hal-00466159, version 2 - 25 Sep 2011

From finite-gap solutions of KdV in terms of theta functions to solitons and positons

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Abstract

We degenerate the finite gap solutions of the KdV equation from the general formulation in terms of abelian functions when the gaps tends to points, to recover solutions of KdV equations given a few years ago in terms of wronskians called solitons or positons. For this we establish a link between Fredholm determinants and Wronskians.

1 The KdV equation and solutions in terms of theta functions

We consider the Riemann surface Γ of the algebraic curve defined by $\omega^2 = \prod_{j=1}^{2g+1}(z-E_j)$, with $E_j \neq E_k$, $j \neq k$. Let D be some divisor $D = \sum_{j=1}^{g} P_j$, $P_j \in \Gamma$. The so-called finite gap solution of the KdV equation

$$u_t = 6uu_x - u_{xxx} \tag{1}$$

can be expressed in the form [7]

$$u(x,t) = -2\frac{d^2}{dx^2}\ln\theta(xg+tv+l)+C. \tag{2}$$

We recall briefly, the notations. In (2), θ is the Riemann function defined by

$$\theta(z) = \sum_{k\in\mathbf{Z}^g}\exp\{\pi i(Bk|k)+2\pi i(k|z)\}, \tag{3}$$

hal-00466159, version 2 - 25 Sep 2011

constructed from the matrix of the B-periods of the surface Γ, and the vectors g, v, l are defined by

$$g_j = 2ic_{j1}, \tag{4}$$

$$v_j = 8i(\frac{c_{j1}}{2}\sum_{k=1}^{2g+1} E_k + c_{j2}), \tag{5}$$

$$l_j = -\sum_{k=1}^{g}\int_{\infty}^{P_k} dU_j + \frac{j}{2} - \frac{1}{2}\sum_{k=1}^{g} B_{kj}, \tag{6}$$

$$C = \sum_{k=1}^{2g+1} E_k - 2\sum_{k=1}^{g}\int_{a_k} z\,dU_k, \tag{7}$$

the coefficients c_{jk} being relating with abelian differential dU_j by

$$dU_j = \frac{\sum_{k=1}^{g} c_{jk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+1}(z-E_k)}}dz, \tag{8}$$

and coefficients c_{jk} can be obtained by solving the system of linear equations

$$\int_{a_k} dU_j = \delta_{jk}, \quad 1 \leq j \leq g, \quad 1 \leq k \leq g.$$

2 Degeneracy of solutions

We suppose that E_j are real, $E_m < E_j$ if $m < j$ and try to evaluate the limits of all objects in formula (2) when E_{2m}, E_{2m+1} tends to $-\alpha_m$, $-\alpha_m = \kappa_m^2$, $\kappa_m > 0$, for $1 \leq m \leq g$, and E_1 tends to 0 (these ideas were first presented by A. Its and V.B. Matveev, exposed for example in [1]).

2.0.1 Limit of $P(z) = \prod_{j=1}^{2g+1}(z - E_j)$

The limit of $P(z) = \prod_{j=1}^{2g+1}(z-E_j)$ is evidently equal to $\tilde{P}(z) = z\prod_{j=1}^{g}(z+\alpha_j)^2$

2.0.2 Limit of $dU_m = \frac{\sum_{k=1}^{g} c_{mk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+1}(z-E_k)}}dz$

The limit of dU_m is equal to $d\tilde{U}_m = \frac{\varphi_m(z)}{\sqrt{z}\prod_{j=1}^{g}(z+\alpha_j)}dz$, where $\varphi_m(z) = \sum_{k=1}^{g}\tilde{c}_{mk} z^{g-k}$. The normalization condition takes the form in the limit

$$\int_{a_k} dU_j \to \frac{2\pi i\varphi_j(-\alpha_k)}{\kappa_k \prod_{m\neq k}(\alpha_m - \alpha_k)} = \delta_{kj}, \tag{9}$$

hal-00466159, version 2 - 25 Sep 2011

which proves that the numbers $-\alpha_m$, $m \neq k$ are the zeros of the polynomials $\varphi_k(z)$, and so $\varphi_k(z)$ can be written as $\varphi_k(z) = \tilde{c}_{k1} \prod_{m\neq k}(z+\alpha_m)$. By (9), we get in the limit

$$\tilde{c}_{k1} = \frac{\kappa_k}{2\pi i}.$$

So

$$d\tilde{U}_k = \frac{\kappa_k}{2\pi i\sqrt{z}(z+\alpha_k)}dz$$

2.0.3 Limit of v_k and g_k

By identification of the powers of z^{g-2} in (10)

$$\tilde{\varphi_k} = \tilde{c_{k1}} \prod_{l\neq k}(z+\alpha_l) = \sum_{j=1}^{g} \tilde{c_{kj}} z^{g-j}, \tag{10}$$

we get in the limit

$$\tilde{c}_{k1} \sum_{l=1}^{g} -\alpha_l + \tilde{c}_{k2} = \frac{\kappa_k^3}{2\pi i}.$$

So we have the limit values of v_k and g_k :

$$\tilde{v}_k = \frac{4}{\pi}\kappa_k^3$$

and

$$\tilde{g}_k = \frac{1}{\pi}\kappa_k.$$

2.0.4 Limit of $U_j(P)$ and B_{mk}

For $\lambda_0 = -\alpha_m = \kappa_m^2$, $I = \int_{\lambda_0}^{0} dU_k \rightarrow \frac{1}{2}\tilde{B}_{mk}$. The integral I can be easily evaluate along the real axis on the upper sheet of surface Γ and we get

$$I \rightarrow \frac{i}{2\pi} \ln\left|\frac{\kappa_m+\kappa_k}{\kappa_m-\kappa_k}\right|.$$

So we have the limit values of matrix B :

$$\tilde{B}_{mk} = \frac{i}{\pi} \ln\left|\frac{\kappa_m+\kappa_k}{\kappa_m-\kappa_k}\right|.$$

So iB_{kk} tends to $-\infty$. As previously, we have

$$\int_{\infty}^{P} dU_j \rightarrow -\frac{i}{2\pi} \ln\left|\frac{\kappa_j-\sqrt{z_P}}{\kappa_j+\sqrt{z_P}}\right|. \tag{11}$$

hal-00466159, version 2 - 25 Sep 2011

2.0.5 Limit of argument of exponential in $\theta(p)$

Let us denote A the argument of $\theta(p) = \sum_{k\in \mathbf{Z}^g} \exp\{\pi i(Bk|k) + 2\pi i(k|p)\}$.
A can be rewritten in the form

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j-1) + 2\pi i \sum_{j>m} B_{mj}k_m k_j + \sum_{j=1}^{g} \pi i(2p_j + B_{jj})k_j. \quad (12)$$

Using the inequality $k_j(k_j-1) \geq 0$ for all $k \in \mathbf{Z}^g$ and the fact that iB_{kk} tends to $-\infty$, we can reduce the limit $\tilde{\theta}$ of $\theta(p)$ to a finite sum taken over vectors $k \in \mathbf{Z}^g$ such that each k_j must be equal to 0 or 1.
So, A can be rewritten in the form

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j-1) + 2\pi i \sum_{j>m} B_{mj}k_m k_j + \sum_{j=1}^{g} k_j[2\pi i(g_j x + v_j t)$$

$$-\pi i(-j + 2\sum_{k=1}^{g}\int_{\infty}^{P_k} dU_j + \sum_{m\neq j} B_{mj})].$$

In other words

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j-1) + 2\pi i \sum_{j>m} B_{mj}k_m k_j + \sum_{j=1}^{g} k_j Q_j,$$

with

$$Q_j = 2\pi i(g_j x + v_j t) + \beta_j$$

and

$$\beta_j = -\pi i(-j + 2\sum_{k=1}^{g}\int_{\infty}^{P_k} dU_j + \sum_{m\neq j} B_{mj}).$$

The quantity β_j has a finite limit value $\tilde{\beta}_j$ independent from x and t.

2.0.6 Limit of $\theta(p)$

By means of the inequality $K_j(K_j-1) \geq 0$ for all $K \in \mathbf{Z^g}$ and the previous relation iB_{kk} tends to $-\infty$, it turns out that the limit $\tilde{\theta}$ of $\theta(xg + tv + l)$ reduce to a finite sum taken over vectors $k \in \mathbf{Z^g}$ with the property that each k_j must be equal to 0 or 1.

$$\tilde{\theta} = \sum_{k\in\mathbf{Z}^g,\, k_j=0\, or\, 1} \exp\{\sum_{m>j} 2\ln\left|\frac{\kappa_m - \kappa_j}{\kappa_m + \kappa_j}\right| k_m k_j + i(\sum_{j=1}^{g} 2\kappa_j x + 8\kappa_j^3 t + 2\kappa_j x_j + \pi j$$

hal-00466159, version 2 - 25 Sep 2011

$$-\sum_{m\neq j} i\ln\left|\frac{\kappa_m+\kappa_j}{\kappa_m-\kappa_j}\right|)k_j\},$$

with

$$x_j = x_j(\kappa_j) = \frac{1}{2i\kappa_j}\sum_{k=1}^{g}\ln\left|\frac{\kappa_j-\sqrt{z_k}}{\kappa_j+\sqrt{z_k}}\right|.$$

It can be rewritten as

$$\tilde{\theta} = \sum_{J\subset\{1,\dots,g\}}\prod_{j\in J}\prod_{j,k\in J\ j<k}\left|\frac{\kappa_j-\kappa_k}{\kappa_j+\kappa_k}\right|^2 \exp 2i\sum_{j\in J}(\kappa_j x+4\kappa_j^3 t+\kappa_j x_j).\prod_{j\in J\ k\neq j}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right|.(13)$$

Using the equality

$$\prod_{j,k\in J\ j<k}\left|\frac{\kappa_j-\kappa_k}{\kappa_j+\kappa_k}\right|^2\prod_{j\in J\ k\neq j}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right| = \prod_{j\in J\ k\notin J}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right|,$$

it can be reduced to

$$\tilde{\theta} = \sum_{J\subset\{1,\dots,g\}}\prod_{j\in J}\prod_{j\in J\ k\notin J}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right|\exp 2i\sum_{j\in J}(\kappa_j x+4\kappa_j^3 t+\kappa_j x_j). \qquad (14)$$

3 1-positon of order 1

A 1-positon of order 1 is given by (see [11]),

$$u = -2\partial_x^2\log W(\phi,\partial_K\phi),$$

where W is the classical wronskian, and ϕ the function

$$\phi = \phi(x,K) = \sin(K(x+x_1(K)+4K^2t))$$

In this case

$$W = \frac{1}{2}(\sin 2\Theta - 2K\gamma),$$

where $\Theta = K(x+x_1(K)+4K^2t)$ and $\gamma = \partial_K\Theta$.
We consider here a Riemann surface of genus 2.
Using the previous section, when we take the limit as E_j tend to K_j^2 the function θ tends to $\tilde{\theta}$ which takes the form

$$\tilde{\theta} = 1 - \tfrac{K_2+K_1}{K_2-K_1}\exp 2i(K_1+4K_1^3t+K_1x_1(K_1)) + \tfrac{K_2+K_1}{K_2-K_1}\exp 2i(K_2+4K_2^3t+K_2x_2(K_2))$$
$$-\exp 2i((K_1+K_2)+4(K_1^3+K_2^3)t+K_1x_1(K_1)+K_2x_2(K_2)).$$

hal-00466159, version 2 - 25 Sep 2011

It can be rewritten as

$$\begin{array}{l}\tilde{\theta} = 1 + (K_2 + K_1)\frac{\exp 2i(K_2+4K_2^3t+K_2x_2(K_2))-\exp 2i(K_1+4K_1^3t+K_1x_1(K_1))}{K_2-K_1} \\ -\exp 2i((K_1 + K_2) + 4(K_1^3 + K_2^3)t + K_1x_1(K_1) + K_2x_2(K_2)).\end{array} \tag{15}$$

Now, it is clear that when K_2 tends to $K_1 = K$, we get

$$\tilde{\theta} = 1 + 2Ki\exp(2i\Theta)2i\partial_K\Theta - \exp(4i\Theta). \tag{16}$$

It can be reduced to

$$\tilde{\theta} = -2i\exp(2i\Theta)(\sin 2\Theta - 2K\gamma) = -4i\exp(2i\Theta) \times W. \tag{17}$$

Since Θ is a linear function of x, we recover exactly the 1-positon

$$u(x,t) = -2\partial_x^2 \ln\tilde{\theta} = -2\partial_x^2 W(\phi,,\partial_K\phi) = \frac{16K^2\sin\Theta(\sin\Theta - K\gamma\cos\Theta)}{(\sin 2\Theta - 2K\gamma)^2}.$$

4 1-positon of order 2

We define a 1-positon of order 2 by (see [11]),

$$u = -2\partial_x^2 \log W(\phi, \partial_K\phi, \partial_K^2\phi),$$

where W is the classical wronskian, and ϕ the function

$$\phi = \phi(x,K) = \sin(K(x + x_1(K) + 4K^2t))$$

In this case

$$W = 4\gamma^2K^2\cos\Theta - 2K\gamma\sin\Theta - 48K^3t\sin\Theta - \sin 2\Theta\sin\Theta,$$

where $\Theta = K(x + x_1(K) + 4K^2t)$ and $\gamma = \partial_K\Theta$.
Here we consider a Riemann surface of genus 3.
We can write the function θ using the previous section. When we take the limit as E_j tend to K_j^2 the function θ tends to $\tilde{\theta}$ which takes the form

$$\begin{array}{l}\tilde{\theta} = 1 - \left|\frac{K_1+K_2}{K_1-K_2}\right|\left|\frac{K_1+K_3}{K_1-K_3}\right|\exp 2i(K_1 + 4K_1^3t + K_1x_1(K)) \\ + \left|\frac{K_2+K_1}{K_2-K_1}\right|\left|\frac{K_2+K_3}{K_2-K_3}\right|\exp 2i(K_2 + 4K_2^3t + K_2x_2(K)) \\ - \left|\frac{K_3+K_1}{K_3-K_1}\right|\left|\frac{K_3+K_2}{K_3-K_2}\right|\exp 2i(K_3 + 4K_3^3t + K_3x_3(K)) \\ - \left|\frac{K_1+K_3}{K_1-K_3}\right|\left|\frac{K_2+K_3}{K_2-K_3}\right|\exp 2i(K_1 + 4K_1^3t + K_1x_1(K) + K_2 + 4K_2^3t + K_2x_2(K)) \\ + \left|\frac{K_1+K_2}{K_1-K_2}\right|\left|\frac{K_3+K_2}{K_3-K_2}\right|\exp 2i(K_1 + 4K_1^3t + K_1x_1(K) + K_3 + 4K_3^3t + K_3x_3(K)) \\ - \left|\frac{K_2+K_1}{K_2-K_1}\right|\left|\frac{K_3+K_1}{K_3-K_1}\right|\exp 2i(K_2 + 4K_2^3t + K_2x_2(K) + K_3 + 4K_3^3t + K_3x_3(K)) \\ + \exp 2i(K_1 + 4K_1^3t + K_1x_1(K) + K_2 + 4K_2^3t + K_2x_{21}(K) + K_3 + 4K_3^3t + K_3x_3(K))\end{array}$$

hal-00466159, version 2 - 25 Sep 2011

Now, it is clear that when K_2 and K_3 tends to $K_1 = K$, we get

$$\tilde{\theta} = 1 - \exp(2i\Theta) - 2K^2(-4\gamma^2 + 48iK\gamma t)\exp(2i\Theta) - 4iK\gamma\exp(2i\Theta) + 16K^2\gamma^2\exp(4i\Theta) \\ - \exp(4i\Theta) + \exp(6i\Theta) + 4iK\gamma\exp(4i\Theta) + 2K^2(-4\gamma^2 + 48i\gamma Kt)\exp(4i\Theta).$$

It can be reduced to

$$\tilde{\theta} = -4\exp(3i\Theta)(4\gamma^2K^2\cos\Theta - 2K\gamma\sin\Theta - 48K^3t\sin\Theta - \sin 2\Theta\sin\Theta). \tag{18}$$

As Θ is linear in x, we recover exactly the 1-positon of order 2

$$u(x,t) = -2\partial_x^2 \ln\tilde{\theta} = -2\partial_x^2 W(\phi, \partial_K\phi, \partial_K^2\phi).$$

5 From Theta to Wronskian

5.1 From Theta to Fredholm

We consider the following matrix $A = (a_{jk})_{1\leq j,k\leq N}$ defined by

$$a_{jk} = \prod_{l\neq k}\left|\frac{K_l + K_j}{K_l - K_k}\right|\exp(i(K_jx + 8K_j^3t + 2K_jx_j), \tag{19}$$

where x_j is an arbitrary parameter. Then $\det(I + A)$ has the following form

$$\det(I + A) = \sum_{J\subset\{1,\ldots,N\}}\prod_{j\in J}\prod_{j\in J\, k\notin J}\left|\frac{K_j + K_k}{K_j - K_k}\right|\exp(2i\sum_{j\in J}(K_jx + 4K_j^3t + K_jx_j). \tag{20}$$

By the previous section,

$$\tilde{\theta} = \sum_{J\subset\{1,\ldots,g\}}\prod_{j\in J}\prod_{j\in J\ k\notin J}\left|\frac{\kappa_j + \kappa_k}{\kappa_j - \kappa_k}\right|\exp 2i\sum_{j\in J}(\kappa_jx + 4\kappa_j^3t + \kappa_jx_j). \tag{21}$$

If we compare the expression (20) to (21), we have clearly the equality

$$\tilde{\theta} = \det(I + A). \tag{22}$$

It remains to find the link between this Fredholm determinant and a certain wronskian.

hal-00466159, version 2 - 25 Sep 2011

5.2 From Fredholm to Wronskians

In this section, we consider the following functions

$$\phi_j(x) = \sin(K_j x + 4K_j^3 t + K_j x_j), \tag{23}$$

where K_j are real numbers such that $K_1 \leq \ldots \leq K_N$, and x_j an arbitrary constant independent of x.
We use the following notations :
$\theta_j = K_j x + 4K_j^3 t + K_j x_j$.
$W = W(\phi_j, \ldots, \phi_N)$ is the classical Wronskian $W = \det[(\partial_x^{j-1}\phi_i)_{i,\, j \in [1,\ldots,N]}]$.
We consider the matrix $A = (a_{jk})_{j,\, k \in [1,\ldots,N]}$ defined by

$$a_{jk} = \prod_{l \neq k} \left| \frac{K_l + K_j}{K_l - K_k} \right| \exp(i(K_j x + 8K_j^3 t + 2K_j x_j). \tag{24}$$

Then we have the following statement

Theorem 5.1

$$\det(I + A) = \frac{2^N i^{\frac{N(N+5)}{2}} \exp(i \sum_{j=1}^N \theta_j)}{\prod_{j=2}^N \prod_{i=1}^{j-1} (K_j - K_i)} W(\phi_1, \ldots, \phi_N) \tag{25}$$

Proof : We start to remove the factor $(2i)^{-1} e^{i\theta_j}$ in each row j in the Wronskian W for $1 \leq \mathrm{j} \leq N$.
Then

$$W = \prod_{j=1}^N e^{i\theta_j} (2i)^{-N} \times W_1, \tag{26}$$

with

$$W_1 = \begin{vmatrix} (1 - e^{-2i\theta_1}) & iK_1(1 + e^{-2i\theta_1}) & \ldots & (iK_1)^{N-1}(1 + (-1)^N e^{-2i\theta_1}) \\ (1 - e^{-2i\theta_2}) & iK_2(1 + e^{-2i\theta_2}) & \ldots & (iK_2)^{N-1}(1 + (-1)^N e^{-2i\theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1 - e^{-2i\theta_N}) & iK_N(1 + e^{-2i\theta_N}) & \ldots & (iK_N)^{N-1}(1 + (-1)^N e^{-2i\theta_{2N}}) \end{vmatrix}$$

The determinant W_1 can be written as

$$W_1 = \det(\alpha_{jk} e_j + \beta_{jk}),$$

hal-00466159, version 2 - 25 Sep 2011

where $\alpha_{jk} = (-1)^k (iK_j)^{k-1}$, $e_j = e^{-2i\theta_j}$, and $\beta_{jk} = (iK_j)^{k-1}$.
Denoting $U = (\alpha_{ij})_{i,\,j\in[1,\ldots,N]}$, $V = (\beta_{ij})_{i,\,j\in[1,\ldots,N]}$, the determinant of U is clearly equal to

$$\det(U) = (-1)^{\frac{N(N+1)}{2}} (i)^{\frac{N(N-1)}{2}} \prod_{N\geq l>m\geq 1} (K_l - K_m). \tag{27}$$

Then we use the following Lemma

Lemma 5.1 *Let* $A = (a_{ij})_{i,\,j\in[1,\ldots,N]}$, $B = (b_{ij})_{i,\,j\in[1,\ldots,N]}$,
$(H_{ij})_{i,\,j\in[1,\ldots,N]}$*, the matrix formed by replacing the jth row of* A *by the ith row of* B
Then

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}) \tag{28}$$

Proof : For $\tilde{A} = (\tilde{a}_{ij})_{i,\,j\in[1,\ldots,N]}$ the matrix of cofactors of A, we have the well known formula $A \times^t \tilde{A} = \det A \times I$.
So it is clear that $\det(\tilde{A}) = (\det(A))^{N-1}$.
The general term of the product $(c_{ij})_{i,j\in[1,..,N]} = (a_{ij}x_i + b_{ij})_{i,j\in[1,..,N]} \times (\tilde{a}_{ij})_{i,j\in[1,..,N]}$ can be written as
$c_{ij} = \sum_{s=1}^{N} (a_{is}x_i + b_{is}) \times \tilde{a}_{js}$
$= x_i \sum_{s=1}^{N} a_{is}\tilde{a}_{js} + \sum_{s=1}^{N} b_{is}\tilde{a}_{js}$
$= \delta_{ij} \det(A) x_i + \det(H_{ij})$.
We get
$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)})$.
Thus $\det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)})$.
□
Using the previous lemma (28), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}),$$

where $(H_{ij})_{i,\,j\in[1,\ldots,N]}$ is the matrix formed by replacing the jth row of U by the ith row of V defined previously.
We compute $\det(H_{ij})$ and we get

$$\det(H_{ij}) = (-1)^{\frac{N(N+1)}{2}+1} (i)^{\frac{N(N-1)}{2}} \prod_{N\geq l>m\geq 1,\, l\neq j,\, m\neq j} (K_l - K_m) \prod_{l<j} (K_k - K_l) \prod_{l>j} (K_k - K_l). \tag{29}$$

hal-00466159, version 2 - 25 Sep 2011

We can simplify the quotient $q = \frac{\det(H_{ij})}{\det(\alpha_{ij})}$:

$$q = \frac{\prod_{l \neq k}(K_l + K_k)}{\prod_{l \neq k}(K_l - K_k)}. \tag{30}$$

So $\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})})$ can be expressed as

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{N} e^{-2i\theta_j} \det(\delta_{jk} + \prod_{l \neq k} \left| \frac{K_l + K_k}{K_l - K_k} \right| e^{2i\theta_j}),$$

and therefore

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{N} e^{-2i\theta_j} \det(I + A).$$

The Wronskian can be written as

$$W(\phi_1, \ldots, \phi_N) = \prod_{j=1}^{N} e^{i\theta_j} (2i)^{-N} (-1)^{\frac{N(N+1)}{2}} (i)^{\frac{N(N-1)}{2}} \prod_{j=2}^{N} \prod_{i=1}^{j-1} (K_j - K_i) \prod_{j=1}^{N} e^{-2i\theta_j} \det(I + A)$$

It follows that

$$\det(I + A) = \frac{e^{i \sum_{j=1}^{N} \theta_j} (2)^N (i)^{\frac{N(N+5)}{2}}}{\prod_{j=2}^{N} \prod_{i=1}^{j-1} (K_j - K_i)} W(\phi_1, \ldots, \phi_N) \tag{31}$$

□

6 Positons of arbitrary order

Now it is clear how to get multi-positons. It is the same strategy used for the preceding examples.
If we want to get the following general positon

$$u = -2\partial_x^2 \ln W(\phi_1, \ldots, \phi_1^{(k_1)}, \phi_2, \ldots, \phi_2^{(k_2)}, \ldots, \phi_l, \ldots, \phi_l^{(k_l)}), \tag{32}$$

we consider a Riemann surface of genus $g = \sum_{j=1}^{l} k_j + l$.
As defined in the first section, we suppose that E_j are real, $E_m \leq E_j$ if $m < j$ and we evaluate the limits of all objects in formula (2).
We consider $K_i > 0$, for $1 \leq m \leq g$ such that $K_m \leq K_j$ if $m < j$.

hal-00466159, version 2 - 25 Sep 2011

First, we choose the following limits :
E_1 tends to 0.
for $1 \leq i \leq k_1+1$, E_{2i}, E_{2i+1} tends to K_i^2;
for $1 \leq i \leq k_2+1$, $E_{2(k_1+1)+2i}$, $E_{2(k_1+1)+2i+1}$ tends to $K_{k_1+1+i}^2$;
we continue until;
for $1 \leq i \leq k_l+1$, $E_{2(k_1+\ldots k_{l-1}+l-1)+2i}$, $E_{2(k_1+\ldots k_{l-1}+l-1)+2i+1}$ tends to $K_{k_1+\ldots+k_{l-1}+l-1+i}^2$.
Then from the results (20), (22) and (25), we get

$$\tilde{\theta} = \det(I+A) = \sum_{J\subset\{1,\ldots,g\}} \prod_{j\in J}(-1)^j \prod_{j\in J\, k\notin J} \left|\frac{K_j+K_k}{K_j-K_k}\right| \exp\sum_{j\in J}(i(2K_jx+8K_j^3t+2K_jx_j).$$

$$= \frac{2^g i^{\frac{g(g+5)}{2}} \exp(i\sum_{j=1}^{g}\theta_j)}{\prod_{j=2}^{g}\prod_{i=1}^{j-1}(K_j-K_i)} W(\varphi_1,\ldots,\varphi_l),$$

with

$$\theta_j = K_jx + 4K_j^3 + K_jx_j,$$

and

$$\varphi_j = \sin(\theta_j).$$

We use the following notations :
for $1 \leq i \leq k_1+1$, $\varphi_i = \varphi(K_i)$;
for $1 \leq i \leq k_2+1$, $\varphi_{k_1+1+i} = \varphi(K_{k_1+1+i})$;
and so on until;
for $1 \leq i \leq k_l+1$, $\varphi_{k_1+\ldots+k_{l-1}+l-1+i} = \varphi(K_{k_1+\ldots+k_{l-1}+l-1+i})$.
We make here the following choice for K_j, $1 \leq j \leq g$:
for $1 \leq i \leq k_1+1$, $K_i = \kappa_1 + (i-1)h$;
for $1 \leq i \leq k_2+1$, $K_{k_1+1+i} = \kappa_2 + (i-1)h)$;
and so on until;
for $1 \leq i \leq k_l+1$, $K_{k_1+\ldots+k_{l-1}+l-1+i} = \kappa_l + (i-1)h$.
Then we consider the the classical difference derivative operator Δ_h defined by the formula :

$$\Delta_h f(x) = \frac{f(x+h)-f(x)}{h}.$$

It is easy to prove that

$$\Delta_h^j f(x) = \frac{1}{h^j}\sum_{k=0}^{j}(-1)^k C_j^k f(x+(j-k)h), \qquad (33)$$

hal-00466159, version 2 - 25 Sep 2011

and it is obvious that for any function $f(x) \in C^j$, (i.e. having j continuous derivatives),

$$\lim_{h \to 0} \Delta_h^j f(x) = f^{(j)}(x). \tag{34}$$

We consider $W := W(\varphi_1, \ldots, \varphi_2, \ldots, \varphi_l)$.
Combining the columns, W can be written as

$$W = \prod_{i=1}^{l} h^{\frac{k_i(k_i+1)}{2}}$$

$$\times W(\varphi(\kappa_1), \Delta\varphi(\kappa_1), \ldots, \Delta^{k_1}\varphi(\kappa_1), \varphi(\kappa_2), \Delta\varphi(\kappa_2), \ldots, \Delta^{k_2}\varphi(\kappa_2), \ldots, \varphi(\kappa_l), \Delta\varphi(\kappa_l), \ldots, \Delta^{k_l}\varphi(\kappa_l)).$$

Then $\tilde{\theta}$ can be expressed as

$$\tilde{\theta} = c \times W(\varphi(\kappa_1), \Delta\varphi(\kappa_1), \ldots, \Delta^{k_1}\varphi(\kappa_1), \varphi(\kappa_2), \Delta\varphi(\kappa_2), \ldots, \Delta^{k_2}\varphi(\kappa_2), \ldots, \varphi(\kappa_l), \Delta\varphi(\kappa_l), \ldots, \Delta^{k_1}\varphi(\kappa_l)),$$

with

$$c = c_1 \times c_2.$$

The coefficient c_1 defined by

$$c_1 = 2^g i^{\frac{g(g+5)}{2}} \exp(i \sum_{j=1}^{g} \theta_j)$$

is such that θ_j is linear in x, and so verify $2\partial_x^2 \ln c_1 = 0$.
The coefficient c_2 is defined by

$$c_2 = \frac{1}{\prod_{j=1}^{l} 2! \ldots k_i! \prod_{m=1}^{k_2+1} \prod_{i=1}^{k_1+1} (K_{k_1+1+m} - K_i) \ldots \prod_{m=1}^{k_l+1} \prod_{i=1}^{k_1+\ldots+k_{l-1}+l-1} (K_{k_1+k_{l-1}+l-1+m} - K_i)}.$$

By definition of the terms K_j, the coefficient c_2 tends to a finite value independent of x when h tends to 0, and so verify $2\partial_x^2 \ln c_2 = 0$.
If we denote $\phi_j = \varphi(\kappa_j)$, then when h tends to 0

$$W(\varphi(\kappa_1), \Delta\varphi(\kappa_1), \ldots, \Delta^{k_1}\varphi(\kappa_1), \varphi(\kappa_2), \Delta\varphi(\kappa_2), \ldots, \Delta^{k_2}\varphi(\kappa_2), \ldots, \varphi(\kappa_l), \Delta\varphi(\kappa_l), \ldots, \Delta^{k_1}\varphi(\kappa_l))$$

tends to

$$W(\phi_1, \ldots, \phi_1^{(k_1)}, \phi_2, \ldots, \phi_2^{(k_2)} \ldots, \phi_l, , \ldots, \phi_l^{(k_l)}).$$

So we get when h tends to 0

$$u(x,t) = -2\partial_x^2 \ln(\tilde{\theta}) = -2\partial_x^2 \ln(W) = -2\partial_x^2 \ln W(\phi_1, \ldots, \phi_1^{(k_1)}, \phi_2, \ldots, \phi_2^{(k_2)} \ldots, \phi_l, , \ldots, \phi_l^{(k_l)}).$$

We get clearly in the limit the positon defined by (32).

hal-00466159, version 2 - 25 Sep 2011

7 Conclusion

- This work takes its origin in the seminal paper of A. Its and V.B. Matveev in 1975 [7], in the study of V.B. Matveev in 1976 of abelian functions and solitons [13] and other papers like [16], [12], [14], [15], [17].

  This result is based on two remarks. First, it was essential to express the degenerate θ function into an explicit Fredholm determinant; this remark was initiated by the works of Kirillov and Van Diejen [22]. The second step was to get the transformation of the Fredholm determinant into a Wronskian.

- As a byproduct, we get the result for the case of a soliton; it is more simple.

  If we want to get the following soliton

  $$u = 2\partial_x^2 \ln W(\phi_1, \ldots, \phi_l), \tag{35}$$

  we consider a Riemann surface of genus $g = l$.
  We choose E_j real such that $E_m < E_j$ if $m < j$.
  We consider $K_i > 0$, for $1 \leq m \leq g$. We choose the following limits :
  E_1 tends to 0.
  for $1 \leq i \leq l$, E_{2i}, E_{2i+1} tends to $-\alpha_i = K_i^2$;
  Then we get clearly in the limit the soliton defined by (35).

- We can also mentioned that we can get the famous DPT potential.
  Let n and d be some non negative integers, $m = n + d$, If we choose the coefficients K_j defined by

  $$K_p = p, \quad \text{if} \quad d \neq 0\,, \quad 1 \leq p \leq m - n,$$

  $$K_j = n - m + 2j, \quad \text{if} \quad n \neq 0\,, \quad m - n + 1 \leq j \leq m,$$

  Taking $t = 0$ and $x_j = 0$, then $u = 2\partial_x^2 \ln W(\phi_1, \ldots, \phi_m)$, is exactly the DPT potential

  $$\frac{m(m+1)}{\sin^2(x)} + \frac{n(n+1)}{\cos^2(x)}.$$

  Its a consequence of a previous work. We refer the reader to the paper [4] for the details or to an another forthcoming publication containing a different approach.

hal-00466159, version 2 - 25 Sep 2011

References

[1] E.D. Belokolos, A.I. Bobenko, A.R. Its, V.Z. Enolskij and V.B. Matveev, Algebro-geometric approach to nonlinear integrable equations, Springer series in nonlinear dynamics, Springer Verlag, 1-360, (1994).

[2] R. Beutler, A. Stahlhofen, V.B. Matveev, What do solitons, breathers and positons have in common? Physica Scripta, V. **50**, 9-12, (1994).

[3] B.A. Dubrovin, V.B. Matveev, S.P. Novikov, in Lond. Math. Soc. Lect. Notes Ser., V. **60**, 55-136, Cambridge University Press, (1981).

[4] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, Max-Planck-Institut für Mathematik, Bonn, MPI **02-31**, 1-17, (2002).

[5] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, Letters in Mathematical Physics, V **68**, 77-90, (2004).

[6] R. Hirota, Phys. Rev. Letters, V. **27**, 1192, (1971).

[7] A.R. Its, V.B. Matveev, Hill's operator with finitely many gaps, Funct. Anal. i Prilozhen, V. **9**., 69-70, (1975).

[8] M. Kovalyov, M.H Ali Abadi, An explicit formula for a class of solutions of KdV equation, University of Alberta, Canada, (1997).

[9] M. Jaworski, J. Zagrodzinski, Positon and positon-like solution of the Korteweg-de Vries and Sine-Gordon equations, Chaos Solitons Fractals, V. **5**, N. 12, 2229-2233, (1995).

[10] V.B. Matveev, Lett. Math. Phy., V. **3**, 213-216, 217-222, (1979).

[11] V.B. Matveev, Asymptotics of the multipositon-soliton τ function of the Korteweg-de Vries equation and the supertransparency, J. Math. Phys., V. **125**, N. 6, 2955-2970, (1994).

[12] V.B. Matveev, Theory of positons, Max Planck Institut für Metallforschung, (1992).

[13] V.B. Matveev, Abelian functions and solitons, Preprint N. **313**, Institute of Th. Physics, Univ. of Wroclow, (1976).

hal-00466159, version 2 - 25 Sep 2011

[14] V.B. Matveev, Multi positons solutions of the KdV equation. Positon-positon and soliton-positon interactions, Preprint, (1998).

[15] V.B. Matveev, Positon-positon and soliton-positon collisions : KdV case, Phys. Let. A, V. **166**, 208-212, (1992).

[16] V.B. Matveev, Asymptotics of mulpositon-solution τ function of the Korteweg-de Vries equation and the supertransparency, J. Math. Phys., V. **35**, N. 6, (1994).

[17] V.B. Matveev, Positon : slowly decreasing analogues of solitons, Th. And Math. Phys. V. **131**, N. 1, (2002).

[18] C. Rasinariu, U. Sukhatme, A. Khare, Negaton and positon solutions of the KdV and the mKdV hierarchy, J. Phys. A, V. **29**, 1803-1823, London, (1996).

[19] A.A. Stahlhofen, V.B. Matveev, Positons for the Toda lattice and related spectral problems, J. Phys. A. : Math. Gen., V **28**, 1957-1965, (1995).

[20] J.F. Van Diejen, A.N. Kirillov, Formulas for q-Spherical functions using inverse scattering theory of reflectionless Jacobi operators, Commun. Math. Phys., V. **210**, 335-369, (2000).

[21] Van Diejen, J. F.: Integrabilty of difference Calogero-Moser systems, Jour. of Math. Physics, V. **35** N. 6, 2983-3004, (1999).

[22] J.F. Van Diejen, A. N. Kirillov Determinantal formulas for zonal spherical functions on hyperboloids, Math. Ann., V. **319**, 215-234, (2001).

[23] A. Zabrodin, Finite-gap difference operators with elliptic coefficients and their spectral curves, in "Physical Combinatorics", M. Kashiwara and T. Miwa (eds), Birkhauser Ser. Progress in Mathematics, V. **191**, ISBN 0-816-4175-0, Birkhauser-Boston, 301-317, (2000).

[24] H. Zhao, Z. Zhu, Multi-soliton, multi-positon, multi-negaton, and multi-periodic solutions of the coupled Volterra lattice equation, arXiv : 0911.3458v1 [nlin.S1], 18 nov 2009