Homogenization at different linear scales, bounded martingales and the Two-Scale Shuffle limit

Kévin Santugini-Repiquet*

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Abstract

In this short paper, we look at two-scale limits of sequences with varying homogenization periods, each period being a multiple of the previous one. We establish that, up to a measure preserving rearrangement, these two-scale limits form a martingale which is bounded: the rearranged two-scale limits themselves converge both strongly in L^2 and almost everywhere when the period tends to $+\infty$. This limit, called the two-scale shuffle limit, contains all the information present in all the two-scale limits in the sequence.

1 Introduction

Homogenization is used to study the solutions to equations when there are multiple scales of interest, usually a microscopic one and a macroscopic one. In particular, one may consider the solutions u_ε to a partial differential equation with quasi ε-periodic coefficient and study their behavior as the small period ε tend to 0. Two-scale convergence, introduced by G. Nguetseng [9], and G. Allaire [1] is suited to study this particular subset of homogenization problems called periodic homogenization. This was later extended to the case of periodic surfaces by M. Neuss Radu[7, 8] and G.Allaire, A. Damlamian and U. Hornung[3]. It can also be used in the presence of periodic holes in the geometry, see [4, 5] or to homogenize multilayers [11, 10].

*Institut Mathématiques de Bordeaux, CNRS UMR5251, MC2, INRIA Bordeaux - Sud-Ouest Kevin.Santugini@math.u-bordeaux1.fr

Intuitively, two-scale convergence introduces the concept of two-scale limit u_0 which is a function of both a macroscopic variable $\boldsymbol{x}$ and a microscopic p-periodic variable $\boldsymbol{y}$ such that, in some "meaning", $\boldsymbol{x} \mapsto u_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ is a good approximation of u_ε.

As indicated by its name, two-scale convergence captures the behavior at two scales: the macroscopic one and the $p\varepsilon$-periodic one. However, two-scale convergence does not capture all phenomena that happens at a scale linear in ε but only those whose length scale is $p\varepsilon/m$ where m is an integer. The two-scale limit of a sequence depends, not only on the asymptotic scale, but also on the precise value of the chosen period. For example, any phenomena happening at the length scale of 2ε will not be fully apparent in the two-scale limit computed with period ε. The two-scale limit computed with period 2ε will contain no less —and might actually contain more— information than the two-scale limit computed with period ε. For example, the homogenization of $\sin(2\pi x/\varepsilon) + \sin(\pi x/\varepsilon)$ gives a two-scale limit of $u_0 : (x, y) \mapsto \sin(2\pi y)$ if computed with the homogenization period ε, *i.e.* when $p = 1$, and $u_0 : (x, y) \mapsto \sin(2\pi y) + \sin(\pi y)$ if computed with the homogenization period 2ε, *i.e.* when $p = 2$. Furthermore, if we choose $p = 1/2$, then the two-scale limit is none other than the null function. Worse, the scale factor p could be irrational.

The choice of the scale factor p used in the homogenization process is therefore of utmost importance in two-scale convergence. Using a badly chosen scale factor p may and will often cause a huge loss of information. At worst, we recover no more information than the one obtained by the standard weak L^2 limit: if $p\varepsilon$ is the correct choice of homogenization period, the two-scale limit computed with period $\lambda p\varepsilon$ where λ is an irrational number should, intuitively, carry no information about what happens at scale $p\varepsilon$.

Fortunately, there is usually a natural choice of period: the coefficients of the partial differential equation are often chosen quasi-periodic in ε. The most natural choice is to choose $p = 1$, *i.e.* to consider the correct microscopic scale for u_ε is ε itself. If there are two important periods to consider ε_1 and ε_2, the intuitive solution is to choose a period that is a multiple of both. However, this can only be done if the ratio $\varepsilon_1/\varepsilon_2$ is a rational number.

When the two-scale limit depends on the fast variable, we may, to increase the information obtained by two-scale convergence, consider an homogenization period of $m\varepsilon$ instead of ε where m is a positive integer. The two-scale limit computed with the homogenization period $m\varepsilon$ contains more information than the two-scale limit computed with the homogenization period ε. It is then natural to study the behavior of the two-scale limit as m tends to $+\infty$. G. Allaire and C. Conca studied in [2] a similar problem and established, for an elliptic problem, the behavior of the spectra of the equation satisfied by the two-scale limit as the scale factor p goes to $+\infty$.

In this paper, we consider various two-scale limits, each computed with a different homogenization period. In particular, we consider a sequence of periods $(p_n)_{n\in\mathbb{N}}$ such that for all integers n, p_{n+1}/p_n is a positive integer and we study the two-scale limit of $(u_\varepsilon)_{\varepsilon>0}$ computed with the homogenization period $p_n\varepsilon$. This two-scale limit, denoted u_{0,p_n}, is p_n-periodic in each component of its fast variable. Since p_{n+1} is always a multiple of p_n, one can always recover the two-scale limit u_{0,p_n} from the two scale limit $u_{0,p_{n+1}}$. If $p_{n+1} = m_n p_n$ and in dimension $d \geq 1$:

$$u_{0,p_{n+1}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{m_n^d} \sum_{\alpha\in[\![0,m_n-1]\!]^d} u_{0,m_n p_n}(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{\alpha}).$$

The sequence of two-scale limits $(u_{0,p_n})_{n\in\mathbb{N}}$ yields increasing information on the asymptotic behavior of $(u_\varepsilon)_{\varepsilon>0}$. A natural question is whether the two-scale limits u_{0,p_n} themselves converge whenever n tends to $+\infty$. *I.E.* does there exist a function that carry the information of all the p_n-two-scale limits? The goal of our paper is to answer this question. The answer is positive. We show in this paper that the sequence of two-scale limits is, after a measure preserving rearrangement, a bounded martingale in L^2 and therefore converges both strongly in L^2 and almost everywhere to a function we call the two-scale shuffle limit.

In §2, we remind the reader about the known theorems of two-scale convergence. In §3, we show how the different two-scale limits are bound by martingale type equalities and explain how to transform these two-scale limits to get a genuine martingale. This leads to our stating of our main theorem: Theorem 3.5 in which we show that in a certain meaning the two-scale limits themselves converge to the two-scale shuffle limit. In §4, we use this result on the heat equation in multilayers with transmission conditions between adjacent layers and establish, for this particular example, the equation satisfied by the two-scale shuffle limit in Theorem 4.1.

2 Notations and the classical notion of two-scale convergence

First, as in [1], we introduce some notations. In this paper, p always refer to a scale factor. It remains constant while taking the two-scale limit. However, the goal of this paper is to observe the behavior of the two-scale limits as p tend to $+\infty$.

By Ω, we denote a bounded open domain of $\mathbb{R}^d$ where $n \geq 1$. By Y_p, we denote the cube $[0,p]^d$. By $\mathrm{L}^2_\#(Y_p)$, we denote the space of measurable functions defined over $\mathbb{R}^d$, that are p-periodic in each variable and that are square integrable over Y_p. By $\mathcal{C}_\#(Y_p)$, we denote the set of continuous functions defined on $\mathbb{R}^d$ that are p-periodic in each variable.

We reproduce the now classical definition of two-scale convergence found in [1, 9]. For convenience, we added the scale factor p.

Definition 2.1 (Two-scale convergence)**.** Let p be a positive real. A sequence $(u_\varepsilon)_{\varepsilon>0}$ belonging to $\mathrm{L}^2(\Omega)$ is said to p-two-scale converge if there exist $u_{0,p}$ in $\mathrm{L}^2(\Omega \times Y_p)$ such that:

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(\boldsymbol{x})\psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x} = \frac{1}{p^d}\int_\Omega\int_{Y_p} u_{0,p}(\boldsymbol{x},\boldsymbol{y})\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}, \tag{2.1}$$

for all ψ in $\mathrm{L}^2(\Omega;\mathcal{C}_\#(Y_p))$.

G. Allaire, see [1], and G.Nguetseng, see [9], proved that any sequence of functions bounded in L^2 has a subsequence that two-scale converges. Let's reproduce this precise compactness result.

Theorem 2.2. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of functions belonging to $\mathrm{L}^2(\Omega)$. Then, there exist $u_{0,p}$ in $\mathrm{L}^2(\Omega\times(0,1))$ and a subsequence ε_k converging to 0 such that*

$$\lim_{k\to\infty} \int_\Omega u_{\varepsilon_k}(\boldsymbol{x})\psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon_k}\right)\,\mathrm{d}\boldsymbol{x} = \frac{1}{p^d}\int_\Omega\int_{Y_p} u_{0,p}(\boldsymbol{x},\boldsymbol{y})\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}, \tag{2.2}$$

for all ψ in $\mathrm{L}^2(\Omega;\mathcal{C}_\#(Y))$.

Proof. See G. Allaire [1] and G. Nguetseng [9]. The presence of the scale factor p has no impact on the proof. □

We also have the classical proposition

Proposition 2.3. *Let u_ε p-two-scale converge to $u_{0,p}$, then*

$$\frac{1}{p^d}\|u_{0,p}\|_{\mathrm{L}^2(\Omega\times Y_p)} \leq \varliminf_{\varepsilon\to 0}\|u_\varepsilon\|_{\mathrm{L}^2(\Omega)}.$$

Proof. See G. Allaire [1] and G. Nguetseng [9]. The presence of the scale factor p has no impact on the proof. □

The next proposition is easy to derive from Theorem 2.2

Proposition 2.4. *Let $(p_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers. Let $(u_\varepsilon)_{\varepsilon>0}$ be a continuous sequence of function belonging to $\mathrm{L}^2(\Omega)$, then there exist a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ converging to 0, and a sequence of functions u_n such that for any integer n the sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ p_n-two-scale converges to u_{0,p_n}. I.E., such that for all integers n:*

$$\lim_{k\to\infty}\int_\Omega u_{\varepsilon_k}(\boldsymbol{x})\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon_k}\right)\,\mathrm{d}\boldsymbol{x} = \frac{1}{p_n^d}\int_\Omega\int_{Y_{p_n}} u_{0,p_n}(\boldsymbol{x},\boldsymbol{y})\psi\left(\boldsymbol{x},\boldsymbol{y}\right)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x},$$

for all ψ in $\mathrm{L}^2(\Omega;\mathcal{C}_\#(Y_{p_n}))$.

Proof. Apply Theorem 2.2 multiple times and proceed via diagonal extraction. □

Our goal in this paper is to study the limit of u_{0,p_n} as p_n tends to $+\infty$.

3 Two-scale limits and bounded martingales

In this section, we always consider a sequence of scale factors $(p_n)_{n\in\mathbb{N}}$ such that p_{n+1} is a multiple of p_n for every non negative integer n. We set for $n\geq 1$, $m_n := p_n/p_{n-1}$ and for $n\geq 0$ $M_n := p_n/p_0$. We always consider a continuous sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ bounded in $\mathrm{L}^2(\Omega)$ and a decreasing sequence of positive $(\varepsilon_k)_{k\in\mathbb{N}}$ such that the sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ p_n-two-scale converges for all integers n. This is justified by Proposition 2.4.

To choose the sequence p_n, one may set p_0 first, then set $p_n = M_n p_0$ and choose the sequence of integers M_n such that M_{n+1} is always a multiple of M_n and such that any integer eventually divide M_n when n is large enough. For example, one may set $M_n = n!$ or let M_n be the smallest multiple of all positive integers smaller than n.

Our goal is to study the convergence of the two-scale limits u_{0,p_n} when n goes to infinity. In this section, we proceed as follows: we begin by establishing a useful equality that looks like a martingale equality in §3.1, then we propose a rearrangement of the two-scale limits in §3.2, and finally propose another rearrangement of the two-scale limits in §3.3, the shuffle, which transform the sequence of two-scale limits into a bounded martingale.

3.1 An almost martingale equality

Consider a sequence $(p_n)_{n\in\mathbb{N}}$ such that p_{n+1}/p_n is a positive integer for all n. We begin by deriving the p-two-scale limit from the mp-two-scale limit when m is an integer.

Proposition 3.1. *Let m be an integer. Let p be a positive scale factor. Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of functions belonging to $\mathrm{L}^2(\Omega)$, and p-two-scale converging to $u_{0,p}$ and mp-two-scale converging to $u_{0,mp}$. Then, for almost all $\boldsymbol{x}$ in Ω and $\boldsymbol{y}$ in Y_p:*

$$u_{0,p}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{m^d} \sum_{\boldsymbol{\alpha}\in[\![0,m-1]\!]^d} u_{0,mp}(\boldsymbol{x},\boldsymbol{y}+\boldsymbol{\alpha}p).$$

Proof. Let ϕ belong to $\mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}^d)$ be p-periodic in the last d variables. Since m is an integer, ϕ is also mp-periodic in the last d variables. We take the limit of $\int_\Omega u_\varepsilon(\boldsymbol{x})\phi(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\,\mathrm{d}\boldsymbol{x}$, as ε tend to 0, in the sense of two-scale convergence for both scale factors mp and p:

$$\begin{aligned}\frac{1}{p^d}\int_\Omega\int_{Y_p} u_{0,p}(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} &=\\ &= \frac{1}{m^dp^d}\int_\Omega\int_{Y_{mp}} \Big(u_{0,mp}(\boldsymbol{x},\boldsymbol{y})\Big)\,\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}, =\\ &= \frac{1}{m^dp^d}\int_\Omega\int_{Y_p} \Big(\sum_{\boldsymbol{\alpha}\in([\![0,m-1]\!]^d} u_{0,mp}(\boldsymbol{x},\boldsymbol{y}+\boldsymbol{\alpha}p)\Big)\,\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y},\end{aligned}$$

for all ϕ in $\mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}^d)$. □

This simple but essential proposition provides all we need to show that up to a rearrangement the sequence of two-scale limits u_{0,p_n} is actually a martingale.

We derive the following corollary.

Corollary 3.2. *For all $\boldsymbol{x}$ in Ω, $\boldsymbol{y}$ in $\mathbb{R}^d$, j in $\mathbb{N}$ and n in $\mathbb{N}$, we have*

$$u_{0,p_n}(\boldsymbol{x},\boldsymbol{y}) = \left(\frac{p_n}{p_{n+j}}\right)^d \sum_{\boldsymbol{\alpha}\in[\![0,p_{n+j}/p_n-1]\!]^d} u_{0,p_{n+j}}(\boldsymbol{x},\boldsymbol{y}+\boldsymbol{\alpha}p_n).$$

Ideally, we would like to consider the limit of u_{0,p_n} as n tend to $+\infty$. Because of this equality, the sequence u_{0,p_n} is “morally” a martingale for the filtration made of these σ-fields:

$$\mathcal{F}_n = \mathcal{B}(\Omega)\times\big(\mathcal{B}(\mathbb{R}^n)+p_n\mathbb{Z}^d\big).$$

Unfortunately, this isn't technically true as all the sets belonging to these σ fields are of infinite measure and the u_{0,p_n} are all periodic. However, the equalities defining what is a martingale are satisfied if one replaces the standard integral of $\mathbb{R}^d$ by the limit of the mean over a ball as its radius tends to $+\infty$. *I.E.*, we have for all F in $\mathcal{F}_n$

$$\lim_{R\to+\infty} \frac{1}{|B(\mathbf{0},R)|} \int_\Omega \int_{B(\mathbf{0},R)} 1\{(\boldsymbol{x},\boldsymbol{y}) \in F\} u_{0,p_n}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \\ = \lim_{R\to+\infty} \frac{1}{|B(\mathbf{0},R)|} \int_\Omega \int_{B(\mathbf{0},R)} 1\{(\boldsymbol{x},\boldsymbol{y}) \in F\} u_{0,p_{n+j}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x},$$

where $B(\mathbf{0},R)$ is the open ball of $\mathbb{R}^d$ centered on 0 and of radius R and where $|A|$ is the Lebesgue measure of set A. Unfortunately, we were unable to derive a direct convergence result using this pseudo martingale equality. To proceed further, we need to rearrange the two-scale limits to get genuine martingales.

3.2 Rearrangement of the two-scale limits with integers

In this subsection, we rearrange the two-scale limits as functions defined over $\Omega \times \mathbb{Z}^d \times Y_{p_0}$. While we were unable to obtain a convergence in that case, the ideas of this section provide insight on the next section in which we prove convergence for another rearrangement. Given the two-scale limits u_{0,p_n}, we define

$$v_{p_n} : \Omega \times \mathbb{Z}^d \times Y_{p_0} \to \mathbb{R}, \\ (\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{y}) \mapsto u_{0,p_n}(\boldsymbol{x},\boldsymbol{y}+p_0\boldsymbol{\alpha}).$$

We have the following proposition

Proposition 3.3. *For all n in $\mathbb{N}$, for almost all $\boldsymbol{x}$ in Ω and $\boldsymbol{y}$ in Y_{p_0}, the sequence $(v_{p_n}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{y}))_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ is p_n/p_0-periodic in each direction. Moreover:*

$$v_{p_n}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{y}) = \left(\frac{M_n}{M_{n+j}}\right)^d \sum_{\boldsymbol{\beta}\in[\![0,M_{n+j}/M_n-1]\!]^d} v_{p_{n+j}}(\boldsymbol{x},\boldsymbol{\alpha}+M_n\boldsymbol{\beta},\boldsymbol{y}). \tag{3.1}$$

Proof. This is a direct consequence of Corollary 3.2. □

This in turn should encourage us to look at the following problem.

Problem 3.4. Let's call sequences $(t_{n,\boldsymbol{\alpha}})_{n\in\mathbb{N},\boldsymbol{\alpha}\in\mathbb{Z}^d}$ that are M_n periodic in each component of $\boldsymbol{\alpha}$ and that satisfy

$$t_{n,\boldsymbol{\alpha}} = \left(\frac{M_n}{M_{n+j}}\right)^d \sum_{\boldsymbol{\beta}\in[\![0,M_{n+j}/M_n-1]\!]^d} t_{n+j,\boldsymbol{\alpha}+M_n\boldsymbol{\beta}}.$$

"imbricated $(M_n)_n$-periodic d-dimensional sequences". Study the convergence of such sequences as n tend to $+\infty$. Under which condition does there exists a sequence $t_{\infty,\boldsymbol{\alpha}}$ such that for all non negative integers n

$$t_{n,\boldsymbol{\alpha}} = \lim_{N\to+\infty} \left(\frac{1}{N^d}\right)^d \sum_{\boldsymbol{\beta}\in[\![0,N-1]\!]^d} t_{\infty,\boldsymbol{\alpha}+M_n\boldsymbol{\beta}}.$$

or such that

$$t_{n,\boldsymbol{\alpha}} = \lim_{N\to+\infty} \left(\frac{1}{2^d N^d}\right)^d \sum_{\boldsymbol{\beta}\in[\![-N,N-1]\!]^d} t_{\infty,\boldsymbol{\alpha}+M_n\boldsymbol{\beta}}.$$

or both?

By Proposition 3.3, for almost all $\boldsymbol{x}$ in Ω and $\boldsymbol{y}$ in Y_{p_0}, the sequences $(v_{p_n}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{y}))_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ are imbricated $(p_n)_n$-periodic d-dimensional sequences. Solving Problem 3.4 would be the first step in having a very elegant limit to the v_{p_n} as a function defined on $\Omega\times\mathbb{Z}^d\times Y_{p_0}$. Unfortunately, we do not have an answer for Problem 3.4. While this sequence is morally a martingale with respect to the filtration made of the σ-fields $\{\boldsymbol{\alpha}+\frac{p_n}{p_0}\mathbb{Z}^d, \boldsymbol{\alpha}\in[\![0,\frac{p_n}{p_0}-1]\!]^d\}$, it technically is not: we have the same problem we had in the previous section. To conclude with bounded martingales on the convergence, we would need a measure μ on $\mathbb{Z}^d$ such that $\mu(\mathbb{Z}^d)=1$, invariant by translation and such that $\mu(m\mathbb{Z}^d)=1/m$ whenever m is an integer different from 0. Such a measure cannot be σ-additive. If we remove the σ additivity constraint, then μ exists: just set

$$\mu(A) = \lim_{N\to+\infty} \frac{\#(A\cap[\![-N,N]\!]^d)}{(2N+1)^d}.$$

It is unknown to the author if bounded martingales converge when they are defined on a non σ-additive measure. To avoid that problem, we introduce, in the next section, a different less natural rearrangement for the u_{0,p_n}, the shuffle, for which we finally prove a convergence result.

3.3 Shuffle rearrangement of two-scale limits

In this section, we finally construct a rearrangement, the shuffle, that results in a bounded martingale. And, since bounded martingales in L^2 converge both strongly in L^2 and almost everywhere, this establishes a convergence result for the u_{0,p_n} as n tend to $+\infty$.

Let for $\boldsymbol{x}$ in Ω, $\boldsymbol{y}$ in $[0,p_0]^d$ and $\boldsymbol{y}'$ in $[0,1]^d$,

$$w_n(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}') = v_{p_n}(\boldsymbol{x},\boldsymbol{\alpha}(\boldsymbol{y}'),\boldsymbol{y}),$$

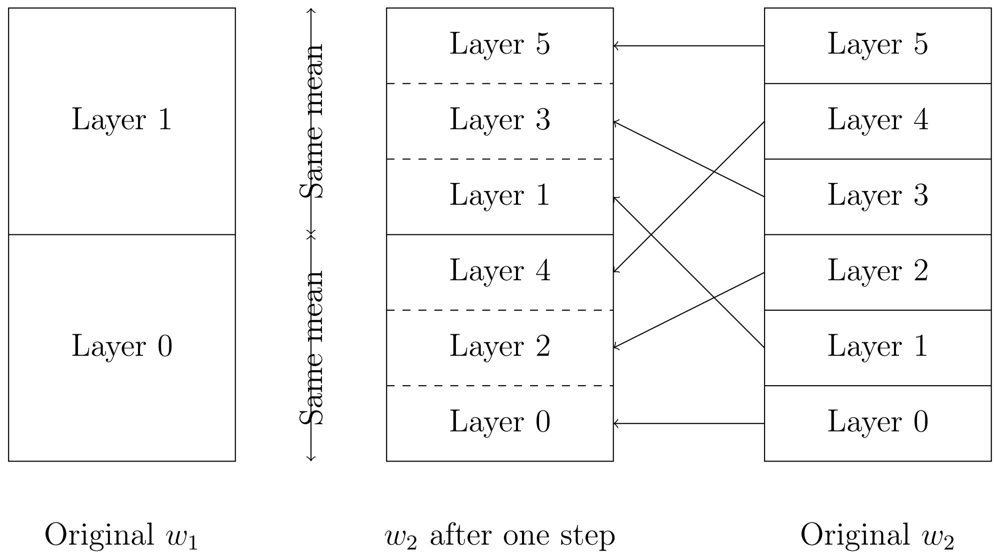

Figure 1: One step of the measure preserving rearrangement $M_1 = 2$ and $M_2 = 6$

where $\boldsymbol{\alpha}(\boldsymbol{y}')_i = \lfloor M_n \boldsymbol{y}'_i \rfloor$ for all integers i in $[\![1, d]\!]$. The variable $\boldsymbol{y}'$ replaces the variable $\boldsymbol{\alpha}$ of the previous section. Using Proposition 3.3, we derive that for almost all $\boldsymbol{x}$ in Ω, $\boldsymbol{y}$ in Y_{p_0}, (j, n) in $\mathbb{N}^2$, and $\boldsymbol{\alpha}$ in $[\![0, M_n - 1]\!]^d$

$$\int_{\prod_{i=1}^d [\frac{\alpha_i}{M_n}, \frac{\alpha_i+1}{M_n})} w_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}') \,\mathrm{d}\boldsymbol{y}' = \\ = \sum_{\boldsymbol{\beta} \in [\![0, \frac{M_{n+j}}{M_n} - 1]\!]^d} \int_{\prod_{i=1}^d [\frac{M_n \beta_i + \alpha_i}{M_{n+j}}, \frac{M_n \beta_i + \alpha_i + 1}{M_{n+j}})} w_{n+j}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}') \,\mathrm{d}\boldsymbol{y}'. \quad (3.2)$$

To transform the w_n into martingales, we need to shuffle the hypercubes as in Figure 1 where, to simplify the drawing, homogenization was only performed on the last component of $\mathbb{R}^d$, hence the presence of layers instead of hypercubes. In that figure, we show one step of the rearrangement. As seen in the drawing, each step of the rearrangement is measure preserving, therefore the full rearrangement is also measure preserving. We need $n - 1$ such steps to fully rearrange w_n.

To define rigorously this rearrangement, we begin by defining the function that maps the rearranged layer index onto the unrearranged layer index:

$$\begin{aligned} R_{M,m}(i) :& [\![0, Mm - 1]\!] \to [\![0, Mm - 1]\!] \\ & i \mapsto M(i \mod m) + \left\lfloor \frac{i}{m} \right\rfloor. \end{aligned} \quad (3.3a)$$

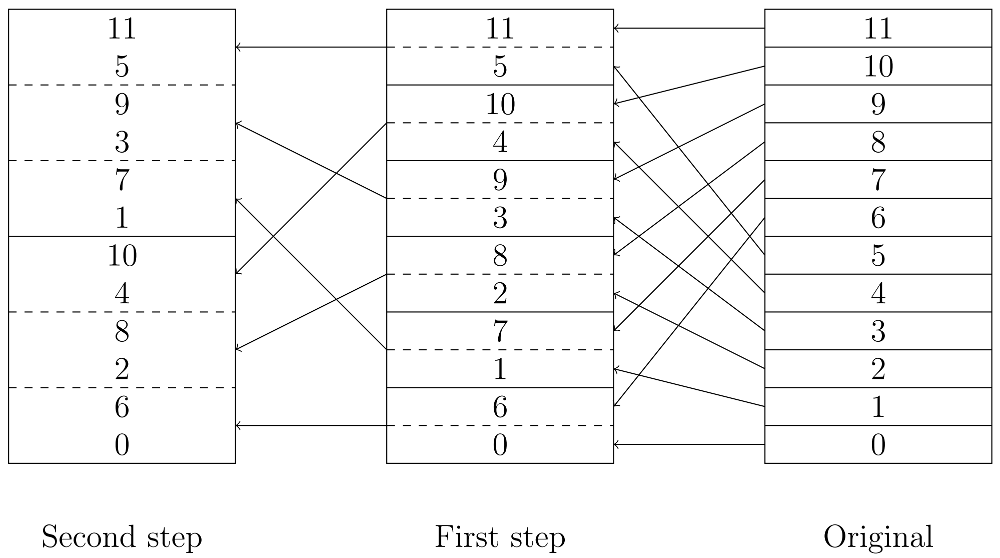

Figure 2: Two steps of the measure preserving rearrangement $M_1 = 2$, $M_2 = 6$ and $M_3 = 12$

for all i in $[\![0, Mm-1]\!]$. We have $R_{M,m} \circ R_{m,M} = R_{m,M} \circ R_{M,m} = \mathrm{Id}$. Then, we set the function that maps the rearranged layer onto the unrearranged one:

$$\begin{aligned} h^*_{M,m} &: [0,1) \to [0,1), \\ y' &\mapsto \frac{R_{M,m}(\lfloor Mmy' \rfloor)}{Mm} + \left(y' - \frac{\lfloor Mmy' \rfloor)}{Mm} \right). \end{aligned} \tag{3.3b}$$

This represents only one step of the rearrangement on one component. For hypercubes, the permutation is the same but is done componentwise: we set

$$\begin{aligned} h_{M,m} &: (0,1)^d \to (0,1)^d, \\ (y'_1, \ldots, y'_n) &\mapsto (h^*_{M,m}(y'_1), \ldots, h^*_{M,m}(y'_n)). \end{aligned}$$

And obtain one step of the rearrangement on all d components. For the complete rearrangement on one component, see Figure 2, we set

$$H^*_n := h^*_{M_{n-1},m_n} \circ \ldots \circ h^*_{M_1,m_2} \circ h^*_{M_0,m_1}. \tag{3.3c}$$

To get the complete rearrangement on all components we set

$$\begin{aligned} H_n &: [0,1)^d \to [0,1)^d, \\ (y'_1, \ldots, y'_n) &\mapsto (H^*_n(y'_1), \ldots, H^*_n(y'_n)). \end{aligned} \tag{3.3d}$$

We also have

$$H_n = h_{M_{n-1},m_n} \circ \ldots \circ h_{M_1,m_2} \circ h_{M_0,m_1}.$$

The function H_n shuffles the hypercubes $\prod_{i=1}^{d}[\beta_i/M_n,(\beta_i+1)/M_n)$, hence we call H_n the shuffle function.

Finally, we define

$$\widetilde{w}_n(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y'}) := w_n(\boldsymbol{x},\boldsymbol{y},H_n(\boldsymbol{y'})). \tag{3.4}$$

This measure preserving rearrangement, the shuffle, is purposefully constructed so the $\widetilde{w}_n$ form a martingale for the following filtration of σ-fields $\mathcal{F}_n = \mathcal{B}(\Omega)\times\mathcal{B}([0,p_0]^d)\times\sigma\left\{\prod_{i=1}^{d}\left[\frac{\beta_i}{M_n},\frac{\beta_i+1}{M_n}\right),\boldsymbol{\beta}\in\mathbb{Z}^d\right\}$.

Remark 1. The above rearrangement of hypercubes is similar to the one used for computing in place the Discrete Fast Fourier transform: the bit reversal. In the special case where $M_n = 2^n$, the rearrangement simply exchanges layers i, *i.e.* $[i/2^n,(i+1)/2^n)$, and i', *i.e.* $[i'/2^n,(i'+1)/2^n)$, when i and i' are bit reversal permutations of each other. *I.E* when $i=\sum_{j=0}^{N-1} b_j 2^j$ and $i'=\sum_{j=0}^{N-1} b_j 2^{N-1-j}$.

Remark 2. For general M_n, the rearrangement of hypercubes is also a bit reversal but for a mixed basis. If $\lfloor M_n y'\rfloor = \sum_{j=1}^{n} b_j M_{j-1}$ with b_j in $[\![0,m_j-1]\!]$, then

$$H_n^*\left(\frac{1}{M_n}\sum_{j=1}^{n} b_j\frac{M_n}{M_j}+(y'-\frac{\lfloor M_n y'\rfloor}{M_n})\right)=y'.$$

We now state our main result as a self contained theorem.

Theorem 3.5 (Two-Scale Shuffle convergence)**.** *Let Ω be a bounded open domain of $\mathbb{R}^d$ with $d\geq 1$. Let $(u_\varepsilon)_{\varepsilon>0}$ be a bounded sequence of functions belonging to $\mathrm{L}^2(\Omega)$. Let (p_n) be an increasing sequence of positive numbers such that for all integers n the ratio p_{n+1}/p_n is an integer. Set for all $n\geq 0$ $M_n := p_n/p_0$ and for all $n\geq 1$ $m_n = p_n/p_{n-1}$. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a decreasing sequence of positive real numbers converging to 0 such that the sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ p_n-two-scale converges to u_{0,p_n} for all non negative integer n.*

Set

$$\begin{aligned}\widetilde{w}_n : \Omega\times[0,p_0]^d\times[0,1]^d &\to \mathbb{R}\\ (\boldsymbol{x},\boldsymbol{y},\boldsymbol{y'}) &\mapsto u_{0,p_n}\left(\boldsymbol{x},p_0\lfloor M_n H_n(\boldsymbol{y'})\rfloor+\boldsymbol{y}\right).\end{aligned}$$

where H_n is defined by Equations (3.3)*.*

Then, the sequence $\widetilde{w}_n$ is a bounded martingale in L^2 for the filtration

$$\mathcal{F}_n = \mathcal{B}(\Omega)\times\mathcal{B}([0,p_0]^d)\times\sigma\left\{\prod_{i=1}^{d}\left[\frac{\beta_i}{M_n},\frac{\beta_i+1}{M_n}\right),\boldsymbol{\beta}\in\mathbb{Z}^d\right\}. \tag{3.5}$$

And, the sequence $\widetilde{w}_n$ converges both strongly in L^2 *and almost everywhere to $\widetilde{w}_\infty$, which we call the two-scale shuffle limit. Moreover,*

$$\iiint_A \widetilde{w}_n(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \iiint_A \widetilde{w}_\infty(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x},$$

for all sets A in $\mathcal{F}_n$. I.E., $\widetilde{w}_n = \mathbb{E}(\widetilde{w}_\infty|\mathcal{F}_n)$.

Proof. The $\widetilde{w}_n$ were constructed specifically so as to be a martingale for the filtration (3.5). To prove they are a martingale for the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$, we only need to prove that for every nonnegative integer n, we have for almost all $\boldsymbol{x}$ in Ω, almost all $\boldsymbol{y}$ in $[0,p_0]$ and for all $\boldsymbol{\beta}$ in $[\![0, M_n-1]\!]^d$ we have

$$\int_{\prod_{i=1}^d [\frac{\beta_i}{M_n},\frac{\beta_i+1}{M_n})} \widetilde{w}_{n+1}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' = \int_{\prod_{i=1}^d [\frac{\beta_i}{M_n},\frac{\beta_i+1}{M_n})} \widetilde{w}_n(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'.$$

I.E., we need to show that

$$\int_{\prod_{i=1}^d [\frac{\beta_i}{M_n},\frac{\beta_i+1}{M_n})} w_n(\boldsymbol{x},\boldsymbol{y},H_n(\boldsymbol{y}'))\,\mathrm{d}\boldsymbol{y}' = \int_{\prod_{i=1}^d [\frac{\beta_i}{M_n},\frac{\beta_i+1}{M_n})} w_{n+1}(\boldsymbol{x},\boldsymbol{y},H_{M_n,m_{n+1}}\circ H_n(\boldsymbol{y}'))\,\mathrm{d}\boldsymbol{y}'.$$

But H_n maps any hypercube $[\beta_i/M_n, (\beta_i+1)/M_n)$ to another hypercube $[\beta_i'/M_n, (\beta_i'+1)/M_n)$ and H_n is measure preserving therefore, we only need to prove that for almost all $\boldsymbol{x}$ in Ω, almost all $\boldsymbol{y}$ in $[0,p_0]$ and for all $\boldsymbol{\beta}$ in $[\![0, M_n-1]\!]^d$

$$\int_{\prod_{i=1}^d [\frac{\beta_i}{M_n},\frac{\beta_i+1}{M_n})} w_n(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' = \int_{\prod_{i=1}^d [\frac{\beta_i}{M_n},\frac{\beta_i+1}{M_n})} w_{n+1}(\boldsymbol{x},\boldsymbol{y},H_{M_n,m_{n+1}}(\boldsymbol{y}'))\,\mathrm{d}\boldsymbol{y}'.$$

is satisfied. But this equality is equivalent to

$$\begin{aligned} \frac{1}{M_n^d} v_{p_n}(\boldsymbol{x},\boldsymbol{\beta},\boldsymbol{y}) &= \\ &= \frac{1}{M_{n+1}^d} \sum_{\boldsymbol{\beta}'\in[\![0,m_{n+1}-1]\!]^d} v_{p_{n+1}}(\boldsymbol{x}, R_{M_n,m_{n+1}}(m_{n+1}\boldsymbol{\beta}+\boldsymbol{\beta}'),\boldsymbol{y}) = \\ &= \frac{1}{M_{n+1}^d} \sum_{\boldsymbol{\beta}'\in[\![0,m_{n+1}-1]\!]^d} v_{p_{n+1}}(\boldsymbol{x}, (\boldsymbol{\beta}+\boldsymbol{\beta}'M_n),\boldsymbol{y}), \end{aligned}$$

which is true by Proposition 3.3. Therefore, the sequence $\widetilde{w}_n$ is a martingale for the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

By Proposition 2.3, this martingale is bounded in L^2. It converges both strongly in $\mathrm{L}^2(\Omega\times[0,p_0]^d\times[0,1]^d)$ and almost everywhere to a function $\widetilde{w}_\infty$, see [6, Corollary 7.22]. □

It is possible to recover u_{0,p_n} from the two-scale shuffle limit $\widetilde{w}_\infty$. First, for all $\boldsymbol{\beta}$ in $[\![0, M_n - 1]\!]^d$, all $\boldsymbol{y}'$ in $\prod_{d=1}^n [\beta_i, \beta_i + 1/M_n)$, and almost all $(\boldsymbol{x}, \boldsymbol{y})$ in $\Omega \times [0, p_0]^d$, we have

$$\widetilde{w}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}') = \frac{1}{M_n^d} \int_{\prod_{d=1}^n [\beta_i, \beta_i + 1/M_n)} \widetilde{w}_\infty(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}') \, \mathrm{d}\boldsymbol{y}'$$

because $\widetilde{w}_n = \mathbb{E}(\widetilde{w}_\infty | \mathcal{F}_n)$. Since the shuffle function H_n is one to one from $[0,1)^d$ to $[0,1)^d$, see Remark 2, we have $w_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}') = \widetilde{w}_n(\boldsymbol{x}, \boldsymbol{y}, H_n^{-1}(\boldsymbol{y}'))$. Finally, $u_{0,p_n}(\boldsymbol{x}, \boldsymbol{y})$ is equal to the value taken by $w_n(\boldsymbol{x}, \boldsymbol{y} - p_0 \lfloor \boldsymbol{y}/p_0 \rfloor, \cdot)$ on the interval $[\lfloor \boldsymbol{y}/p_0 \rfloor / M_n, (\lfloor \boldsymbol{y}/p_0 \rfloor + 1)/M_n)$.

4 Application: heat equation in multilayers

In [11], the author established the equations satisfied by the two-scale limit of the heat equation in multilayers with transmission conditions between adjacent layers.

As noted in [11, Remark 6.2] the equation satisfied by the two-scale limit depends on the number of layers present in the homogenization period. In this section, our goal is to establish the equation satisfied by the limit of two-scale limits in the sense of Theorem 3.5 ?

In this section, we only homogenize in the last space variable. We also use the following notations, $\triangle_\mathrm{T} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and $\nabla_\mathrm{T} = [\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}]^\mathrm{T}$

To avoid unecessary complications, we consider here a simpler problem than the one considered in [11, System (4.1)]. Let Ω be $B \times (0,1)$ where B is a convex bounded open subset of $\mathbb{R}^+$ with smooth boundary. Let δ, $0 < \delta < 1/2$. For all N, let I be the interval $(\delta, 1-\delta)$. Let I_N be $\bigcup_{j=0}^{N-1}((j+\delta)/n, (j+1-\delta)/n)$. Let Ω^N be the domain $B \times I_N$. Let $\Gamma_j^{N,+} = B \times \{(j+\delta)/N\}$ and $\Gamma_j^{N,-} = B \times \{(j-\delta)/N\}$. Let $\Gamma^{N,+} = \bigcup_{j=1}^{N-1} \Gamma_j^{N,+}$ and $\Gamma^{N,-} = \bigcup_{j=1}^{N-1} \Gamma_j^{N,-}$. Let $\Gamma_e = \partial B \times (0,1) \cup \Gamma_0^{N,+} \cup \Gamma_N^{N,-}$. Let γ be the application on $\partial \Omega^N$ that maps u in $\mathrm{H}^1(\Omega^N)$ to its trace on $\partial\Omega^N$. Let $\gamma' u$ be the trace swapped between $\Gamma_j^{N,+}$ and $\Gamma_j^{N,-}$.

For all positive integer N, we consider the heat equation

$$\frac{\partial u_N}{\partial t} - A \triangle \boldsymbol{u_N} = 0 \text{ in } \Omega^N \times \mathbb{R}^+ \tag{4.1a}$$

with the boundary conditions

$$A \frac{\partial u_N}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \Gamma_e \times \mathbb{R}^+ \\ -\frac{K}{N} \gamma u_N + \frac{J}{N}(\gamma' u_N - \gamma u_N) & \text{on } (\Gamma^{N,+} \cup \Gamma^{N,+}) \times \mathbb{R}^+. \end{cases} \tag{4.1b}$$

and the initial condition

$$u(\cdot, 0) = u_0. \tag{4.1c}$$

Let $(M_n)_{n\geq 0}$ be a sequence of positive integers such that $M_0 = 1$ and such that M_{n+1} is always a multiple of M_n. Using two-scale convergence[1, 9] and its variant on periodic surfaces[7, 8, 3], the properties of u_{0,M_n}, the M_n-two-scale limit of $(u^N)_{N\in\mathbb{N}}$, were established in [11, Theorem 6.1]. First, the two-scale limit $u_{0,M_n}(\boldsymbol{x}, t, \cdot)$ is constant on each interval $(j+\delta, j+1-\delta)$. We note $u_{0,M_n,j}(\boldsymbol{x}, t)$ the value of $u_{0,M_n}(\boldsymbol{x}, t, \cdot)$ on this interval. These functions satisfies, for all j in $\mathbb{Z}/M_n\mathbb{Z}$, the weak forumulation of

$$\frac{\partial u_{0,M_n,j}}{\partial t} - A \triangle_{\mathrm{T}} u_{0,M_n,j} + \frac{2K}{1-2\delta} u_{0,M_n,j} + \frac{J}{1-2\delta}(2u_{0,M_n,j} - u_{0,M_n,j+1} - u_{0,M_n,j-1}) = 0, \tag{4.2a}$$

in $\Omega^N \times \mathbb{R}^+$, with boundary conditions

$$\frac{\partial u_{0,M_n,j}}{\partial \boldsymbol{\nu}} = 0 \text{ on } \partial B \times \mathbb{R}^+ \times (0, 1). \tag{4.2b}$$

We now use Theorem 3.5 to have the two-scale limits themselves converge. We choose $M_n = 2^n$ to avoid complications at first. We establish the following:

Theorem 4.1. *Let $\widetilde{w_n}$ be the two-scale shuffle limit defined from $u_{0,2^n}$ as in Theorem 3.5. For all $\boldsymbol{x}$ in $\Omega \times \mathbb{R}^+$, and y' in $[0, 1]$. The function $\widetilde{w}_n(\boldsymbol{x}, t, \cdot, y')$ is constant on $(\delta, 1-\delta)$. If we denote by $\widetilde{w}_n(\boldsymbol{x}, t, y')$ the value of $\widetilde{w}_n(\boldsymbol{x}, t, \cdot, y')$ on $(\delta, 1-\delta)$, the two-scale shuffle limit $\widetilde{w}_\infty$ is a weak solution to:*

$$\begin{aligned}\frac{\partial \widetilde{w}_\infty}{\partial t}(\boldsymbol{x}, t, y') - A \triangle_{\mathrm{T}} \widetilde{w}_\infty(\boldsymbol{x}, t, y') + \frac{2K}{1-2\delta}\widetilde{w}_\infty(\boldsymbol{x}, t, y') \\ + \frac{J}{1-2\delta}(2\widetilde{w}_\infty(\boldsymbol{x}, t, y') - \widetilde{w}_\infty(\boldsymbol{x}, t, \tau^+(y')) - \widetilde{w}_\infty(\boldsymbol{x}, t, \tau^-(y'))) = 0,\end{aligned} \tag{4.3a}$$

in $\Omega^N \times \mathbb{R}^+ \times (0, 1)$, and where, for all non negative integers j:

$$\tau^+(y') = y' + 3 \cdot 2^{-(j+1)} - 1 \text{ when } 1 - 2^{-j} \leq y' < 1 - 2^{-(j+1)},$$
$$\tau^-(y') = y' - 3 \cdot 2^{-(j+1)} + 1 \text{ when } 2^{-(j+1)} \leq y' < 2^{-j}.$$

with boundary conditions

$$\frac{\partial u_{0,M_n,j}}{\partial \boldsymbol{\nu}} = 0 \text{ on } \partial B \times \mathbb{R}^+ \times (0, 1). \tag{4.3b}$$

Proof. Consider a test function φ belonging to $\mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}^+)$. Let n in $\mathbb{N}$ and β belongs to $[\![0,2^n-1]\!]$. Set $\psi(\boldsymbol{x},t,y)=\phi(x,t)1\{y\in[\frac{\beta}{2^n},\frac{\beta+1}{2^n}\}$. Then

$$\begin{aligned}
&\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}\frac{\partial w_n}{\partial t}(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+A\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}\nabla_{\mathrm{T}}\partial w_n(\boldsymbol{x},t,y')\cdot\nabla_{\mathrm{T}}\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\frac{2K}{1-2\delta}\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}w_n(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\frac{J}{1-2\delta}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}2w_n(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&-\frac{J}{1-2\delta}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}(w_n(\boldsymbol{x},t,y'+2^{-n})+w_n(\boldsymbol{x},t,y'-2^{-n})\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0,
\end{aligned}$$

where, to simplify notations, we consider the function w_n to be 1-periodic in y'. Therefore, for all β in $[\![0,2^n-1]\!]$,

$$\begin{aligned}
&\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}\frac{\partial\widetilde{w}_n}{\partial t}(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+A\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}\nabla_{\mathrm{T}}\partial\widetilde{w}_n(\boldsymbol{x},t,y')\cdot\nabla_{\mathrm{T}}\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\frac{2K}{1-2\delta}\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}\widetilde{w}_n(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\frac{J}{1-2\delta}\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}2\widetilde{w}_n(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&-\frac{J}{1-2\delta}\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}\widetilde{w}_n(\boldsymbol{x},t,H_n^{*-1}(H_n^*(y')+2^{-n}))\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&-\frac{J}{1-2\delta}\iint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})}\widetilde{w}_n(\boldsymbol{x},t,H_n^{*-1}(H_n^*(y')-2^{-n}))\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0
\end{aligned}\tag{4.4}$$

Here H_n^* is simply the bit reversal of the first n coefficients in the binary expansion. Thus:

$$H_n^{*-1}(H_n^*(y'+2^{-n}))=\begin{cases}y'+3\cdot2^{-(j+1)}-1 & \text{if } 1-2^{-j}\le y'<1-2^{-(j+1)},\ 0\le j\le n-1,\\ y'-1+2^{-n} & \text{if } 1-2^{-n}\le y'<1.\end{cases}\tag{4.5}$$

And

$$H_n^{*-1}(H_n^*(y')-2^{-n}) = \begin{cases} y' - 3\cdot 2^{-(j+1)} + 1 & \text{if } 2^{-(j+1)} < y' \le 2^{-j},\ 0 \le j \le n-1, \\ y' + 1 - 2^{-n} & \text{if } 0 < y' \le 2^{-n}. \end{cases} \tag{4.6}$$

SInce $\phi(\boldsymbol{x},t)1\{y' \in [\frac{\beta}{2^n}, \frac{\beta+1}{2^n}\}$ is $\mathcal{F}_n$-measurable and $\widetilde{w}_n = \mathbb{E}(\widetilde{w}_\infty|\mathcal{F}_n)$, Equality (4.4) remains valid after replacing $\widetilde{w}_n$ by $\widetilde{w}_\infty$. Therefore,

$$\begin{aligned} &\iiint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})} \frac{\partial \widetilde{w}_\infty}{\partial t}(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+A\iiint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})} \nabla_{\mathrm{T}}\widetilde{w}_\infty(\boldsymbol{x},t,y')\cdot\nabla_{\mathrm{T}}\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+\frac{2K}{1-2\delta}\iiint_{Q_T}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})} \widetilde{w}_\infty(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+\frac{J}{1-2\delta}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})} 2\widetilde{w}_\infty(\boldsymbol{x},t,y')\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &-\frac{J}{1-2\delta}\int_{[\frac{\beta}{2^n},\frac{\beta+1}{2^n})} (\widetilde{w}_\infty(\boldsymbol{x},t,\tau^+(y')) - \widetilde{w}_\infty(\boldsymbol{x},t,\tau^-(y')))\cdot\phi(\boldsymbol{x},t)\,\mathrm{d}y'\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0 \end{aligned}$$

for all n in $\mathbb{N}$ and β in $[\![1, 2^n-2]\!]$. Choose y' in $(0,1)$, for any positive integer n, set $\beta = \lfloor 2^n y' \rfloor$ and take the limit in the above equality divided by 2^{-n} as n tends to $+\infty$. □

If instead of setting $M_n = 2^n$, we consider a general sequence $(M_n)_{n\in\mathbb{N}}$, the same reasonning holds. When M_n is 2^n, the shuffling of layers is the bit reversal of the first n coefficients of the binary representation of y, thus involutive. This is not the case for general M_n and we must use Remark 2. Therefore, utmost care must be taken to compute the analogue of (4.5) and (4.6). We provide the limit in the general case without proof. In that case, we have

$$H_n^{*-1}(H_n^*(y')+\frac{1}{M_n}) = \begin{cases} y' - \sum_{l=1}^{j}\frac{1}{M_l} + \frac{1}{M_{j+1}} & \text{if } 1-\frac{1}{M_j} \le y' < 1 - \frac{1}{M_{j+1}},\ 0\le j\le n-1, \\ y' - 1 + \frac{1}{M_n} & \text{if } 1-\frac{1}{M_n} \le y' < 1. \end{cases}$$

$$H_n^{*-1}(H_n^*(y')-\frac{1}{M_n}) = \begin{cases} y' + \sum_{l=1}^{j}\frac{1}{M_l} - \frac{1}{M_{j+1}} & \text{if } 1-\frac{1}{M_j} \le y' < 1 - \frac{1}{M_{j+1}},\ 0\le j\le n-1, \\ y' + 1 - \frac{1}{M_n} & \text{if } 0 < y' \le \frac{1}{M_n}. \end{cases}$$

and the limit equation (4.3a) remains valid if we set instead

$$\tau^{+}(y') = y' - \sum_{l=1}^{j} \frac{1}{M_l} + \frac{1}{M_{j+1}} \text{ when } 1 - \frac{1}{M_j} \leq y' < 1 - \frac{1}{M_{j+1}} \tag{4.7a}$$

$$\tau^{-}(y') = y' + \sum_{l=1}^{j} \frac{1}{M_l} - \frac{1}{M_{j+1}} \text{ when } \frac{1}{M_j} < y' \leq 1 - \frac{1}{M_{j+1}}, \tag{4.7b}$$

for all non negative integer j.

5 Conclusion

We have proven in this paper that the two-scale limits of a given sequence of functions computed for periods that are multiple of the previous ones, form a bounded martingale and thus converge both strongly in L^2 and almost everywhere. From the limit, called the two-scale shuffle limit, one can recover any element in the sequence of two-scale limits: this limit contains all the information contained in the whole sequence of two-scale limits. For a good choice of increasing periods, this limits captures everything that happens at any length scale that is a multiple of ε. We established the equation satisfied by the two-scale shuffle limit for the solution to the heat equation in multilayers with transmission conditions between layers.

Unfortunately, this limit does not capture all phenomena with a period linear in ε: it cannot capture phenomena with an irrational scale factor. The construction of the martingale depends on the assumption that p_{n+1} is always a multiple of p_n. If there are two interesting scales whose ratio is irrational then no choice of periodic scale carry the information for both scale.

While we were able to conclude using the rearrangement described in §3.3, we feel results on the existence of the limit in the setting of §3.2 would be more satisfying. Solving Problem 3.4 would be a first step to obtain a limit in this setting.

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