DENSITY ESTIMATES FOR SOLUTIONS TO ONE DIMENSIONAL SDE'S AND BACKWARD SDE'S

OMAR ABOURA AND SOLESNE BOURGUIN

Abstract. In this paper, we give sufficient conditions for the solutions of stochastic differential equations and backward stochastic differential equations to have a density for which we give upper and lower estimates. In the case of backward SDEs, the density estimates we derive are Gaussian.

1. Introduction

In [2], I. Nourdin and F.G. Viens have introduced sufficient conditions to prove the existence of a density for a Malliavin differentiable random variable and to provide upper and lower Gaussian estimates for this density.

This result has lead to several research papers, such as those by D. Nualart and L. Quer-Sardanyons ([4], [5]), in which these authors applied Nourdin and Viens result to solutions of quasi-linear stochastic partial differential equations and to a class of stochastic equations with additive noise.

In this paper, we use Nourdin and Viens's approach to prove that, under proper conditions on the coefficients, each component of the solution (X_t, Y_t, Z_t) to a backward stochastic differential equation

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s & (1.1) \\ Y_t = \phi(X_T) + \int_t^T f(X_s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s & (1.2) \end{cases}$$

has a density for which upper and lower Gaussian bounds can be derived. This implies to study the diffusion equation (1.1) (which stands for itself) and provide upper and lower bounds for its density on one hand, and the backward SDE (1.2) on an other hand.

Our paper is organized in two main parts, the first one dealing with diffusions and the second one with backward SDEs. The question of the existence of a density for the solution to an SDE of the type (1.1) and the properties of this density has been intensively studied and we refer the reader to [3] for an extensive survey of the existing litterature and results on this topic.

We establish that under a sign condition on σ and a growth condition on the Lie bracket of b and σ (see Hypotheses (**H1**) and (**H2**)), (1.1) has a density for which upper and lower estimates can be derived. We also study the same question in the backward SDEs setting, where we consider equations of the type (1.2). These equations introduced in [6], which are closely related with viscosity solution to PDEs, have been intensively studied and have many applications in control theory and financial methods among others.

Date: September 25, 2011.

2000 *Mathematics Subject Classification.* 60H10, 60H07.

Key words and phrases. Backward stochastic differential equations, Malliavin calculus, density estimates.

The existence of the density for the random variable Y_t at a fixed time $t \in (0, T)$, as well as upper bounds for its tail behavior, have been proven by F. Antonelli and A. Kohatsu-Higa [1], using a different approach (Bouleau-Hirsch Theorem). We retrieve Antonelli and Kohatsu-Higa's existence result for the density of Y_t, and we also derive Gaussian estimates for it. In order to provide (additionally to the existence result itself) estimates for the density of Y_t, we need to slightly strengthen the hypotheses of Antonelli and Kohatsu-Higa.

We also address the question of the existence of a density for the random variable Z_t as well as the possibility of deriving Gaussian estimates for it. This question has not been solved in [1]. We need the same hypotheses as in the case of Y_t, as well as additional ones, since Z_t can be expressed as a function of the Malliavin derivative of Y_t.

In order to be self contained, we at first give an overview of some elements of Malliavin calculus in Section 2, and the corresponding notations. In Section 3, we study the case of a diffusion and give sufficient conditions for the density of the solution to exist and admit upper and lower estimates (which need not be Gaussian, except in some particular cases). Section 4 is dedicated to the backward SDE case and is organized in two subsections, dealing respectively with the question of the existence of a density, as well as its Gaussian upper and lower estimates for Y_t and Z_t.

2. Framework, main tools and notations

2.1. **Elements of Malliavin calculus.** Consider the real separable Hilbert space $L^2([0,T])$ and $(W(\varphi), \varphi \in L^2([0,T]))$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{A}, P)$, that is a centered Gaussian family of random variables such that $\mathbf{E}\left(W(\varphi)W(\psi)\right) = \langle \varphi, \psi \rangle_{L^2([0,T])}$. For any integer $n \geq 1$, denote by I_n the multiple stochastic integral with respect to W (see [3] for an extensive survey on Malliavin calculus). The map I_n is actually an isometry between the Hilbert space $L^2([0,T]^n)$ equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{L^2([0,T]^n)}$ and the Wiener chaos of order n, which is defined as the closed linear span of the random variables $H_n(W(\varphi))$ where $\varphi \in L^2([0,T])$, $\|\varphi\|_{L^2([0,T])} = 1$ and H_n is the Hermite polynomial of degree $n \geq 1$, that is defined by

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \qquad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as follows: for positive integers m, n,

$$\begin{aligned} \mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle f, g\rangle_{L^2([0,T]^n)} \quad \text{if } m = n, \\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0 \quad \text{if } m \neq n. \end{aligned}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n \geq 0} I_n(f_n) \tag{2.1}$$

where $f_n \in L^2([0,T]^n)$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

Let L be the Ornstein-Uhlenbeck operator defined by $LF = -\sum_{n\geq 0} nI_n(f_n)$ if F is given by (2.1). For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables (see (1.28) in [3]) with respect to the norm defined by

$$\|F\|_{\alpha,p} = \|(I-L)^{\frac{\alpha}{2}} F\|_{L^p(\Omega)},$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(W(\varphi_1), \ldots, W(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in L^2([0,T])$) as follows:

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_n))\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}\left(L^2([0,T])\right)$. The adjoint of D is denoted by δ and is called the divergence (or Skorohod) integral. It is a continuous operator from $\mathbb{D}^{\alpha,p}\left(L^2([0,T])\right)$ into $\mathbb{D}^{\alpha+1,p}$. More generally, we can introduce iterated weak derivatives of order k. If F is a smooth random variable and k is a positive integer, we set

$$D^k_{t_1,\ldots,t_k} F = D_{t_1} D_{t_2} \ldots D_{t_k} F.$$

We have the following duality relationship between D and δ

$$\mathbf{E}(F\delta(u)) = \mathbf{E}\langle DF, u\rangle_{L^2([0,T])} \text{ for every smooth } F.$$

For adapted integrands, the divergence integral coincides with the classical Itô integral. We will use the notation

$$\delta(u) = \int_0^T u_s dW_s.$$

Note that the following integration by parts relationship between D and δ holds

$$D_t(\delta(u)) = u_t + \int_0^T D_t u_s dW_s,$$

with $u \in \mathbb{D}^{1,2}(L^2([0,T]))$ such that $\delta(u) \in \mathbb{D}^{1,2}$.

2.2. **Density existence and Gaussian estimates.** In [2], Corollary 3.5, Nourdin and Viens have given the following sufficient condition for a weakly differentiable random variable to have a density with lower and upper Gaussian estimates.

Proposition 2.1. *Let F be in $\mathbb{D}^{1,2}$ and let the function g be defined for all $x \in \mathbb{R}$ by*

$$g(x) = \mathbf{E}\left(\langle DF, -DL^{-1}F\rangle_{L^2([0,T])} \Big| F - \mathbf{E}(F) = x\right). \tag{2.2}$$

If there exist positive constants $\gamma_{\min}, \gamma_{\max}$ such that, for all $x \in \mathbb{R}$, almost surely

$$0 < \gamma^2_{\min} \leqslant g(x) \leqslant \gamma^2_{\max}$$

then F has a density ρ satisfying, for almost all $z \in \mathbb{R}$

$$\frac{\mathbf{E}|F-\mathbf{E}(F)|}{2\,\gamma^2_{\max}} \exp\left(-\frac{(z-\mathbf{E}(F))^2}{2\gamma^2_{\min}}\right) \leqslant \rho(z) \leqslant \frac{\mathbf{E}|F-\mathbf{E}(F)|}{2\,\gamma^2_{\min}} \exp\left(-\frac{(z-\mathbf{E}(F))^2}{2\gamma^2_{\max}}\right).$$

Furthermore, Nourdin and Viens have also provided the following useful result, which gives some rather explicit description of $g(x)$. Recall that $W = \left(W(\phi), \phi \in L^2([0,T])\right)$.

Proposition 2.2. *Let F be in $\mathbb{D}^{1,2}$ and write $DF = \Phi_F(W)$ with a measurable function $\Phi_F : \mathbb{R}^{L^2([0,T])} \to L^2([0,T])$. Then, if $g(x)$ is defined by (2.2), we have*

$$g(x) = \int_0^\infty e^{-u}\, \mathbf{E}\left(\mathbf{E}'\big(\langle \Phi_F(W), \widetilde{\Phi^u_F}(W)\rangle_{L^2([0,T])}\big) | F - \mathbf{E}(F) = x\right) du,$$

where $\widetilde{\Phi_F^u}(W) = \Phi_F(e^{-u}W + \sqrt{1-e^{-2u}}W')$, *$W'$ stands for an independent copy of W, and is such that W and W' are defined on the product probability space* $(\Omega\times\Omega', \mathcal{F}\otimes\mathcal{F}', \mathbb{P}\times\mathbb{P}')$ *and $\mathbf{E}'$ denotes the mathematical expectation with respect to $\mathbb{P}'$.*

2.3. **Notations.** Let f be a twice differentiable function of two variables x and y. We will use the following notations : $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2}(x,y) = f_{yy}$, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, $\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$. We will also use the following notation for the Lie bracket : $[f,g] = fg' - gf'$.

In the whole paper, c and C will denote constants that may vary from line to line.

3. Density estimates for one dimensional SDEs

Consider the following one dimensional stochastic differential equation

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{3.1}$$

where $x_0 \in \mathbb{R}$, b and σ are appropriately smooth functions to ensure the existence and uniqueness of solutions and $(W_t)_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$. In this section, we establish under what conditions the solution to (3.1) has a density for which upper and lower estimates can be derived. Expect for some particular cases such as the Ornstein-Uhlenbeck process, these estimates are not Gaussian and are defined on the support of the density. We will start by giving the hypotheses we will be working with before stating the main result of this section, i.e., the upper and lower estimates of the density of the solution to (3.1).

3.1. **Hypotheses and examples.** We consider b and σ to be C^2 Lipschitz functions, which ensures the existence and uniqueness of solutions to (3.1). In addition to that, we impose the following conditions:

$$\begin{cases} \mathbf{H0}: \begin{cases} \text{For every } t>0,\ \sigma(x)>0 \text{ a.e. on the support of } X_t \\ \text{Moreover we suppose that } supp(X_t) \text{ is an interval independent of } t>0 \end{cases} \\ \mathbf{H1}: \exists\, M_l \geq 0, |\,[b,\sigma]\,| \leq M_l|\sigma| \\ \mathbf{H2}: \exists\, M_{\sigma\sigma''} \geq 0, |\sigma\sigma''| \leq M_{\sigma\sigma''} \end{cases} \tag{3.2}$$

where $[b,\sigma]$ denotes the Lie bracket of b and σ.

Remark 3.1. Hypothesis (**H0**) on the positivity of σ on the support of X_t is not a loss of generality. In fact, we only need σ to keep the same sign on the support of X_t. The case where σ is negative was included neither in the proofs nor in the hypotheses for the sake of clarity and readability of the paper. However this case can be addressed (without any further difficulties) by using the following transformations: $\sigma \to \tilde{\sigma} := -\sigma$ and $W \to \tilde{W} := -W$. After performing those tranformations, it suffices to consider X to be the solution of

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \tilde{\sigma}(X_s)d\tilde{W}_s.$$

This brings the problem back to the above set of hypotheses and it can be dealt with by the exact same arguments.

Here are some examples of coefficients b and σ satisfying hypotheses (**H0**) – (**H2**).

Example 3.2. *Consider the particular case where $(X_t)_{t\geq 0}$ is the drifted Brownian motion, i.e. $x_0 = 0$, $b(x) = b$ and $\sigma(x) = \sigma \neq 0$. It is clear that $[b,\sigma] = 0 = \sigma\sigma''$, $supp(X_t) = \mathbb{R}$, and that hypotheses (**H0**) – (**H2**) are satisfied.*

Example 3.3. *Consider the particular case where $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process, i.e. $b(x) = bx$, $b \in \mathbb{R}$ and $\sigma(x) = \sigma \neq 0$ Thus, $[\sigma, b] = b\sigma$, $\sigma\sigma'' = 0$, $supp(X_t) = \mathbb{R}$, and hypotheses (**H0**) – (**H2**) are satisfied.*

Example 3.4. *Consider the particular case where $(X_t)_{t\geq 0}$ is a geometric Brownian motion, i.e. $x_0 \neq 0$, $b(x) = bx$, $b \in \mathbb{R}$ and $\sigma(x) = \sigma x$, $\sigma \neq 0$ with $\sigma x_0 > 0$. Thus, $[b, \sigma] = 0 = \sigma\sigma''$, if $x_0 > 0$ then $supp(X_t) = [0, \infty)$ and we suppose that $\sigma > 0$ (resp. if $x_0 < 0$, then $supp(X_t) = (-\infty, 0]$ and we suppose that $\sigma < 0$). The hypotheses (**H0**) – (**H2**) are satisfied.*

3.2. **Main result (Existence and estimates for the density of X).** The following result provides upper and lower estimates for the solutions to (3.1).

Theorem 3.5. *Consider equation (3.1) and let G be an antiderivative of $\frac{1}{\sigma}$. Under the hypotheses of Subsection 3.1, for $t \in (0, T]$ the random variable X_t has a density ρ_{X_t}. Furthermore, there exist strictly positive constants c and C such that, for almost all $x \in \mathbb{R}$, ρ_{X_t} satisfies the following:*

$$\rho_{X_t}(x) \geq \mathbf{1}_{supp(X_t)}(x) \frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)Ct} e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2ct}} \tag{3.3}$$

and

$$\rho_{X_t}(x) \leq \mathbf{1}_{supp(X_t)}(x) \frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)ct} e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2Ct}}. \tag{3.4}$$

Note that $G|_{supp(X_t)}$ is invertible and that $supp(X_t) = Im(\{G|_{supp(X_t)}\}^{-1})$ does not depend on the antiderivative G.

Remark 3.6. Note that the support of the density ρ_{X_t} is not necesseraly $\mathbb{R}$, but $supp(X_t)$.

Here are some examples of bounds derived on classical processes using Theorem 3.5.

Example 3.7. *Consider the particular case where $X_t = x_0 + \sigma W_t + bt$, i.e. $x_0 \in \mathbb{R}$, $b(x) = b$ and $\sigma(x) = \sigma$. We have $G(x) = \frac{x}{\sigma} + cst$ and $X_t(\Omega) = \mathbb{R}$. Thus the bounds (3.3) and (3.4) become*

$$\frac{1}{2C\sigma t} e^{-\frac{(x-bt-x_0)^2}{2c\sigma^2 t}} \leq \rho_{X_t}(x) \leq \frac{1}{2c\sigma t} e^{-\frac{(x-bt-x_0)^2}{2C\sigma^2 t}}$$

Example 3.8. *Consider the particular case where $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process, i.e. $b(x) = bx$, $b \in \mathbb{R}^*$ and $\sigma(x) = \sigma \in \mathbb{R}$. Then $X_t \sim \mathcal{N}\left(x_0 e^{bt}, \frac{\sigma^2}{2b}(e^{2bt-1})\right)$. We have $G(x) = \frac{x}{\sigma} + cst$ and $X_t(\Omega) = \mathbb{R}$. Thus the bounds (3.3) and (3.4) become*

$$\frac{\sqrt{e^{2bt}-1}}{2\sigma Ct\sqrt{b}} e^{-\frac{(x-x_0e^{bt})^2}{2\sigma^2 ct}} \leq \rho_{X_t}(x) \leq \frac{\sqrt{e^{2bt}-1}}{2\sigma ct\sqrt{b}} e^{-\frac{(x-x_0e^{bt})^2}{2\sigma^2 Ct}}$$

Example 3.9. *Consider the particular case where $(X_t)_{t\geq 0}$ is a geometric Brownian motion, i.e. $x_0 \neq 0$, $b(x) = bx$, $b \in \mathbb{R}$ and $\sigma(x) = \sigma x$, $\sigma \neq 0$ with $\sigma x_0 > 0$. We have, for $x \neq 0$, $G(x) = \frac{\ln(|x|)}{\sigma} + cst$ and if $x_0 > 0$, $X_t(\Omega) =]0, +\infty[$ (resp. $X_t(\Omega) =]-\infty, 0[$ if $x_0 < 0$). Thus the bounds (3.3) and (3.4) become*

$$\frac{\mathbf{1}_{supp(X_t)}(x)}{4\sigma x Ct} e^{-\frac{\left(\ln(|x|)-\ln(|x_0|)-\left(b-\frac{\sigma^2}{2}\right)t\right)^2}{2c\sigma^2 t}} \leq \rho_{X_t}(x) \leq \frac{\mathbf{1}_{supp(X_t)}(x)}{4\sigma x ct} e^{-\frac{\left(\ln(|x|)-\ln(|x_0|)-\left(b-\frac{\sigma^2}{2}\right)t\right)^2}{2C\sigma^2 t}}.$$

Let us first prove the following Lemma that will be useful for the proof of Theorem 3.5.

Lemma 3.10. *For every $T > 0$, $x \in \mathbb{R}^*$, there exist positive constants c and C such that for every $t \in [0,T]$,*

$$ct \leq \frac{e^{xt}-1}{x} \leq Ct.$$

Proof: If $x > 0$, Taylor's formula implies that for $t \in [0,T]$,

$$t \leq \frac{e^{xt}-1}{x} = \int_0^t e^{xs} ds \leq te^{xT}.$$

Similarly, if $x < 0$, we have, for $t \in [0,T]$, $te^{xT} \leq \frac{e^{xt}-1}{x} \leq t$. □

Proof of Theorem 3.5: Recall that G denotes an antiderivative of $\frac{1}{\sigma}$. Then G is strictly increasing on $supp(X_t)$ and we denote by G^{-1} the inverse map of $G : supp(X_t) \to G(supp(X_t))$. Let U_t be defined by

$$U_t = G(X_t) \Leftrightarrow X_t = G^{-1}(U_t). \tag{3.5}$$

Remark 3.11. Note at first that G does not depend on t (as an antiderivative of $\frac{1}{\sigma}$), and that (**H0**) implies that the restriction of G to the support of X_t is invertible since $supp(X_t)$ is assumed to be an interval independent of t. The invertibility of G reduced to the interior of the support of X_t, $\overset{\circ}{supp}(X_t)$, is the only assumption that is required for the proof.

Applying Itô's formula to $G(X_t)$ and using the identity $G'(x) = \frac{1}{\sigma(x)}$, we obtain

$$\begin{aligned} dU_t =& G'(X_t)dX_t + \frac{1}{2}G''(X_t)d\langle X\rangle_t \\ =& \left[G'(X_t)b(X_t) + \frac{1}{2}G''(X_t)\sigma^2(X_t)\right]dt + G'(X_t)\sigma(X_t)dW_t \\ =& \beta \circ G^{-1}(U_t)dt + dW_t, \end{aligned}$$

where β is defined by

$$\beta(x) = G'(x)b(x) + \frac{1}{2}G''(x)\sigma^2(x) = \frac{b}{\sigma}(x) - \frac{\sigma'(x)}{2}. \tag{3.6}$$

Thus,

$$U_t = G(x_0) + \int_0^t \beta \circ G^{-1}(U_s)ds + W_t$$

and for $\theta \in [0,t]$ we have

$$D_\theta U_t = 1 + \int_\theta^t (\beta \circ G^{-1})'(U_s)D_\theta U_s ds = \exp\left[\int_\theta^t (\beta \circ G^{-1})'(U_s)ds\right]. \tag{3.7}$$

Deriving the identity $G \circ G^{-1}(x) = x$ on $G(supp(X_t))$ yields $(G^{-1})'(x) = \sigma \circ G^{-1}(x)$. Using this fact we get $(\beta \circ G^{-1})'(x) = \beta' \circ G^{-1}(x)(G^{-1})'(x) = (\beta'\sigma) \circ G^{-1}(x)$. In addition, it is easy to check that on $G(supp(X_t))$,

$$(\beta'\sigma)(x) = \frac{[\sigma,b](x)}{\sigma(x)} - \frac{(\sigma\sigma'')(x)}{2}. \tag{3.8}$$

Gathering those results and using hypotheses (**H1**) and (**H2**) of Subsection 3.1 immediatly yields on $G(supp(X_t))$

$$-\left(M_l + \frac{M_{\sigma\sigma''}}{2}\right) \leq (\beta \circ G^{-1})' \leq \left(M_l + \frac{M_{\sigma\sigma''}}{2}\right).$$

Using (3.7), we deduce, $\mathbb{P}$-a.s,

$$0 < e^{-\left(M_l + \frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)} \le D_\theta U_t \le e^{\left(M_l + \frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)}. \tag{3.9}$$

Write $D_\bullet U_t = \Phi^\bullet_{U_t}(W)$ with a measurable function $\Phi^\bullet_{U_t} : \mathbb{R}^{L^2([0,T])} \longrightarrow L^2([0,T])$. Then (3.9) becomes, for $\theta < t$,

$$0 < e^{-\left(M_l + \frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)} \le \Phi^\theta_{U_t}(W) \le e^{\left(M_l + \frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)}. \tag{3.10}$$

Define $\widetilde{\Phi^{\bullet,u}_{U_t}}(W) = \Phi^\bullet_{U_t}(e^{-u}W + \sqrt{1-e^{-2u}}W')$ for $u \in [0,+\infty[$, where W' stands for an independent copy of W and is such that W and W' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P}')$. Using (3.9), it is clear that the following holds, for $\theta < t$, $u \in [0,\infty)$

$$0 < e^{-\left(M_l + \frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)} \le \widetilde{\Phi^{\theta,u}_{U_t}}(W) \le e^{\left(M_l + \frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)}. \tag{3.11}$$

Using Proposition 2.2 for $g(x) = \mathbf{E}\left(\langle DU_t, -DL^{-1}U_t\rangle_{L^2([0,T])} \middle| U_t - \mathbf{E}(U_t) = x\right)$, we have

$$\begin{aligned} g(x) &= \int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'\big(\langle \Phi^\bullet_{U_t}(W), \widetilde{\Phi^{\bullet,u}_{U_t}}(W)\rangle_{L^2([0,T])}\big)|U_t - \mathbf{E}(U_t) = x\right)du \\ &= \int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'\left(\int_0^t \Phi^\theta_{U_t}(W)\widetilde{\Phi^{\theta,u}_{U_t}}(W)d\theta\right)|U_t - \mathbf{E}(U_t) = x\right)du. \end{aligned}$$

Using the bounds in (3.10) and (3.11), we obtain, $\mathbb{P}$-a.s,

$$0 < \int_0^\infty e^{-u}\int_0^t e^{-(2M_l + M_{\sigma\sigma''})(t-\theta)}d\theta du \le g(x) \le \int_0^\infty e^{-u}\int_0^t e^{(2M_l + M_{\sigma\sigma''})(t-\theta)}d\theta du,$$

which leads to, $\mathbb{P}$-a.s,

$$0 < \frac{1 - e^{-(2M_l + M_{\sigma\sigma''})t}}{2M_l + M_{\sigma\sigma''}} \le g(x) \le \frac{e^{(2M_l + M_{\sigma\sigma''})t} - 1}{2M_l + M_{\sigma\sigma''}}.$$

Lemma 3.10 implies the existence of strictly positive constants c and C such that, for $t \in (0,T]$,

$$0 < ct \le g(x) \le Ct \quad \mathbb{P} - a.s.$$

Using (3.9) we deduce that $U_t \in \mathbb{D}^{1,2}$. Hence Proposition 2.1 implies that U_t has a density ρ_{U_t} and that there exist constants c and C such that $0 < c < C$ and for $u \in G(supp(X_t))$,

$$\frac{\mathbf{E}|U_t - \mathbf{E}(U_t)|}{2Ct}\exp\left(-\frac{(u-\mathbf{E}(U_t))^2}{2ct}\right) \le \rho_{U_t}(u) \le \frac{\mathbf{E}|U_t - \mathbf{E}(U_t)|}{2ct}\exp\left(-\frac{(u-\mathbf{E}(U_t))^2}{2Ct}\right).$$

We now prove that for any $t \in (0,T]$, X_t has a density, which we compare to that of U_t. For any bounded Borel function f, (using the change of variable $x = G^{-1}(u)$) for all $x \in supp(X_t)$, we deduce

$$\mathbf{E}(f(X_t)) = \mathbf{E}\left(f \circ G^{-1}(U_t)\right) = \int_{G(supp(X_t))} f \circ G^{-1}(u)\rho_{U_t}(u)du = \int_{supp(X_t)} f(x)\frac{\rho_{U_t} \circ G(x)}{\sigma(x)}dx.$$

Using this, we can recover that X_t has a density ρ_{X_t} such that

$$\rho_{X_t}(x) = \frac{\rho_{U_t} \circ G(x)}{\sigma(x)}\mathbf{1}_{supp(X_t)}(x).$$

Hence, the upper and lower estimates of ρ_{U_t} yield

$$\rho_{X_t}(x) \ge \mathbf{1}_{supp(X_t)}(x)\frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)Ct}e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2ct}}$$

and

$$\rho_{X_t}(x) \leq \mathbf{1}_{supp(X_t)}(x)\frac{\mathbf{E}|G(X_t)-\mathbf{E}(G(X_t))|}{2\sigma(x)ct}e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2Ct}}.$$

This concludes the proof of Theorem 3.5. □

4. Gaussian density estimates for one dimensional backward SDEs

4.1. **Preliminaries.** The following backward stochastic differential equation was introduced in Pardoux and Peng [6] (see also [7]) and was also studied in [1]:

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \\ Y_t = \phi(X_T) + \int_t^T f(X_s, Y_s, Z_s)\,ds - \int_t^T Z_s dW_s. \end{cases} \tag{4.1}$$

In this section, we give conditions for the random variables Y_t and Z_t to have a density which can be bounded from above and below by Gaussian ones. We will first focus on the case of Y_t for which we have to impose strict ellipticity conditions on the coefficients of X along with some additional hypotheses on the coefficients of Y. The case of Z_t requires stronger assumptions on the coefficients of X and Y that will be detailed in the dedicated subsection. Note that, as opposed to the case of the diffusion process X_t, the estimates we obtain for the backward part (Y, Z) of equation (4.1) are always Gaussian.

4.2. **Notations.** We introduce here some notations in use in this section. Let $B_0^{n,+}(\mathbb{R})$ be the space of bounded $\mathcal{C}^n(\mathbb{R})$ functions which are bounded positively away from 0 such that their derivatives up to order n are bounded, i.e., $f \in B_0^{n,+}(\mathbb{R})$ if and only if there exist positive constants c, C and $M_{f^{(i)}}$, $i = 1, ..., n$, such that $0 < c \leq f \leq C$ and for each $i = 1, ..., n$, $\left|f^{(i)}\right| \leq M_{f^{(i)}}$. Let $B_0^{n,-}(\mathbb{R})$ be the space of $\mathcal{C}^n(\mathbb{R})$ functions such that $-f \in B_0^{n,+}(\mathbb{R})$ and $B_0^n(\mathbb{R}) = B_0^{n,+}(\mathbb{R}) \cup B_0^{n,-}(\mathbb{R})$.

4.3. **Density of Y_t : existence and Gaussian estimates.** We first give the set of hypotheses (in terms of the diffusion's coefficients and the backward equation's coefficients) that will be needed in the main theorem on the density of Y_t.

4.3.1. *Hypotheses.* Consider equation (4.1). On the diffusion part, we still consider b and σ to be appropriately smooth functions to ensure the existence and uniqueness of solutions to the first equation in (4.1). We also need to impose these two additional conditions on b and σ:

$$\begin{cases} \mathbf{H3}: \exists\, M_l \geq 0, \quad |[b,\sigma]| \leq M_l \\ \mathbf{H4}: \sigma \in B_0^{2,+}(\mathbb{R}) \end{cases}$$

where $[b, \sigma]$ denotes the Lie bracket between b and σ.

Remark 4.1. Recall that $\sigma \in B_0^{2,+}(\mathbb{R})$ implies that there exist two strictly positive constants that will be refered to as m_σ and M_σ such that $0 < m_\sigma \leq \sigma \leq M_\sigma$.

Hence if $\sigma \in B_0^{2,+}(\mathbb{R})$, the assumption (**H1**) is equivalent to $|[b, \sigma]| \leq M$ for some positive constant M. Clearly, (**H0**) and (**H2**) are also satisfied if $\sigma \in B_0^{2,+}(\mathbb{R})$.
On the backward part of (4.1), we make the following assumptions:

$$\begin{cases} \mathbf{H5}: \exists\, c_{\phi'}, C_{\phi'}, \quad 0 < c_{\phi'} \leq |\phi'| \leq C_{\phi'} \\ \mathbf{H6}: \exists\, c_{f_x}, C_{f_x}, M_{f_y}, M_{f_z}, \quad \left\{ \begin{array}{l} 0 < c_{f_x} \leq |f_x| \leq C_{f_x} \\ |f_y| \leq M_{f_y} \quad |f_z| \leq M_{f_z} \end{array} \right\} \\ \mathbf{H7}: \forall u, v, \quad \phi'(u) f_x(v) > 0 \end{cases}$$

4.3.2. *Main result (Existence and estimates for the density of Y_t).* Under the above assumptions, we have the following Gaussian estimates for the density of Y_t.

Theorem 4.2. *Under the hypotheses of Subsection 4.3.1, for $t \in (0,T)$ the random variable Y_t defined in (4.1) has a density ρ_{Y_t}. Furthermore, there exist strictly positive constants c and C such that, for almost all $y \in \mathbb{R}$, ρ_{Y_t} satisfies the following:*

$$\frac{\mathbf{E}|Y_t - \mathbf{E}(Y_t)|}{2ct}\exp\left(-\frac{(y-\mathbf{E}(Y_t))^2}{2Ct}\right) \le \rho_{Y_t}(y) \le \frac{\mathbf{E}|Y_t - \mathbf{E}(Y_t)|}{2Ct}\exp\left(-\frac{(y-\mathbf{E}(Y_t))^2}{2ct}\right).$$

Before proving Theorem 4.2, we will first prove the following Proposition that will play a key role in the upcoming proof of this Theorem.

Proposition 4.3. *Suppose that the conditions (**H3**)−(**H7**) hold. Then for $0 < \theta < t \le T$, there exist some strictly positive constants $k_{Y,1}(\theta,t), k_{Y,2}(\theta,t)$ such that $\mathbb{P}$-a.s,*

$$0 < k_{Y,1}(\theta,t) \le |D_\theta Y_t| \le k_{Y,2}(\theta,t) \tag{4.2}$$

with

$$\begin{aligned} k_{Y,1}(\theta,t) =& c_{\phi'} m_\sigma e^{m_{b,\sigma}(T-\theta) - M_{f_y}(T-t)} \\ &+ \frac{c_{f_x} m_\sigma e^{M_{f_y}t - m_{b,\sigma}\theta}}{m_{b,\sigma} - M_{f_y}}\left(e^{(m_{b,\sigma}-M_{f_y})T} - e^{(m_{b,\sigma}-M_{f_y})t}\right) \end{aligned}$$

and

$$\begin{aligned} k_{Y,2}(\theta,t) =& C_{\phi'} M_\sigma e^{M_{b,\sigma}(T-\theta) + M_{f_y}(T-t)} \\ &+ \frac{C_{f_x} M_\sigma e^{-M_{f_y}t - M_{b,\sigma}\theta}}{M_{b,\sigma} + M_{f_y}}\left(e^{(M_{b,\sigma}+M_{f_y})T} - e^{(M_{b,\sigma}+M_{f_y})t}\right), \end{aligned}$$

where $m_{b,\sigma}$ and $M_{b,\sigma}$ are constants depending only on b and σ.

Proof: We at first represent $D_\theta Y_t$ by means of an equivalent probability; this is similar to [1] and the proof is included for the sake of completeness. It is well known (see for example Theorem 2.2 in [1]) that, for every $t \in (0,T]$, $Y_t \in \mathbb{D}^{1,2}$ and $Z \in L^2\left(0,T;\mathbb{D}^{1,2}\right)$. Furthermore, since $\theta < t$, we have

$$\begin{aligned} D_\theta Y_t =& \phi'(X_T) D_\theta X_T - \int_t^T D_\theta Z_s dW_s \\ &+ \int_t^T \left[f_x(X_s,Y_s,Z_s) D_\theta X_s + f_y(X_s,Y_s,Z_s) D_\theta Y_s + f_z(X_s,Y_s,Z_s) D_\theta Z_s\right] ds. \end{aligned} \tag{4.3}$$

The product $e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} D_\theta Y_t$ yields a more suitable representation of $D_\theta Y_t$; indeed, for $t \in (0,T]$, and $0 \le \theta < t$

$$\begin{aligned} d\left[e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} D_\theta Y_t\right] =& \Big[D_\theta Y_t e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} f_y(X_t,Y_t,Z_t) \\ & - e^{\int_0^t f_y(X_s,Y_s,Z_s)ds}\left(f_x(X_t,Y_t,Z_t) D_\theta X_t + f_y(X_t,Y_t,Z_t) D_\theta Y_t\right. \\ & \left. + f_z(X_t,Y_t,Z_t) D_\theta Z_t\right)\Big] dt + e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} D_\theta Z_t dW_t. \end{aligned}$$

Integrating from t to T yields, for $\theta < t$,

$$\begin{aligned} e^{\int_0^T f_y(X_s,Y_s,Z_s)ds} D_\theta Y_T - e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} D_\theta Y_t =& -\int_t^T e^{\int_0^s f_y(X_r,Y_r,Z_r)dr}\left[f_x(X_s,Y_s,Z_s) D_\theta X_s\right. \\ & \left. + f_z(X_s,Y_s,Z_s) D_\theta Z_s\right] ds + \int_t^T e^{\int_0^s f_y(X_r,Y_r,Z_r)dr} D_\theta Z_s dW_s. \end{aligned}$$

Note that $D_\theta Y_T = \phi'(X_T) D_\theta X_T$; therefore, for $t \in (0,T]$,

$$D_\theta Y_t = e^{\int_t^T f_y(X_s,Y_s,Z_s)ds}\phi'(X_T) D_\theta X_T + \int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr}\left[f_x(X_s,Y_s,Z_s) D_\theta X_s + f_z(X_s,Y_s,Z_s) D_\theta Z_s\right] ds - \int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr} D_\theta Z_s dW_s.$$

Let $\widetilde{W}_t = W_t - \int_0^t f_z(X_s,Y_s,Z_s)\,ds$. Because $|f_z| \leq M_{f_z}$, Novikov's condition is verified and $\widetilde{W}$ is a Brownian motion under some equivalent probability $\widetilde{\mathbb{P}}$. Girsanov's theorem yields

$$D_\theta Y_t = e^{\int_t^T f_y(X_s,Y_s,Z_s)ds}\phi'(X_T) D_\theta X_T + \int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr} f_x(X_s,Y_s,Z_s) D_\theta X_s ds - \int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr} D_\theta Z_s d\widetilde{W}_s.$$

Conditionning by $\mathcal{F}_t$ under $\widetilde{\mathbb{P}}$ and taking into account the fact that Y_t and $D_\theta Y_t$ are adapted with respect to $\mathcal{F}_t$ while $\int_t^s f_y(X_r,Y_r,Z_r)\,dr$ and $D_\theta Z_s$ are $\mathcal{F}_s$-adapted for $\theta < t \leq s \leq T$, we obtain

$$D_\theta Y_t = \widetilde{\mathbf{E}}\left(e^{\int_t^T f_y(X_s,Y_s,Z_s)ds}\phi'(X_T) D_\theta X_T \middle| \mathcal{F}_t\right) + \widetilde{\mathbf{E}}\left(\int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr} f_x(X_s,Y_s,Z_s) D_\theta X_s ds \middle| \mathcal{F}_t\right). \tag{4.4}$$

At this point, we need to make use of the random variable U_t introduced in the proof of Theorem 3.5. Recall that $U_t = G(X_t)$ where G is an antiderivative of $\frac{1}{\sigma}$. We have established in (3.7) that if G^{-1} is the inverse function of G restricted to $supp(X_t)$,

$$D_\theta U_t = \exp\left[\int_\theta^t (\beta \circ G^{-1})'(U_s) ds\right], \tag{4.5}$$

where β is defined by (3.6). Finally, recall that $(\beta \circ G^{-1})' = (\beta'\sigma) \circ G^{-1}$ where $\beta'\sigma$ is given by (3.8). Using hypotheses (**H3**) and (**H4**) of Subsection 4.3.1 as well as (3.9), we deduce the existence of two constants $m_{b,\sigma}$ and $M_{b,\sigma}$ such that for $0 < \theta < t \leq T$,

$$0 < e^{m_{b,\sigma}(t-\theta)} \leq D_\theta U_t \leq e^{M_{b,\sigma}(t-\theta)}. \tag{4.6}$$

Futhermore, as $X_t = G^{-1}(U_t)$, it holds that, for $0 < \theta < t \leq T$,

$$D_\theta X_t = (G^{-1})'(U_t) D_\theta U_t = \sigma \circ G^{-1}(U_t) D_\theta U_t. \tag{4.7}$$

Combining (4.6) and (4.7) with the fact that $0 < m_\sigma \leq \sigma \leq M_\sigma$ yields, $\mathbb{P}$-a.s (and $\widetilde{\mathbb{P}}$-a.s since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent),

$$0 < m_\sigma e^{m_{b,\sigma}(t-\theta)} \leq D_\theta X_t \leq M_\sigma e^{M_{b,\sigma}(t-\theta)}. \tag{4.8}$$

For every $t \in [0,T)$, $D_\theta X_t$ is positive and using (**H7**), we deduce that $\phi'(X_T)$ and $f_x(X_s,Y_s,Z_s)$ have the same sign. Hence both terms in the right hand side of (4.4) have the same sign. Using this fact along with hypotheses (**H5**), (**H6**) we estimate both terms in the right hand side of (4.4) and hence their sum; this yields $\widetilde{\mathbb{P}}$-a.s for $0 < \theta < t \leq T$,

$$0 < k_{Y,1}(\theta,t) \leq |D_\theta Y_t| \leq k_{Y,2}(\theta,t) \tag{4.9}$$

with

$$\begin{aligned} k_{Y,1}(\theta,t) =& c_{\phi'} m_\sigma e^{m_{b,\sigma}(T-\theta)-M_{f_y}(T-t)} \\ &+ \frac{c_{f_x} m_\sigma e^{M_{f_y}t - m_{b,\sigma}\theta}}{m_{b,\sigma}-M_{f_y}} \left(e^{(m_{b,\sigma}-M_{f_y})T} - e^{(m_{b,\sigma}-M_{f_y})t} \right) \end{aligned}$$

and

$$\begin{aligned} k_{Y,2}(\theta,t) =& C_{\phi'} M_\sigma e^{M_{b,\sigma}(T-\theta)+M_{f_y}(T-t)} \\ &+ \frac{C_{f_x} M_\sigma e^{-M_{f_y}t - M_{b,\sigma}\theta}}{M_{b,\sigma}+M_{f_y}} \left(e^{(M_{b,\sigma}+M_{f_y})T} - e^{(M_{b,\sigma}+M_{f_y})t} \right). \end{aligned}$$

This concludes the proof of Proposition 4.3. □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2: Write $D_\bullet Y_t = \Phi^\bullet_{Y_t}(W)$ with a measurable function $\Phi^\bullet_{Y_t} : \mathbb{R}^{L^2([0,T])} \longrightarrow L^2([0,T])$. Then Proposition 4.3 yields, for $\theta < t$,

$$0 < k_{Y,1}(\theta,t) \le \left|\Phi^\theta_{Y_t}(W)\right| \le k_{Y,2}(\theta,t). \tag{4.10}$$

Define $\widetilde{\Phi^{\bullet,u}_{Y_t}}(W) = \Phi^\bullet_{Y_t}(e^{-u}W + \sqrt{1-e^{-2u}}W')$ for $u \in [0,+\infty[$, where W' stands for an independent copy of W and is such that W and W' are defined on the product probability space $(\Omega\times\Omega', \mathcal{F}\otimes\mathcal{F}', \mathbb{P}\times\mathbb{P}')$. Using Proposition 4.3, it is clear that, for $\theta < t$, we have for any $u \in [0,\infty)$, $0 < k_{Y,1}(\theta,t) \le \left|\widetilde{\Phi^{\theta,u}_{Y_t}}(W)\right| \le k_{Y,2}(\theta,t)$. Noticing that $\Phi^\theta_{Y_t}(W)$ and $\widetilde{\Phi^{\theta,u}_{Y_t}}(W)$ have the same sign and combining the two previous bounds yields, for $\theta < t$, $u \in [0,\infty)$,

$$0 < k^2_{Y,1}(\theta,t) \le \Phi^\theta_{Y_t}(W)\widetilde{\Phi^{\theta,u}_{Y_t}}(W) \le k^2_{Y,2}(\theta,t). \tag{4.11}$$

Using the notation from Propositions 2.1 and 2.2,

$$g(y) = \int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'\left(\int_0^t \Phi^\theta_{Y_t}(W)\widetilde{\Phi^{\theta,u}_{Y_t}}(W)d\theta\right)\Big|Y_t - \mathbf{E}(Y_t) = y\right)du.$$

The bounds obtained in (4.11) immediatly yield

$$0 < \int_0^t k^2_{Y,1}(\theta,t)d\theta \le g(y) \le \int_0^t k^2_{Y,2}(\theta,t)d\theta. \tag{4.12}$$

We will now give lower (resp. upper) estimates of $A_1 = \int_0^t k^2_{Y,1}(\theta,t)d\theta$ (resp. $A_2 = \int_0^t k^2_{Y,2}(\theta,t)d\theta$). The constants c and C appearing in the calculations may change from line to line. We start by calculating a lower bound of A_1. Since both summands in $k_{Y,1}$ are positive, we have

$$k^2_{Y,1}(\theta,t) \ge c e^{2m_{b,\sigma}(T-\theta)-2M_{f_y}(T-t)}.$$

Thus,

$$A_1 \ge c e^{-2M_{f_y}(T-t)}\left[\frac{e^{2m_{b,\sigma}T} - e^{2m_{b,\sigma}(T-t)}}{2m_{b,\sigma}}\right] \ge \frac{e^{(2m_{b,\sigma}-2M_{f_y})(T-t)}}{2m_{b,\sigma}}(2m_{b,\sigma}ct),$$

where we used the fact that $e^y(x-y) \le e^x - e^y$ if $x \ge y$. Because $e^{(2m_{b,\sigma}-2M_{f_y})(T-t)}$ is lower bounded by $e^{-2|M_{f_y}-m_{b,\sigma}|T}$, we finally obtain for some constant c depending on b,

σ, f_y and T,

$$A_1 = \int_0^t k_{Y,1}^2(\theta,t)d\theta \geq ct. \tag{4.13}$$

It remains to prove that $A_2 \leq Ct$ for some constant C. Using Young's lemma, we can write

$$\begin{aligned} A_2 &\leq Ce^{2M_{f_y}(T-t)}\left[\frac{e^{2M_{b,\sigma}T}-e^{2M_{b,\sigma}(T-t)}}{2M_{b,\sigma}}\right] + Ce^{2(M_{b,\sigma}+M_{f_y})T}\int_0^t e^{-2M_{b,\sigma}\theta-2M_{f_y}t}d\theta \\ &\leq \frac{e^{2M_{f_y}(T-t)}}{2M_{b,\sigma}}e^{2M_{b,\sigma}T}(2M_{b,\sigma}Ct) + C\left[\frac{e^{-2M_{f_y}t}-e^{-2(M_{b,\sigma}+M_{f_y})t}}{2M_{b,\sigma}}\right]. \end{aligned}$$

This upper estimate and the fact that $e^x - e^y \leq e^x(x-y)$ for $x \geq y$ yields

$$A_2 = \int_0^t k_{Y,2}^2(\theta,t)d\theta \leq Ct. \tag{4.14}$$

The inequalities (4.12) – (4.14) yield, $\mathbb{P}$-a.s,

$$0 < ct \leq g(y) \leq Ct,$$

with strictly positive constants c and C. Thus, Propositions 2.1 and 2.2 conclude the proof of Theorem 4.2. □

4.4. **Density of Z_t : existence and Gaussian estimates.** In this subsection, we will prove that under some conditions on the coefficients, Z_t has a density with Gaussian upper and lower bounds. We begin by listing those conditions in the upcoming subsection.

4.4.1. *Hypotheses.* We need to make additional assumptions with respect to those in subsections 3.1 and 4.3. More precisely, we assume (**H1**) and that the following holds on the diffusion process X_t,

$$\begin{cases} \textbf{H8}: \ \sigma \in B_0^{3,+}(\mathbb{R}), \quad \sigma' \geq 0. \\ \textbf{H9}: \ \exists\ M_l, M_{dl} \geq 0, \quad |[b,\sigma]| \leq M_l\sigma, \quad 0 \leq [\sigma,[\sigma,b]] \leq M_{dl}\sigma. \end{cases}$$

where $[\phi,\psi]$ denotes the Lie bracket between ϕ and ψ.

Remark 4.4. Recall that $\sigma \in B_0^{3,+}(\mathbb{R})$ implies that there exist strictly positive constants that will be refered to as m_σ and M_σ such that $0 < m_\sigma \leq \sigma \leq M_\sigma$. It also implies, along with the fact that $\sigma' \geq 0$, that there exist strictly positive constants that will be refered to as $M_{\sigma'}$, $M_{\sigma''}$ and $M_{\sigma^{(3)}}$ such that $0 \leq \sigma' \leq M_{\sigma'}$, $|\sigma''| \leq M_{\sigma''}$ and $|\sigma^{(3)}| \leq M_{\sigma^{(3)}}$.

On the backward process (Y,Z), we need the following conditions on the functions ϕ and f, where $f:\mathbb{R}^2 \to \mathbb{R}$ does not depend on z:

$$\begin{cases} \textbf{H10}: \ \text{There exist constants } c_{\phi'}, C_{\phi'}, C_{\phi''} \text{ such that } 0 < c_{\phi'} \leq \phi' \leq C_{\phi'},\ 0 < c_{\phi''} \leq \phi'' \leq C_{\phi''} \\ \textbf{H11}: \ \text{There exist constants } m_{f_x}, M_{f_x}, M_{f_y}, M_{f_{xx}}, M_{f_{xy}}, M_{f_{yx}}, M_{f_{yy}} \text{ such that} \\ \qquad 0 < m_{f_x} \leq f_x \leq M_{f_x}, |f_y| \leq M_{f_y}, 0 \leq f_{xx} \leq M_{f_{xx}}, 0 \leq f_{xy} \leq M_{f_{xy}}, 0 \leq f_{yy} \leq M_{f_{yy}} \end{cases}$$

Note that (**H10**) and (**H11**) imply (**H5**)-(**H7**).

4.4.2. *Main result (Existence and estimates for the density of Z_t).* We consider equation (4.1) with a function $f^\star$ that only has a linear dependency on Z, i.e.

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \\ Y_t = \phi(X_T) + \int_t^T f^\star(X_s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s \end{cases}$$

where $f^\star(x,y,z) = f(x,y) + \alpha z$, $\alpha \in \mathbb{R}$.

Remark 4.5. Because of the dependency of f on Z, the Malliavin derivative DZ will depend on D^2Z, which is not suitable for analyzing it within our framework. One can circumvent the above mentioned issue by using the Girsanov theorem to dispose of the impeding terms (similarly as done in the proof of Propositon 4.3). In order to clarify the proofs and to improve readability, we will consider that this step has already been performed in all of our proofs. This procedure leaves us with an equation of the type

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \\ Y_t = \phi(X_T) + \int_t^T f(X_s, Y_s)\, ds - \int_t^T Z_s dW_s, \end{cases}$$

which is the one that will be referred to in the proofs of the upcoming results.

The following theorem provides Gaussian estimates for the density of Z_t.

Theorem 4.6. *Under the hypotheses of Subsection 4.4.1, for $t \in (0,T)$ the random variable Z_t has a density ρ_{Z_t}. Furthermore, there exist strictly positive constants c and C such that, for almost all $z \in \mathbb{R}$, ρ_{Z_t} satisfies the following:*

$$\frac{\mathbf{E}|Z_t - \mathbf{E}(Z_t)|}{2ct}\exp\left(-\frac{(z-\mathbf{E}(Z_t))^2}{2Ct}\right) \le \rho_{Z_t}(z) \le \frac{\mathbf{E}|Z_t - \mathbf{E}(Z_t)|}{2Ct}\exp\left(-\frac{(z-\mathbf{E}(Z_t))^2}{2ct}\right).$$

Before proving Theorem 4.6, we will first give a technical Lemma and a Proposition which will play a key role in the upcoming proof of this Theorem. First recall a lemma used to calculate the Malliavin derivative of a product of random variables in $\mathbb{D}^{1,2}$ (for example, see [3], p.36, exercice 1.2.12).

Lemma 4.7. *(i) Let $s,t \in [0,T]$ and $F \in \mathbb{D}^{1,2}$; then we have $\mathbf{E}(F|\mathcal{F}_t) \in \mathbb{D}^{1,2}$ and*

$$D_s\mathbf{E}(F|\mathcal{F}_t) = \mathbf{E}(D_sF|\mathcal{F}_t)\, 1_{s\le t}.$$

(ii) If $F, G \in \mathbb{D}^{1,2}$ are such that F and $\|DF\|_{L^2([0,T])}$ are bounded, then $FG \in \mathbb{D}^{1,2}$ and

$$D(FG) = FDG + GDF.$$

The following Proposition ensures that under the hypotheses of Subsection 4.4.1, the second order Malliavin derivatives of X and Y are positive and bounded from above.

Proposition 4.8. *Under the assumptions of section 4.4.1, there exist two positive constants M_{D^2X} and M_{D^2Y} such that for $0 < \theta < t < s \le T$, $\mathbb{P}$-a.s,*

$$0 \le D^2_{\theta,t}X_s \le M_{D^2X} \quad \text{and} \quad 0 \le D^2_{\theta,t}Y_s \le M_{D^2Y}.$$

Remark 4.9. Here we obtain large inequalities since the basic example of standard Brownian motion shows that the second Malliavin derivative of X_t may be null.

Proof: We start by proving the inequalities on $D^2_{\theta,t}X_s$. Applying the Malliavin derivative to (4.7) and using the second point in Lemma 4.7, we deduce for $\theta, t \le s \le T$, since $U_s = G(X_s)$,

$$\begin{aligned} D^2_{\theta,t}X_s =& (\sigma \circ G^{-1})'(U_s)D_\theta U_s D_t U_s + (\sigma \circ G^{-1})(U_s)D^2_{\theta,t}U_s \\ =& (\sigma'\sigma)(X_s)D_\theta U_s D_t U_s + \sigma(X_s)D^2_{\theta,t}U_s. \end{aligned} \tag{4.15}$$

The hypotheses (**H8**) and (**H9**) along with (3.9) ensure that the term $(\sigma'\sigma)(X_s)D_\theta U_s D_t U_s$ is non negative and can be bounded from above by a constant given by

$$0 \leq (\sigma'\sigma)(X_s)D_\theta U_s D_t U_s \leq M_{\sigma'} M_\sigma e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(2s-t-\theta)}. \tag{4.16}$$

It remains to prove that the second summand in (4.15) is also non negative and bounded from above. As σ is non negative and bounded, we focus on proving that $D^2_{\theta,t}U_s$ is too. Applying once again the Malliavin derivative operator to (3.7) and using the second point in Lemma 4.7 as well as (4.5), we deduce for $\theta < t \leq s$,

$$\begin{aligned} D^2_{\theta,t}U_s &= \int_t^s (\beta \circ G^{-1})''(U_r)D_tU_rD_\theta U_r dr + \int_t^s (\beta \circ G^{-1})'(U_r)D^2_{\theta,t}U_r dr \\ &= \int_t^s e^{\int_r^s (\beta\circ G^{-1})'(U_v)dv}(\beta \circ G^{-1})''(U_r)D_tU_rD_\theta U_r dr \\ &= \int_t^s (\beta \circ G^{-1})''(U_r)D_rU_sD_tU_rD_\theta U_r dr. \end{aligned}$$

Further calculations yield the following expression

$$\begin{aligned} \left(\beta \circ G^{-1}\right)''(x) &= \left(\sigma\left(\frac{[\sigma,b]'}{\sigma} - \frac{[\sigma,b]\,\sigma'}{\sigma^2}\right) - \frac{1}{2}\left(\sigma''\sigma\right)'\sigma\right) \circ G^{-1}(x) \\ &= \left(\frac{[\sigma,[\sigma,b]]}{\sigma} - \frac{1}{2}\left(\sigma''\sigma\right)'\sigma\right) \circ G^{-1}(x). \end{aligned}$$

Using hypotheses (**H8**), (**H9**), the fact that $D_aU_b > 0$ for $a < b$ and (3.9), we immediatly obtain for $\theta < t \leq s$,

$$0 \leq \sigma(X_s)D^2_{\theta,t}U_s \leq \frac{2\left(M_{dl} + \frac{1}{2}M_{(\sigma''\sigma)'}M_\sigma\right)}{2M_l + M_\sigma M_{\sigma''}}\left[e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(2s-\theta-t)} - e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(s-\theta)}\right]. \tag{4.17}$$

Combining (4.15) and (4.17), it is clear that there exists a positive constant M_{D^2X} such that $0 \leq D^2_{\theta,t}X_s \leq M_{D^2X}$ with, for $\theta < t \leq s$,

$$\begin{aligned} M_{D^2X} = &\left(M_{\sigma'}M_\sigma + \frac{2\left(M_{dl} + \frac{1}{2}M_{(\sigma''\sigma)'}M_\sigma\right)}{2M_l + M_\sigma M_{\sigma''}}\right)e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(2s-t-\theta)} \\ &- \frac{2\left(M_{dl} + \frac{1}{2}M_{(\sigma''\sigma)'}M_\sigma\right)}{2M_l + M_\sigma M_{\sigma''}}e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(s-\theta)}. \end{aligned}$$

We will now address the second part of the Proposition, i.e., the inequalities on $D^2_{\theta,t}Y_s$. Let $\theta < t \leq s$. Applying once more the Malliavin derivative operator to $D_\theta Y_s$ in (4.3) and using the second point in Lemma 4.7, since f does not depend on Z we obtain, for $0 \leq \theta < t \leq s \leq T$,

$$\begin{aligned} D^2_{\theta,t}Y_s =& \phi'\left(X_T\right)D^2_{\theta,t}X_T + \phi''(X_T)D_\theta X_T D_t X_T - \int_s^T D^2_{\theta,t}Z_r dW_r \\ &+ \int_s^T \Big\{ f_{xx}\left(X_r,Y_r\right)D_\theta X_r D_t X_r + f_x\left(X_r,Y_r\right)D^2_{\theta,t}X_r \\ &\quad + f_{yx}\left(X_r,Y_r\right)\left(D_\theta Y_r D_t X_r + D_\theta X_r D_t Y_r\right) \\ &\quad + f_{yy}\left(X_r,Y_r\right)D_\theta Y_r D_t Y_r + f_y\left(X_r,Y_r\right)D^2_{\theta,t}Y_r \Big\} dr. \end{aligned}$$

Since $D^2_{\theta,t}Y_s$ solves a linear equation and is $\mathcal{F}_s$-measurable, we have that, for $0 \leq \theta < t \leq s \leq T$,

$$\begin{aligned} D^2_{\theta,t}Y_s =& \mathbf{E}\Big(e^{\int_s^T f_y(X_r,Y_r)dr}\{\phi'(X_T)\,D^2_{\theta,t}X_T + \phi''(X_T)D_\theta X_T D_t X_T\}\Big|\mathcal{F}_s\Big) \\ &+ \mathbf{E}\Big(\int_s^T e^{\int_s^T f_y(X_u,Y_u)du}\Big\{f_{xx}(X_r,Y_r)\,D_\theta X_r D_t X_r + f_x(X_r,Y_r)\,D^2_{\theta,t}X_r \\ &+ f_{yx}(X_r,Y_r)\,(D_\theta Y_r D_t X_r + D_\theta X_r D_t Y_r) + f_{yy}(X_r,Y_r)\,D_\theta Y_r D_t Y_r\Big\}dr\Big|\mathcal{F}_s\Big). \end{aligned}$$

Since $\sigma \geq c > 0$, (4.8) proves $D_u X_v \geq 0$ for $u \leq v$. Furthermore, (4.4) and (**H10**)–(**H11**) prove that $D_u Y_v \geq 0$ for $u \leq v$. Since (**H8**)–(**H11**) imply (**H5**)–(**H7**), the results in (4.2) and (4.8) remain valid. Thus, we immediatly obtain the positivity and an upper bound for $D^2_{\theta,t}Y_s$. This concludes the proof. □

Proof of Theorem 4.6: The outline of the proof is as follows: using a representation of Z, we compute its Malliavin derivative and show that under the hypotheses of Subsection 4.4.1, it is strictly bounded away from zero. This allows us to conclude using Proposition 2.1. We begin by giving a representation of Z. However, we do not use the one from [7] in terms of gradient, that is $Z_t = \sigma(X_t)(\nabla X_t)^{-1}\nabla Y_t$, but rather use the fact that Z_t can be represented by use of the Clark-Ocone formula. Indeed, by the uniqueness of the solution (Y,Z), Z_t can be written as

$$Z_t = \mathbf{E}\left(D_t\phi(X_T) + D_t\int_0^T f(X_s,Y_s)\,ds\Big|\mathcal{F}_t\right) \in \mathbb{D}^{1,2}. \tag{4.18}$$

Using this fact, we get for $t \in [0,T]$

$$Z_t = \mathbf{E}\left(\phi'(X_T)\,D_t X_T + \int_t^T \{f_x(X_s,Y_s)\,D_t X_s + f_y(X_s,Y_s)\,D_t Y_s\}\,ds|\mathcal{F}_t\right).$$

Let $\theta \leq t$. We use both points of Lemma 4.7 and Proposition 4.8 in order to calculate the first order Malliavin derivative of Z_t. This leads, for $\theta \leq t$:

$$\begin{aligned} D_\theta Z_t =& \mathbf{E}\Big(\phi''(X_T)\,D_\theta X_T D_t X_T + \phi'(X_T)\,D^2_{\theta,t}X_T \\ &+ \int_t^T \Big\{f_{xx}(X_s,Y_s)\,D_\theta X_s D_t X_s + f_{yx}(X_s,Y_s)\,(D_\theta Y_s D_t X_s + D_\theta X_s D_t Y_s) \\ &+ f_{yy}(X_s,Y_s)\,D_\theta Y_s D_t Y_s + f_x(X_s,Y_s)\,D^2_{\theta,t}X_s + f_y(X_s,Y_s)\,D^2_{\theta,t}Y_s\Big\}ds\Big|\mathcal{F}_t\Big). \end{aligned} \tag{4.19}$$

We now need to bound from above each summand of this expression; in what follows, c and C denote strictly positive constants that may vary from line to line. Recall that under the assumptions (**H8**)–(**H11**) using (4.7) and (4.4), we deduce that $D_u X_v \geq c > 0$ and $D_u Y_v \geq 0$ for $u \leq v$. The hypothesis (**H10**) on ϕ (along with (**H8**) and (**H9**) on the diffusion X), (4.7) and Proposition 4.8 ensure that there exist strictly positive constants such that $0 < c \leq \phi''(X_T)\,D_\theta X_T D_t X_T \leq C$ and $0 \leq \phi'(X_T)\,D^2_{\theta,t}X_T \leq C$. Using hypothesis (**H11**) on f and its derivatives and Proposition 4.8 again allows us to bound the remaining terms in (4.19) by positive constants, i.e.

$$\begin{aligned} &0 \leq f_x(X_s,Y_s)\,D^2_{\theta,t}X_s \leq C, \quad 0 \leq f_y(X_s,Y_s)\,D^2_{\theta,t}Y_s \leq C, \\ &0 \leq f_{xx}(X_s,Y_s)\,D_\theta X_s D_t X_s \leq C, \quad 0 \leq f_{yy}(X_s,Y_s)\,D_\theta Y_s D_t Y_s \leq C, \\ &0 \leq f_{xy}(X_s,Y_s)\,D_\theta X_s D_t Y_s \leq C, \quad 0 \leq f_{yx}(X_s,Y_s)\,D_\theta Y_s D_t X_s \leq C. \end{aligned}$$

Gathering all of these immediatly gives us the existence of two strictly positive constants m_{DZ} and M_{DZ} such that for $0 < \theta < t \leq T$, $\mathbb{P} - a.s$,

$$0 < m_{DZ} \leq D_\theta Z_t \leq M_{DZ}. \tag{4.20}$$

Write $D_\bullet Z_t = \Phi^\bullet_{Z_t}(W)$ with a measurable function $\Phi^\bullet_{Z_t} : \mathbb{R}^{L^2([0,T])} \longrightarrow L^2([0,T])$. Then (4.20) yields, for $\theta < t$, $0 < m_{DZ} \leq \Phi^\theta_{Z_t}(W) \leq M_{DZ}$. As previously done, define $\widetilde{\Phi^{\bullet,u}_{Z_t}}(W) = \Phi^\bullet_{Z_t}(e^{-u}W + \sqrt{1-e^{-2u}}W')$ for $u \in [0,+\infty[$. Using (4.20), it is clear that, for $\theta < t$, we have for $u \in [0,+\infty)$, $0 < m_{DZ} \leq \widetilde{\Phi^{\theta,u}_{Z_t}}(W) \leq M_{DZ}$. Combining the bounds on $\Phi^\theta_{Z_t}$ and $\widetilde{\Phi^{\theta,u}_{Z_t}}$ yields, for $\theta < t$ and $u \in [0,+\infty)$,

$$0 < m^2_{DZ} \leq \Phi^\theta_{Z_t}(W)\widetilde{\Phi^{\theta,u}_{Z_t}}(W) \leq M^2_{DZ}. \tag{4.21}$$

Finally, let

$$\begin{aligned} g(z) &= \int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'(\langle \Phi^\bullet_{Z_t}(W), \widetilde{\Phi^{\bullet,u}_{Z_t}}(W)\rangle_{L^2([0,T])})\Big| Z_t - \mathbf{E}(Z_t) = z\right) du \\ &= \int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'\left(\int_0^t \Phi^\theta_{Z_t}(W)\widetilde{\Phi^{\theta,u}_{Z_t}}(W) d\theta\right)\Big| Z_t - \mathbf{E}(Z_t) = z\right) du. \end{aligned}$$

The bounds obtained in (4.21) immediatly yield $0 < m^2_{DZ}t \leq g(z) \leq M^2_{DZ}t$. Thus, Proposition 2.1 concludes the proof of Theorem 4.6. □

Remark 4.10. Theorem 4.6 has been proved under a set of hypotheses (those of Subsection 4.4.1) based on the fact that σ is positive. The case where σ is negative was included neither in the proof nor in the hypotheses for the sake of clarity and readability of the paper. However, as already mentioned in Remark 3.1, this case can be addressed (without any further difficulties) by using the following transformations: $\sigma \to \tilde{\sigma} := -\sigma$ and $W \to \tilde{W} := -W$. After performing those tranformations, it suffices to consider $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X, Y, -Z)$ to be the solution of

$$\begin{cases} d\tilde{X}_t = & b\left(\tilde{X}_t\right) dt + \tilde{\sigma}\left(\tilde{X}_t\right) d\tilde{W}_t \\ d\tilde{Y}_t = & \phi\left(\tilde{X}_T\right) + \int_t^T f\left(\tilde{X}_r, \tilde{Y}_r\right) dr - \int_t^T \tilde{Z}_r d\tilde{W}_r \end{cases}$$

This brings the problem back to the set of hypotheses of Subsection 4.4.1 and it can be dealt with using the techniques presented in the last section.

Acknowledgments: We would like to thank F.G. Viens for introducing us to this topic as well as A. Millet and C.A. Tudor for helpful comments.

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E-mail address: omar.aboura@malix.univ-paris1.fr

E-mail address: solesne.bourguin@univ-paris1.fr

SAMM, EA 4543, Université Paris 1 Panthéon Sorbonne, 90 Rue de Tolbiac, 75634 Paris Cedex France