

Editorial Manager(tm) for Theory and Decision
Manuscript Draft

Manuscript Number:

Title: A representation of preferences by the Choquet integral with respect to a 2-additive capacity

Article Type: Original Research

Keywords: Capacity; Möbius transform; Choquet integral; Multiple criteria decision analysis; k-monotone function; belief function

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Theory and Decision manuscript No. (will be inserted by the editor)
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A representation of preferences by the Choquet integral with respect to a 2-additive capacity

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the date of receipt and acceptance should be inserted later

Abstract In the context of Multiple criteria decision analysis, we present the necessary and sufficient conditions allowing to represent an ordinal preferential information provided by the decision maker by a Choquet integral w.r.t a 2-additive capacity. We provide also a characterization of this type of preferential information by a belief function which can be viewed as a capacity. These characterizations are based on three axioms, namely strict cycle-free preferences and some monotonicity conditions called MOPI and 2-MOPI.

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JEL Classification: C43; C51; C52; D63

1 Introduction

Since its introduction by Schmeidler [1986], the Choquet integral (see Choquet [1953]) has become a popular and flexible tool in decision making. In decision under uncertainty, as shown by the well-known Ellsberg's paradox (see Ellsberg [1961], Chateauneuf [1995]), the use of the expected utility model (see Von Neuman and Morgenstern [1944]) is limited. Therefore some non-additive

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models like Choquet expected utility (see Schmeidler [1989], Chateauneuf et al. [2004]) have been proposed in order to overcome the limitations of the expected utility model. In social welfare, it generalizes the Gini index (see Weymark [1981]). In multicriteria decision making, it generalizes the weighted arithmetic mean, so far the main tool to aggregate criteria, allowing to take into account interaction between criteria (see Grabisch [1996]).

The Choquet integral is defined w.r.t. a capacity (or nonadditive monotonic measure, or fuzzy measure), and can be thought as a generalization of the expected value, the capacity playing the role of a probability measure. Supposing a finite set of criteria denoted by N and having at disposal some piece of information on the preference of the decision maker, a fundamental question in practice is to determine a capacity so that the Choquet integral can represent the revealed preference of the decision maker, if such a capacity exists at all. The fact that a capacity is defined on the power set of N makes the problem exponentially complex, and thus practically intractable for large sets of criteria. For this reason, many simpler models have been proposed, where the capacity is restricted to some particular subspace. These are for example symmetric capacities (depending solely on the cardinality of subsets of N), as well as their generalization called p -symmetric capacities (see Miranda et al. [2002]), k -additive capacities (see Grabisch [1997]), whose Möbius transform lives on subsets of at most k elements, decomposable capacities, (see Weber [1984]) (including distorted probabilities), etc.

The concept of k -additive capacity seems to be of particular interest, since the value of k is directly related to the complexity of the model (the number of subsets of at most k elements) and it has a clear interpretation in many domains of decision making. In social welfare, the generalized Gini index proposed by Weymark [1981] corresponds in fact to a Choquet integral w.r.t. a symmetric k -additive capacity, and has a very natural interpretation in terms of the weight the decision maker puts on envy in the society (see Gajdos [2002], Miranda et al. [2005]). For the 2-additive case, it corresponds to a decision maker (DM) who is inequality averse in the sense that any Pigou-Dalton transfer increases his measure of welfare, wherever this transfer is applied on the income distribution. In multicriteria decision making, any interaction between two criteria can be represented and interpreted by a Choquet integral w.r.t. a 2-additive capacity, but not more complex interaction. The Choquet integral w.r.t. a 2-additive capacity is very used in many applications such that the evaluation of discomfort in sitting position (see Grabisch et al. [2002]), the construction of performance measurement systems model in a supply chain context (see Berrah and Clivillé [2007], Clivillé et al. [2007]) and complex system design (see Pignon and Labreuche [2007]).

The above development suggests that the Choquet integral w.r.t. a 2-additive capacity seems to be of particular interest, and offers a good compromise between flexibility of the model and complexity. Therefore, we focus in this paper on the 2-additive model.

Usually the DM is supposed to be able to express his preference over the set of all alternatives X . Because this is not feasible in most of practical situation (the cardinality of X may be very large), the DM is asked to give, using pairwise comparisons, an ordinal information (a preferential information containing only a strict preference and an indifference relations) on a subset $X' \subseteq X$ called reference set. When X' is given by the DM, it is seldom the case that the so-obtained preferential information are sufficient to specify all the parameters of the model and especially the interaction between criteria. Therefore, we propose him to reason on some fictitious alternatives which represent concrete situations. These alternatives are simple and easy to use, and are directly related to the parameters of the model. This approach is present in the MACBETH methodology for the computation of the weights for the arithmetic mean (see Bana e Costa et al. [2005]).

The use of Choquet integral requires to ensure the commensurateness between criteria i.e. one shall be able to compare any element of one point of view with any element of any other point of view. The only way to construct the utility functions with the Choquet integral uses the reference levels (see Labreuche and Grabisch [2003], Grabisch et al. [2003]). The existence of reference levels has been justified by psychologists (Simon [1956], Slovic et al. [2002], Hsee [1996]).

The set X' we use in this paper is the set of binary alternatives or binary actions denoted by \mathcal{B} . A binary action is an (fictitious) alternative representing a prototypical situation where on a given subset of criteria, the attributes reach a satisfactory level $\mathbf{1}$ for at most two criteria, while on the remaining ones, they are at a neutral level (neither satisfactory nor unsatisfactory) $\mathbf{0}$. The binary actions are used in many applications through the MACBETH methodology (see Bana e Costa et al. [2001, 2002], Clivillé et al. [2007]). Since these alternatives have a very simple structure and make sense for the DM, he should have no difficulty to express preference on them. Our aim is to solve the following fundamental problem: *Is the preference of the decision maker representable by a Choquet integral w.r.t. a 2-additive capacity?* If the answer is positive, one can extend the preference relation over the whole set X .

We provide necessary and sufficient conditions on the preference of the decision maker so that a Choquet integral w.r.t. a 2-additive capacity can represent the preference. The main condition, called MOPI (MONotonicity of Preferential Information) is related to the special kind of monotonicity induced by a 2-additive capacity. The proof of this result being constructive, we are even able to provide an example of such a capacity.

We study also the case of belief functions (see Shafer [1976]), which are particular capacities whose Möbius transform is nonnegative. They have been well studied in decision making (see, e.g., Chateauneuf and Jaffray [1989], Jaffray [1989], Jaffray and Wakker [1993]). We prove that surprisingly any ordinal information representable by a belief function is also representable by a Choquet integral w.r.t. a 2-additive capacity (but not the converse). We find the necessary and sufficient conditions so that a Choquet integral w.r.t. a

(2-additive) belief function can represent the preference of the DM. The main condition is a strong version of the MOPI condition, called 2-MOPI. We give an interpretation of the 2-MOPI and MOPI conditions through the notion of importance of a criterion. These two conditions are very important in order to deal with inconsistencies when they occur in the ordinal information provided by the DM.

The paper is organized as follows. Section 2 sets the framework, and gives the basic material on capacities, 2-additive capacities and the Choquet integral. Section 3 introduces various concepts useful for the characterization, such as cycles, and the MOPI property. In Section 4, we introduce the 2-MOPI property for the characterization through belief functions. All proofs are given in the last section.

2 Preliminaries

2.1 Notations and aim

Let us denote by $N = \{1, \dots, n\}$ the set of n criteria and by 2^N the set of all subsets of N . An action (or alternative, option) $x = (x_1, \dots, x_n)$ is identified to an element of the Cartesian product $X = X_1 \times \dots \times X_n$, where X_1, \dots, X_n represent the set of points of view or attributes. For a subset A of N , the notation $z = (x_A, y_{N-A})$ means that z is defined by $z_i = x_i$ if $i \in A$, and $z_i = y_i$ otherwise. For all i, j in N , the element $i \vee j$ denotes one of the elements i, j .

Our aim is to construct a preference relation over X . In practice (see Chateaneuf et al. [2008], Marchant [2003]) one can only ask to the DM pairwise comparisons of alternatives on a finite subset X' of X , X' having a small size. Hence we get a preference relation $\succsim_{X'}$ on X' . The question is then: how to construct a preference relation \succsim_X on X , so that \succsim_X is an extension of $\succsim_{X'}$? To this end, people usually suppose that \succsim_X is representable by an overall utility function:

$$x \succsim_X y \Leftrightarrow F(U(x)) \geq F(U(y)) \quad (1)$$

where $U(x) = (u_1(x_1), \dots, u_n(x_n))$, $u_i : X_i \rightarrow \mathbb{R}$ is called a utility function, and $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is an aggregation function. Usually, we consider a family of aggregation functions characterized by a parameter vector θ (e.g., a weight distribution over the criteria). The parameter vector θ can be deduced from the knowledge of $\succsim_{X'}$, that is, we determine the possible values of θ for which (1) is fulfilled over X' . We study the case where F is the Choquet integral w.r.t. a 2-additive capacity, thus in this case the parameter vector is the 2-additive capacity and X' is the set of binary actions. The aim of this paper is to give necessary and sufficient conditions on $\succsim_{X'}$ to be represented by a Choquet integral w.r.t. a 2-additive capacity. The model obtained in X' will be then automatically extended to X .

2.2 Capacities and the Choquet integral

The Choquet integral w.r.t. a capacity is a well known tool in multicriteria decision making (see Schmeidler [1986], Grabisch and Labreuche [2004], Gajdos [2002], Chateauneuf [1994]). It generalizes the arithmetic mean. We consider here a particular case of the Choquet integral, where the capacity is 2-additive. We define this notion below.

Definition 21

1. A capacity on N is a set function $\mu : 2^N \rightarrow [0, 1]$ such that:
 - (a) $\mu(\emptyset) = 0$
 - (b) $\mu(N) = 1$
 - (c) $\forall A, B \in 2^N, [A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).
2. The Möbius transform (see Chateauneuf and Jaffray [1989]) of a capacity μ on N is a function $m : 2^N \rightarrow \mathbb{R}$ defined by:

$$m(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K), \forall T \in 2^N. \quad (2)$$

When m is given, it is possible to recover the original μ by the following expression:

$$\mu(T) := \sum_{K \subseteq T} m(K), \forall T \in 2^N. \quad (3)$$

Definition 22 A capacity μ on N is said to be 2-additive if

- For all subset T of N such that $|T| > 2$, $m(T) = 0$;
- There exists a subset B of N such that $|B| = 2$ and $m(B) \neq 0$.

Notations We simplify our notation for a capacity μ and its Möbius transform m by using the following shorthand: $\mu_i := \mu(\{i\})$, $\mu_{ij} := \mu(\{i, j\})$, $m_i := m(\{i\})$, $m_{ij} := m(\{i, j\})$, for all $i, j \in N$, $i \neq j$. Whenever we use i and j together, it always means that they are different.

The following important Lemma shows that a 2-additive capacity is entirely determined by the value of the capacity on the singletons $\{i\}$ and pairs $\{i, j\}$ of 2^N :

Lemma 1

1. Let μ be a 2-additive capacity on N . We have

$$\mu(K) = \sum_{\{i,j\} \subseteq K} \mu_{ij} - (|K| - 2) \sum_{i \in K} \mu_i, \forall K \subseteq N, |K| \geq 2. \quad (4)$$

2. If the coefficients μ_i and μ_{ij} are given for all $i, j \in N$, then the necessary and sufficient conditions that μ is a 2-additive capacity are:

$$\sum_{\{i,j\} \subseteq N} \mu_{ij} - (n - 2) \sum_{i \in N} \mu_i = 1 \quad (5)$$

$$\mu_i \geq 0, \forall i \in N \quad (6)$$

$$\sum_{i \in A \setminus \{k\}} (\mu_{ik} - \mu_i) \geq (|A| - 2)\mu_k, \forall A \subseteq N, |A| \geq 2, \forall k \in A. \quad (7)$$

Proof See Grabisch [1997].

For an alternative $x := (x_1, \dots, x_n) \in X$, the expression of the Choquet integral w.r.t. a capacity μ is given by:

$$C_\mu(U(x)) := \sum_{i=1}^n (u_{\tau(i)}(x_{\tau(i)}) - u_{\tau(i-1)}(x_{\tau(i-1)}))\mu(\{\tau(i), \dots, \tau(n)\}) \quad (8)$$

where τ is a permutation on N such that $u_{\tau(1)}(x_{\tau(1)}) \leq u_{\tau(2)}(x_{\tau(2)}) \leq \dots \leq u_{\tau(n-1)}(x_{\tau(n-1)}) \leq u_{\tau(n)}(x_{\tau(n)})$, and $u_{\tau(0)}(x_{\tau(0)}) := 0$.

A Choquet integral w.r.t. a 2-additive capacity μ is called for short a *2-additive Choquet integral*. Given an alternative $x := (x_1, \dots, x_n) \in X$, the 2-additive Choquet integral can be written also as follows (see Grabisch and Labreuche [2008]):

$$C_\mu(U(x)) = \sum_{i=1}^n v_i u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij} |u_i(x_i) - u_j(x_j)| \quad (9)$$

where $v_i = \sum_{K \subseteq N \setminus i} \frac{(n - |K| - 1)! |K|!}{n!} (\mu(K \cup i) - \mu(K))$ is the importance of criterion i corresponding to the Shapley value of μ (see Shapley [1953]) and $I_{ij} = \mu_{ij} - \mu_i - \mu_j$ is the interaction index between the two criteria i and j (see Grabisch [1997], Murofushi and Soneda [1993]).

2.3 Binary actions and relations

We assume that the DM is able to identify for each criterion i two reference levels:

1. A reference level $\mathbf{1}_i$ in X_i which he considers as good and completely satisfying if he could obtain it on criterion i , even if more attractive elements could exist. This special element corresponds to the *satisficing level* in the theory of bounded rationality of Simon [1956].
2. A reference level $\mathbf{0}_i$ in X_i which he considers neutral on i . The neutral level is an element which is thought by the DM to be neither good nor bad, neither attractive nor repulsive relatively to his concerns with respect to the criterion i . The existence of this neutral level has roots in psychology (see Slovic et al. [2002]), and is used in bipolar models like Cumulative Prospect Theory (see Tversky and Kahneman [1992]).

We set for convenience $u_i(\mathbf{1}_i) = 1$ and $u_i(\mathbf{0}_i) = 0$. The use of Choquet integral requires to ensure the commensurateness between criteria. Therefore the previous reference levels can be used in order to define the same scale on each criterion (see Labreuche and Grabisch [2003], Grabisch et al. [2003]). For more details about these reference levels, see Grabisch and Labreuche [2008, 2004].

We call a *binary action or binary alternative*, an element of the set

$$\mathcal{B} = \{\mathbf{0}_N, (\mathbf{1}_i, \mathbf{0}_{N-i}), (\mathbf{1}_{ij}, \mathbf{0}_{N-ij}), i, j \in N, i \neq j\} \subseteq X$$

where

- $\mathbf{0}_N = (\mathbf{1}_\emptyset, \mathbf{0}_N) =: a_0$ is an action considered neutral on all criteria.
- $(\mathbf{1}_i, \mathbf{0}_{N-i}) =: a_i$ is an action considered satisfactory on criterion i and neutral on the other criteria.
- $(\mathbf{1}_{ij}, \mathbf{0}_{N-ij}) =: a_{ij}$ is an action considered satisfactory on criteria i and j and neutral on the other criteria.

In MACBETH methodology (see Bana e Costa et al. [2005]), only two types of binary actions are used in order to determine the weights of the arithmetic mean: a_0 and $a_i, \forall i \in N$.

Using the Choquet integral, we get the following consequences:

1. For any capacity μ ,

$$C_\mu(U((\mathbf{1}_A, \mathbf{0}_{N-A}))) = \mu(A), \forall A \subseteq N. \quad (10)$$

2. For any 2-additive capacity, we have

$$C_\mu(U(a_0)) = 0 \quad (11)$$

$$C_\mu(U(a_i)) = \mu_i = v_i - \frac{1}{2} \sum_{k \in N, k \neq i} I_{ik} \quad (12)$$

$$C_\mu(U(a_{ij})) = \mu_{ij} = v_i + v_j - \frac{1}{2} \sum_{k \in N, k \notin \{i,j\}} (I_{ik} + I_{jk}) \quad (13)$$

The last two equations come from general relations between the capacity μ and interaction (see Grabisch [1997] for details). Generally the DM knows how to compare some alternatives using his knowledge of the problem, his experience, etc. These alternatives form a set of reference alternatives and allow to determine the parameters of a model (utility functions, subjective probabilities, weights, ...) in the decision process (see Marchant [2003] for more details). As shown by the previous equations (11), (12), (13) and Lemma 1, it should be sufficient to get some preferential information from the DM only on binary actions. To entirely determine the 2-additive capacity this information is expressed by the following relations:

$P = \{(x, y) \in \mathcal{B} \times \mathcal{B} : \text{DM strictly prefers } x \text{ to } y\}$, $I = \{(x, y) \in \mathcal{B} \times \mathcal{B} : \text{DM is indifferent between } x \text{ and } y\}$. The relation P is irreflexive and asymmetric while I is reflexive and symmetric.

Definition 23 *The ordinal information on \mathcal{B} is the structure $\{P, I\}$.*

Now we will suppose P to be nonempty for any ordinal information $\{P, I\}$ (“non triviality axiom”). Before we end this section, let us introduce another relation M which completes the ordinal information $\{P, I\}$ given by the DM and models the natural relations of monotonicity between binary actions. For $(x, y) \in \{(a_i, a_0), i \in N\} \cup \{(a_{ij}, a_i), i, j \in N, i \neq j\}$,

$$x M y \text{ if not}(x (P \cup I) y).$$

The relation M models the monotonicity conditions $\mu(\{i\}) \geq 0$ and $\mu(\{i, j\}) \geq \mu(\{i\})$ for a capacity μ .

Example 1 *If we consider $N = \{1, 2, 3\}$, $\mathcal{B} = \{a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}\}$, $P = \{(a_{13}, a_3), (a_2, a_3), (a_{23}, 0)\}$, $I = \{(a_{12}, a_1)\}$, then the relation M is $M = \{(a_1, a_0), (a_2, a_0), (a_3, a_0), (a_{12}, a_2), (a_{13}, a_1), (a_{23}, a_2), (a_{23}, a_3)\}$.*

Remark 1

For any 2-additive capacity μ , by the definition of the relation M , we have for all $x, y \in \mathcal{B}$, $x M y \Rightarrow C_\mu(U(x)) \geq C_\mu(U(y))$.

3 The representation of the ordinal information by the Choquet integral

An ordinal information $\{P, I\}$ is said to be *representable by a 2-additive Choquet integral* if there exists a 2-additive capacity μ such that:

1. $\forall x, y \in \mathcal{B}$, $x P y \Rightarrow C_\mu(U(x)) > C_\mu(U(y))$
2. $\forall x, y \in \mathcal{B}$, $x I y \Rightarrow C_\mu(U(x)) = C_\mu(U(y))$.

Given an ordinal information $\{P, I\}$, we look for the necessary and sufficient conditions on \mathcal{B} for which $\{P, I\}$ is representable by a 2-additive Choquet integral. To do it, we need to define first the notion of strict cycle of the relation $(P \cup I \cup M)$.

3.1 Cycle of $(P \cup I \cup M)$

For a binary relation \mathcal{R} on \mathcal{B} and x, y elements of \mathcal{B} , $\{x_1, x_2, \dots, x_p\} \subseteq \mathcal{B}$ is a *path* of \mathcal{R} from x to y if $x = x_1 \mathcal{R} x_2 \mathcal{R} \dots \mathcal{R} x_{p-1} \mathcal{R} x_p = y$. A path of \mathcal{R} from x to x is called a *cycle* of \mathcal{R} .

- A path $\{x_1, x_2, \dots, x_p\}$ of $(P \cup I \cup M)$ is said to be a *strict path* from x to y if there exists i in $\{1, \dots, p-1\}$ such that $x_i P x_{i+1}$. In this case, we will write $x TC_P y$.
- We write $x \sim y$ if there exists a nonstrict cycle of $(P \cup I \cup M)$ (hence a cycle of $(I \cup M)$) containing x and y .

Hence we deduce the following easy proposition:

Proposition 1 Let μ be a 2-additive capacity. Let $\{P, I\}$ be an ordinal information representable by a 2-additive Choquet integral and x_1, x_2, \dots, x_p be elements of \mathcal{B} .

If (x_1, x_2, \dots, x_p) is a nonstrict cycle of $(P \cup I \cup M)$ then $C_\mu(U(x_1)) = C_\mu(U(x_2)) = \dots = C_\mu(U(x_p))$.

Proof (x_1, x_2, \dots, x_p) is a nonstrict cycle of $(P \cup I \cup M)$ means $x_1 (I \cup M) x_2 (I \cup M) \dots (I \cup M) x_{p-1} (I \cup M) x_p (I \cup M) x_1$. So using the definition of a representation of an ordinal information on \mathcal{B} and the Remark 1, we have $C_\mu(U(x_1)) \geq C_\mu(U(x_2)) \geq \dots \geq C_\mu(U(x_p)) \geq C_\mu(U(x_1))$.

3.2 MOPI property and theorem of characterization

Before defining the property we call MOPI, let us explain this new condition through a simple example:

Example 2 Suppose that the DM says : $a_{12} I a_3$, $a_{13} I a_2$ and $a_1 P a_0$. Using the relation M , we have $a_{12} M a_2 I a_{13} M a_3 I a_{12}$. Remark that $(a_{12}, a_2, a_{13}, a_3, a_{12})$ forms a nonstrict cycle of $(P \cup I \cup M)$. If $\{P, I\}$ is representable by a 2-additive Choquet integral C_μ , this implies $\mu_{12} = \mu_{13} = \mu_2 = \mu_3$ and $\mu_1 > 0$. However, we get a contradiction with the monotonicity constraint $\mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3$ of a 2-additive capacity with the subset $A = \{1, 2, 3\}$, $k = 1$ (see Equation (7) in Lemma 1).

This type of inconsistency is defined in general form by:

Definition 31 (MOPI property) Let $i, j, k \in N$, i fixed.

1. We call Monotonicity of Preferential Information in $\{i, j, k\}$ w.r.t. i the following property (denoted by $(\{i, j, k\}, i)$ -MOPI):

$$\left. \begin{array}{l} a_{ij} \sim a_{i \vee j} \\ a_{ik} \sim a_{i \vee k} \\ i \vee j \neq i \vee k \end{array} \right\} \Rightarrow [\text{not}(a_l TC_P a_0), l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\}]$$

We recall that $i \vee j = i$ or j . If the property $(\{i, j, k\}, i)$ -MOPI is satisfied then the elements $a_l, l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\}$ are called the neutral binary actions of $\{i, j, k\}$ w.r.t. i . The set of all such elements is denoted by NBA.

2. We say that $\{i, j, k\}$ satisfies the property of Monotonicity of Preferential Information (MOPI) if $\forall l \in \{i, j, k\}$, $(\{i, j, k\}, l)$ -MOPI is satisfied.

Example 3 Let $N = \{1, 2, 3, 4\}$ and $i = 1$ fixed. The property $(\{1, 2, 3\}, 1)$ -MOPI reads as follows:

$$\begin{array}{l} \left\{ \begin{array}{l} a_{12} \sim a_2 \\ a_{13} \sim a_1 \end{array} \right\} \Rightarrow \text{not}(a_3 TC_P a_0) \\ \text{and} \\ \left\{ \begin{array}{l} a_{12} \sim a_1 \\ a_{13} \sim a_3 \end{array} \right\} \Rightarrow \text{not}(a_2 TC_P a_0) \\ \text{and} \\ \left\{ \begin{array}{l} a_{12} \sim a_2 \\ a_{13} \sim a_3 \end{array} \right\} \Rightarrow \text{not}(a_1 TC_P a_0). \end{array}$$

The MOPI condition given in this paper is in fact equivalent to the more complex MOPI property presented in Mayag et al. [2008]. We give below the main result of the paper which is a theorem of characterization of consistent ordinal information $\{P, I\}$ representable by a 2-additive Choquet integral:

Theorem 1 *An ordinal information $\{P, I\}$ is representable by a 2-additive Choquet integral on \mathcal{B} if and only if the following conditions are satisfied:*

1. $(P \cup I \cup M)$ contains no strict cycle;
2. Any subset K of N such that $|K| = 3$ satisfies the MOPI property.

Theorem 1 characterizes the 2-additive Choquet integral in terms of preference, and shows that only two types of inconsistencies can occur in an ordinal information given by a DM in order to represent it by a 2-additive Choquet integral. The first one is a very classical condition (no cycle in strict preferences), while the second is inherent to the special monotonicity conditions implied by a 2-additive capacity (see equations (6) and (7)). Observe however that the MOPI condition involves only subsets of three criteria, and thus remains simple. As it is shown in the proof of the theorem, this condition suffices to infer monotonicity conditions given in (7) for all subsets.

When $I = \emptyset$ in the ordinal information, the MOPI condition becomes void, hence always satisfied. Therefore, we have the following result:

Corollary 1

Any ordinal information $\{P, I\}$ on \mathcal{B} such that $I = \emptyset$ is representable by a 2-additive Choquet integral if and only if $(P \cup M)$ has no strict cycle.

Furthermore any ordinal information with empty indifference for which $(P \cup M)$ has no strict cycle, can be represented by a 2-additive capacity with nonnegative interaction indices.

Is it possible to represent an ordinal information by another aggregation function instead of the 2-additive Choquet integral? If the answer is yes, can we give a similar characterization like in Theorem 1? In the next section, we will show that it is possible by using for instance belief functions.

4 The representation of ordinal information by belief functions

4.1 General definitions

Beliefs functions are one of the fundamental concepts used in the theory of evidence of Shafer [1976], and have been well studied and used in decision making (see, e.g., Jaffray [1989], Jaffray and Wakker [1993], Smets [2005]). A belief function Bel is defined as a capacity whose Möbius transform $m : 2^N \rightarrow \mathbb{R}$, called in this case *mass distribution* or *basic belief assignment* is nonnegative:

$$m(A) \geq 0, \forall A \in 2^N.$$

Remark 2

- The sets A such that $m(A) > 0$ are called the focal elements;
- If all focal elements are singletons then the mass distribution can be considered as a probability distribution;

We can have a definition of the representation of ordinal information by a belief function which is similar to the same representation by a Choquet integral (see Section 3).

Definition 41 An ordinal information $\{P, I\}$ is said to be representable by a belief function if there exists a belief function Bel such that

1. $\forall x, y \in \mathcal{B}, x P y \Rightarrow C_{Bel}(U(x)) > C_{Bel}(U(y))$
2. $\forall x, y \in \mathcal{B}, x I y \Rightarrow C_{Bel}(U(x)) = C_{Bel}(U(y))$.

By using Definition 22, a 2-additive belief function has a mass distribution m characterized by:

1. $\exists i, j \in N$ such that $m(\{i, j\}) \neq 0$;
2. $\forall K \in 2^N$ such that $|K| \geq 3, m(K) = 0$.

In the next section, Theorem 2 provides a relation between k -monotone functions (see Grabisch [1997], Chateauneuf and Jaffray [1989]) and belief functions, and a relation between k -monotone functions and the previous MOPI property.

4.2 k -monotone functions and belief functions

Given an integer $k \geq 2$, a set function $\mu : 2^N \rightarrow [0, 1]$ is k -monotone (shorthand for: monotone of order k) if for each family $\{A_1, A_2, \dots, A_k\} \subseteq 2^N$, we have

$$\mu\left(\bigcup_{i=1}^k A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1, \dots, k\}} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} A_i\right). \quad (14)$$

A simpler characterization of k -monotone functions by their Möbius inversion is given by the following proposition:

Proposition 2 Let k be an integer, $k \geq 2$. Let $\mu : 2^N \rightarrow [0, 1]$ be a set function, and m be its Möbius transform. μ is k -monotone if and only if

$$\sum_{A \subseteq L \subseteq B} m(L) \geq 0 \quad \forall A, B \subseteq N, A \subseteq B \text{ and } 2 \leq |A| \leq k. \quad (15)$$

Proof See Chateauneuf and Jaffray [1989]

It is well known that $\mu : 2^N \rightarrow [0, 1]$ is a belief function if and only if μ is a k -monotone capacity for all $k \geq 2$. The following result gives another sufficient condition to obtain a belief function from a k -monotone and 2-additive capacity, and relates belief function with the MOPI condition translated in terms of capacity.

Theorem 2

Let $\mu : 2^N \rightarrow [0, 1]$ be a set function and k be an integer such that $k \geq 2$.

1. If μ is monotone, k -monotone and 2-additive, then μ is a belief function (precisely a 2-additive belief function);
2. If μ is monotone and k -monotone then μ satisfies the following property: for all $i, j, k \in N$, i fixed

$$\left. \begin{array}{l} \mu_{ij} = \mu_{i \vee j} \\ \mu_{ik} = \mu_{i \vee k} \\ i \vee j \neq i \vee k \end{array} \right\} \Rightarrow [\mu_l = 0, l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\}].$$

We end the paper by a characterization of ordinal information by belief functions.

4.3 A link between belief functions and the 2-additive Choquet integral

In this section, we give through the MOPI property (see Section 3) a link between belief functions and the 2-additive Choquet integral.

Proposition 3 Let $\{P, I\}$ be an ordinal information on \mathcal{B} .

If there exist $i, j, k \in N$, i fixed such that the property $(\{i, j, k\}, i)$ -MOPI is violated, then there is no belief function Bel which represents $\{P, I\}$.

Corollary 2 Every ordinal information $\{P, I\}$ on \mathcal{B} representable by a belief function $Bel : 2^N \rightarrow [0, 1]$ is representable also by a 2-additive Choquet integral.

Proof Because it is obvious that an ordinal information $\{P, I\}$ for which $(P \cup I \cup M)$ contains a strict cycle is not representable by a belief function, the proof of the Corollary 2 is a consequence of the Theorem 1 and Proposition 3.

The inverse of Corollary 2 is false. If we suppose $P = \{(a_2, a_0)\}$, $I = \{(a_{12}, a_1)\}$ and μ a 2-additive capacity, we will have $\{P, I\}$ representable by a 2-additive Choquet integral and $I_{12} = m_{12} = \mu_{12} - \mu_1 - \mu_2 < 0$. So no belief function can represent $\{P, I\}$ in this case. Then it is interesting to look for the class of 2-additive capacities which are belief functions. In order to characterize them, we introduce a new fundamental property called 2-MOPI property:

Definition 42 An ordinal information $\{P, I\}$ satisfies the 2-MOPI property if

$$\forall i, j \in N, i \neq j, [a_{ij} \sim a_i \Rightarrow \text{not}(a_j \text{ TC}_P a_0)]. \quad (16)$$

The relation between the 2-MOPI property and the MOPI property is given by the following proposition:

Proposition 4 Let $\{P, I\}$ an ordinal information on \mathcal{B} .

$$\begin{array}{c} \{P, I\} \text{ satisfies the 2-MOPI property} \\ \Downarrow \\ \forall i, j, k \in N, \{i, j, k\} \text{ satisfies the MOPI property} \end{array}$$

Now we can state the main result of this section:

Theorem 3

$\{P, I\}$ is representable by a 2-additive belief function if and only if the two following conditions are satisfied:

1. $(P \cup I \cup M)$ contains no strict cycle;
2. $\{P, I\}$ satisfies the 2-MOPI property.

4.4 Interpretation of 2-MOPI and MOPI properties

We try to give an interpretation in terms of decision behavior of the two main conditions introduced in this paper. We assume here for clarity that the reference level **1** is a satisfactory level for the DM, while the reference level **0** is neither bad nor good (neutral).

Facing a situation where for two criteria i and j the DM is indifferent between the two alternatives a_{ij} and a_i , the 2-MOPI property says that alternative a_j is equivalent to alternative a_0 . Hence in such a situation, the DM thinks that the criterion j is not important. This is a strong condition, since it suffices that one such criterion i exists to infer the “nullity” of the criterion j . This condition can be related to the notion of null set in generalized measure theory (see, e.g., Pap [1995]): a set $A \subseteq N$ is said to be *null* for capacity μ if $\mu(B \cup A) = \mu(B)$, $\forall B \subseteq N \setminus A$. Taking $A = \{j\}$ and $B = \{i\}$ gives our condition 2-MOPI. Observe that for the nullity condition, $\{j\}$ would be null if for *all* subsets B not containing j we would have $\mu(B \cup j) = \mu(B)$, but the 2-MOPI condition asks to find only one singleton satisfying this equality.

The MOPI property is a weakening of the above one, and can be interpreted in a similar way. Let us consider now three criteria i, j and k . The MOPI condition can be translated as follows (see Example 3, with $i = 1, j = 2$, and $k = 3$). Suppose that a_{ij} and a_j are indifferent. As above, this would suggest that i is not important for the DM, but this is *relatively to j* , or put differently, i is much less important than j . Suppose in addition that a_{ik} is indifferent to a_i . Again, this suggests that k is much less important than i . Since i is much less important than j , the conclusion is that k is very unimportant, hence a_k is indifferent to a_0 . This explains the first case in the MOPI condition. The second case (indifference between a_{ik} and a_k , and between a_{ij} and a_i) works exactly the same way. The third case says that a_{ij} and a_j are indifferent (i is much less important than j) as well as a_{ik} and a_k (i is much less important than k). Since i is much less important than both j and k , the conclusion is that i is very unimportant, so that a_i is indifferent with a_0 .

5 Proofs

We present here the proofs of our results given in the previous sections. Some notions will be defined here to support our purpose.

Definition 51

- $\text{NBA} = \{x \in \mathcal{B} \mid \exists i, j, k \in N \text{ such that } x \in (\{i, j, k\}, i)\text{-NBA}\}$.
- $\forall x, y \in \mathcal{B}, x \mathcal{Z} y \Leftrightarrow x = a_0 \text{ and } y \in \text{NBA}$.
- $\forall x, y \in \mathcal{B}, x \text{TC} y \Leftrightarrow \text{there exists a path of } (P \cup I \cup M) \text{ from } x \text{ to } y$.
- $\forall x, y \in \mathcal{B}, x \text{TC}' y \Leftrightarrow \text{there exists a path of } (P \cup I \cup M \cup \mathcal{Z}) \text{ from } x \text{ to } y$.
- $\forall x, y \in \mathcal{B} \ x \sim' y \Leftrightarrow (x = y) \text{ or } (x \text{TC}' y \text{ and } y \text{TC}' x)$.
- *The map ϕ will indicate the bijection between \mathcal{B} and $\mathcal{P}^2(N) = \{S \subseteq N : |S| \leq 2\}$ defined by, for all $S \in \mathcal{P}^2(N)$, $\phi((\mathbf{1}_S, \mathbf{0}_{N-S})) := S$.*

The “zero” relation, \mathcal{Z} , has been introduced to indicate that if an ordinal information is representable by a 2-additive Choquet integral C_μ , then any neutral binary action a_m has the same value of μ as a_0 , i.e, $\mu(\phi(a_m)) = \mu(\phi(a_0)) = 0$. The relations TC and TC' are respectively the transitive closure of $(P \cup I \cup M)$ and $(P \cup I \cup M \cup \mathcal{Z})$. We will denote by $\overline{\mathcal{B}}$ the set of all equivalence classes of the equivalence relation \sim' , and by $[x]$ the equivalence class of an element x of \mathcal{B} .

Lemma 2 *Let $\mu : 2^N \rightarrow \mathbb{R}_+$ be a mapping such that*

$$\sum_{j \in A \setminus \{i\}} (\mu_{ij} - \mu_j) \geq (|A| - 2)\mu_i, \quad \forall A \subseteq N, |A| \geq 2, \forall i \in A. \quad (17)$$

If $\exists i, j \in N$ such that $\mu_i > 0$ or $\mu_{ij} > 0$ then we have the following results:

1. $\sum_{\{i,j\} \subseteq N} \mu_{ij} - (n-2) \sum_{i \in N} \mu_i > 0$.
2. *If we introduce*

$$\begin{aligned} \alpha &:= \sum_{\{i,j\} \subseteq N} \mu_{ij} - (n-2) \sum_{i \in N} \mu_i \\ \mu'_0 &:= 0 \\ \mu'_i &:= \frac{\mu_i}{\alpha} \quad \forall i \in N \\ \mu'_{ij} &:= \frac{\mu_{ij}}{\alpha} \quad \forall i, j \in N \\ \mu'(K) &:= \sum_{\{i,j\} \subseteq K} \mu'_{ij} - (|K| - 2) \sum_{i \in K} \mu'_i, \quad \forall K \subseteq N, |K| > 2. \end{aligned}$$

Then μ' is a 2-additive capacity on N .

Proof Assume that equation (17) is satisfied.

1. For $A = N$, we have

$$\sum_{j \in N \setminus \{i\}} (\mu_{ij} - \mu_j) \geq (n-2)\mu_i \quad \forall i \in N.$$

Hence

$$\sum_{i \in N} \sum_{j \in N \setminus \{i\}} (\mu_{ij} - \mu_j) \geq \sum_{i \in N} (n-2)\mu_i,$$

which means that

$$\sum_{i \in N} \sum_{j \in N \setminus \{i\}} \mu_{ij} \geq \sum_{i \in N} (n-2)\mu_i + \sum_{i \in N} \sum_{j \in N \setminus \{i\}} \mu_j.$$

Therefore,

$$2 \sum_{\{i,j\} \subseteq N} \mu_{ij} \geq (n-2) \sum_{i \in N} \mu_i + (n-1) \sum_{i \in N} \mu_i.$$

– If there exists $k \in N$ with $\mu_k > 0$, then

$$\sum_{\{i,j\} \subseteq N} \mu_{ij} \geq \frac{(2n-3)}{2} \sum_{i \in N} \mu_i \geq (n-2) \sum_{i \in N} \mu_i > 0.$$

– If $\forall i \in N, \mu_i = 0$ and there exists $k, l \in N$ with $\mu_{kl} > 0$ then

$$\sum_{\{i,j\} \subseteq N} \mu_{ij} - (n-2) \sum_{i \in N} \mu_i \geq \mu_{kl} > 0.$$

2. Using Lemma 1 we easily prove that μ' is a 2-additive capacity.

5.1 An equivalent property to the MOPI property

Let $K \subseteq N$ and $|K| = k \geq 2$. Let i be a fixed element of K . Let us consider the multiset or bag K^i of \mathcal{B} in which a repetition of the element a_i is allowed.

$$K^i = \underbrace{\{a_i, a_i, \dots, a_i\}}_{(k-2) \text{ times}} \cup \left(\bigcup_{j \in K \setminus \{i\}} \{a_j\} \right).$$

Definition 52 Let $K \subseteq N$ such that $|K| = k \geq 3$. Let i be a fixed element of K . Let us set $K \setminus \{i\} := \{j_1, j_2, \dots, j_{k-1}\}$.

1. We call Monotonicity of Preferential Information-1 in K w.r.t. i the following property (denoted by (K, i) -**MOPI-1**):

$$\left. \begin{array}{l} a_{ij_1} \sim a_{i_1} \\ a_{ij_2} \sim a_{i_2} \\ a_{ij_3} \sim a_{i_3} \\ \vdots \\ a_{ij_{k-1}} \sim a_{i_{k-1}} \\ \{a_{i_1}, a_{i_2}, \dots, a_{i_{k-1}}\} \subseteq K^i \end{array} \right\} \Rightarrow [\text{not}(a_{i_h} \text{TC}_P a_0), \forall a_{i_h} \in K^i \setminus \{a_{i_1}, a_{i_2}, \dots, a_{i_{k-1}}\}]$$

The elements $a_{i_1}, a_{i_2}, \dots, a_{i_{k-1}}$ are distinct in the sense of the definition of the multiset K^i . We say that K satisfies the property of Monotonicity of Preferential Information-1 (**MOPI-1**) if $\forall i \in K, (K, i)$ -**MOPI-1** is satisfied.

2. We call Monotonicity of Preferential Information-2 in K w.r.t. i the following property (denoted by (K, i) -**MOPI-2**):

$$\left. \begin{array}{l} a_{ij_1} \sim a_{i \vee j_1} \\ a_{ij_2} \sim a_{i \vee j_2} \\ a_{ij_3} \sim a_{i \vee j_3} \\ \vdots \\ a_{ij_{k-1}} \sim a_{i \vee j_{k-1}} \\ \{a_{i \vee j_1}, \dots, a_{i \vee j_{k-1}}\} \subseteq K^i \end{array} \right\} \Rightarrow [\text{not}(a_{l_h} TC_P a_0), \forall a_{l_h} \in K^i \setminus \{a_{i \vee j_1}, \dots, a_{i \vee j_{k-1}}\}]$$

The elements $a_{i \vee j_1}, \dots, a_{i \vee j_{k-1}}$ are distinct in the sense of the definition of the multiset K^i . We say that K satisfies the property of Monotonicity of Preferential Information-2 (MOPI-2) if $\forall i \in K$, (K, i) -MOPI-2 is satisfied.

Remark 3 Let $K \subseteq N$ such that $|K| = 3$. Using the definitions of the MOPI and MOPI-2 properties, it is obvious that K satisfies MOPI if and only if K satisfies MOPI-2.

Proposition 5 Let $K \subseteq N$ such that $|K| = k \geq 3$. Let i be a fixed element of K . Let us set $K \setminus \{i\} := \{j_1, j_2, \dots, j_{k-1}\}$. If we have the following system

$$\left\{ \begin{array}{l} a_{ij_1} \sim a_{l_1} \\ a_{ij_2} \sim a_{l_2} \\ a_{ij_3} \sim a_{l_3} \\ \vdots \\ a_{ij_{k-1}} \sim a_{l_{k-1}} \\ \{a_{l_1}, a_{l_2}, \dots, a_{l_{k-1}}\} \subseteq K^i \end{array} \right. \quad (18)$$

then $\forall j \in K \setminus i$, $\begin{cases} a_{ij} \sim a_j & \text{if } j \in \{l_1, l_2, \dots, l_{k-1}\} \\ a_{ij} \sim a_i & \text{otherwise} \end{cases}$

In other terms, the elements $\{a_{l_1}, a_{l_2}, \dots, a_{l_{k-1}}\}$ in the MOPI-1 condition correspond to the elements $\{a_{i \vee j_1}, \dots, a_{i \vee j_{k-1}}\}$ in the MOPI-2 condition using a suitable permutation.

Proof We suppose that the system (18) is satisfied and denote by π a mapping defined by

$$\begin{array}{ccc} \pi : K \setminus \{i\} & \rightarrow & \{l_1, l_2, \dots, l_{k-1}\} \\ j_p & \mapsto & l_p \end{array}$$

For $j \in K \setminus \{i\}$ we write the following sequence of relations:

$$\begin{array}{l} a_{ij} \sim a_{\pi(j)} \\ a_{i\pi(j)} \sim a_{\pi^2(j)} \\ a_{i\pi^2(j)} \sim a_{\pi^3(j)} \\ \vdots \\ a_{i\pi^l(j)} \sim a_{\pi^{l+1}(j)} \\ \vdots \\ 1 \leq l \leq k-2 \end{array}$$

where $\pi^l(j) := \underbrace{(\pi \circ \pi \circ \dots \circ \pi)}_{l \text{ times}}(j)$

1. *The case $j \in \{l_1, l_2, \dots, l_{k-1}\}$:*

Let us vary l from 1 until we encounter either $\pi^l(j) = i$ or $\pi^l(j) = j$. It is always possible to do so by using the equations (18) since the l_h 's are different when they are in $K \setminus i$ and we may have one $l_h = i$. In both cases we will have $\pi(j) \neq \pi^2(j) \neq \dots \neq \pi^l(j)$.

- If we encounter $\pi^l(j) = i$, then $a_{ij} \sim a_i$ and the element $\pi^{l+1}(j)$ is not defined. Since $j \in \{l_1, l_2, \dots, l_{k-1}\}$, there exists $k \in \{l_1, l_2, \dots, l_{k-1}\}$ such that $a_{ik} \sim a_j$. Hence we have $a_j \lesssim a_{ij} \sim a_i \lesssim a_{ik} \sim a_j$ i.e. $a_{ij} \sim a_j$, where $\lesssim := M^{-1}$, where $M^{-1} = \{(x, y) \in \mathcal{B} : (y, x) \in M\}$.
- If we encounter $\pi^l(j) = j$, then we have $a_{ij} \sim a_j$.

2. *The case $j \notin \{l_1, l_2, \dots, l_{k-1}\}$:*

Let us vary l from 1 until we encounter $\pi^l(j) = i$. It is always possible because $j \notin \{l_1, l_2, \dots, l_{k-1}\} \Rightarrow i \in \{l_1, l_2, \dots, l_{k-1}\}$ (by using the definition of K^i) and the equation (18) is satisfied. In this case we will have $\pi(j) \neq \pi^2(j) \neq \dots \neq \pi^l(j)$. Hence we have $a_{ij} \sim a_i$ and the element $\pi^{l+1}(j)$ is not defined.

Corollary 3 *Let $K \subseteq N$ such that $|K| = k \geq 3$. Let i be a fixed element of K .*

(K, i) -MOPI-2 is satisfied if and only if (K, i) -MOPI-1 is satisfied.

Proof It is a consequence of Proposition 5.

Proposition 6 *Every subset $L \subseteq N$ such that $|L| = 3$ satisfies the MOPI property if and only if every subset K of N such that $|K| = k \geq 3$ satisfies the MOPI-1 property.*

Proof

- (\Leftarrow) This statement is true by using Corollary 3 and Remark 3.
- (\Rightarrow) Let us suppose that every subset $L \subseteq N$ such that $|L| = 3$ satisfies the MOPI property. Let $K \subseteq N$ such that $|K| = k \geq 3$. Let i be a fixed element of K . Let us set $K \setminus \{i\} := \{j_1, j_2, \dots, j_{k-1}\}$. Let us suppose

$$\left\{ \begin{array}{l} a_{ij_1} \sim a_{l_1} \\ a_{ij_2} \sim a_{l_2} \\ a_{ij_3} \sim a_{l_3} \\ \vdots \\ a_{ij_{k-1}} \sim a_{l_{k-1}} \\ \{a_{l_1}, a_{l_2}, \dots, a_{l_{k-1}}\} \subseteq K^i \end{array} \right.$$

and let us show that $[not(a_{l_h} TCP a_0), \forall a_{l_h} \in K^i \setminus \{a_{l_1}, a_{l_2}, \dots, a_{l_{k-1}}\}]$.

Let $a_{l_h} \in K^i \setminus \{a_{l_1}, a_{l_2}, \dots, a_{l_{k-1}}\}$.

1. *The case $l_h \neq i$:* Since $l_h \in K \setminus i$ and $l_h \notin \{l_1, l_2, \dots, l_{k-1}\}$, we have using Proposition 5

$$a_{il_h} \sim a_i. \quad (19)$$

Furthermore, we have at most $k - 2$ elements a_i contained in the set $\{a_{l_1}, a_{l_2}, \dots, a_{l_{k-1}}\}$. So there exists $j_p \in K \setminus i$, $j_p \neq l_h$ such that by Proposition 5

$$a_{ij_p} \sim a_{j_p}. \quad (20)$$

Considering the set of three elements $L = \{i, l_h, j_p\}$ where i is fixed and the system formed by the two equations (19) et (20), we have $\text{not}(a_{l_h} TC_P a_0)$ because $L = \{i, l_h, j_p\}$ satisfies the MOPI property.

2. *The case $l_h = i$:* Because this i is not in the set $\{l_1, l_2, \dots, l_{k-1}\}$, we have by definition of K^i at most $k - 3$ elements a_i among the elements of $\{a_{l_1}, a_{l_2}, \dots, a_{l_{k-1}}\}$. So we can find two elements a_{j_p} and a_{j_q} of $\{a_{l_1}, a_{l_2}, \dots, a_{l_{k-1}}\}$ such that $j_p, j_q \in K \setminus i$. Using Proposition 5 we get the following system:

$$\begin{cases} a_{ij_p} \sim a_{j_p} \\ a_{ij_q} \sim a_{j_q} \end{cases} \quad (21)$$

Hence we have $\text{not}(a_i TC_P a_0)$ because $L = \{i, j_p, j_q\}$ satisfies the MOPI property.

5.2 Topological sorting

In this section we suppose the following two conditions:

1. $(P \cup I \cup M)$ contains no strict cycle.
2. Any subset K of N such that $|K| = k \geq 3$ is MOPI-1.

Lemma 3

1. $\forall x \in \mathcal{B} \setminus \{a_o\}, x TC' a_o$.
2. $\forall x, y \in \mathcal{B}, x TC' y \Leftrightarrow [x TC y]$ or $[\exists a_m \in \text{NBA such that } a_m TC y]$.
3. $\forall x \in \mathcal{B}, x \notin [a_0] \Rightarrow (\forall z, t \in [x], z \sim' t \Leftrightarrow z \sim t)$.

Proof

1. The first point of the lemma is true because $\forall x \in \mathcal{B} \setminus \{a_o\}$, we have $x (P \cup I \cup M) a_o$ by the definition of $(P \cup I \cup M)$.
2. If $x TC' y$ and $\text{not}(x TC y)$ then by the definition of the relations TC , \mathcal{Z} and TC' (see Definition 51), there exists $a_m \in \text{NBA}$ such that $x TC a_0 \mathcal{Z} a_m TC y$ with $a_m \in \text{NBA}$. Hence we easily prove the second point of the lemma.
3. Let $x \in \mathcal{B}$ such that $x \notin [a_0]$ and $z, t \in [x]$. If $z TC t$ then $z TC' t$. If $z TC' t$ and $\text{not}(z TC t)$ then by using the second point of the lemma, there exists $a_m \in \text{NBA}$ such that $a_m TC t$. So, we have $z TC' a_0 \mathcal{Z} a_m TC' t TC' a_0$ i.e $t \in [a_0]$. This leads to a contradiction with $t \notin [a_0]$. Hence $z TC' t \Leftrightarrow z TC t$, and $z \sim' t \Leftrightarrow z \sim t$ because $(P \cup I \cup M)$ contains no strict cycle.

Lemma 4

- i) $\forall x, y \in \mathcal{B}$, if $x, y \in [a_0]$ then $\text{not}(x P y)$.
- ii) $\forall x \in [a_0], \forall y \notin [a_0], x TC' y$ is false.

Proof

- i) Let $x, y \in [a_0]$ such that $x P y$. We have $a_0 TC' x$. Using the second point of Lemma 3, we have $[a_0 TC x]$ or $[\exists a_m \in \text{NBA such that } a_m TC x]$.
 If $a_0 TC x$, then $a_0 TC x P y M a_0$. So, (a_0, x, y) forms a strict cycle of $(P \cup I \cup M)$. This is impossible because $(P \cup I \cup M)$ contains no strict cycle.
 If $\exists a_m \in \text{NBA such that } a_m TC x$, then $a_m TC x P y M a_0$. Hence, $a_m TC_P a_0$. This leads to a contradiction with $\text{not}(a_m TC_P a_0)$ because $a_m \in \text{NBA}$.
- ii) Let $x \in [a_0]$ and $y \notin [a_0]$ such that $x TC' y$. Using the first point of the Lemma 3 and the definition of $[a_0]$, we have $a_0 TC' x TC' y TC' a_0$. Hence $y \in [a_0]$, a contradiction.

Let us define the following binary relation P_{\sim} by: $\forall [x], [y] \in \overline{\mathcal{B}}, [x] P_{\sim} [y] \Leftrightarrow \exists z \in [x], \exists t \in [y]$ such that $z (P \cup M) t$. The relation P_{\sim} is an asymmetric and acyclic relation because P and M are asymmetric and $(P \cup I \cup M)$ contains no strict cycle. Let us consider the following sets:

$$\begin{aligned} \overline{\mathcal{B}}_0 &= \{a \in \overline{\mathcal{B}} : \forall b \in \overline{\mathcal{B}}, \text{not}(a P_{\sim} b)\} \\ \overline{\mathcal{B}}_1 &= \{a \in \overline{\mathcal{B}} \setminus \overline{\mathcal{B}}_0 : \forall b \in \overline{\mathcal{B}} \setminus \overline{\mathcal{B}}_0, \text{not}(a P_{\sim} b)\} \\ &\vdots \\ \overline{\mathcal{B}}_i &= \{a \in \overline{\mathcal{B}} \setminus (\overline{\mathcal{B}}_0 \cup \dots \cup \overline{\mathcal{B}}_{i-1}) : \forall b \in \overline{\mathcal{B}} \setminus (\overline{\mathcal{B}}_0 \cup \dots \cup \overline{\mathcal{B}}_{i-1}), \text{not}(a P_{\sim} b)\} \end{aligned}$$

Each set $\overline{\mathcal{B}}_k$ exists since P_{\sim} is asymmetric and acyclic. Let $m+1$ be the number of sets $\overline{\mathcal{B}}_i$ which are built, $m+1$ is finite because the cardinality of $\overline{\mathcal{B}}$ is finite. The set $\{\overline{\mathcal{B}}_0, \overline{\mathcal{B}}_1, \dots, \overline{\mathcal{B}}_m\}$ is called a *topological sorting* of $(P \cup I \cup M \cup \mathcal{Z})$, which is a classical notion often used in graph theory.

Lemma 5 *The sets $\overline{\mathcal{B}}_0, \overline{\mathcal{B}}_1, \dots, \overline{\mathcal{B}}_m$ form a partition of $\overline{\mathcal{B}}$.*

Proof The proof comes from the construction of the sets $\overline{\mathcal{B}}_0, \overline{\mathcal{B}}_1, \dots, \overline{\mathcal{B}}_m$.

Let us define for each set $\overline{\mathcal{B}}_i, i \in \{0, \dots, m\}$, the set \mathcal{B}_i by

$$\mathcal{B}_i = \{x \in a : a \in \overline{\mathcal{B}}_i\}$$

One may have incomparable elements in a some class \mathcal{B}_i . The sets $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m$ form a partition of \mathcal{B} .

Lemma 6 *Let $x, y \in \mathcal{B}$ and $i, j \in N$*

1. $\mathcal{B}_0 = [a_0]$.
2. *If $x \sim y$ then $\exists i \in \{0, \dots, m\}$ such that $x, y \in \mathcal{B}_i$.*
3. $\{a_{ij}, a_i\} \subseteq \mathcal{B}_s, s \in \{1, \dots, m\} \Rightarrow a_{ij} \sim a_i$.
4. $a_{ij} \in \mathcal{B}_0 \Rightarrow \{a_i, a_j\} \subseteq \mathcal{B}_0$.
5. $[a_{ij} \in \mathcal{B}_s, s \in \{1, \dots, m\}] \Rightarrow [a_i \in \mathcal{B}_r, r \leq s \text{ and } a_j \in \mathcal{B}_q, q \leq s]$.
6. $[a_i \in \mathcal{B}_r, r \in \{0, \dots, m\}] \Rightarrow [\forall k \neq i \in N, a_{ik} \in \mathcal{B}_s, r \leq s]$.

Proof

1. Let us prove that $\forall x \notin [a_0], [x] P_{\sim} [a_0]$: if $x \notin [a_0]$, then $\forall y \in [a_0], x TC' y$ and $\text{not}(y TC' x)$ by using the first point of Lemma 3 and the second point of Lemma 4. Hence $[x] P_{\sim} [a_0]$.
Then $[a_0] \in \overline{\mathcal{B}}_0$ and no other class $[x] \in \overline{\mathcal{B}}_0$ because $[x] P_{\sim} [a_0]$. Therefore $\mathcal{B}_0 = [a_0]$.
2. If $x \sim y$ then $x \sim' y$. Hence $[x] = [y]$ and then $\exists i \in \{0, \dots, m\}$ such that $x, y \in \mathcal{B}_i$ by the definitions of the relation \sim, \sim' and the sets $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m$.
3. If $\text{not}(a_{ij} \sim a_i)$ then $a_{ij} (P \cup M) a_i$ and $\text{not}(a_i TC a_{ij})$. Using the second point of Lemma 3, we have $\text{not}(a_i TC' a_{ij})$ because $s \in \{1, \dots, m\}$. Hence $[a_{ij}] P_{\sim} [a_i]$ and $\{a_{ij}, a_i\} \not\subseteq \mathcal{B}_s$, a contradiction.
Let us remark that, if $s = 0$, we can have $\text{not}(a_{ij} \sim a_i)$ and $a_i TC a_0$ $Z a_m TC a_{ij}$ with $a_m \in \text{NBA}$, i.e. $\{a_{ij}, a_i\} \subseteq \mathcal{B}_0$.
4. Property 4 is true because if $a_{ij} \in \mathcal{B}_0$ then $a_0 TC' a_{ij} TC' a_i TC' a_0$ and $a_0 TC' a_{ij} TC' a_j TC' a_0$. Hence $\{a_i, a_j\} \subseteq [a_0] = \mathcal{B}_0$.
5. If $a_{ij} \in \mathcal{B}_s, s \in \{1, \dots, m\}$ then we have $[a_{ij}] P_{\sim} [a_i]$ or $[a_{ij}] = [a_i]$ because $(P \cup I \cup M)$ contains no strict cycle and by the definitions of P_{\sim} and \sim . Hence, $a_i \in \mathcal{B}_r, r \leq s$. The same argument is applied to a_j .
6. This property is the consequence of the previous properties 4 and 5.

Lemma 7 *Let $K \subseteq N$ such that $|K| = k \geq 3$ and i a fixed element of K . Let us denote $K \setminus \{i\} = \{j_1, j_2, \dots, j_{k-1}\}$ and $H(i) = \{a_{ij_1}, a_{ij_2}, \dots, a_{ij_{k-1}}\}$.*

If there exists $B \subseteq H(i)$ and $C \subseteq K^i$ such that the following two conditions are satisfied:

1.
$$\left\{ \begin{array}{l} a_{ij_1} \sim a_{l_1} \\ a_{ij_2} \sim a_{l_2} \\ a_{ij_3} \sim a_{l_3} \\ \vdots \\ a_{ij_{p'}} \sim a_{l_{p'}} \\ B = \{a_{ij_1}, \dots, a_{ij_{p'}}\} \subseteq H(i) \\ C = \{a_{l_1}, \dots, a_{l_{p'}}\} \subseteq K^i \\ 1 \leq p' < k-1 \end{array} \right.$$
2. $\forall a_{ij} \in H(i) \setminus B, \forall a_l \in K^i \setminus C, \text{not}(a_{ij} \sim a_l)$

then

$$\begin{cases} l_0 < k_0 & \text{if } k_0 \neq 0 \\ l_0 = k_0 & \text{if } k_0 = 0 \end{cases}$$

where

$$\begin{cases} k_0 = \max\{s \in \{0, 1, \dots, m\} : \mathcal{B}_s \cap (H(i) \setminus B) \neq \emptyset\} \\ l_0 = \max\{s \in \{0, 1, \dots, m\} : \mathcal{B}_s \cap (K^i \setminus C) \neq \emptyset\} \end{cases}$$

with $m+1$ the number of sets $\overline{\mathcal{B}}_i$ which form the topological sorting of $(P \cup I \cup M \cup Z)$.

Proof $H(i) \setminus B \neq \emptyset$ since $p' < k-1$. Hence k_0 and l_0 are well-defined. Let $a_{ij_0} \in \mathcal{B}_{k_0}$ and $a_{h_0} \in \mathcal{B}_{l_0}$. We have $a_{ij_0} \in H(i) \setminus B$ and $a_{h_0} \in K^i \setminus C$.

1. Let us suppose that $l_0 > k_0$.

$a_{ij_0} \in \mathcal{B}_{k_0} \Rightarrow a_i \in \mathcal{B}_t, t \leq k_0$ by Lemma 6, points 4 and 5. Hence $a_{h_0} \neq a_i$. Using Lemma 6 point 6, $a_{h_0} \in \mathcal{B}_{l_0} \Rightarrow a_{ih_0} \in \mathcal{B}_r, r \geq l_0$. Since $r \geq l_0 > k_0$, by definition of k_0 , $a_{ih_0} \in B$. Hence there exists $a_{h_1} \in C$ such that $a_{ih_0} \sim a_{h_1}$, and $a_{h_1} \neq a_i$ because $r > t$. Also, $a_{h_1} \neq a_{h_0}$ because $a_{h_0} \notin C$ and $a_{h_1} \in C$. Then $a_{ih_1} \in \mathcal{B}_{r_1}, r_1 \geq r$, moreover $a_{ih_1} \in B$ since $r_1 > k_0$. Since $h_1 \neq h_0$, we have $a_{ih_1} \in \mathcal{B}_{r_1} \cap (B \setminus \{a_{ih_0}\})$.

$a_{ih_1} \in B \Rightarrow \exists a_{h_2} \in C \setminus \{a_i, a_{h_1}\}$ such that $a_{ih_1} \sim a_{h_2}$ and $a_{ih_2} \in \mathcal{B}_{r_2} \cap (B \setminus \{a_{ih_0}, a_{ih_1}\})$ with $r_2 \geq r_1 \geq r \geq l_0 > k_0$.

$a_{ih_2} \in B \Rightarrow \exists a_{h_3} \in C \setminus \{a_i, a_{h_1}, a_{h_2}\}$ such that $a_{ih_2} \sim a_{h_3}$ and $a_{ih_3} \in \mathcal{B}_{r_3} \cap (B \setminus \{a_{ih_0}, a_{ih_1}, a_{ih_2}\})$ with $r_3 \geq r_2 \geq r_1 \geq r \geq l_0 > k_0$.

We repeat the same process until exhaustion of all the elements of C . In the last step we will have $a_{ih_{p'-1}} \sim a_{h_{p'}}, a_{h_{p'}} \in C \setminus \{a_i, a_{h_1}, a_{h_2}, \dots, a_{h_{p'-1}}\}$ and $a_{ih_{p'}} \in \mathcal{B}_{r_{p'}} \cap (B \setminus \{a_{ih_0}, a_{ih_1}, a_{ih_2}, \dots, a_{ih_{p'-1}}\})$ with $r_{p'} \geq \dots \geq r_2 \geq r_1 \geq r \geq l_0 > k_0$. Hence we obtain $p' + 1$ elements of B , $a_{ih_0}, a_{ih_1}, \dots, a_{ih_{p'}}$ such that $a_{ih_0} \sim a_{h_1}, a_{ih_1} \sim a_{h_2}, \dots, a_{ih_{p'-1}} \sim a_{h_{p'}}$ and $a_{ih_{p'}} \in B$. This is a contradiction with the hypothesis $|B| = p'$. Hence $0 \leq l_0 \leq k_0$ and $l_0 = 0$ if $k_0 = 0$.

2. We suppose that $k_0 \neq 0$ and $l_0 = k_0$.

If $a_{h_0} = a_i$ then $a_{ij_0} \sim a_i$ by using Lemma 6 point 3. It is a contradiction with the hypothesis $\forall a_{ij} \in H(i) \setminus B, \forall a_l \in K^i \setminus C, \text{not}(a_{ij} \sim a_l)$. Hence $a_{h_0} \neq a_i$ and $a_i \in \mathcal{B}_t$ with $t < k_0$ because $a_{ij_0} \in H(i) \setminus B$ and $a_{h_0} \in K^i \setminus C$ by the definition of k_0 and l_0 .

$a_{h_0} \in \mathcal{B}_{l_0} \Rightarrow a_{ih_0} \in \mathcal{B}_r, r \geq l_0 = k_0$.

If $r > l_0 = k_0$ then $a_{ih_0} \in B$ by the definition of k_0 . Using the reasoning of the previous point (1), we find $p' + 1$ elements of B , $a_{ih_0}, a_{ih_1}, \dots, a_{ih_{p'}}$ such that $a_{ih_0} \sim a_{h_1}, a_{ih_1} \sim a_{h_2}, \dots, a_{ih_{p'-1}} \sim a_{h_{p'}}$ and $a_{ih_{p'}} \in B$. It is impossible because $|B| = p'$.

If $r = l_0 = k_0$ then by using the point 3 of the Lemma 6, we have $a_{ih_0} \sim a_{h_0}$ and $a_{ih_0} \in B$ because $a_{h_0} \notin C$. $a_{ih_0} \in B \Rightarrow \exists a_{h_1} \in C$ such that $a_{ih_0} \sim a_{h_1}$ and $a_{ih_1} \in \mathcal{B}_{r_1}$ with $r_1 \geq r = l_0 = k_0$.

If $r_1 > r = l_0 = k_0$, we apply the reasoning used in step 1 to obtain a contradiction with $|B| = p'$.

If $r_1 = r = l_0 = k_0$ then $a_{h_0} \sim a_{ih_0} \sim a_{h_1} \sim a_{ih_1}$. Because $a_{h_0} \notin C$, we have necessarily $a_{ih_1} \in B \setminus \{a_{ih_0}\}$.

We repeat this process with a_{ih_1} until the construction of $p' + 1$ elements of B , $a_{ih_0}, a_{ih_1}, \dots, a_{ih_{p'}}$, such that $a_{h_0} \sim a_{ih_0} \sim a_{h_1} \sim a_{ih_1} \sim a_{h_2} \sim a_{ih_2} \sim a_{h_3} \sim a_{ih_3} \sim \dots \sim a_{h_{p'-1}} \sim a_{ih_{p'-1}} \sim a_{h_{p'}}$. That leads to the contradiction with $|B| = p'$. Hence if $k_0 \neq 0$ then $l_0 < k_0$.

5.3 Proof of Theorem 1

Proposition 7 *Let $\{P, I\}$ be an ordinal information. If the two following conditions are satisfied*

1. $(P \cup I \cup M)$ contains no strict cycle

2. Any subset K of N such that $|K| = k \geq 3$ is MOPI-1, then $\{P, I\}$ is representable by a 2-additive Choquet integral.

Proof

1. Let $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$ be the partition of \mathcal{B} built in Section 5.2. Let us define the mapping $\mu : \mathcal{B} \rightarrow \mathbb{R}$ as follows: For $i \in \{0, \dots, m\}$,

$$\forall x \in \mathcal{B}_i, \mu(\phi(x)) = \begin{cases} 0 & \text{if } i = 0 \\ (2n)^i & \text{otherwise.} \end{cases}$$

Let $K \subseteq N$ such that $|K| = k \geq 2$, $i \in K$ being fixed, $K \setminus \{i\} = \{j_1, j_2, \dots, j_{k-1}\}$ and $H(i) = \{a_{ij_1}, a_{ij_2}, \dots, a_{ij_{k-1}}\}$. We will prove that

$$\sum_{j \in K \setminus \{i\}} \mu_{ij} \geq (k-2)\mu_i + \sum_{j \in K \setminus \{i\}} \mu_j. \quad (22)$$

Using the points 4 and 5 of the Lemma 6, the case $k = 2$ is obviously satisfied. Let us suppose that $k \geq 3$.

- (a) If there exists $a_{ij} \in H(i)$ and $a_l \in K^i$ such that $a_{ij} \sim a_l$ then there exists p' elements of $H(i)$ (without loss of the generality we are going to take the first p' elements), $B \subseteq H(i)$ and $C \subseteq K^i$ such that:

$$(S) \left\{ \begin{array}{l} a_{ij_1} \sim a_{l_1} \\ a_{ij_2} \sim a_{l_2} \\ a_{ij_3} \sim a_{l_3} \\ \vdots \\ a_{ij_{p'}} \sim a_{l_{p'}} \\ B = \{a_{ij_1}, \dots, a_{ij_{p'}}\} \subseteq H(i) \\ C = \{a_{l_1}, \dots, a_{l_{p'}}\} \subseteq K^i \\ 1 \leq p' \leq k-1 \end{array} \right.$$

and $\forall a_{ij} \in H(i) \setminus B, \forall a_l \in K^i \setminus C, \text{not}(a_{ij} \sim a_l)$.

By application of the point 2 of Lemma 6 and the definition of μ , the equation (22) becomes

$$\sum_{a_{ij} \in H(i) \setminus B} \mu_{ij} \geq \sum_{a_l \in K^i \setminus C} \mu_l \quad (23)$$

- (i) If $p' = k-1$ then we have $\forall a_{l_k} \in K^i \setminus C, a_{l_k} \in \text{NBA}$ because K is MOPI-1 and $a_{l_k} \in [a_0] = \mathcal{B}_0$ and $\mu_{l_k} = 0$ by the definitions of $\mu, [a_0]$ and Lemma 6. Hence equation (22) is satisfied.
(ii) Let us assume that $p' < k-1$ and let

$$\begin{aligned} k_0 &= \max\{s \in \{0, 1, \dots, m\} : \mathcal{B}_s \cap (H(i) \setminus B) \neq \emptyset\} \\ l_0 &= \max\{s \in \{0, 1, \dots, m\} : \mathcal{B}_s \cap (K^i \setminus C) \neq \emptyset\}. \end{aligned}$$

Let $a_{ij_0} \in \mathcal{B}_{k_0}$ and $a_{l_0} \in \mathcal{B}_{l_0}$.

- If $k_0 \neq 0$ then $l_0 < k_0$ (see Lemma 7), $\mu(\phi(a_{ij_0})) = \mu_{ij_0} = (2n)^{k_0}$ and $\forall a_l \in K^i \setminus C$, $\mu(\phi(a_l)) = \mu_l \leq (2n)^{l_0}$. Because

$$l_0 < k_0 \Rightarrow (2n)^{k_0} \geq 2n(2n)^{l_0} = (n+n)(2n)^{l_0},$$

we will have

$$\sum_{a_{ij} \in H(i) \setminus B} \mu_{ij} \geq \mu_{ij_0} = (2n)^{k_0} \geq (n+n)(2n)^{l_0} \geq \sum_{a_l \in K^i \setminus C} \mu_l.$$

Hence (23) is satisfied.

- If $k_0 = 0$ then $l_0 = 0$ (see Lemma 7) and (23) is obviously satisfied by application of Lemma 6.

- (b) Let us suppose that $\forall a_{ij} \in H(i)$, $\forall a_l \in K^i$, $\text{not}(a_{ij} \sim a_l)$. Let

$$k_1 = \min\{s \in \{0, 1, \dots, m\} : a_{ij} \in \mathcal{B}_s \cap H(i)\},$$

$a_{ij_0} \in \mathcal{B}_{k_1}$, $a_i \in \mathcal{B}_r$ and $a_{j_0} \in \mathcal{B}_t$. Using Lemma 6, we have $r = t = 0$ if $k_1 = 0$ and $r, t < k_1$ if $k_1 \neq 0$ because we have supposed that $\text{not}(a_{ij_0} \sim a_i)$ and $\text{not}(a_{ij_0} \sim a_{j_0})$. According to the definition of μ , we will have $\mu(\phi(a_{ij_0})) = \mu_{ij_0} \geq (k-2)\mu_i + \mu_{j_0}$. Furthermore $\forall j \in K \setminus \{i, j_0\}$, $\mu_{ij} \geq \mu_j$ (see the case $k = 2$). Hence

$$\sum_{j \in K \setminus \{i\}} \mu_{ij} = \sum_{j \in K \setminus \{i, j_0\}} \mu_{ij} + \mu_{ij_0} \geq \sum_{j \in K \setminus \{i, j_0\}} \mu_j + (k-2)\mu_i + \mu_{j_0},$$

and

$$\sum_{j \in K \setminus \{i, j_0\}} \mu_j + (k-2)\mu_i + \mu_{j_0} = (k-2)\mu_i + \sum_{j \in K \setminus \{i\}} \mu_j.$$

That means that equation (22) is satisfied.

2. Let us build a 2-additive capacity on 2^N

Let $\alpha = \sum_{\{i,j\} \subseteq N} \mu_{ij} - (n-2) \sum_{i \in N} \mu_i$ and the mapping $\nu : 2^N \rightarrow [0, 1]$ defined

by :

$$\begin{cases} \nu_\emptyset = 0 \\ \nu_i = \frac{\mu_i}{\alpha}, \forall i \in N \\ \nu_{ij} = \frac{\mu_{ij}}{\alpha}, \forall i, j \in N \\ \nu(K) = \sum_{\{i,j\} \subseteq K} \nu_{ij} - (|K| - 2) \sum_{i \in K} \nu_i, \forall K \subseteq N, |K| > 2. \end{cases}$$

Because P is not empty (“nontriviality axiom”) and μ satisfies the equation (22) in the previous step, we have by applying Lemma 2, $\alpha > 0$ and ν is a 2-additive capacity.

3. Let $x, y \in \mathcal{B}$.

If $x P y$ then by the definition of the sets $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_m$, we have $x \in \mathcal{B}_i$ and $y \in \mathcal{B}_j$ with $0 \leq j < i$. Thus $C_\nu(U(x)) = \frac{(2n)^i}{\alpha} > \frac{(2n)^j}{\alpha} = C_\nu(U(y))$ if $j > 0$ or $C_\nu(U(x)) > 0 = C_\nu(U(y))$ if $j = 0$.

If $x I y$ then $x \sim y$, i.e., $x, y \in \mathcal{B}_i$, $i \in \{0, \dots, m\}$. Thus $C_\nu(U(x)) = C_\nu(U(y))$.

Conclusion: ν is a 2-additive capacity such that $\{P, I\}$ is representable by a Choquet integral w.r.t. ν .

Corollary 4 *Let $\{P, I\}$ be an ordinal information. If the two following conditions are satisfied*

1. $(P \cup I \cup M)$ contains no strict cycle
2. Any subset K of N such that $|K| = 3$ satisfies MOPI,

then $\{P, I\}$ is representable by a 2-additive Choquet integral.

Proof The proof is a consequence of Propositions 6 and 7.

Proposition 8 *If an ordinal information $\{P, I\}$ is representable by a 2-additive Choquet integral then the two following conditions are satisfied:*

1. $(P \cup I \cup M)$ contains no strict cycle
2. Any subset K of N such that $|K| = 3$ satisfies MOPI.

Proof Let us suppose that μ is a 2-additive capacity such that $\{P, I\}$ is representable by a Choquet integral w.r.t. μ .

1. If $x_1, x_2, \dots, x_p \in \mathcal{B}$ such that (x_1, x_2, \dots, x_p) is a strict cycle of $(P \cup I \cup M)$ then $\exists i \in \{1, \dots, p\} : x_i P x_{i+1}$ (with $x_{p+1} = x_1$). This implies $\mu(\phi(x_i)) > \mu(\phi(x_{i+1}))$. By applying Proposition 1, we have $\mu(\phi(x_1)) = \mu(\phi(x_2)) = \dots = \mu(\phi(x_p))$, a contradiction.
2. Let $i, j, k \in N$, i fixed such that the following system is satisfied:

$$\begin{cases} a_{ij} \sim a_{i \vee j} \\ a_{ik} \sim a_{i \vee k} \\ i \vee j \neq i \vee k \end{cases}$$

If there exists $l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\}$ such that $a_l TC_P a_0$ then we have $\mu(\phi(a_l)) > 0$, $\mu(\phi(a_{ij})) = \mu(\phi(a_{i \vee j}))$ and $\mu(\phi(a_{ik})) = \mu(\phi(a_{i \vee k}))$, by using Proposition 1. Hence the monotonicity constraint of a 2-additive capacity for $\{i, j, k\}$, with i fixed, $\mu_{ij} + \mu_{ik} \geq \mu_{i \vee j} + \mu_{i \vee k} + \mu_l$ becomes $0 \geq \mu_l > 0$, a contradiction.

The proof of Theorem 1 is a consequence of Corollary 4 and Proposition 8.

5.4 Proof of Corollary 1

Remark 4 *Let us suppose $I = \emptyset$. Let $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$ be the partition of \mathcal{B} built in Section 5.2.*

1. $I = \emptyset \Rightarrow \forall x, y \in \mathcal{B}$, $\text{not}(x \sim y)$ by the definition of \sim . Thus any subset K of N such that $|K| = 3$ is obviously MOPI. Hence if $(P \cup I \cup M)$ contains no strict cycle then the two conditions of Theorem 1 are satisfied.
2. $[a_0] = \mathcal{B}_0 = \{a_0\}$ because NBA is empty.
3. $\forall i, j \in N$, $a_{ij} \in \mathcal{B}_s$, $a_i \in \mathcal{B}_r$, $a_j \in \mathcal{B}_t \Rightarrow r, t < s$ with $s, r, t \in \{1, \dots, m\}$ by applying Lemma 6 and by the definition of the sets $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$.

The first part of Corollary 1 is a consequence of Theorem 1 and Remark 4. For the second part of this corollary, by using the definition of the 2-additive capacity ν built in the proof of Theorem 1 (precisely in the proof of Proposition 7), we have $\forall i, j \in N$ such that $a_{ij} \in \mathcal{B}_s$, $a_i \in \mathcal{B}_r$ and $a_j \in \mathcal{B}_t$, $\mu_{ij} = (2n)^s \geq (n+n)(2n)^{\max(r,t)} \geq \mu_i + \mu_j$ because $r, t < s$ (see Remark 4). Hence $I_{ij} = \mu_{ij} - \mu_i - \mu_j \geq 0$.

5.5 Proof of Theorem 2

Proof Let m be the Möbius transform of $\mu : 2^N \rightarrow [0, 1]$ and k be an integer such that $k \geq 2$.

1. $-\mu$ k -monotone $\Rightarrow \forall i, j \in N, i \neq j, m_{ij} \geq 0$ (for $A = B = L$ and $|A| = 2$ in the equation (15));
 - $-\mu$ is monotone then $\mu_i = m_i \geq 0$;
 - furthermore, using the definition of a 2-additive belief function, we have: μ is 2-additive $\Rightarrow \mu$ is a 2-additive belief function.
2. Let us suppose μ monotone et k -monotone and the existence of $i, j, k \in N$, i fixed such that :

$$\begin{cases} \mu_{ij} = \mu_{i \vee j} \\ \mu_{ik} = \mu_{i \vee k} \\ i \vee j \neq i \vee k \end{cases}$$
 and there exists $l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\} \mid \mu_l > 0$.

Then we have

$$\begin{cases} m_{ij} + m_i + m_j = m_{i \vee j} \\ m_{ik} + m_i + m_k = m_{i \vee k} \\ i \vee j \neq i \vee k \end{cases}$$

and there exists $l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\} \mid m_l > 0$.

In this case we have this equality:

$$m_{ij} + m_{ik} + 2m_i + m_j + m_k = m_{i \vee j} + m_{i \vee k} \quad (24)$$

Because $i \vee j \in \{i, j\}$, $i \vee k \in \{i, k\}$, $i \vee j \neq i \vee k$ and $l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\}$, we have

$$m_{i \vee j} + m_{i \vee k} + m_l = m_i + m_j + m_k \quad (25)$$

Therefore equation (24) becomes

$$m_{ij} + m_{ik} + m_i + m_l = 0 \quad (26)$$

which leads to a contradiction with $m_l > 0$.

5.6 Proof of Proposition 3

Proof Let $\{P, I\}$ be an ordinal information on \mathcal{B} . Let us suppose that there exists $i, j, k \in N$, i fixed such that the property $(\{i, j, k\}, i)$ -MOPI is violated i.e.

$$\begin{cases} a_{ij} \sim a_{i \vee j} \\ a_{ik} \sim a_{i \vee k} \\ i \vee j \neq i \vee k \end{cases}$$

and there exists $l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\} \mid a_l \text{ } TC_P \text{ } a_0$. If there exists a belief function Bel which represents $\{P, I\}$, we will have:

$$\begin{cases} Bel_{ij} = Bel_{i \vee j} \\ Bel_{ik} = Bel_{i \vee k} \\ i \vee j \neq i \vee k \end{cases}$$

and there exists $l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\} \mid Bel_l > 0$.

By replacing μ by Bel in the second part of the proof of Theorem 2 (see Section 5.5) and using the same reasoning, we obtain a contradiction.

5.7 Proof of Proposition 4

Proof Let $i, j, k \in N$, i fixed such that

$$\begin{cases} a_{ij} \sim a_{i \vee j} \\ a_{ik} \sim a_{i \vee k} \\ i \vee j \neq i \vee k \end{cases}$$

Let be l an element of $\{i, j, k\} \setminus \{i \vee k, i \vee j\}$.

1. If $l \in \{j, k\}$ then $[i \vee j = i \text{ if } l = j]$ or $[i \vee k = i \text{ if } l = k]$. So we have $a_{il} \sim a_i$. Hence $not(a_l \text{ } TC_P \text{ } a_0)$ because $\{P, I\}$ satisfies 2-MOPI.
2. If $l = i$ then $i \vee j = j$ and $i \vee k = k$. So $a_{ij} \sim a_j$ and $a_{ik} \sim a_k$. Therefore we have $not(a_l \text{ } TC_P \text{ } a_0)$ because $\{P, I\}$ satisfies 2-MOPI.

5.8 Proof of Theorem 3

Proof Let $\{P, I\}$ an ordinal information.

1. (\Rightarrow) We suppose that $\{P, I\}$ is representable by a 2-additive belief function Bel .
 - (a) It is easy to show that $(P \cup I \cup M)$ contains no strict cycle;
 - (b) If there exists $i, j \in N, i \neq j$ such that $a_{ij} \sim a_j$ and $a_i \text{ } TC_P \text{ } a_0$ then $m_{ij} < 0$ with m the mass distribution of Bel . That is impossible because Bel is a belief function.
2. (\Leftarrow) Let us suppose that
 - $(P \cup I \cup M)$ contains no strict cycle;
 - $\{P, I\}$ satisfies the 2-MOPI property.

Using the Proposition 4, the conditions of Theorem 1 are satisfied. Let $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$ be a partition of \mathcal{B} built like in Section 5.2. Let us define the mapping $f : \mathcal{B} \rightarrow \mathbb{R}$ and $Bel : 2^N \rightarrow [0, 1]$ as follows: For $i \in \{0, \dots, m\}$,

$$\forall x \in \mathcal{B}_i, f(\phi(x)) = \begin{cases} 0 & \text{if } i = 0 \\ (2n)^i & \text{otherwise.} \end{cases}$$

$$\begin{cases} Bel_\emptyset = 0 \\ Bel_i = \frac{f_i}{\alpha}, \forall i \in N \\ Bel_{ij} = \frac{f_{ij}}{\alpha}, \forall i, j \in N \\ Bel(K) = \sum_{\{i,j\} \subseteq K} f_{ij} - (|K| - 2) \sum_{i \in K} Bel_i, \forall K \subseteq N, |K| > 2. \end{cases}$$

$$\text{where } \alpha = \sum_{\{i,j\} \subseteq N} f_{ij} - (n - 2) \sum_{i \in N} f_i$$

Bel define like this is 2-additive (the proof is similar to the proof of Theorem 1 in Section 5.3). The last thing we have to prove is $\forall i, j \in N, i \neq j, m_{ij} = Bel_{ij} - Bel_i - Bel_j \geq 0$ where m is the mass distribution of Bel . Let $i, j \in N, i \neq j$.

- If $a_{ij} \sim a_i$ then $m_{ij} = 0$ because $\{P, I\}$ satisfies the 2-MOPI property. Using the same reasons if $a_{ij} \sim a_j$ then $a_{ji} \sim a_j$ and $m_{ij} = 0$;
- If $a_{ij} \text{ } TC_P \text{ } a_i$ and $a_{ij} \text{ } TC_P \text{ } a_j$ then there exists $p, q, s \in \{0, \dots, m\}$ such that $a_{ij} \in \mathcal{B}_p, a_i \in \mathcal{B}_q, a_j \in \mathcal{B}_s$ with $p > q$ and $p > s$. So we have $f_{ij} = (2n)^p, f_i = (2n)^q, f_j = (2n)^s$ and then $Bel_{ij} \geq Bel_i + Bel_j$.

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