CHEN/RUAN ORBIFOLD COHOMOLOGY OF THE BIANCHI GROUPS

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Abstract. We give formulae for the Chen / Ruan orbifold cohomology for the orbifolds given by a Bianchi group acting on a model for its classifying space for proper actions: complex hyperbolic space.
The Bianchi groups are the arithmetic groups $\mathrm{PSL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in an imaginary quadratic number field. The underlying real orbifolds which help us in our study, have applications in Physics [2].

Contents

1. Introduction

Denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by $\mathcal{O}_{-m}$ its ring of integers. The *Bianchi groups* are the groups $(\mathrm{P})\mathrm{SL}_2(\mathcal{O}_{-m})$. The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian* groups, which date back to work of Henri Poincaré [15]. In fact, each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [12]. A wealth of information on the Bianchi groups can be found in the monographs [6, 8, 12]. These groups act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space associated to them. The kernel of this action is the centre $\{\pm 1\}$ of the groups. Thus it is useful to study the quotient of $\mathrm{SL}_2(\mathcal{O}_{-m})$ by its centre, namely $\mathrm{PSL}_2(\mathcal{O}_{-m})$. In 1892, Luigi Bianchi [3] computed fundamental domains for some of these groups. Such a fundamental domain has the shape of a hyperbolic polyhedron (up to a missing vertex at certain cusps, which represent the ideal classes of $\mathcal{O}_{-m}$), so we will call it the *Bianchi fundamental polyhedron*.

The orbifold structure obtained by our group action is determined by the Bianchi fundamental polyhedron and its stabilisers and identifications. The computation of this information has been implemented for all Bianchi groups [17] in the language Pari/GP [1].

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We complexify our orbifolds by complexifying the real hyperbolic three-space. We obtain orbifolds given by the induced action of the Bianchi groups on complex hyperbolic three-space. Then we compute the Chen/Ruan Orbifold Cohomology for these complex orbifolds. We can determine its product structure with theorem 5.

As a result of theorems 23 and 24, we can express the vector space structure of the orbifold cohomology of any Bianchi group $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$ with units $\{\pm 1\}$ in terms of Krämer's numbers λ_4, λ_4^*, λ_6 and λ_6^* of conjugacy classes of certain finite subgroups of Γ (see section 5 for details) and the first Betti number β_1 of the quotient space $\mathcal{H}/\Gamma$. The precise expression is

$$\mathrm{H}^d_{orb}\left(\mathcal{H}//\mathrm{PSL}_2(\mathcal{O}_{-m})\right) \cong \begin{cases} \mathbb{Q}, & d = 0 \\ \mathbb{Q}^{\beta_1}, & d = 1 \\ \mathbb{Q}^{\beta_1 - 1 + \lambda_4 + 2\lambda_6 - \lambda_6^*}, & d = 2 \\ \mathbb{Q}^{\lambda_4 - \lambda_4^* + 2\lambda_6 - \lambda_6^*}, & d = 3 \\ 0 & \text{otherwise.} \end{cases}$$

The values which the Krämer numbers take, are given for a range of Bianchi groups in [18], and the Betti number has been computed in [21] and [10].

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2. The vector space structure of Chen/Ruan Orbifold Cohomology

Let Γ be a discrete group acting *properly*, i.e. with finite stabilisers, by diffeomorphisms on a manifold Y. For any element $g \in \Gamma$, denote by $C_\Gamma(g)$ the centraliser of g in Γ. Denote by Y^g the subset of Y consisting of the fixed points of g.

Definition 1. *Let $T \subset \Gamma$ be a set of representatives of the conjugacy classes of elements of finite order in Γ. Then we set*

$$\mathrm{H}^*_{orb}(Y//\Gamma) := \bigoplus_{g \in T} \mathrm{H}^*\left(Y^g / C_\Gamma(g); \mathbb{Q}\right).$$

It can be checked that this definition gives the vector space structure of the orbifold cohomology defined by Chen and Ruan [5], if we forget the grading of the latter. We can verify this analogously to the case where Γ is a finite group, treated by Fantechi and Göttsche [7]. The additional argument needed when considering some element g in Γ of infinite order, is the following. As the action of Γ on Y is proper, g does not admit any fixed point in Y. Thus,

$$\mathrm{H}^*\left(Y^g / C_\Gamma(g); \mathbb{Q}\right) = \mathrm{H}^*\left(\emptyset; \mathbb{Q}\right) = 0.$$

3. The orbifold cohomology product

In order to equip the orbifold cohomology vector space with the Chen/Ruan product structure, we need an almost complex orbifold structure on $Y//\Gamma$.
Let Y be a complex manifold of dimension D with a proper action of a discrete group Γ by diffeomorphisms, the differentials of which are holomorphic. For any $g \in \Gamma$ and $y \in Y^g$, we consider the eigenvalues $\lambda_1, \ldots, \lambda_D$ of the action of g on the tangent space $\mathrm{T}_y Y$. As the action of g on $\mathrm{T}_y Y$ is complex linear, its eigenvalues are roots of unity.

Definition 2. *Write $\lambda_j = e^{2\pi i r_j}$, where r_j is a rational number in the interval* $[0, 1[$. *The* degree shifting number *of g in y is the rational number* $\mathrm{shift}(g, y) := \sum_{j=1}^{D} r_j$.

We see in [7] that the degree shifting number agrees with the one defined by Chen and Ruan. It is also called the fermionic shift number in [22]. The degree shifting number of an element g is constant on a connected component of its fixed point set Y^g. For the groups under our consideration, Y^g is connected, so we can omit the argument y. Details for this and the explicit value of the degree shifting number are given in lemma 4. Then we can define the graded vector space structure of the orbifold cohomology as

$$\mathrm{H}^d_{orb}(Y//\Gamma) := \bigoplus_{g \in T} \mathrm{H}^{d-2\,\mathrm{shift}(g)}\left(Y^g/C_\Gamma(g); \mathbb{Q}\right). \tag{1}$$

Denote by g, h two elements of finite order in Γ, and by $Y^{g,h}$ their common fixed point set. Chen and Ruan construct a certain vector bundle on $Y^{g,h}$ we call the *obstruction bundle*. We denote by $c(g,h)$ its top Chern class. In our cases, $Y^{g,h}$ is a connected manifold. In the general case, the fibre dimension of the obstruction bundle can vary between the connected components of $Y^{g,h}$, and $c(g,h)$ is the cohomology class restricting to the top Chern class of the obstruction bundle on each connected component. The obstruction bundle is at the heart of the construction [5] of the Chen/Ruan orbifold cohomology product. In [7], this product, when applied to a cohomology class associated to Y^g and one associated to Y^h, is described as a push-forward of the cup product of these classes restricted to $Y^{g,h}$ and multiplied by $c(g,h)$. The following statement is made for global quotient orbifolds, but it is a local property, so we can apply it in our proper actions case.

Lemma 3 (Fantechi/Göttsche)**.** *Let $Y^{g,h}$ be connected. Then the obstruction bundle on it is a vector bundle of fibre dimension*

$$\mathrm{shift}(g) + \mathrm{shift}(h) - \mathrm{shift}(gh) - \mathrm{codim}_{\mathbb{C}}\left(Y^{g,h} \subset Y^{gh}\right).$$

In [7], a proof is given in the more general setting that $Y^{g,h}$ needs not be connected.
Examples where the product structure is worked out in the non-global quotient case, are for instance given in [5, 5.3] and [4].

3.1. **Groups of hyperbolic motions.** A class of examples with complex structures admitting the grading (1) is given by the discrete subgroups Γ of the orientation preserving isometry group $\mathrm{PSL}_2(\mathbb{C})$ of real hyperbolic 3-space $\mathcal{H}^3_{\mathbb{R}}$. The Lobachevski model of $\mathcal{H}^3_{\mathbb{R}}$ gives a natural identification of the orientation preserving isometries of $\mathcal{H}^3_{\mathbb{R}}$ with matrices in $\mathrm{PSO}(3,1)$. By the subgroup inclusion $\mathrm{PSO}(3,1) \hookrightarrow \mathrm{PSU}(3,1)$, these matrices specify isometries of the complex hyperbolic space $\mathcal{H}^3_{\mathbb{C}}$.

Lemma 4. *The degree shifting number of any rotation of $\mathcal{H}^3_{\mathbb{C}}$ on its fixed points set is 1.*

Proof. For any rotation $\hat{\theta}$ of angle θ around a geodesic line in $\mathcal{H}^3_{\mathbb{R}}$, there is a basis for the construction of the Lobachevski model such that the matrix of $\hat{\theta}$ takes the shape

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{PSO}(3,1).$$

This matrix, considered as an element of $\mathrm{PSU}(3,1)$, performs a rotation of angle θ around the "complexified geodesic line" with respect to the inclusion $\mathcal{H}^3_{\mathbb{R}} \hookrightarrow \mathcal{H}^3_{\mathbb{C}}$. The fixed points of this rotation are exactly the points p lying on this complexified geodesic line, and the action on their tangent space $\mathrm{T}_p\mathcal{H}^3_{\mathbb{C}} \cong \mathbb{C}^3$ is again a rotation of angle θ. Hence we can choose a basis of this tangent space such that this rotation is expressed by the matrix

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{C}).$$

Therefore the degree shifting number of the rotation $\hat{\theta}$ at p is 1. □

Theorem 5. *Let Γ be a group generated by translations and rotations of $\mathcal{H}^3_{\mathbb{C}}$. Then all obstruction bundles of the orbifold $\mathcal{H}^3_{\mathbb{C}}//\Gamma$ are of fibre dimension zero.*

Proof. Non-trivial obstruction bundles can only appear for two elements of Γ with common fixed points, and such that one of these is not a power of the other one. The translations of $\mathcal{H}^3_{\mathbb{C}}$ have their fixed point on the boundary and not in $\mathcal{H}^3_{\mathbb{C}}$. So let b and c be non-trivial hyperbolic rotations around distinct axes intersecting in the point $p \in \mathcal{H}^3_{\mathbb{C}}$. Then bc is again a hyperbolic rotation around a third distinct axis passing through p. Obviously, these rotation axes constitute the fixed point sets Y^b, Y^c and Y^{bc}. Hence the only fixed point of the group generated by b and c is p. Now lemma 3 yields the following fibre dimension for the obstruction bundle on $Y^{b,c}$:

$$\mathrm{shift}(b) + \mathrm{shift}(c) - \mathrm{shift}(bc) - \mathrm{codim}_{\mathbb{C}}\left(Y^{b,c} \subset Y^{bc}\right).$$

After computing degree shifting numbers using lemma 4, we see that this fibre dimension is zero. □

Hence the obstruction bundle is trivial, and its top Chern class is the neutral element of the cohomological cup product. By Fantechi/Göttsche's description, the Chen/Ruan orbifold cohomology product is then a push-forward of the cup product of the cohomology classes restricted to the intersection of the fixed points sets.

4. The conjugacy classes of finite order elements in the Bianchi groups

Let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$ be a Bianchi group. Then any element of Γ fixing a point inside real hyperbolic 3-space $\mathcal{H}^3_{\mathbb{R}}$ acts as a rotation of finite order. Hence the induced action of Γ on complex hyperbolic 3-space $\mathcal{H}^3_{\mathbb{C}}$ is proper. For the remainder of this article, we will reduce all our considerations to the action on real hyperbolic 3-space $\mathcal{H}^3_{\mathbb{R}}$.

Let Z be the refined cellular complex obtained from the action of Γ on hyperbolic 3-space as described in [16], namely we subdivide $\mathcal{H}^3_{\mathbb{R}}$ until the stabiliser in Γ of any cell σ fixes σ pointwise. We achieve this by computing Bianchi's fundamental polyhedron for the action of Γ, taking as preliminary set of 2-cells its facets lying on the Euclidean hemispheres and vertical planes of the upper-half space model for $\mathcal{H}^3_{\mathbb{R}}$, and then subdividing along the rotation axes of the elements of Γ. Let ℓ be a prime number.

Definition 6. *The* ℓ-torsion subcomplex *is the subcomplex of* $_\Gamma\backslash Z$ *consisting of all the cells, the pre-images of which have stabilisers in* Γ *containing elements of order* ℓ.

For ℓ being one of the two occurring primes 2 and 3, this subcomplex is a finite graph, because the cells of dimension greater than 1 are trivially stabilised in the refined cellular complex. We reduce this subcomplex with the procedure of [16], which consists in taking the pairs of edges with a common endpoint such that no further edge is adjacent to this endpoint, and replacing them together with this endpoint by a single edge.

We make the following group-theoretic construction in order to build a bridge between the ℓ–torsion subcomplexes and the Krämer numbers of section 5. For a circle to become a graph, we identify the two endpoints of a single edge.

Definition 7. *The* ℓ–conjugacy classes graph *of an arbitrary group* Γ *is given by the following construction.*

- We take as vertices the conjugacy classes of finite subgroups G of Γ containing elements γ of order ℓ such that the normaliser of $\langle\gamma\rangle$ in G is not $\langle\gamma\rangle$ itself.
- We connect two vertices by an edge if and only if they admit representatives sharing a common subgroup of order ℓ.
- For every pair of subgroups of order ℓ in G, which are conjugate in Γ but not in G, we draw a circle attached to the vertex labelled by G.
- For every conjugacy class of subgroups of order ℓ which are not properly contained in any finite subgroup of Γ, we add a disjoint circle.

Except for the Gaußian and Eisenstein integers, which have to be treated separately, all the rings of integers of imaginary quadratic number fields admit as only units $\{\pm 1\}$. In the latter case, we call $\mathrm{PSL}_2(\mathcal{O})$ a Bianchi group with units $\{\pm 1\}$.

Theorem 8 ([18]). *Let* Γ *be any Bianchi group with units* $\{\pm 1\}$ *and* ℓ *any prime number. Then the* ℓ–conjugacy classes graph *and the reduced* ℓ–torsion subcomplex *of* Γ *are isomorphic graphs.*

We recall the lemmata which have been used for the proof of this theorem. The first ingredient is the following classification of Felix Klein [9].

Lemma 9 (Klein). *The finite subgroups in* $\mathrm{PSL}_2(\mathcal{O})$ *are exclusively of isomorphism types the cyclic groups of orders one, two and three, the Klein four-group* $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$*, the symmetric group* $\mathcal{S}_3$ *and the alternating group* $\mathcal{A}_4$.

Lemma 10 ([16])**.** *Let v be a non-singular vertex in the refined cell complex. Then the number* $\mathbf{n}$ *of orbits of edges in the refined cell complex adjacent to v, with stabiliser in* $\mathrm{PSL}_2(\mathcal{O}_{-m})$ *isomorphic to $\mathbb{Z}/\ell$, is given as follows for $\ell = 2$ and $\ell = 3$.*

Isomorphism type of the vertex stabiliser	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
$\mathbf{n}$ for $\ell = 2$	0	2	0	3	2	1
$\mathbf{n}$ for $\ell = 3$	0	0	2	0	1	2.

Now we investigate the associated normaliser groups. Straight-forward verification using the multiplication tables of the implied finite groups yields the following.

Lemma 11. *Let G be a finite subgroup of* $\mathrm{PSL}_2(\mathcal{O}_{-m})$*. Then the type of the normaliser of any subgroup of type $\mathbb{Z}/\ell$ in G is given as follows for $\ell = 2$ and $\ell = 3$, where we print only cases with existing subgroup of type $\mathbb{Z}/\ell$.*

Isomorphism type of G	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
normaliser of $\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathcal{D}_2$	$\mathbb{Z}/2$	$\mathcal{D}_2$
normaliser of $\mathbb{Z}/3$			$\mathbb{Z}/3$		$\mathcal{S}_3$	$\mathbb{Z}/3$.

The final ingredient in the proof of theorem 8 is the following.

Theorem 12 ([18])**.** *There is a natural bijection between conjugacy classes of subgroups of* $\mathrm{PSL}_2(\mathcal{O}_{-m})$ *of order ℓ and edges of the reduced ℓ–torsion subcomplex. It is given by considering the stabiliser of a representative edge in the refined cell complex.*

In order to prove the latter theorem, we need several lemmata, and we recall them now.

Lemma 13 ([18])**.** *Consider two adjacent edges E, E' of the non-reduced torsion subcomplex. Then for any representative e of E, there is an adjacent representative e' of E' on the same geodesic line as e.*

Corollary 14 ([18])**.** *Any edge of the reduced torsion subcomplex can be represented by a chain of edges on the intersection of one geodesic line with a strict fundamental domain for Γ in $\mathcal{H}$.*

Corollary 15. *Any edge of the reduced torsion subcomplex admits only representatives with stabiliser in the same conjugacy class.*

Lemma 16 ([18])**.** *Let α and γ be elements of* $\mathrm{PSL}_2(\mathbb{C})$*. Then the fixed point set in $\mathcal{H}$ of α is identified by γ with the fixed point set of $\gamma\alpha\gamma^{-1}$.*

Lemma 17 ([18])**.** *Let $v \in \mathcal{H}^3_{\mathbb{R}}$ be a vertex with stabiliser in Γ of type $\mathcal{D}_2$ or $\mathcal{A}_4$. Let γ in Γ be a rotation of order 2 around an edge e adjacent to v. Then the centraliser $C_\Gamma(\gamma)$ reflects $\mathcal{H}^\gamma$ — which is the geodesic line through e — onto itself at v.*

Let α be any torsion element in Γ. We construct a *chain of edges for α* as follows. Consider the edge of the reduced torsion subcomplex to which the edge stabilised by α belongs. Use corollary 14 to represent it by a connected chain of edges on a geodesic line. Now, α is conjugate to an element $\gamma\alpha\gamma^{-1}$ of the stabiliser of one of the edges in the chain. By lemma 16, the element $\gamma^{-1} \in \Gamma$ maps the mentioned geodesic line to the rotation axis of α. The image under γ^{-1} of the chain of edges under consideration is the desired chain for α. So the chain of edges for α exists and is unique up to translation on the rotation axis of α.

Lemma 18 ([18])**.** *Let α be any 2–torsion element in Γ. Then the chain of edges for α is a fundamental domain for the centraliser of α on the rotation axis of α.*

Lemma 19 ([18])**.** *Let α be any non-trivial torsion element in a Bianchi group Γ. Then the Γ–image of the chain of edges for α contains the rotation axis of α.*

This completes the tools for proving theorem 12 and hence theorem 8.

Furthermore, the following easy-to-check statement will be useful for our orbifold cohomology computations.

Lemma 20. *There is only one conjugacy class of elements of order 2 in $\mathcal{S}_3$ as well as in $\mathcal{A}_4$. In $\mathcal{S}_3$, there is also only one conjugacy class of elements of order 3, whilst in $\mathcal{A}_4$ there is an element γ such that γ and γ^2 represent the two conjugacy classes of elements of order 3.*

Proof. In cycle type notation, we can explicitly establish the multiplication tables of $\mathcal{S}_3$ and $\mathcal{A}_4$, and compute the conjugacy classes. ☐

Corollary 21 (Corollary to lemma 20)**.** *Let γ be an element of order 3 in a Bianchi group Γ with units $\{\pm 1\}$. Then, γ is conjugate in Γ to its square γ^2 if and only if there exists a group $G \cong \mathcal{S}_3$ with $\langle\gamma\rangle \subsetneq G \subsetneq \Gamma$.*

5. The Krämer numbers and orbifold cohomology of the Bianchi groups

Krämer [11] has determined number-theoretic formulae for the numbers of conjugacy classes of finite subgroups in the Bianchi groups. These formulae apply to the following types of subgroups, where the symbols in the first row are Krämer's notations for the number of their conjugacy classes:

μ_2	μ_T	μ_3	λ_{2n}	$\lambda_4^T = l_4^T$	λ_4^*	λ_6^*	μ_2^-
$\mathcal{D}_2$	$\mathcal{A}_4$	$\mathcal{S}_3$	$\mathbb{Z}/n$	$\mathbb{Z}/2 \hookrightarrow \mathcal{A}_4$	$\mathbb{Z}/2 \hookrightarrow \mathcal{D}_2$	$\mathbb{Z}/3 \hookrightarrow \mathcal{S}_3$	$\mathcal{D}_2 \not\subseteq \mathcal{A}_4$

Here, the inclusion arrows mean that we only consider copies of $\mathbb{Z}/n$ admitting the specified inclusion in the given Bianchi group and $\mathcal{D}_2 \not\subseteq \mathcal{A}_4$ means that we only consider copies of $\mathcal{D}_2$ not admitting any inclusion into a subgroup of type $\mathcal{A}_4$ of the Bianchi group. The values given by Krämer's formulae are matching with the values computed with [17].

***Observation* 22.** The Krämer numbers determine the 3-conjugacy classes graph and hence the reduced 3–torsion subcomplex for all Bianchi groups with units $\{\pm 1\}$, as we can see immediately from the description of the reduced 3–torsion subcomplex in [16].

Our main results on the vector space structure of the Chen/Ruan orbifold cohomology of the Bianchi groups are the following.

Theorem 23. *For any element* γ *of order* 3 *in a Bianchi group* Γ *with units* $\{\pm 1\}$*, the quotient space* $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ *of the rotation axis modulo the centraliser of* γ *is homeomorphic to a circle.*

Proof. By lemma 19, the Γ–image of the chain of edges for γ contains the rotation axis $\mathcal{H}^\gamma$. Now we can observe two cases.

- ○ First, assume that the rotation axis of γ does not contain any vertex of stabiliser type $\mathcal{S}_3$. Then in the 3-conjugacy classes graph, the class of $\langle\gamma\rangle$ is represented by a disjoint circle. And by theorem 8 we know that this gives us a circle in the 3-torsion subcomplex. There can be no reflection of $\mathcal{H}^\gamma$ onto itself by an element of Γ, because such a reflection would fix a point on $\mathcal{H}^\gamma$ and so this point would have a stabiliser of type $\mathcal{S}_3$ (the normaliser of $\langle\gamma\rangle$ in this stabiliser would contain the reflection, which we can exclude by lemma 11 for the other types). As Γ acts by CAT(0) isometries, every element $g \in \Gamma$ sending an edge of the chain for γ to an edge on $\mathcal{H}^\gamma$ outside the fundamental domain, must perform a translation on $\mathcal{H}^\gamma$. A translation along the rotation axis of γ commutes with γ, so $g \in C_\Gamma(\gamma)$. Hence the quotient space $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ is homeomorphic to a circle.
- •–• If $\mathcal{H}^\gamma$ contains a point with stabiliser in Γ of type $\mathcal{S}_3$, then there are exactly two Γ-orbits of such points. The elements of order 2 do not commute with the elements of order 3 in $\mathcal{S}_3$, so the centraliser of γ does not contain the former ones. Hence, $C_\Gamma(\gamma)$ does not contain any reflection of $\mathcal{H}^\gamma$ onto itself. Denote by α and β elements of order 2 of each of the stabilisers of the two endpoints of a chain of edges for γ. Then $\alpha\beta$ performs a translation on $\mathcal{H}^\gamma$ and hence commutes with γ. A fundamental domain for the action of $\langle\alpha\beta\rangle$ on $\mathcal{H}^\gamma$ is given by the chain of edges for γ united with its reflection through one of its endpoints. As no such reflection belongs to the centraliser of γ and the latter endpoint is the only one on its Γ-orbit in this fundamental domain, the quotient $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ matches with the quotient $\mathcal{H}^\gamma/_{\langle\alpha\beta\rangle}$, which is homeomorphic to a circle.

□

Theorem 24. *Let* γ *be an element of order* 2 *in a Bianchi group* Γ *with units* $\{\pm 1\}$*. Then, the homeomorphism type of the quotient space* $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ *is*

- •–• an edge without identifications, if there exists a finite group G such that $\langle\gamma\rangle \subsetneq G \subsetneq \Gamma$ and
- ○ a circle, otherwise.

Proof. By lemma 18, the chain of edges for γ is a fundamental domain for $C_\Gamma(\gamma)$ on $\mathcal{H}^\gamma$. Again, we have two cases.

- •–• If there exists a finite group G such that $\langle\gamma\rangle \subsetneq G \subsetneq \Gamma$, then any chain of edges for γ admits endpoints of stabiliser types $\mathcal{D}_2$ or $\mathcal{A}_4$. As $\mathcal{D}_2$ is an Abelian group and the reflections in $\mathcal{A}_4$ are contained in the normal subgroup $\mathcal{D}_2$, the reflections in these endpoint stabilisers commute with γ, so the quotient space $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ can be identified with a chain of edges for γ. By corollary 14, this chain of edges for γ represents a reduced edge in the 3-torsion subcomplex with distinct endpoints, so especially there is no identification on this chain by Γ. So, the homeomorphism type of $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ is an edge without identifications.
- ○ The other case is analogous to the first case of the proof of theorem 23, the rôle of $\mathcal{S}_3$ being played by $\mathcal{D}_2$ and $\mathcal{A}_4$.

□

As a result of theorems 23 and 24, the orbifold cohomology of any Bianchi group $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$ with units $\{\pm 1\}$ is (abbreviating conjugacy class as "c.c.") given as

$$\mathrm{H}^d_{orb}(\mathcal{H}//\Gamma) \cong \mathrm{H}^d\left(\mathcal{H}/\Gamma;\mathbb{Q}\right) \bigoplus_{g\,\text{c.c. of order 2}} \mathrm{H}^{d-2}\left(\mathcal{H}^g/C_\Gamma(g);\mathbb{Q}\right) \bigoplus_{\text{c.c.}'\text{s of order 3}} \mathrm{H}^{d-2}(\bigcirc\!\!\!\!\bullet\,;\mathbb{Q}).$$

With the above Krämer numbers and β_1 the first Betti number of the quotient space $\mathcal{H}/\Gamma$, we can state the above term more explicitly as

$$\mathrm{H}^d_{orb}\left(\mathcal{H}//\mathrm{PSL}_2(\mathcal{O}_{-m})\right) \cong \begin{cases} \mathbb{Q}, & d=0 \\ \mathbb{Q}^{\beta_1}, & d=1 \\ \mathbb{Q}^{\beta_1-1+\lambda_4+2\lambda_6-\lambda_6^*}, & d=2 \\ \mathbb{Q}^{\lambda_4-\lambda_4^*+2\lambda_6-\lambda_6^*}, & d=3 \\ 0 & \text{otherwise,} \end{cases}$$

where we use that by corollary 21, there are $2\lambda_6 - \lambda_6^*$ conjugacy classes of elements of order 3. The values which the Krämer numbers take, are given for a range of Bianchi groups in [18], and the Betti number has been computed in [21] and [10].

As we can calculate the Bredon homology $\mathrm{H}_0^{\mathfrak{Fin}}(G; R_\mathbb{C})$ of the Bianchi groups, the following lemma provides a check on our computations.

Lemma 25 (Mislin [13])**.** *Let G be an arbitrary group and write* $\mathrm{FC}(G)$ *for the set of conjugacy classes of elements of finite order in G. Then there is an isomorphism*

$$\mathrm{H}_0^{\mathfrak{Fin}}(G; R_\mathbb{C}) \otimes_\mathbb{Z} \mathbb{C} \cong \mathbb{C}[\mathrm{FC}(G)].$$

6. SAMPLE ORBIFOLD COHOMOLOGY COMPUTATIONS FOR THE BIANCHI GROUPS

We will carry out our computations in the upper-half space model

$$\{x + iy + rj \in \mathbb{C} \oplus \mathbb{R}j \mid r > 0\}$$

for $\mathcal{H}^3_\mathbb{R}$. Details on how to compute Chen/Ruan orbifold cohomology can be found in [14].

The case $\Gamma = \mathbf{PSL}_2(\mathbb{Z}[\sqrt{-2}\,])$. Let $\omega := \sqrt{-2}$. A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\omega])$ in real hyperbolic 3-space has been found by Luigi Bianchi [3]. We can obtain it by taking the geodesic convex envelope of of its lower boundary (half of which is depicted in figure 1) and the vertex ∞, and then removing the vertex ∞, making it noncompact. The other half of the lower boundary consists of one isometric Γ-image of each of the depicted 2-cells (in fact, the depicted 2-cells are a fundamental domain for a Γ-equivariant retract of $\mathcal{H}$, which is described in [19]). The coordinates of the vertices of figure 1 in the upper-half space model are $(1) = j$, $(1)' = \omega + j$, $(2) = \frac{1}{2}\omega + \sqrt{\frac{1}{2}}j$, $(7) = \frac{1}{2} + \sqrt{\frac{3}{4}}j$, $(7)' = \frac{1}{2} + \omega + \sqrt{\frac{3}{4}}j$, $(8) = \frac{1}{2} + \frac{1}{2}\omega + \sqrt{\frac{1}{4}}j$.

The 2-torsion subcomplex (- -) and the 3-torsion subcomplex (•••••) are coloured in the figure. The set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id},\, \alpha,\, \gamma,\, \beta,\, \beta^2\},$$

with $\alpha = \pm\begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}$, $\beta = \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so α and γ are of order 2, and β is of order 3. Using lemma 25 and with the help of our Bredon homology computations, we check the cardinality of T. The fixed point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}^3_\mathbb{C}$:
$\mathcal{H}^{\mathrm{Id}} = \mathcal{H}$,
$\mathcal{H}^\alpha =$ the complex geodesic line through (2) and (8),
$\mathcal{H}^\gamma =$ the complex geodesic line through (1) and (2),
$\mathcal{H}^\beta = \mathcal{H}^{\beta^2} =$ the complex geodesic line through (7) and (8).

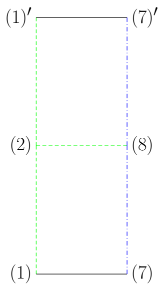

FIGURE 1. Fundamental domain in the case $m = 2$.

The matrix $g = \pm \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}$ performs a translation preserving the j-coordinate and sends the edge (1)(7) onto the edge $(1)'(7)'$, so the orbit space $\mathcal{H}/\Gamma$ is homotopy equivalent to a circle and we obtain

$$\mathrm{H}^{d-0}\left(\mathcal{H}_{\mathbb{C}}^{\mathrm{Id}}/C_\Gamma(\mathrm{Id});\mathbb{Q}\right) = \mathrm{H}^d\left(\mathcal{H}_{\mathbb{C}}/\Gamma;\mathbb{Q}\right) \cong \mathbb{Q} \text{ for } d = 0, 1; \text{ and zero otherwise.}$$

Consider the real geodesic line $\mathcal{H}_{\mathbb{R}}^{\gamma}$ on the unit circle of real part zero. The edge $g^{-1} \cdot ((2)(1)')$ $= \left(g^{-1}(2)\right)(1)$ lies on $\mathcal{H}_{\mathbb{R}}^{\gamma}$ and is not Γ–equivalent to the edge (1)(2). Because of lemma 17, the centralizer $C_\Gamma(\gamma)$ reflects the line $\mathcal{H}_{\mathbb{R}}^{\gamma}$ onto itself at (2), and again at $g^{-1}(2)$. Furthermore, none of the four elements of Γ sending (2) to $g^{-1}(2)$ belongs to $C_\Gamma(\gamma)$. Hence the quotient space $\mathcal{H}_{\mathbb{R}}^{\gamma}/C_\Gamma(\gamma)$ consists of a contractible segment of two adjacent edges. Thus $\mathrm{H}^{d-2}\left(\mathcal{H}_{\mathbb{C}}^{\gamma}/C_\Gamma(\gamma);\mathbb{Q}\right) \cong \begin{cases} \mathbb{Q}, & d = 2 \\ 0 & \text{else} \end{cases}$ is contributed to the orbifold cohomology.

Next, consider the real geodesic line $\mathcal{H}_{\mathbb{R}}^{\beta}$ on the circle of constant real coordinate $\frac{1}{2}$, of center $\frac{1}{2}$ and radius $\sqrt{\frac{3}{4}}$. The edge $g^{-1} \cdot ((8)(7)') = \left(g^{-1}(8)\right)(7)$ lies on $\mathcal{H}_{\mathbb{R}}^{\beta}$ and is not Γ–equivalent to the edge (7)(8). The centraliser of β contains the matrix $V := \pm \begin{pmatrix} 21-\omega & \\ \omega-1 & 1+\omega \end{pmatrix}$ of infinite order, which sends the edge $\left(g^{-1}(8)\right)(7)$ to $(8)z$ with $z = \frac{1}{2} + \frac{3}{5}\omega + \sqrt{\frac{3}{100}}j$. We conclude that the translation action of the group $\langle V \rangle$ on the line $\mathcal{H}_{\mathbb{R}}^{\beta}$ is transitive, with quotient space represented by the circle $\left(g^{-1}(8)\right)(7) \cup (7)(8)$, first and last vertex identified. Thus $\mathrm{H}^{d-2}\left(\mathcal{H}_{\mathbb{C}}^{\beta}/C_\Gamma(\beta);\mathbb{Q}\right) \cong \mathrm{H}^{d-2}\left(\mathcal{H}_{\mathbb{C}}^{\beta^2}/C_\Gamma(\beta^2);\mathbb{Q}\right) \cong \begin{cases} \mathbb{Q}, & d = 2, 3 \\ 0 & \text{else} \end{cases}$ is contributed to the orbifold cohomology.

Because of lemma 17, the centralizer $C_\Gamma(\alpha)$ reflects the line $\mathcal{H}_{\mathbb{R}}^{\alpha}$ onto itself at (2), and again at (8). So, the quotient space $\mathcal{H}_{\mathbb{R}}^{\alpha}/C_\Gamma(\alpha)$ is the single contractible edge (2)(8). This yields that $\mathrm{H}^{d-2}\left(\mathcal{H}_{\mathbb{C}}^{\alpha}/C_\Gamma(\alpha);\mathbb{Q}\right) \cong \begin{cases} \mathbb{Q}, & d = 2 \\ 0 & \text{else} \end{cases}$ is contributed to the orbifold cohomology.

Summing up over T, we obtain

$$\mathrm{H}^d_{orb}\left(\mathcal{H}_{\mathbb{C}}//\mathrm{PSL}_2(\mathbb{Z}[\omega])\right) \cong \begin{cases} \mathbb{Q}, & d = 0 \\ \mathbb{Q}, & d = 1 \\ \mathbb{Q}^4, & d = 2 \\ \mathbb{Q}^2, & d = 3 \\ 0 & \text{otherwise.} \end{cases}$$

The case $\Gamma = \mathbf{PSL}_2(\mathcal{O}_{-11})$.

Let $\mathcal{O}_{-11}$ be the ring of integers in $\mathbb{Q}(\sqrt{-11}\,)$. Then $\mathcal{O}_{-11} = \mathbb{Z}[\omega]$ with $\omega = \frac{-1+\sqrt{-11}}{2}$.

A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-11})$ in real hyperbolic 3-space has been found by Luigi Bianchi [3]. Half of its lower boundary given in figure 2. The coordinates of the vertices of figure 2 in the upper-half space model are $(3) = j$, $(3)' = 1 + \omega + j$, $(6) = \frac{1}{2} + \sqrt{\frac{3}{4}}j$, $(6)' = \frac{1}{2} + \omega + \sqrt{\frac{3}{4}}j$,

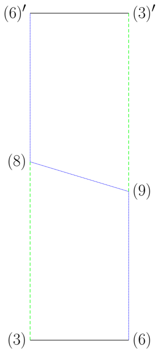

FIGURE 2. Fundamental domain in the case $m = 11$.

$(8) = \frac{3}{11} + \frac{3}{11}\omega + \sqrt{\frac{2}{11}}j$, $(9) = \frac{8}{11} + \frac{5}{11}\omega + \sqrt{\frac{2}{11}}j$. The set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id},\, \gamma,\, \beta,\, \beta^2\},$$

with $\beta = \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,
so γ is of order 2, and β is of order 3. Using lemma 25 and with the help of our Bredon homology computations, we check the cardinality of T. That we have one less conjugacy class of finite order elements than in the case $\mathcal{O}_{-2}$, comes from the fact that by lemma 20, there is only one conjugacy class of order–2–elements in $\mathcal{A}_4$.

The fixed point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}^3_{\mathbb{C}}$:
$\mathcal{H}^{\mathrm{Id}} = \mathcal{H}$,
$\mathcal{H}^{\gamma} =$ the complex geodesic line through (3) and (8),
$\mathcal{H}^{\beta} = \mathcal{H}^{\beta^2} =$ the complex geodesic line through (6) and (9).

The 2–torsion subcomplex is of homeomorphism type •—• and the 3–torsion subcomplex is of homeomorphism type ○•. Therefore, we obtain

$$\mathrm{H}^d_{orb}\left(\mathcal{H}//\mathrm{PSL}_2(\mathcal{O}_{-11})\right) \cong \begin{cases} \mathbb{Q}, & d = 0 \\ \mathbb{Q}, & d = 1 \\ \mathbb{Q}^{1+2}, & d = 2 \\ \mathbb{Q}^2, & d = 3 \\ 0 & \text{otherwise.} \end{cases}$$

The case $\Gamma = \mathbf{PSL}_2(\mathcal{O}_{-191})$.

Let $\mathcal{O}_{-191}$ be the ring of integers in $\mathbb{Q}(\sqrt{-191}\,)$. Again, the set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id},\, \gamma,\, \beta,\, \beta^2\},$$

with $\beta = \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so γ is of order 2, and β is of order 3. Both the 2– and the 3–torsion subcomplexes are of homeomorphism type ○•. Then,

$$\mathrm{H}^d_{orb}\left(\mathcal{H}//\mathrm{PSL}_2(\mathcal{O}_{-191})\right) \cong \begin{cases} \mathbb{Q}, & d = 0 \\ \mathbb{Q}^{15}, & d = 1 \\ \mathbb{Q}^{14+1+2}, & d = 2 \\ \mathbb{Q}^{1+2}, & d = 3 \\ 0 & \text{otherwise.} \end{cases}$$

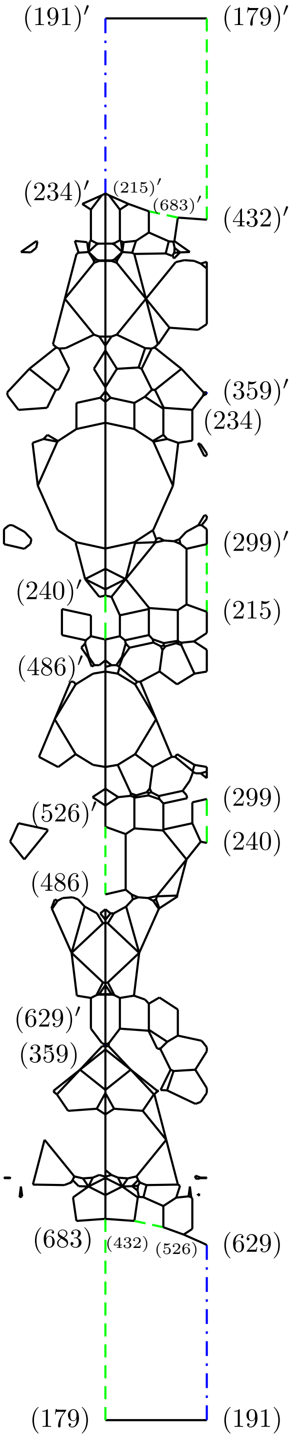

FIGURE 3. Fundamental domain in the case $m = 191$.

The author would like to add the following explanation why in our fundamental domain diagrams, there occurs only one representative per torsion-stabilised edge.

***Remark* 26.** Let e be a non-trivially stabilised edge in the fundamental domain for the refined cell complex. Then the fundamental domain for the 2–dimensional retract can be chosen such that it contains e as the only edge on its orbit.

Sketch of proof. Observe that the inner dihedral angle $\frac{2\pi}{q}$ of the Bianchi fundamental polyhedron is $\frac{2\pi}{\ell}$ or $\frac{\pi}{\ell}$ at its edges admitting a rotation of order ℓ from the Bianchi group. We can verify this in the vertical half-plane where the action of $\mathrm{PSL}_2(\mathbb{Z})$ is embedded into the action of the Bianchi group, for the generators of orders $\ell = 2$ and $\ell = 3$ of $\mathrm{PSL}_2(\mathbb{Z})$ which fix edges orthogonal to the vertical half-plane. These angles are transported to all edges stabilised by Bianchi group elements conjugate to these two rotations. Poincaré [15] partitions the edges of the Bianchi fundamental polyhedron into cycles, consisting of the edges on the same orbit, of length $\frac{q}{\ell} = 1$ or 2. In the case of length 2, Poincaré's description implies that each of the two 2–cells separated by the first edge of the cycle, is respectively on the same orbit as one of the 2–cells separated by the second edge of the cycle. As the fundamental domain for the 2–dimensional retract is strict with respect to the 2–cells, it can be chosen such that it contains e as the only edge on its orbit. ☐

7. Appendix: Computation of subgroups in the centralisers

We can check our computations using the following algorithm for the computation of subgroups in the centralisers. We start by computing a subgroup in the centraliser of β, for β running through the representatives of conjugacy classes of elements of finite order in Γ. For an arbitrary matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \beta$, the centraliser elements are of the form $\begin{pmatrix} a & b \\ \frac{g}{f}b & a + b\frac{h-e}{f} \end{pmatrix}$, and the determinant 1 equation splits as follows into real and imaginary part. Assume that m is congruent to 1 or 2 $\mod 4$, so the ring of integers of $\mathbb{Q}(\sqrt{-m})$ is given by $\mathbb{Z}[\sqrt{-m}]$. Let $\omega := \sqrt{-m}$, and write $a =: j + k\omega$, $b =: \ell + n\omega$ with j, k, ℓ, $n \in \mathbb{Z}$, and set $\frac{g}{f} =: R + \omega J$ as well as $\frac{h-e}{f} =: \rho + \omega\iota$. Then

$$(\mathrm{Re}): \qquad j^2 - mk^2 + (j\ell - mkn)\rho - (jn + k\ell)m\iota + (-l^2 + mn^2)R + 2mJln = 1,$$

$$(\mathrm{Im}): \qquad 2jk + (jn - k\ell)\rho + (j\ell - mkn)\iota - 2R\ell n - J\ell^2 + Jmn^2 = 0.$$

In the special case of $\beta = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, a matrix which is contained in all the Bianchi groups via the inclusion $\mathrm{PSL}_2(\mathbb{Z}) < \Gamma$, these equations reduce to

$$(\mathrm{Re}): \qquad j^2 - mk^2 + \ell^2 - mn^2 - j\ell + mkn = 1,$$

$$(\mathrm{Im}): \qquad (2k - n)j + 2\ell n - k\ell = 0.$$

First case: $n = 2k$. The equation (Im) yields $k\ell = 0$. If $k = 0$, we obtain the matrix group $\langle\beta\rangle \cong \mathbb{Z}/3$ generated by our centralised matrix β.
Otherwise $\ell = 0$, and equation (Re) gives

$$j^2 = 3mk^2 + 1. \tag{2}$$

Second case: $n \neq 2k$. This means that we can transform the equation (Im) into

$$j = \ell \frac{k-2n}{2k-n},$$

which we insert into the equation (Re):

$$\ell^2 \left(\frac{k-2n}{2k-n}\right)^2 - mk^2 + \ell^2 - mn^2 - \ell^2 \frac{k-2n}{2k-n} + mkn = 1.$$

We solve for ℓ^2 and find

$$\ell^2 = \frac{mk^2 + mn^2 - mkn + 1}{1 + \frac{k-2n}{2k-n}\left(\frac{k-2n}{2k-n} - 1\right)}. \tag{3}$$

As m is fixed, we can numerically compute the integer solutions of equations (2) and (3), up to a chosen bound for the absolute value of k, which shall also bound the absolute value of n in the second equation. We obtain a subset S of $C_\Gamma(\beta)$ that contains all the matrices in $C_\Gamma(\beta)$ the entries of which have absolute value at most the chosen bound. Then we calculate the group $\langle S \rangle$ generated by S. Once the quotient space $\mathcal{H}^\beta / \langle S \rangle$ contains no more than one representative for any Γ-orbit of cells, we know that we have obtained the quotient space $\mathcal{H}^\beta / C_\Gamma(\beta)$, because of the subgroup inclusions $\langle S \rangle < C_\Gamma(\beta) < \Gamma$.

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