ for any $X \in G^\Theta$.*

2.3 Combination rules

A main contribution of the theory of belief functions consists in the combination rules. It is assumed that two or more sources of information are providing a viewpoint about the universe. These viewpoints are described by specific bba for each source. The question then is to make a unique representation of the information, *i.e.* a unique bba, from these several bbas.

2.3.1 Conjunctive rule

The conjunctive rule belief functions is a rule of reference for combining belief functions with the *open world hypothesis*. It is applied for fusing independent sources of information.

Definition 6 (conjunctive combination). *Let m_1 and m_2 be 2 bbas over G^Θ . The conjunctive combination $m_1 \oplus_\cap m_2$ of m_1 and m_2 is defined by:*

$$m_1 \oplus_\cap m_2(X) = \sum_{Y_1, Y_2 \in G^\Theta: Y_1 \cap Y_2 = X} m_1(Y_1) m_2(Y_2). \quad (2)$$

It is proved that $m_1 \oplus_\cap m_2$ is a bba when m_1 and m_2 are bbas. However, the quantity $m_1 \oplus_\cap m_2(\emptyset)$, called *degree of conflict*, is not necessary zero, even if both m_1 and m_2 meet the constraint $m_1(\emptyset) = m_2(\emptyset) = 0$. Thus, \oplus_\cap is an acceptable combination rule when the open world hypothesis is made, like in the TBM, but it is not acceptable in general when the closed world hypothesis is done. However, there are classes of bbas for which the conjunctive rule is admissible for the closed world hypothesis, that is bbas restricted to Boolean substructures for which contradiction \emptyset cannot be derived from the conjunction \cap . In [21,25], Dezert and Smarandache proposed such a structure, the hyperpower set. On such structure, the conjunctive rule does not generate conflict and is admissible even with the closed world hypothesis.

2.3.2 Conflict management with the closed world hypothesis

In the general case, the conjunctive rule generates a degree of conflict, which is not compliant with the closed world hypothesis. While many rules are based on the conjunctive rule, a management of the conflict is necessary and is implemented specifically by each rule.

Dempster-Shafer rule. The historical rule of Dempster-Shafer [3,4] is based on the conjunctive rule, but handles the conflict as a normalization parameter.

Definition 7 (Dempster-Shafer). *Let m_1 and m_2 be 2 bbas over G^Θ . The Dempster-Shafer combination $m_1 \oplus_{DS} m_2$ of m_1 and m_2 is defined by:*

$$m_1 \oplus_{DS} m_2(X) = \frac{I[X \neq \emptyset] m_1 \oplus_\cap m_2(X)}{1 - Z}, \quad (3)$$

where $Z = m_1 \oplus_{\cap} m_2(\emptyset)$ is the degree of conflict and:

$$I[X \neq \emptyset] = 1 \text{ if } X \neq \emptyset, \text{ and } I[X \neq \emptyset] = 0 \text{ if } X = \emptyset.$$

Rules defined from a conflict redistribution. Variety of rules have been proposed recently [5, 8, 10, 11, 13, 14], especially based on a conflict redistribution [9]. We will not give here a description of all these rules, but a principle of construction is proposed.

The principle is to construct a combination rule from the conjunctive rule by redistributing the conflict. Being given a redistribution function r , it is defined the combination rule \oplus_r by:

$$m_1 \oplus [r]m_2(X) = m_1 \oplus_{\cap} m_2(X) + r(X|m_1, m_2) m_1 \oplus_{\cap} m_2(\emptyset), \quad (4)$$

for any $X \in G^{\ominus}$. The redistribution function r should be typically such that:

$$r(\emptyset|m_1, m_2) = -1, \quad \sum_{Y \in G^{\ominus}} r(Y|m_1, m_2) = 0, \quad (5)$$

and:

$$r(X|m_1, m_2) \geq 0, \text{ for any } X \in G^{\ominus} \setminus \{\emptyset\}. \quad (6)$$

But there are examples of redistribution laws [8] where the constraint (6) is relaxed. However, in this approach of Florea, Jousselme and *al*, the redistribution function is conditioned by the conflict $Z = m_1 \oplus_{\cap} m_2(\emptyset)$, which is a more specific conditionner than m_1, m_2

There are many possible rules deduced from the redistribution principle. Dempster-Shafer rule is among these rules:

$$\oplus_{\text{DS}} = \oplus[r_{\text{DS}}] \text{ where } r_{\text{DS}}(X|m_1, m_2) = \frac{m_1 \oplus_{\cap} m_2(X)}{1 - Z}, \text{ for any } X \in G^{\ominus} \setminus \{\emptyset\}.$$

In this case, the redistribution is proportionnal to the conjunctively fused belief.

The PCR5 (Proportional Conflict Redistribution no. 5) rule [13, 14] is another rule deduced from a redistribution of the conflict, *i.e.* $\oplus_{\text{PCR5}} = \oplus[r_{\text{PCR5}}]$ where:

$$r_{\text{PCR5}}(X|m_1, m_2) = \frac{1}{Z} \left(\sum_{Y \in G^{\ominus} \setminus \{X\}} \frac{m_1(X)}{m_1(X) + m_2(Y)} m_1(X) m_2(Y) + \sum_{Y \in G^{\ominus} \setminus \{X\}} \frac{m_2(X)}{m_2(X) + m_1(Y)} m_1(Y) m_2(X) \right).$$

for any $X \in G^{\ominus} \setminus \{\emptyset\}$. Under that form, it appears clearly that PCR5 is based on a local conflict redistribution, attributed proportionnaly to the input basic belief.

The principle of conflict redistribution implies a potentially infinite variety of rules. The question now is *how to decide for a rule or another?* This choice is certainly dependent of the structure of the fusion problem. A theoretical criterion for such a choice is still an issue.

In the next sections, we derive a theoretically grounded ruling method, for which the conflict redistribution is implied by a new concept of independence of the sources. The new rules are essentially computed from an entropic optimization problem.

3 Entropic approach for the rule definition

To begin with this new rule concept, we directly settle the concrete optimization principles of our method. The logical justification comes later in part II.

3.1 Independent sources and entropy

The idea is not completely new, and the entropy maximization principle has been linked to the conjunctive rule in [25]. It is noticed, that \oplus_{\cap} could be derived from an entropic maximization:

$$m_1 \oplus_{\cap} m_2(X) = \sum_{Y \cap Z = X} f_o(Y, Z), \text{ for any } X \in G^{\Theta},$$

where:

$$f_o \in \arg \max_f - \sum_{Y, Z \in G^{\Theta}} f(Y, Z) \ln f(Y, Z)$$

under constraints:

$$\begin{aligned} f &\geq 0, \\ \sum_{Y \in G^{\Theta}} f(Y, Z) &= m_2(Z), \text{ for any } Z \in G^{\Theta}, \\ \sum_{Z \in G^{\Theta}} f(Y, Z) &= m_1(Y), \text{ for any } Y \in G^{\Theta}. \end{aligned} \tag{7}$$

Equation (7) has a particular interpretation in the paradigm of information theory: f_o is the law with maximal ignorance of the interaction between the marginal variables Y and Z , owing to the fact that the marginal law are m_1 and m_2 . By the way, independent sources of information should indeed provide a maximal ignorance of interaction between the marginal variables, so that the maximization of entropy appears as the good way to characterize independent sources. Since, in this case, the constraints are limited to the marginalizations, the solution to this maximization is:

$$f_o(Y, Z) = m_1(Y)m_2(Z),$$

where is recognized the joint law of two independent variables. The definition (7) works when the *open world hypothesis* is done. But when the *closed world* is hypothesized, new constraints are

implied and have to be added to the optimization problem. In regards to these considerations, a generalization of the notion of independance between variables is introduced now, and will be used for the definition of a new rule of combination.

3.2 Independance of constrained variables

Definition 8 (constrained independance). *Let be given the variables $X_{1:s}$ defined over the finite sets $E_{1:s}$ respectively. Let ρ_o be a probability density defined over the joint variable $(X_{1:s})$ and let $\rho_{1:s}$ be the marginals of ρ_o over the variables $X_{1:s}$ respectively. Let $\vec{\varphi}(\cdot)$ be the constraint defined by $\vec{\varphi} : (\rho(X_{1:s}))_{(X_{1:s}) \in \prod_{i=1:s} E_i} \mapsto \{false, true\}^M$ is a vectorial Boolean mapping defined on any joint probability density ρ . Then, the variables $X_{1:s}$ are independant in regards to the probability density ρ_o and to the constraint $\vec{\varphi}(\cdot) \geq \vec{0}$ if and only if ρ_o is the solution of the following optimization:*

$$\begin{aligned} \rho_o \in \arg \max_{\rho} \sum_{(X_{1:s}) \in \prod_{i=1:s} E_i} -\rho(X_{1:s}) \ln \rho(X_{1:s}), \\ \text{with respect to :} \\ \rho \geq 0, \\ \text{the marginality constraints : } \sum_{(X_{1:i-1}) \in \prod_{j=1:i-1} E_j} \sum_{(X_{i+1:s}) \in \prod_{j=i+1:s} E_j} \rho(X_{1:s}) = \rho_i(X_i), \\ \text{the constraint : } \vec{\varphi}(\rho) = \vec{true}. \end{aligned} \quad (8)$$

It is noticed that this concept is dependent of the variables set. For example, it is not necessary true that the variables of the subsequence X_{i_1}, \dots, X_{i_k} are independant in regards to the constraints, when $X_{1:s}$ are. There is an exception however, when there is no constraint, *i.e.* $\vec{\varphi}(\rho)$ is always \vec{true} . In such a case, it is proven by applying Kuhn-Tucker theorem that $\rho(X_{1:s}) = \prod_{i=1:s} \rho_i(X_i)$.

The notion of *constrained independance* is now applied to the definition of a combination rule; the sets are $E_i = G^\ominus$, and the constraint may be defined by $\vec{\varphi} = (\varphi_{X_{1:s}})_{X_{1:s} \in G^\ominus}$, where:

$$\begin{aligned} \varphi_{X_{1:s}}(\rho) = false \text{ if } \rho(X_{1:s}) > 0 \text{ and } \bigcap_{i=1:s} X_i = \emptyset, \\ \varphi_{X_{1:s}}(\rho) = true \text{ otherwise.} \end{aligned}$$

After simplification, this construction results in the following definition of the fusion rule.

3.3 Entropy maximizing rule for a closed world

Definition 9 (entropy maximizing rule). *Let $m_{1:s}$ be s bbas defined on G^\ominus , compliant with the closed world hypothesis. The combination rule \oplus_H , named *Entropy Maximizing Rule (EMR)*, is*

defined by:

$$m_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} m_s(X) = \sum_{\bigcap_{i=1:s} Y_i = X} f_o(Y_{1:s}), \text{ for any } X \in G^\Theta,$$

where:

$$f_o \in \arg \max_f - \sum_{Y_{1:s} \in G^\Theta} f(Y_{1:s}) \ln f(Y_{1:s})$$

under constraints:

$$f \geq 0,$$

$$\sum_{Y_{1:i-1}, Y_{i+1:s} \in G^\Theta} f(Y_{1:s}) = m_i(Y_i), \text{ for any } i = 1 : s \text{ and } Y_i \in G^\Theta,$$

$$\bigcap_{i=1:s} Y_i = \emptyset \Rightarrow f(Y_{1:s}) = 0, \text{ for any } Y_{1:s} \in G^\Theta.$$

(9)

Corollary of the definition 10. \oplus_{H} is compliant with the closed world hypothesis.

Conflict management. In the definition of rule \oplus_{H} , the independence of the sources has been implemented practically by an entropic maximization principle. From this principle, the conflict management is implied automatically. Nevertheless, the rule \oplus_{H} is also able to reject the sources, when they are considered as definitively non compatible. The fundamental incompatibility of the sources is related to the feasibility of the constraints in (9).

A glimpse of the logical background. The definition of $m_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} m_s$ based on f_o is the expression of hypotheses on the logical background, which require some caution when applying the rule. Especially, it is the joint result of a conservation of the beliefs during the combination and of a logical independence hypothesis. The constraints in (9) are also the consequence of the logical background. In particular, the constraint $\bigcap_{i=1:s} Y_i = \emptyset \Rightarrow f(Y_{1:s}) = 0$ is related to the hypothesis that the sources are in coherence with the objective world. In the most general case, the Belief functions may be used for fusing subjective information, and different logical background and constraints should be considered for such information. However, the entropy maximization principle may be combined with many kinds of constraints, and we have potentially a general principle for defining rules.

4 Properties and implementation

This section is devoted to the development of basic properties of EMR and to practical implementation on examples. The following definitions and notations are used in this section.

Definition 11 (probabilistic bba). A bba m is said to be probabilistic if there are propositions $X_1, \dots, X_n \in G^\Theta \setminus \{\emptyset\}$ such that:

$$\sum_{j=1}^n m(X_j) = 1 \quad \text{and} \quad \begin{cases} i \neq j \Rightarrow X_i \cap X_j = \emptyset, \\ \bigcup_{j=1}^n X_j = \Omega. \end{cases}$$

Moreover, when X_1, \dots, X_m are minimal non empty elements, that is:

$$Y \subsetneq X_i \text{ implies } Y = \emptyset \text{ for any } i = 1 : n,$$

then m is called a probability density over G^Θ and the credibility Bel related to m is called the probability measure on G^Θ derived from m .

Notations. Subsequently, m_1, \dots, m_s are bbas on G^Θ compliant with the closed world hypothesis. It is denoted by $\text{Bel}_i \oplus_{\text{H}} \text{Bel}_j$ the credibility function related to $m_i \oplus_{\text{H}} m_j$. It is denoted by ρ a probability density over G^Θ and by P the probability measure on G^Θ derived from ρ .

Now, we have first to explain some notions about belief sharpening and its converse, belief weakening. Belief weakening is commonly used by the community, especially when a reliability degree of the information is taken into account. These notions are introduced now, with some refinements. They are necessary subsequently.

4.1 Belief sharpening

Definition 12 (belief sharpening). A sharpening from m_1 to m_2 is a mapping $r_{12} : G^\Theta \times G^\Theta \rightarrow \mathbb{R}$, such that $r_{12} \geq 0$, and:

- $\sum_{Y \in G^\Theta} r_{12}(X, Y) = \sum_{Y: Y \subseteq X} r_{12}(X, Y) = m_1(X)$, for any $X \in G^\Theta$,
- $\sum_{X \in G^\Theta} r_{12}(X, Y) = \sum_{X: Y \subseteq X} r_{12}(X, Y) = m_2(Y)$, for any $Y \in G^\Theta$.

It is noticed that this definition implies:

$$r_{12}(X, Y) = 0 \quad \text{for any } (X, Y) \in (G^\Theta \times G^\Theta) \text{ such that } Y \not\subseteq X. \quad (10)$$

A sharpening will move the basic belief from propositions to subpropositions, thus resulting in a more accurate assignment of the basic belief. We will also say that m_2 is a sharpening of m_1 , or conversely that m_1 is a *weakening* of m_2 , when exists a sharpening from m_1 to m_2 . We will equivalently write the \triangleleft relation for credibility functions:

$$\text{Bel}_1 \triangleleft \text{Bel}_2 \iff m_1 \triangleleft m_2.$$

It is noticed that the sharpening implies an order relation on belief functions.

Property 13 (equivalence relation). Define the relation \triangleleft on belief functions by:

$\text{Bel}_1 \triangleleft \text{Bel}_2$ if and only if there is a sharpening r_{12} from Bel_1 to Bel_2 .

Then, the relation \triangleleft is an order relation, and in particular:

\triangleleft is **transitive**. If r_{12} and r_{23} are sharpenings respectively from Bel_1 to Bel_2 , and from Bel_2 to Bel_3 , then r_{13} defined by:²

$$r_{13}(X_1, X_3) \triangleq \sum_{X_2 \in G^\Theta} \frac{r_{12}(X_1, X_2)r_{23}(X_2, X_3)}{m_2(X_2)}, \quad (11)$$

is a sharpening from Bel_1 to Bel_3 ,

\triangleleft is **reflexive**. More precisely, there is only one sharpening r from Bel to Bel , which is defined by $r(X, X) = m(X)$ and $r(X, Y) = 0$ for any $X, Y \in G^\Theta$ such that $Y \neq X$,

\triangleleft is **antisymmetric**. If r_{12} and r_{21} are sharpenings respectively from Bel_1 to Bel_2 , and from Bel_2 to Bel_1 , then:

$$r_{12}(X, X) = r_{21}(X, X) = m_1(X) = m_2(X) \quad \text{and} \quad r_{12}(X, Y) = r_{21}(X, Y) = 0,$$

for any $X, Y \in G^\Theta$ such that $X \neq Y$.

Proof is done in appendix A.

Property 14. $\text{Bel}_1 \triangleleft \text{Bel}_2$ implies $\text{Bel}_1 \leq \text{Bel}_2$.

Proof is done in appendix A. The converse is false.

Example 15 (counterexample). Assume:

$$G^\Theta = \{\emptyset, A, B, C, A \cup B, B \cup C, C \cup A, \emptyset\},$$

and define:

$$m_1(A \cup B) = m_1(B \cup C) = 0.5 = m_2(B) = m_2(\Omega).$$

It is obvious that $\text{Bel}_1 \leq \text{Bel}_2$, but it is also clear that $\text{Bel}_1 \not\triangleleft \text{Bel}_2$.

Property 16. Probability measures are minimal for \triangleleft . More precisely:

$$P \triangleleft \text{Bel} \quad \text{implies} \quad P = \text{Bel}.$$

The result is a direct consequence of the definitions.

²Since $r_{12}(X_1, X_2) \leq m_2(X_2)$ and $r_{23}(X_2, X_3) \leq m_2(X_2)$, it is noticed that (11) is well defined by setting $\frac{r_{12}(X_1, X_2)r_{23}(X_2, X_3)}{m_2(X_2)} \triangleq 0$ when $m_2(X_2) = 0$.

4.2 Properties of EMR

Property 17 (commutativity). *Let $\sigma : \llbracket 1, s \rrbracket \rightarrow \llbracket 1, s \rrbracket$ be a permutation on $\llbracket 1, s \rrbracket$. Then:*

$$m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s \text{ exists if and only if } m_{\sigma(1)} \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_{\sigma(s)} \text{ exists ,}$$

and:

$$m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s = m_{\sigma(1)} \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_{\sigma(s)} .$$

The result is a direct consequence of the definitions.

Property 18 (neutral element). *Define the bba of total ignorance ν by $\nu(\Omega) = 1$. Then:*

$$m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s \oplus_{\mathbb{H}} \nu \text{ exists if and only if } m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s \text{ exists ,}$$

and:

$$m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s \oplus_{\mathbb{H}} \nu = m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s .$$

Proof is given in appendix A.

Property 19 (conservation of belief). *Assume that $m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s$ exists. Then:*

$$\text{Bel}_i \triangleleft \text{Bel}_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} \text{Bel}_s \text{ for any } i = 1 : s . \quad (12)$$

Proof is given in appendix A.

Corollary 20.

$$\text{Bel}_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} \text{Bel}_s \geq \max_{i=1:s} \text{Bel}_i .$$

Corollary 21. *Let $X_1, \dots, X_n \in G^\Theta$ be such that $X_i \cap X_j = \emptyset$ for any $i \neq j$. Then:*

$$\text{The existence of } m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s \text{ implies } \sum_{j=1}^n \max_{i=1:s} \text{Bel}_i(X_j) \leq 1 .$$

This result is a sufficient criterion for proving the non-existence of $m_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} m_s$.

Property 22 (combination of dominated belief functions). *Assume that $\text{Bel}_i \triangleleft \text{Bel}$ for any $i = 1 : s$.*

Then $\text{Bel}_1 \oplus_{\mathbb{H}} \cdots \oplus_{\mathbb{H}} \text{Bel}_s$ exists.

Proof. Let r_i be a sharpening from Bel_i to Bel , for $i = 1 : s$. Since $r_i(X_i, X) \leq m(X)$, it is set:

$$m(X)^{1-s} \prod_{i=1:s} r_i(X_i, X) \stackrel{\Delta}{=} 0 \quad \text{when } m(X) = 0 .$$

Then, define:

$$f(X_{1:s}) = \sum_{X \in G^\Theta} m(X)^{1-s} \prod_{i=1:s} r_i(X_i, X) .$$

From the definition of $r_i(X_i, X)$, it is clear that:

$$\prod_{i=1:s} r_i(X_i, X) = 0 \quad \text{when} \quad X \not\subseteq \bigcap_{i=1:s} X_i .$$

As a consequence of this and of $m(\emptyset) = 0$, it comes that:

$$m(X)^{1-s} \prod_{i=1:s} r_i(X_i, X) = 0 \quad \text{when} \quad \bigcap_{i=1:s} X_i = \emptyset .$$

Then:

$$\bigcap_{i=1:s} X_i = \emptyset \quad \text{implies} \quad f(X_{1:s}) = 0 .$$

Now, it is also derived:

$$\begin{aligned} \sum_{X_{1:i-1}, X_{i+1:s}} f(X_{1:s}) &= \sum_{X \in G^\Theta} r_i(X_i, X) m(X)^{1-s} \prod_{j=1:i-1, i+1:s} \sum_{X_j \in G^\Theta} r_j(X_j, X) \\ &= \sum_{X \in G^\Theta} r_i(X_i, X) m(X)^{1-s} m(X)^{s-1} = \sum_{X \in G^\Theta} r_i(X_i, X) = m_i(X_i) . \end{aligned}$$

Since it is also clear that $f \geq 0$, so there is a solution to the optimization (9). \square

Property 23 (probabilistic bound). *EMR is coherent with the probabilistic bounds:*

$$\text{Bel}_i \triangleleft P, \text{ for any } i=1:s, \text{ implies } \begin{cases} \text{Bel}_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} \text{Bel}_s \oplus_{\text{H}} P \text{ exists,} \\ \text{Bel}_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} \text{Bel}_s \oplus_{\text{H}} P = P . \end{cases} \quad (13)$$

Proof is given in appendix A.

The interpretation of this result may not be obvious. It says that, when the sources of information are weakenings of a probabilistic law, then their fusion with this law exists and confirms the law. However, the fused bba $\text{Bel}_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} \text{Bel}_s$ obtained from the sources $\text{Bel}_{1:s}$ alone is not necessary compatible with a probability P such that $\text{Bel}_i \triangleleft P$ for any $i = 1 : s$. In particular:

$$\begin{cases} \text{Bel}_i \triangleleft P, \text{ for any } i=1:s, \text{ does not imply } \text{Bel}_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} \text{Bel}_s \triangleleft P, \\ \text{Bel}_i \triangleleft P, \text{ for any } i=1:s, \text{ does not imply } (\text{Bel}_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} \text{Bel}_s) \oplus_{\text{H}} P \text{ exists.} \end{cases}$$

This result is an indirect consequence of the fact that \oplus_{H} will add information by means of the independence principle. This is illustrated by the following counterexample.

Example 24 (counterexample). *Let m_1, m_2 and ρ be defined of over G^Θ :*

$$G^\Theta = \{\emptyset, A, B, C, A \cup B, B \cup C, C \cup A, \emptyset\} ,$$

by: $m_1(A \cup B) = m_2(A \cup B) = \rho(A) = m_1(B \cup C) = m_2(B \cup C) = \rho(C) = 0.5$.

It comes out that $\text{Bel}_1 \triangleleft P$ and $\text{Bel}_2 \triangleleft P$ by redistributing the masses from $A \cup B$ to A and from

$B \cup C$ to C . Since the combination does not generate any conflict, then $m_1 \oplus_{\text{H}} m_2$ coincides with $m_1 \oplus_{\cap} m_2$ so that:

$$m_1 \oplus_{\text{H}} m_2(A \cup B) = m_1 \oplus_{\text{H}} m_2(B \cup C) = 0.25 \quad \text{and} \quad m_1 \oplus_{\text{H}} m_2(B) = 0.5 .$$

Obviously, $\text{Bel}_1 \oplus_{\text{H}} \text{Bel}_2 \not\prec P$ and $(\text{Bel}_1 \oplus_{\text{H}} \text{Bel}_2) \oplus_{\text{H}} P$ does not exist, since otherwise $(m_1 \oplus_{\text{H}} m_2) \oplus_{\text{H}} \rho(X) \geq 0.5$ for $X = A, B, C$.

Property 25 (associativity). *EMR is not associative.*

The counterexample of section 23 turns out to be a counterexample for the associativity.

As a consequence, one has to distinguish between $(m_1 \oplus_{\text{H}} m_2) \oplus_{\text{H}} m_3$, $m_1 \oplus_{\text{H}} (m_2 \oplus_{\text{H}} m_3)$ and $m_1 \oplus_{\text{H}} m_2 \oplus_{\text{H}} m_3$, this last one being a simultaneous combination of $m_{1:3}$.

Property 26 (idempotence). *EMR is not idempotent, but it is proved:*

$$m \text{ is probabilistic} \Rightarrow m \oplus_{\text{H}} \cdots \oplus_{\text{H}} m = m . \quad (14)$$

Proof is given in appendix A.

4.3 Algorithm

The optimization (9) is convex and is not difficult. However, its size increases exponentially. The gradient of $H(f) = \sum_{X_{1:s}} -f(X_{1:s}) \ln f(X_{1:s})$ is:

$$D_f H(f) = \sum_{X_{1:s}} -(1 + \ln f(X_{1:s})) df(X_{1:s}) .$$

The following algorithm, based on the projected gradient, has been implemented:

1. Initialize $t = 0$ and the convergence step θ_0 ,
2. Find a feasible solution f_0 to the constraints of (9); a simplex is solved. If such a solution does not exist, then stop: *the combination is not possible*. Otherwise, continue on next step.
3. Build Δf_t by solving the quadratic program:

$$\min_{\Delta f_t} \sum_{X_{1:s}} (\theta_t (1 + \ln f_t(X_{1:s})) + \Delta f_t(X_{1:s}))^2 ,$$

under constraints:

$$f_t + \Delta f_t \geq 0 , \quad \sum_{X_{1:i-1} X_{i+1:s}} \Delta f_t(X_{1:s}) = 0 , \quad \text{for any } i = 1 : s \text{ and any } X_i$$

$$\text{and } \bigcap_{i=1}^s X_i = \emptyset \Rightarrow \Delta f_t(X_{1:s}) = 0 \text{ for any } X_{1:s} .$$

Table 1: Example of EMR implementation.

α_1	γ_1	β_2	γ_2	$m = m_1 \oplus m_2$
0.99	0.01	0.99	0.01	Rejection[Zadeh]
0.501	0	0.501	0	Rejection
0.499	0	0.499	0	$m(a) = m(b) = 0.499, m(\{a, b, c\}) = 0.002$
0.3	0.1	0.3	0.1	$m(a) = m(b) = 0.3, m(c) = 0.175, m(\{a, b, c\}) = 0.225$
0.3	0.05	0.3	0.05	$m(a) = m(b) = 0.3, m(c) = 0.09375, m(\{a, b, c\}) = 0.30625$
0.3	0.01	0.3	0.01	$m(a) = m(b) = 0.3, m(c) = 0.01975, m(\{a, b, c\}) = 0.38025$

4. If $H(f_t + \Delta f_t) < H(f_t)$, then set $\theta_{t+1} = \theta_t/2$ and $f_{t+1} = f_t$,
5. Else set $f_{t+1} = f_t + \Delta f_t$,
6. Set $t := t + 1$ and reiterate from step 3 until full convergence.

4.4 Experimentation

We will consider the fusion of m_1 and m_2 defined on the power set of $\{a, b, c\}$ by:

$$\begin{cases} m_1(a) = \alpha_1, m_1(c) = \gamma_1, m_1(\{a, b, c\}) = 1 - \alpha_1 - \gamma_1, \\ m_2(b) = \beta_2, m_2(c) = \gamma_2, m_2(\{a, b, c\}) = 1 - \beta_2 - \gamma_2. \end{cases}$$

Notice that *Zadeh's example* is a subcase where defined by $\alpha_1 = \beta_2 = 0,99$ and $\gamma_1 = \gamma_2 = 0,01$.

The table 1 summarize the results of combination by \oplus_H for various cases. These examples express clearly the property of belief conservation, and the capacity of EMR to reject any combination which is not compatible with this property.

5 Conclusion

The rule EMR is a theoretically grounded combination rule, defined on the basis of:

- constraints expressing the belief conservation, which is a desirable property,
- an hypothesis of sources independence, which is generalized by means of a principle of information maximization.

The good news is that such a rule exists. Especially, it is compliant with the probabilistic bound property as expressed in section 23. The bad news is that such a rule is quite constraining and cannot be associative. Here comes a fundamental point related to the use of the hypothesis of independence. This hypothesis introduces information which may imply some conflicts with the *ground truth* expressed by means of a probability of reference P . The properties of sections 22 (combination of dominated belief functions) and 23 express that. More precisely, it is possible to

combine as many sources of information (even P itself!) which is compatible with P , so as to make the best approximation of P :

$$\text{Bel}_i \triangleleft P, \text{ for any } i=1:s, \text{ implies } \text{Bel}_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} \text{Bel}_s \text{ exists .}$$

But as soon as the rule has been applied, the introduction of information by means of the independence principle may implies conflicts with the *ground truth*:

$$\text{Bel}_i \triangleleft P, \text{ for any } i=1:s, \text{ does not imply } \text{Bel}_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} \text{Bel}_s \triangleleft P .$$

This finding leads us to question about the hypotheses of implementation of belief combination rules, at least in the context of a closed universe. We could give up with the property of belief conservation: this is actually what most rules do in practice, when redistributing the conjunctive conflict. We could resign ourself to implement the combination rule just as a final step of the fusion process, as this is suggested by the properties of EMR. However, such approach is practically impossible in regards to the combinatorics. But there is perhaps a third way: we could manipulate conditionnal propositions instead of absolute propositions, as is usually done: conditional propositions are efficient tools for handling the independence hypotheses. Conditionnal beliefs are not a new concept; the theoretical tools exist. Such an approach would lead to define some forms of reasoning similar to the Bayesian reasoning.

Part II

A multimodal logic for modelling beliefs

Part I introduced the rule EMR by applying a principle of sources independence based on a maximization of the information. While the resulting optimization (9) appears as a natural extension in regards to the conjunctive rule, this problem and especially its constraints have not been justified clearly. The prupose of this second part is to address the question of belief combination from a logical point of view. Based on this logical interpretation, the definition (9) is implied.

6 Belief functions, modality and probability

Belief functions may be seen as a generalization of probabilistic distributions by combining imprecisions within the uncertainty. Belief functions are characterized by the notion of basic belief assignment which will assign a basic belief to any element of the Boolean algebra. In this way, bbas may be seen as a generalization of probabilistic densities, which assign probabilistic densities to the minimal elements of the Boolean algebra. This difference implies the ability of belief functions for

manipulating imprecise information. It has also an algebraic consequence, by replacing the additivity property by the *sub-additivity* property. The qualification of belief functions as a generalization of probabilities has been questioned, by arguing that belief functions were subcases of random sets. While this is true in some manner, an elementary interpretation of this theory by means of random sets is not able to seize some semantic subtleties behind belief functions.

However, there is another way to concile probabilities and belief functions. This could be done by introducing the semantic at a logical level. While probabilities are constructed on a classical logical framework (Boolean algebras), it is still possible to implement probabilities on a non classical logic, and this how we will interpret belief functions. In [26], Ruspini derived a logical counterpart of the credibility function by means of an epistemic operator K representing the knowledge of the source of information. This operator was in fact related to the modal operator of system S5. He proposed a logical counterpart of basic belief assignment, *i.e.* the *epistemic* proposition denoted $e(X)$, derived from logical combinations of the modal propositions as follows:

$$e(X) = KX \setminus \bigcup_{Y \subset X; Y \neq X} KY . \quad (15)$$

Subsequently, an equivalent construction is proposed on the basis of a multimodal logic. In our model, $\text{Bel}_i(X)$, the credibility of X according to sensor i , is interpreted as the probability $P([i]X \wedge \neg[i]\perp)$ of the modal proposition $[i]X \wedge \neg[i]\perp$.

Ruspini also introduced the principle of a logical combination rule:

$$e(X) = \bigcup_{Y \cap Z = X} [e_1(Y) \cap e_2(Z)] . \quad (16)$$

This combination rule is in fact an immediate consequence of an *elaborated* axiom directly inspired from the conjunctive combination rule formulated on the credibility functions:

$$KX \text{ is true } \iff \text{ there is } Y, Z \text{ such that } Y \cap Z \subseteq X \text{ and } K_1Y, K_2Z \text{ are true.} \quad (17)$$

Subsequently, we will go a step further in this way by deriving the logical combination rule (16) from axioms of a multimodal logic. These axioms will be related to elementary properties of our deduction system.

From (16), Ruspini derived back Dempster-Shafer combination by applying the property $P(e_1(Y) \cap e_2(Z)) = P(e_1(Y))P(e_2(Z))$, which depicts a rough independence hypothesis of the sources. Since $e_1(Y) \cap e_2(Z) = \emptyset$ whenever $Y \cap Z = \emptyset$, this approach generates conflict and needs a normalization of the combination rule. In particular, it is possible to have $P(\emptyset) > 0$; this is clearly not satisfying from a logical viewpoint.

Again, we will go further with this idea of Ruspini, but, this time, by implementing our principle of sources independence based on the information maximization. This principle takes care of the logical exclusions. Nevertheless, we will also handle the distinction between the open world and the closed world, by considering different axioms. Especially, closed worlds will be such that the modalities of the contradiction are necessarily a contradiction, *i.e.* $[i]\perp \equiv \perp$, while open worlds will accept a relaxation of this constraint.

7 Definition of a multimodal logic

This section introduces a multimodal logic which is the backbone for a logical interpretation of the rule EMR. This logic has been previously introduced in [18].

7.1 Language definition

It is defined a *finite* set, \mathbf{P} , called the set of atomic propositions. Atomic propositions of \mathbf{P} are denoted p, q, r, \dots . For simplicity, it will be assumed subsequently that *the atomic propositions describe a partition*, that is the conjunction of different atomic propositions is the contradiction and the disjunction of all atomic propositions is a tautology. Although these strong hypotheses depart from usual axiomatization of classical logic, it is not a restriction, since it is known that is possible to construct a free Boolean algebra by means of a power set.

It is defined the finite set $I = \{1, \dots, n\}$, describing n sources of information.

Definition 27 (classical propositions). *The set \mathcal{L}_C of classical propositions is defined recursively by $\perp \in \mathcal{L}_C$, $\mathbf{P} \subseteq \mathcal{L}_C$ and $X \rightarrow Y \in \mathcal{L}_C$ for any $X, Y \in \mathcal{L}_C$.*

The classical propositions describe the logical information on the real world.

Definition 28 (simple modal propositions). *The set $\mathcal{L}_{C[C]}$ of simple modal propositions are defined recursively by $\perp \in \mathcal{L}_{C[C]}$, $\mathbf{P} \subseteq \mathcal{L}_{C[C]}$, $X \rightarrow Y \in \mathcal{L}_{C[C]}$ for any $X, Y \in \mathcal{L}_{C[C]}$, and $[J]X \in \mathcal{L}_{C[C]}$ for any $X \in \mathcal{L}_C$ and any non empty $J \subseteq I$.*

In $\mathcal{L}_{C[C]}$, modalities are only applied to classical propositions, resulting in a language free of nested modalities. This language contains all necessary elements for defining the belief reasoning at a logical level. Nevertheless, it is not closed under the modal operators, and the language of all modal propositions is preferred for this logical study.

Definition 29 (modal propositions). *The set \mathcal{L} of modal propositions is defined recursively by $\perp \in \mathcal{L}$, $\mathbf{P} \subseteq \mathcal{L}$, $X \rightarrow Y \in \mathcal{L}$ for any $X, Y \in \mathcal{L}$, and $[J]X \in \mathcal{L}$ for any $X \in \mathcal{L}$ and any non empty $J \subseteq I$.*

Propositions of \mathcal{L} are denoted X, Y, Z, \dots . It is noticed that the *empty* modality $[\emptyset]X$ is not defined in \mathcal{L} . The modal propositions describe the logical information according to the sources of information. Subsequently, the modal operator $[J]$ describes logically the information received and interpreted by the set J of sources of information working conjointly. Practically, we are essentially interested by the information on the classical propositions, so that, although we authorize nested modal propositions like $[J][K]X$, we will not deal with them in practice.

The abbreviations $\neg X \triangleq X \rightarrow \perp$, $X \vee Y \triangleq \neg X \rightarrow Y$, $X \wedge Y \triangleq \neg(X \rightarrow \neg Y)$, $\top \triangleq \neg\perp$ are used, for any $J \subseteq I$ and $X, Y \in \mathcal{L}$. The dual of $[J]X$ is defined as $\langle J \rangle X \triangleq \neg[J]\neg X$.

A multimodal logic is defined subsequently, by means of axioms and deductions rules. A proposition which is deduced from these axioms and rules is called a *theorem*. The following definition will be usefull subsequently:

Definition 30. For any finite set of propositions $E \subset \mathcal{L}$, it is defined:

$$\mathbf{v}E = \bigvee_{X \in E} X. \quad (18)$$

is often used subsequently.

Definition 31 (logical equivalence). The logical equivalence \equiv is defined by:

$$X \equiv Y \text{ if and only if } X \rightarrow Y \text{ and } Y \rightarrow X \text{ are theorems.}$$

7.2 Axioms.

The multimodal logic is defined by the following rules and axioms defined for any propositions $X, Y \in \mathcal{L}$ and subsets $J, K \subseteq I$:

Classical axioms.

Atomic propositions form a partition. \mathbf{vP} is an axiom and $(p \wedge q) \rightarrow \perp$ is an axiom, for any atomic propositions $p, q \in \mathbf{P}$ such that $p \neq q$,

Modus ponens. If X and $X \rightarrow Y$ are theorems, then Y is a theorem,

Modal rules and axioms:

N. If X is a theorem, then $[J]Y$ is a theorem,

K. $[J](X \rightarrow Y) \rightarrow ([J]X \rightarrow [J]Y)$ is an axiom,

T. $[J]X \rightarrow X$ is an axiom (optional),

Inc. $[J]X \rightarrow [J \cup K]X$ is an axiom,

Ind. Let $E \subseteq \mathbf{P}$ be a subset of atomic propositions. Using the notation (18), the following proposition is an axiom:

$$[J \cup K]\mathbf{v}E \rightarrow \bigvee_{\substack{F, G \subseteq \mathbf{P} \\ F \cap G \subseteq E}} (([J]\mathbf{v}F) \wedge ([K]\mathbf{v}G)) .$$

The hypothesis, that *atomic propositions form a partition*, is used for simplicity. On the first hand, it allows the introduction of general Boolean structures and not only free Boolean structures for characterizing the model of the world. On the second hand, it makes much more simple the definition of axiom *Ind* as well as the subsequent definition of the basic logical assignment. The rule *N* and axiom *K* are minimal requirements of modal systems, and express that the sources are logically coherent. The axiom *T* is optional and expresses that the sources provide information which are coherent with the ground truth. We will see that this axiom is in relation with the closed world hypothesis. The axiom *Inc* expresses that the sources are always contributing positively to the global knowledge. Incidentally, this axiom also expresses that the sources are working in coherence with each other at a the logical level. The axiom *Ind* expresses at the logical level that the sources are contributing *independently* to the global knowledge of the classical propositions, that is the joint knowledge $[J \cup K]X$ does not contains more information than the disjoint knowledges $[J]Y \wedge [K]Z$, where X, Y, Z are classial propositions such that $(Y \cap Z) \rightarrow X$.

It is assumed subsequently that the reader is familiar with classical propositional logic and its main results. Some results are recalled subsequently about the logical equivalence.

7.3 Logical equivalence

Owing to the axioms and rules of classical logic, \equiv is an equivalence relation on the propositions of \mathcal{L} . The equivalence class of a proposition $X \in \mathcal{L}$ for the logical equivalence is denoted $[X]$. Let $\mathcal{F} \subseteq \mathcal{L}$. The set of equivalence classes of propositions $X \in \mathcal{F}$ is also denoted $[F]$.

The logical operators are naturally infered on the classes of propositions. For example, it is defined $[X] \wedge [Y] \triangleq [X \wedge Y]$, $[J][X] \triangleq [[J]X]$, ... An order relation is also defined over $[\mathcal{L}]$.

Definition 32 (logical order).

$$[X] \subseteq [Y] \text{ if and only if } X \rightarrow Y \text{ is a theorem.}$$

The notation $[X] \subsetneq [Y]$ is used when $[X] \subseteq [Y]$ and $[X] \neq [Y]$.

Owing to the axioms, there is an morphism between $2^{\mathbf{P}}$ and $[\mathcal{L}_C]$. :

Definition 33 (morphism). For any classical proposition $X \in \mathcal{L}_C$, it is defined pX be the set of atomic propositions such that $[p] \subseteq [X]$.

$$pX = \{p \in \mathbf{P} / [p] \subseteq [X]\} . \quad (19)$$

Property 34 (morphism). Any classical proposition is equivalent to proposition $\mathbf{v}pX$:

$$\forall X \in \mathcal{L}_C , [X] = [\mathbf{v}pX] .$$

For any $E, F \subseteq \mathbf{P}$, it is also derived:

$$(\mathbf{v}E) \wedge (\mathbf{v}F) \equiv \mathbf{v}(E \cap F), (\mathbf{v}E) \vee (\mathbf{v}F) \equiv \mathbf{v}(E \cup F) . \text{ and } \neg \mathbf{v}E \equiv \mathbf{v}(\mathbf{P} \setminus E) .$$

These are clear consequences of classical propositional logic.

8 Deriving the combination rules

8.1 Preliminary results

Some useful logical theorems are now derived. These theorems are deduced without the help of the optional axiom T .

Property 35 (modality is non-decreasing).

$$\text{If } X \rightarrow Y \text{ is a theorem, then } [J]X \rightarrow [J]Y \text{ is a theorem.} . \quad (20)$$

This result is a direct consequence of N and K .

Property 36 (modality and conjunction).

$$([J]X \wedge [J]Y) \equiv [J](X \wedge Y) . \quad (21)$$

This result is a consequence of the theorems $(X \wedge Y) \rightarrow X$, $(X \wedge Y) \rightarrow Y$, $X \rightarrow (Y \rightarrow (X \wedge Y))$, N and K .

Property 37 (modality and disjunction).

$$([J]X \vee [J]Y) \rightarrow [J](X \vee Y) \text{ is a theorem.} \quad (22)$$

This result is a consequence of the theorems $X \rightarrow (X \vee Y)$, $Y \rightarrow (X \vee Y)$, N and K .

Property 38 (conjunction of heterogeneous modalities).

$$([J]X \wedge [K]Y) \rightarrow [J \cup K](X \wedge Y) . \quad (23)$$

Deduced from theorems $[J]X \rightarrow [J \cup K]X$ and $[K]Y \rightarrow [J \cup K]Y$, derived from *Inc*.

Property 39 (disjunction of heterogeneous modalities).

$$([J]X \vee [K]X) \rightarrow [J \cup K]X . \quad (24)$$

Immediate consequence of *Inc*.

8.2 Logical combination rule

8.2.1 Preliminary definitions

The notion of partition will be useful for defining the logical combination.

Definition 40 (partition). *A set of propositions, $\Pi \subseteq \mathcal{L}$ is a partition of \top if it satisfies:*

- *The propositions of Π are exclusive : $X \wedge Y \equiv \perp$ for any $X, Y \in \Pi$ such that $X \neq Y$,*
- *The propositions of Π are exhaustive: $\bigvee_{X \in \Pi} X \equiv \top$.*

Notice that Π may contain \perp , in this definition. The following property is a direct consequence of the definition.

Property 41. *Let Π and Λ be two partitions of \top . Then:*

- $\left(\bigvee_{X \in A} X \right) \wedge \neg \left(\bigvee_{X \in B} X \right) \equiv \bigvee_{X \in A \setminus B} X$, for any $A, B \subseteq \Pi$,
 - $X_1 \wedge Y_1 \equiv X_2 \wedge Y_2$ and $(X_1, Y_1) \neq (X_2, Y_2)$ imply $X_1 \wedge Y_1 \equiv X_2 \wedge Y_2 \equiv \perp$,
for any $X_1, Y_1 \in \Pi$ and $X_2, Y_2 \in \Lambda$,
 - *The set of joint propositions $\Gamma = \{X \wedge Y / X \in \Pi \text{ and } Y \in \Lambda\}$ is a partition.*
- (25)

8.2.2 Logical combination

Principle of the logical combination. Let $J, K \subseteq I$ be sources subsets, and X be any classical proposition. Arises the following issue: *How to characterize the fused information $[J \cup K]X$ from the primary information $[J]X$ and $[K]X$.* In order to solve this question, we introduce first the notion of *basic logical assignments* (bla) which constitute the elementary logical components of the information.

Definition 42 (basic logical assignments). *Let $E \subseteq \mathbf{P}$ The basic logical assignment (bla) related to the classical proposition $\mathbf{v}E$ and to sources subset J is the modal proposition $(J)\mathbf{v}E$ defined by:*

$$(J)\mathbf{v}E \triangleq [J]\mathbf{v}E \wedge \neg \left(\bigvee_{F \subsetneq E} [J]\mathbf{v}F \right) . \quad (26)$$

The bla $(J)\mathbf{v}E$ is the logical information, which the sources of set J attribute to proposition $\mathbf{v}E$ intrinsically. The information of $(J)\mathbf{v}E$ cannot be attributed to *classical* propositions smaller than $\mathbf{v}E$. From the morphism property 34, the definition extends to $[\mathcal{L}_C]$:

Definition 43 (bla on classes). *For any $[X] \in [\mathcal{L}_C]$, the basic logical assignment for $[X]$ is defined by $(J)[X] \triangleq (J)[\mathbf{v}pX]$. The bla $(J)[X]$ is also alternatively defined by:*

$$(J)[X] = [J][X] \wedge \neg \left(\bigvee_{[Y] \in [\mathcal{L}_C]: [Y] \subsetneq [X]} [J][Y] \right). \quad (27)$$

The blas have interesting properties, which will imply the definition of a logical combination.

Property 44 (exclusivity). *The blas $(J)\mathbf{v}E$, where $E \subseteq \mathbf{P}$, are exclusive:*

$$\forall E, F \subseteq \mathbf{P}, E \neq F \Rightarrow (J)\mathbf{v}E \wedge (J)\mathbf{v}F \equiv \perp. \quad (28)$$

Proof is given in appendix B.

Property 45 (exhaustivity). *The blas $(J)\mathbf{v}E$, where $E \subseteq \mathbf{P}$, are exhaustive:*

$$\bigvee_{F \subseteq E} (J)\mathbf{v}F \equiv [J]\mathbf{v}E, \text{ and in particular: } \bigvee_{F \subseteq \mathbf{P}} (J)\mathbf{v}F \equiv \top. \quad (29)$$

Proof is given in appendix B.

Corollary 46 (inversion). *It is possible to define $[J][X]$ from the blas $(J)[Y]$:*

$$[J][X] = \bigvee_{[Y] \in [\mathcal{L}_C]: [Y] \subseteq [X]} (J)[Y], \text{ for any } [X] \in [\mathcal{L}_C]. \quad (30)$$

It is deduced from these results that bbas constitute partitions:

Corollary 47 (bla partition). *The blas $(J)\mathbf{v}E$, where $E \subseteq \mathbf{P}$, constitute a partition of \top .*

Corollary 48 (joint bla partition). *Let $J, K \subseteq I$. The propositions $(J)\mathbf{v}E \wedge (K)\mathbf{v}F$, where $E, F \subseteq \mathbf{P}$, constitute a partition of \top .*

Based on these partitions, it is possible to express the logical combination:

Property 49 (computing the combination).

$$(J \cup K)\mathbf{v}E \equiv \bigvee_{F \cap G = E} ((J)\mathbf{v}F \wedge (K)\mathbf{v}G), \text{ for any } E \subseteq \mathbf{P}. \quad (31)$$

Corollary 50.

$$(J \cup K)[X] = \bigvee_{[Y], [Z] \in [\mathcal{L}_C]: [Y] \wedge [Z] = [X]} ((J)[Y] \wedge (K)[Z]), \text{ for any } [X] \in [\mathcal{L}_C]. \quad (32)$$

The proof of the main property will apply the following lemma.

Lemma 51.

$$[J \cup K]_{\mathbf{v}E} \equiv \bigvee_{F \cap G \subseteq E} ([J]_{\mathbf{v}F} \wedge [K]_{\mathbf{v}G}) \equiv \bigvee_{F \cap G \subseteq E} ((J)_{\mathbf{v}F} \wedge (K)_{\mathbf{v}G}) .$$

Proofs are given in appendix B.

Property 52 (generalized combination). *The combination (32) happens to be associative and may be extended to s sources:*

$$(J_1 \cup \dots \cup J_s)[X] = \bigvee_{[Y_{1:s}] \in [\mathcal{L}_C] : \bigwedge_{i=1:s} [Y_i] = [X]} \bigwedge_{i=1:s} (J_i)[Y_i] , \text{ for any } [X] \in [\mathcal{L}_C] . \quad (33)$$

8.2.3 Processus of logical fusion

As a conclusion, the fused sources $J \cup K$ is computed from sources J and K as follows:

- Build $(H)[X]$ by means of (27) for $H = J, K$,
- Compute $(J \cup K)[X]$ by means of (32)
- Derive $[J \cup K][X]$ by means of (30),

for any $[X] \in [\mathcal{L}_C]$.

9 Model and consequences

In this section, a model is constructed for the logic. This model is not complete, when embeded modalities are considered. However, if we restrict to the set of simple modal propositions, $\mathcal{L}_{C[C]}$, the model has some completeness properties, which will be useful for deriving the belief combination from the logical combination. The model is not based on a Kripke construction.

9.1 Model

9.1.1 Definition

Two models are constructed for the logic respectively without and with the optional axiom T . The models are quite similar, except for the minimal elements, denoted $\phi(\dots)$. The elements $\phi(p_0, E_{1:n})$ are defined for any $p_0 \in \mathbf{P}$ and any $E_{1:n} \subseteq \mathbf{P}$ by:

- *Without* axiom T :

$$\phi(p_0, E_{1:n}) = \{(p_0, E_{1:n})\} , \quad (34)$$

- With axiom T :

$$\begin{aligned} \phi(p_0, E_{1:n}) &= \{(p_0, E_{1:n})\}, \text{ if } p_0 \in \bigcap_{i=1}^n E_i, \\ \phi(p_0, E_{1:n}) &= \{\} \text{ otherwise.} \end{aligned} \quad (35)$$

For any proposition $p \in \mathbf{P}$, it is defined:

$$\bar{p} = \bigcup_{E_{1:n} \subseteq \mathbf{P}} \phi(p, E_{1:n}).$$

For any $E \subseteq \mathbf{P}$, it is also defined: $\bar{E} = \bigcup_{p \in E} \bar{p}$.

Then, the model \underline{X} of any proposition $X \in \mathcal{L}$ is defined recursively by:

- $\underline{p} = \bar{p}$ for any $p \in \mathbf{P}$ and $\underline{\perp} = \bar{\mathbf{P}}$,
- $\underline{X \wedge Y} = \underline{X} \cap \underline{Y}$, $\underline{X \vee Y} = \underline{X} \cup \underline{Y}$ and $\underline{\neg X} = \underline{\perp} \setminus \underline{X}$, for any $X, Y \in \mathcal{L}$,
- For any non empty $J \subseteq I$ and $X \in \mathcal{L}$, it is defined:

$$\underline{[J]X} = \bigcup_{p_0 \in \mathbf{P}} \bigcup_{E_{I \setminus J} \subseteq \mathbf{P}} \bigcup_{E_J \subseteq \mathbf{P}: \bigcap_{j \in J} E_j \subseteq X} \phi(p_0, E_{1:n}) \quad (36)$$

Other connectors are defined from \vee , \neg and $[J]$, e.g. $\underline{X \rightarrow Y} = \underline{\neg X \vee Y}$ and $\underline{\leq J > X} = \underline{\neg [J] \neg X}$.

Property 53 (classical propositions). *Let $X \in \mathcal{L}_C$ be a classical proposition. Then, it is clear from the definition, that there is $E \subset \mathbf{P}$ such that $\underline{X} = \bar{E}$.*

9.1.2 Interpretation

Subsequently, we are assuming $X, X_i, Y \in \mathcal{L}_C$ and $p \in \mathbf{P}$. It is assumed $E_i \subset X_i$ such that $\underline{X_i} = \bar{E_i}$.

It is easily shown that:

$$p \wedge \bigwedge_{i=1:n} (i)X_i = \phi(p, E_{1:n}). \quad (37)$$

In fact, it is implied from propositions (29) and (33) of the previous section that any combination of non-nested modal propositions, like $[J]X$ with $X \in \mathcal{L}_C$, by means of classical operators ($\wedge, \vee, \neg, \rightarrow$) is equivalent to a disjunction of bla like $\bigwedge_{i=1:n} (i)Y$ with $Y \in \mathcal{L}_C$. As a consequence, any propositions of $\mathcal{L}_{C[C]}$, the set of simple modal propositions, is equivalent to a disjunction of propositions like $p \wedge \bigwedge_{i=1:n} (i)X_i$. Now, in the case where axioms T also holds, it is easily derived that:

$$p \wedge \bigwedge_{i=1:n} (i)\mathbf{v}E_i \equiv \perp, \text{ for any } E_{1:n} \subseteq \mathbf{P} \text{ such that } p \notin \bigcap_{i=1:n} E_i,$$

which is the logical counterpart of the definition of (35). It follows from these remarks that our model (assuming that it fulfills actually the axioms) have *completude property while restricted to $\mathcal{L}_{C[C]}$* , so that:

$$[X] = [Y] \text{ if and only if } \underline{X} = \underline{Y}, \text{ for any } X, Y \in \mathcal{L}_{C[C]}. \quad (38)$$

In the case, where axiom T does hold, it is also noticed that the following characteristic constraints is obtained for the simple modal propositions:

$$\bigwedge_{i=1:s} Y_i = \perp \text{ implies } \bigwedge_{i=1:s} (J_i)[Y_i] = \perp, \text{ for any } Y_{1:s} \in \mathcal{L}_C. \quad (39)$$

From the model definition, it is also clear that this constraint is sufficient to infer the consequence of axiom T on the modal propositions. This discussion about the models is now completed by verifying the axioms.

9.1.3 Axioms checking

The classical axioms are implied since the model is a power set. Other axioms are derived as follows.

Property 54 (axioms verification).

Partition: $\bigvee_{p \in \mathbf{P}} p = \top$ and $\underline{p \wedge q} = \perp = \emptyset$, for $p, q \in \mathbf{P}$ such that $p \neq q$,

N: $\underline{[J]\top} = \top$,

K: $\underline{[J](\neg X \vee Y)} \cap \underline{[J]X} \subseteq \underline{[J]Y}$, for any $X, Y \in \mathcal{L}$,

Inc: $\underline{[J](X)} \subseteq \underline{[J \cup K]X}$, for any $X, Y \in \mathcal{L}$,

Ind: $\underline{[J \cup K]\mathbf{v}E} \subseteq \bigcup_{\substack{F, G \subseteq \mathbf{P} \\ F \cap G \subseteq E}} (\underline{[J]\mathbf{v}F} \cap \underline{[K]\mathbf{v}G})$.

T: $\underline{[J](X)} \subseteq \underline{X}$, when the model follows definition (35).

Proofs indications are given in appendix B.

9.2 From logical combination to belief combination

The interpretation of belief functions and belief combinations by means of modal propositions needs the concept of probability over logical propositions.

Definition 55 (probability on logical propositions). *Let $[\mathcal{F}] \subset [\mathcal{L}]$ be a finite set of propositions of $[\mathcal{L}]$, closed under the classical operators (i.e. \rightarrow and \perp and derived operators). A probability distribution over $[\mathcal{F}]$ is a mapping $P : [\mathcal{F}] \rightarrow \mathbb{R}$ such that:*

$$P \geq 0, \quad P(\top) = 1 \quad \text{and} \quad P([X]) + P([Y]) = P([X] \wedge [Y]) + P([X] \vee [Y]).$$

Now, it is assumed s sensors characterized by the subsets $J_{1:s} \subseteq I$, such that $J_i \cap J_j = \emptyset$ for any $i \neq j$. The logical framework $[\mathcal{F}]$ is then constructed as a classical closure of the sets of simple $[J_i]$ -modalities, i.e.:

$$\{[J_i][X] / i = 1 : s, [X] \in [\mathcal{L}_C]\} \subseteq [\mathcal{F}], .$$

The set $\lfloor \mathcal{F} \rfloor$ is a finite set. It happens from previous properties that the propositions $\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor$, for $\lfloor Y_{1:s} \rfloor \in \lfloor \mathcal{L}_C \rfloor$, constitute a partition generating $\lfloor \mathcal{F} \rfloor$ and that $\lfloor \bigcup_{i=1:s} J_i \rfloor \lfloor X \rfloor \in \lfloor \mathcal{F} \rfloor$. It is then possible to define the notion of belief, based on a probability defined over the framework $\lfloor \mathcal{F} \rfloor$.

Definition 56 (belief as probabilistic modality). *It is defined $\text{Bel}_i(\lfloor X \rfloor) \triangleq P(\lfloor J_i \rfloor \lfloor X \rfloor) - P(\lfloor J_i \rfloor \lfloor \perp \rfloor)$ and:*

$$\text{Bel}_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} \text{Bel}_s(\lfloor X \rfloor) = P \left(\left[\bigcup_{i=1:s} J_i \right] \lfloor X \rfloor \right) - P \left(\left[\bigcup_{i=1:s} J_i \right] \lfloor \perp \rfloor \right).$$

for $i = 1 : s$ and for any $\lfloor X \rfloor \in \lfloor \mathcal{L}_C \rfloor$.

From this definition and the properties of (J) , especially property (33), a definition of the bbas and of their combinations is recovered.

Property 57 (logical definition of bba). $m_i(\lfloor X \rfloor) = P(\lfloor J_i \rfloor \lfloor X \rfloor)$, and:

$$m_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} m_s(\lfloor X \rfloor) = \sum_{\lfloor Y_{1:s} \rfloor \in \lfloor \mathcal{L}_C \rfloor : \bigwedge_{i=1:s} \lfloor Y_i \rfloor = \lfloor X \rfloor} P \left(\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor \right). \quad (40)$$

In order to reduce $P(\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor)$, the independence of the *disjoint* sources $J_{1:s}$ is instrumental in a probabilistic paradigm. The idea is to compute $P(\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor)$ by maximizing the entropy of P over the logical framework $\lfloor \mathcal{F} \rfloor$:

$$P \in \arg \max_Q \sum_{\lfloor Y_{1:s} \rfloor \in \lfloor \mathcal{L}_C \rfloor} -Q \left(\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor \right) \ln Q \left(\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor \right), \quad (41)$$

$$\text{u.c. } Q((J_i) \lfloor Y_i \rfloor) = m_i((J_i) \lfloor Y_i \rfloor) \text{ for any } i = 1 : s \text{ and } \lfloor Y_{1:s} \rfloor \in \lfloor \mathcal{L}_C \rfloor.$$

Combining (40) and (41), it becomes possible to derive $m_1 \oplus_{\text{H}} \cdots \oplus_{\text{H}} m_s$ from $m_{1:s}$. Two cases arise, which are consequences of the model constructed in previous section:

- Axiom T is removed. Then, it is possible to have $\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor \neq \perp$ even if $\bigwedge_{i=1:s} Y_i = \perp$, as seen in the model construction. As a consequence, the optimization implies the conjunctive rule. We are in the hypothesis of an *open world*.
- Axiom T is used. As seen in the model construction, this axiom reduces in the constraint (39) for the simple modal propositions. As a consequence, it is necessary to consider the constraint $\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor = \perp$, when $\bigwedge_{i=1:s} \lfloor Y_i \rfloor = \perp$. This results in the following implied constraint in the optimization (41):

$$\bigwedge_{i=1:s} \lfloor Y_i \rfloor = \perp \text{ implies } Q \left(\bigwedge_{i=1:s} (J_i) \lfloor Y_i \rfloor \right) = 0.$$

The rule EMR is thus derived, from a logical interpretation. Moreover, we are able to distinguish between the open and closed world hypotheses. From a modal point of view, the

closed world (*resp.* open world) hypothesis means that $[J]\perp \rightarrow \perp$ is necessary (*resp.* is false in general). From this modal interpretation, it appears, that the experts from an *open world* may imply contradictions from factual information (typically because his knowledge of the real world is not exhaustive).

10 Conclusion

A new combination rule have been defined for combining belief functions. This rule is derived from a logical derivation, based on axioms of a multimodal logic, and from a principle of sources independence, based on the maximization of the system entropy. This contribution implies a versatile framework for defining the combination rules and their implied intuition. Indeed, it is possible to derive many other logics for defining the notion of belief and their combination at a logical level.

In this contribution especially, we proposed a logic with strong but desirable properties, and we derived some conclusions about the feasibility of the related combinations. These properties are typically the conservation of belief during the combination, and the compliance with a probabilistic ground truth. It appeared that such a rule is theoretically constructible. But however, it appeared that as soon as the principle of independence is used, it introduces information which imply a bias which is conflicting in general with the ground truth. This information should be used at the final stage of the fusion, which is a strong constraint on the fusion process. We then questioned about the need to release the principle of belief conservation (as do most rules with conflict redistribution) or to abandon the compliance with a ground truth reference. We also suggested that these issues may be solved by using conditional belief functions more thoroughly instead of absolute belief functions.

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A Proofs of section 4

For the sake of simplicity, some proofs are done for $s = 1$ or $s = 2$. But these results are easily generalized to $s \geq 1$ sources.

Belief sharpening – property 13 – \triangleleft is an order relation.

Proof.

Proof of transitivity. Let r_{13} be defined according to (10).

Then, it is clear that $r_{13} \geq 0$.

It also comes:

$$\begin{aligned} \sum_{X_1 \in G^\ominus} r_{13}(X_1, X_3) &= \sum_{X_2 \in G^\ominus} \frac{r_{23}(X_2, X_3)}{m_2(X_2)} \sum_{X_1 \in G^\ominus} r_{12}(X_1, X_2) = \\ &= \sum_{X_2 \in G^\ominus} \frac{r_{23}(X_2, X_3)}{m_2(X_2)} m_2(X_2) = \sum_{X_2 \in G^\ominus} r_{23}(X_2, X_3) = m_3(X_3), \end{aligned}$$

and

$$\begin{aligned} \sum_{X_3 \in G^\ominus} r_{13}(X_1, X_3) &= \sum_{X_2 \in G^\ominus} \frac{r_{12}(X_1, X_2)}{m_2(X_2)} \sum_{X_3 \in G^\ominus} r_{23}(X_2, X_3) = \\ &= \sum_{X_2 \in G^\ominus} \frac{r_{12}(X_1, X_2)}{m_2(X_2)} m_2(X_2) = \sum_{X_2 \in G^\ominus} r_{12}(X_1, X_2) = m_1(X_1). \end{aligned}$$

Now, since $r_{12}(X_1, X_2) = 0$ for any $X_1, X_2 \in G^\ominus$ such that $X_2 \not\subseteq X_1$, and $r_{23}(X_2, X_3) = 0$ for any $X_2, X_3 \in G^\ominus$ such that $X_3 \not\subseteq X_2$, it comes that $r_{12}(X_1, X_2)r_{23}(X_2, X_3) = 0$ for any $X_1, X_2, X_3 \in G^\ominus$ such that $X_3 \not\subseteq X_1$. As a consequence:

$$r_{13}(X_1, X_3) = 0 \text{ for any } X_1, X_3 \in G^\ominus \text{ such that } X_3 \not\subseteq X_1,$$

which also results in:

$$\sum_{X_1 \in G^\ominus} r_{13}(X_1, X_3) = \sum_{X_1: X_3 \subseteq X_1} r_{13}(X_1, X_3),$$

and

$$\sum_{X_3 \in G^\ominus} r_{13}(X_1, X_3) = \sum_{X_3: X_3 \subseteq X_1} r_{13}(X_1, X_3).$$

Then, r_{13} is indeed an sharpening from Bel_1 to Bel_3 .

Proof of reflexivity.

The reflexivity itself is obvious, since r , defined by $r(X, X) = m(X)$ and $r(X, Y) = 0$ for any $X, Y \in G^\ominus$ such that $X \not\subseteq Y$, will check all requirements for a sharpening from Bel to itself.

Now, assume another sharpening r' from Bel to itself. Then, there are propositions $X \in G^\ominus$ such that $r'(X, X) < m(X)$. Let $X_M \in G^\ominus$ be a maximal proposition, in regards to the order \subseteq , such that $r'(X_M, X_M) < m(X_M)$. It comes from this definition:

$$r'(X_M, X_M) < m(X_M) \text{ and } r'(X, X_M) = 0,$$

for any $X \in G^\ominus \setminus \{X_M\}$ such that $X_M \subset X$. As a consequence:

$$\sum_{X \in G^\ominus} r'(X, X_M) < m(X_M),$$

which is a contradiction with the hypothesis.

Proof of antisymmetry.

It comes from the transitivity that:

$$r_{11}(X_1, Y_1) = \sum_{X_2 \in G^\ominus} \frac{r_{12}(X_1, X_2)r_{21}(X_2, Y_1)}{m_2(X_2)}$$

is a sharpening from Bel_1 to Bel_1 . From the unicity of this sharpening, it comes:

$$\sum_{X_2 \in G^\ominus} \frac{r_{12}(X_1, X_2)r_{21}(X_2, X_1)}{m_2(X_2)} = m_1(X_1), \quad (42)$$

and

$$\sum_{X_2 \in G^\Theta} \frac{r_{12}(X_1, X_2)r_{21}(X_2, Y_1)}{m_2(X_2)} = 0, \quad (43)$$

for any $X_1 \in G^\Theta$ and $Y_1 \in G^\Theta \setminus \{X_1\}$. Similarly, it is obtained:

$$\sum_{X_1 \in G^\Theta} \frac{r_{21}(X_2, X_1)r_{12}(X_1, X_2)}{m_1(X_1)} = m_2(X_2), \quad (44)$$

for any $X_2 \in G^\Theta$.

Since $r_{12}(X_1, X_2) = 0$ or $r_{21}(X_2, X_1) = 0$ when $X_1 \neq X_2$, it comes from equations (42) and (44):

$$\frac{r_{12}(X_1, X_1)r_{21}(X_1, X_1)}{m_2(X_1)} = m_1(X_1) \quad \text{for any } X_1 \in G^\Theta,$$

and

$$\frac{r_{21}(X_2, X_2)r_{12}(X_2, X_2)}{m_1(X_2)} = m_2(X_2) \quad \text{for any } X_2 \in G^\Theta.$$

As a corollary, and since $r_{ij} \leq m_i$ and $r_{ij} \leq m_j$, it comes:

$$m_2(X) \neq 0 \iff m_1(X) \neq 0 \iff r_{12}(X, X) \neq 0 \iff r_{21}(X, X) \neq 0.$$

Since $r_{12} \geq 0$ and $r_{21} \geq 0$, it comes from equation (43):

$$r_{12}(X_1, X_2) = 0 \quad \text{or} \quad r_{21}(X_2, Y_1) = 0,$$

for any $X_1, X_2 \in G^\Theta$ and $Y_1 \in G^\Theta \setminus \{X_1\}$. By applying this result to $X_2 = Y_1$ and $X_2 \neq X_1$, it is derived $r_{12}(X_1, X_2) = 0$ when $m_2(X_2) \neq 0$. Since anyway $0 \leq r_{12}(X_1, X_2) \leq m_2(X_2)$, it follows the result:

$$r_{12}(X_1, X_2) = 0 \quad \text{for any } X_1, X_2 \in G^\Theta \text{ such that } X_1 \neq X_2.$$

Similarly, it may be proved:

$$r_{21}(X_2, X_1) = 0 \quad \text{for any } X_1, X_2 \in G^\Theta \text{ such that } X_1 \neq X_2.$$

As a consequence:

$$\begin{aligned} r_{12}(X, X) &= \sum_{X_1 \in G^\Theta} r_{12}(X_1, X) = m_2(X) = \sum_{X_1 \in G^\Theta} r_{21}(X, X_1) \\ &= r_{21}(X, X) = \sum_{X_2 \in G^\Theta} r_{21}(X_2, X) = m_1(X). \end{aligned}$$

□

Belief sharpening – property 14 – $\text{Bel}_1 \triangleleft \text{Bel}_2$ implies $\text{Bel}_1 \leq \text{Bel}_2$.

Proof. Let r_{12} be a sharpening from Bel_1 to Bel_2 . Since:

$$\{(Y_1, Y_2) / Y_2 \subseteq Y_1 \subseteq X\} \subseteq \{(Y_1, Y_2) / Y_2 \subseteq Y_1 \text{ and } Y_2 \subseteq X\},$$

it comes:

$$\text{Bel}_1(X) = \sum_{Y_1: Y_1 \subseteq X} \sum_{Y_2: Y_2 \subseteq Y_1} r_{12}(Y_1, Y_2) \leq \sum_{Y_2: Y_2 \subseteq X} \sum_{Y_1: Y_2 \subseteq Y_1} r_{12}(Y_1, Y_2) = \text{Bel}_2(X).$$

□

Neutral element – property 18 – $m \oplus_{\text{H}} \nu$ exists, and $m \oplus_{\text{H}} \nu = m$.

Proof. Since $\sum_Y f_o(Y, Z) = \nu(Z)$ and $f_o \geq 0$, it is deduced $f_o(Y, Z) = 0$ unless $Z = \Omega$. Now, since $\sum_Z f_o(Y, Z) = m(Y)$, it is concluded:

$$f_o(Y, \Omega) = m(Y) \quad \text{and} \quad f_o(Y, Z) = 0 \quad \text{for } Z \neq \Omega.$$

Since this function obviously satisfies the constraints, the existence of $m \oplus_{\text{H}} \nu$ is implied. Then the result $m \oplus_{\text{H}} \nu = m$ is immediate. □

Conservation of belief – property 19 – $\text{Bel}_1 \triangleleft \text{Bel}_1 \oplus_{\text{H}} \text{Bel}_2$.

Proof. By definition, $m_1 \oplus_{\text{H}} m_2(X) = \sum_{V,W:V \cap W=X} f_o(V,W)$, where f_o is an optimal solution to (9). As a consequence, $m_1 \oplus_{\text{H}} m_2(X) = \sum_V r(V,X)$, where:

$$r(V,X) = \sum_{W:V \cap W=X} f_o(V,W).$$

Now r verifies $r(V,X) = 0$ for $X \not\subseteq V$ and, since $\sum_W f_o(V,W) = m_1(V)$, it comes:

$$\sum_X r(V,X) = \sum_{X,W:V \cap W=X} f_o(V,W) = \sum_W f_o(V,W) = m_1(V).$$

□

Corollaries are obvious.

Probabilistic bound – property 23 –

Proof. The existence of $\text{Bel}_1 \oplus_{\text{H}} \dots \oplus_{\text{H}} \text{Bel}_s \oplus_{\text{H}} P$ is a consequence of the combination of dominated belief function, since, indeed, $P \triangleleft P$.

From the conservation of belief, it happens:

$$P \triangleleft \text{Bel}_1 \oplus_{\text{H}} \dots \oplus_{\text{H}} \text{Bel}_s \oplus_{\text{H}} P.$$

Since P is a probability measure, it is minimal and the property is implied.

□

Idempotence – property 26 – m is probabilistic $\Rightarrow m \oplus_{\text{H}} \dots \oplus_{\text{H}} m = m$.

Proof. Since m is probabilistic, there are $X_{1:n} \in G^{\Theta} \setminus \{\emptyset\}$ such that:

$$\sum_{j=1}^n m(X_j) = 1 \quad \text{and} \quad \begin{cases} i \neq j \Rightarrow X_i \cap X_j = \emptyset, \\ \bigcup_{j=1}^n X_j = \Omega. \end{cases}$$

As a consequence, the only function f , which fulfills the constraint of (9), is defined by:

$$f(X, \dots, X) = m(X) \quad \text{for any } X \in G^{\Theta}.$$

Of this, the property is an immediate consequence.

□

B Proofs of section 8 and 9

Bla exclusivity – property 44 –

Proof. It is noticed that $E \cap F \subsetneq E$ or $E \cap F \subsetneq F$ when $E \neq F$. Then, the proof is achieved by deriving:

$$(J)\mathbf{v}E \wedge (J)\mathbf{v}F \equiv [J]\mathbf{v}E \wedge [J]\mathbf{v}F \wedge \left(\bigwedge_{G \subsetneq E} \neg [J]\mathbf{v}G \right) \wedge \left(\bigwedge_{H \subsetneq F} \neg [J]\mathbf{v}H \right),$$

and combining it with $[J]\mathbf{v}E \wedge [J]\mathbf{v}F \equiv [J](\mathbf{v}E \wedge \mathbf{v}F) \equiv [J]\mathbf{v}(E \cap F)$.

□

Bla exhaustivity – property 45 –

Proof. The proof is recursive.

It is first noticed that $[J]\mathbf{v}\emptyset = (J)\mathbf{v}\emptyset$.

Now, let $E \subseteq \mathbf{P}$ and assume $\bigvee_{G \subsetneq F} (J)\mathbf{v}G = [J]\mathbf{v}F$ for any $F \subsetneq E$.

Then:

$$\bigvee_{F \subsetneq E} (J)\mathbf{v}F \equiv (J)\mathbf{v}E \vee \left(\bigvee_{F \subsetneq E} \bigvee_{G \subsetneq F} (J)\mathbf{v}G \right) \equiv (J)\mathbf{v}E \vee \left(\bigvee_{F \subsetneq E} [J]\mathbf{v}F \right).$$

It follows $\bigvee_{F \subsetneq E} (J)\mathbf{v}F \equiv [J]\mathbf{v}E \vee \left(\bigvee_{F \subsetneq E} [J]\mathbf{v}F \right)$, from the definition of $(J)\mathbf{v}E$.

Since the modality is non-decreasing, it is deduced $\bigvee_{F \subsetneq E} (J)\mathbf{v}F \equiv [J]\mathbf{v}E$.

□

Computing the combination – property 49 and lemma 51 –

Proof of lemma. From (23), (20) and axiom *Ind*, applied on $J \cup K$, it comes:

$$\bigvee_{F \cap G \subseteq E} ([J] \mathbf{v}F \wedge [K] \mathbf{v}G) \equiv [J \cup K] \mathbf{v}E .$$

Then, by $[J] \mathbf{v}F \equiv \bigvee_{F_2 \subseteq F} (J) \mathbf{v}F_2$ and $[K] \mathbf{v}G \equiv \bigvee_{G_2 \subseteq G} (K) \mathbf{v}G_2$, it comes:

$$[J \cup K] E \equiv \bigvee_{F \cap G \subseteq E} \bigvee_{F_2 \subseteq F} \bigvee_{G_2 \subseteq G} ((J) \mathbf{v}F_2 \wedge (K) \mathbf{v}G_2) \equiv \bigvee_{F_2 \cap G_2 \subseteq E} ((J) \mathbf{v}F_2 \wedge (K) \mathbf{v}G_2) .$$

□

Main proof. From the definition of the bla and lemma, it is deduced:

$$(J \cup K) E \equiv \bigvee_{F \cap G \subseteq E} ((J) \mathbf{v}F \wedge (K) \mathbf{v}G) \wedge \neg \bigvee_{F \cap G \subsetneq E} ((J) \mathbf{v}F \wedge (K) \mathbf{v}G) .$$

The propositions $(J) \mathbf{v}F \wedge (K) \mathbf{v}G$ constituting a partition, it comes:

$$(J \cup K) E \equiv \bigvee_{F \cap G = E} ((J) \mathbf{v}F \wedge (K) \mathbf{v}G) .$$

□

Axioms verification – property 54 –

Proof sketch. *Partition*, *K*, *N* and *Inc* are almost immediate from the definition. Axiom *K* is derived from the definition of $[J] \dots$ and from the property:

$$\overline{\bigcap_{j \in J} E_j} \subseteq (\mathbb{I} \setminus \underline{X}) \cup \underline{Y} \text{ and } \overline{\bigcap_{j \in J} E_j} \subseteq \underline{X} \text{ imply } \overline{\bigcap_{j \in J} E_j} \subseteq \underline{X} \cap \underline{Y} \subseteq \underline{Y} .$$

Axiom *Ind* follows from the following derivation:

$$\begin{aligned} \overline{[J \cup K] \mathbf{v}E} &= \bigcup_{p_0 \in \mathbf{P}} \bigcup_{E_{I \setminus (J \cup K)}} \bigcup_{E_{J \cup K} : \overline{\bigcap_{j \in J \cup K} E_j} \subseteq \overline{E}} \phi(p_0, E_{1:n}) \\ &= \bigcup_{p_0 \in \mathbf{P}} \bigcup_{E_{I \setminus (J \cup K)}} \bigcup_{F \cap G \subseteq E} \bigcup_{E_{J \cup K} : \begin{cases} \overline{\bigcap_{j \in J} E_j} \subseteq \overline{F} \\ \overline{\bigcap_{j \in K} E_j} \subseteq \overline{G} \end{cases}} \phi(p_0, E_{1:n}) . \end{aligned}$$

At last, axiom *T* is derived from the following derivation based on hypothesis (35):

$$\begin{aligned} \overline{[J] \mathbf{v}E} &= \bigcup_{p_0 \in \mathbf{P}} \bigcup_{E_{I \setminus J}} \bigcup_{E_J : \overline{\bigcap_{j \in J} E_j} \subseteq \overline{E}} \phi(p_0, E_{1:n}) = \bigcup_{p_0 \in E} \bigcup_{E_{I \setminus J}} \bigcup_{E_J : \overline{\bigcap_{j \in J} E_j} \subseteq \overline{E}} \phi(p_0, E_{1:n}) \\ &\subseteq \bigcup_{p_0 \in E} \bigcup_{E_I} \phi(p_0, E_{1:n}) = \underline{\mathbf{v}E} , \end{aligned}$$

and from the property $\overline{[J] X} = \overline{[J] \mathbf{v}\pi(X)}$, where $\pi(X) = \bigvee_{E \subseteq \mathbf{P} : \overline{E} \subseteq \underline{X}} E$.

□