Conditional Laplace formula in regime switching model: Application to defaultable bond.

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Abstract

We give two different formulas to evaluate the conditional Laplace transform of a regime switching Cox Ingersoll Ross model. One using the property of semi-affine of this model and the other one using analytic approximation. Then we study the pricing of bonds issued by two firms considering the default and the correlation risk. In fact, we consider two firms with correlated default times and we obtain numerical formula for the bonds prices considering the regime switching market credit notations and the correlation between the two firms. Finally we give some numerical illustrations.

Keywords Conditional Laplace Transform; Default and Correlation risk; Zero coupon bond; Regime switching; Credit migration.

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Introduction

In a crisis context where the credit notation of a country or a firm imply financial and economics repercussion in other one, it is interesting to study the correlation and the impact of the change of this notation on other country or firm. In the literature, models for pricing defaultable securities have been introduced by Merton [23]. It consists of explicitly linking the risk of firm default and the value of the firm. Although this model is a good issue to understand the default risk, it is less useful in practical applications since it is too difficult to capture all the macroeconomics factors which appear in the dynamics of the firm's value. Hence, Duffie and Singleton [9] introduced the reduced form modeling which has been followed by Madan and Unal [22], Jeanblanc and Rutkowski [20] and others. The main tool of this approach is the "default intensity process" which describes

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in short terms the instantaneous probability of default. We should manage the default risk considering the financial market as a network where every default can affect another one and the propagation spread as far as the connections exist. In the literature, to deal with this correlation risk, the most popular approach is the copula. It consists of defining the joint distribution of the firms on the financial network considered given the marginal distribution of each firm. El Karoui and al. developed in [13] a conditional density approach. Given this density, we can compute explicitly the default intensity processes of firms considered. We will follow this approach and work without losing any generality in the explicit case where financial network is defined only with two firms denoted by A and B. We define default times by general Cox modeling and since density approach is satisfied, we can use all results in [13]. We really don't describe the conditional density but we describe how the default intensity processes have been affected by a common factor which represents the economic state of the firms. In fact, we define default intensity process by a Cox-Ingersoll-Ross (CIR) model with regime switching parameter values. The Cox-Ingersoll-Ross model was first considered to model the term structure of interest rate by Cox and al. in [7]. The study of this class of processes was caution by the fact that it allows us closed form expression of Laplace transform (see Duffie and al. [8]) and model well the default intensity (Alfonsi and Brigo [1]). We choose to take parameter value of the CIR process which depend of a regime switching process, typically a continuous time Markov chain on finite space. Choi in [5] shows that regime switching CIR process capture more short term interest rate than non switching model. Indeed in a econometric point of view, regime switching model were introduced by Hamilton in [16]. Many authors used regime switching model to obtain option price formula as example american option in Black and Scholes model was done by Elliott and Buffington in [12]. Our paper allow us to obtain in a first time a formula which evaluate the value of the conditional Laplace transform of a regime switching Cox Ingersoll Ross regarding to the semi affine property of the regime switching Cox Ingersoll Ross model. This approach is based so on the semi affine property and on solving Riccati's equations. We extend in a second time the analytic approximation found in Choi and Wirjanto [6]. Indeed Choi and Wirjanto in [6] give an analytic approximation of the value of a zero coupon bond price with constant CIR parameter and with constant time step model discretization. We extend this result in three way: first to the case of evaluating conditional Laplace transform of a regime switching Cox Ingersoll Ross, secondly to price defaultable regime switching zero coupon bond price and third to the case of non uniform deterministic time step model discretization. Indeed in our case, the time step model discretization will depend of the regime switching stopping time. We apply this two formula to evaluate defaultable regime switching zero coupon bond price. As said before, the regime switching will represent an economic state of the firm A or B. For this, we will use a continuous time Markov chain called credit migration process studied by Bielecki and Rutkowski in [4].

Hence in a first section, we will introduce the regime switching Cox-Ingersoll-Ross model. Then we will give the two formulas to evaluate the conditional Laplace transform of this model. Finally we will give the financial application by studying the price of a Defaultable zero coupon bond price. For this we will introduce Markov copula and credit migration process. We will do some simulation to compare the formula result and illustrate the model.

1 Regime switching model

Let $T > 0$ be a fixed maturity time and denote by $(\Omega, \overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{[0,T]}, \mathbb{P})$ an underlying probability space. We recall that a Cox Ingersoll Ross (CIR) process is the solution for all $t \in [0, T]$, of the stochastic differential equation given by

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t \tag{1.1}$$

where W is a real one dimensional Brownian motion and κ, θ and σ are constant which satisfy the condition $\sigma > 0$ and $\kappa\theta > 0$. We will assume that $\lambda_0 \in \mathbb{R}^+$ and that $2\kappa\theta \geq \sigma^2$. This is to ensure that the process (λ_t) is strictly positive. We will now define the notion of CIR process with each parameters values depend on the value of a Markov chain.

Definition 1.1. *Let $(X)_t$ be a d-dimensional continuous time Markov chain on finite space $\mathcal{S}^d := \{1, \ldots, K\}^d$ for all $t \in [0, T]$. We will call a* **Regime switching CIR** *the process (λ_t) which is the solution of the stochastic differential equation given for all $t \in [0, T]$ by*

$$d\lambda_t = \kappa(X_t)(\theta(X_t) - \lambda_t)dt + \sigma(X_t)\sqrt{\lambda_t}dW_t \tag{1.2}$$

For all $j \in \{1, \ldots, K\}^d$, we have that $\kappa(j)\theta(j) > 0$ and $2\kappa(j)\theta(j) \geq \sigma(j)^2$

For simplicity, we will denote the values $\kappa(X_t)$, $\theta(X_t)$ and $\sigma(X_t)$ by κ_t, θ_t and σ_t. We will denote by $\mathcal{F}_t^X := \{\sigma(X_s); 0 \leq s \leq t\}$, the natural filtration generated by the continuous Markov chain X.

Assumption 1.1. *We assume that*

(a) we know all the trajectory of the stochastic process $(X_t)_{t\in[0,T]}$.

(b) X is independent of W

There exists an increasing $\mathbb{F}^X$-stopping time in interval $[0, T]$, where the value of the Markov chain change. We denote by Γ this subdivision

$$0 = \tau_0 < \tau_1 < \cdots < \tau_M = T$$

So in each time intervals $[\tau_k, \tau_{k+1}[$, $k \in \{1, \ldots n\}$ the process X is constant. And so the CIR regime switching process λ has constant parameter on this each time interval.

Our aim is to find formula to evaluate the conditional Laplace transform of λ with respect to X denoted by Φ. It is for all $u \in \mathbb{C}$ the expectation given by

$$\Phi_{0,T,\lambda,X}(u) = \mathbb{E}\left[\exp\left(-u\int_0^T \lambda_s ds\right)|\lambda_0 = \lambda, \mathcal{F}_T^X\right] = \mathbb{E}_{\lambda,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right] \tag{1.3}$$

2 Conditional Laplace transform formulas

2.1 A Ricatti approach

Remark 2.1. *Assumption 1.1 implies that we know the sequence of increasing time* Γ*:*

$$0 = \tau_0 < \tau_1 < \cdots < \tau_M = T$$

Proposition 2.1. *The conditional Laplace transform of the regime switching CIR process (for* $u = 1$*) between time* $[\tau_k, \tau_{k+1}[$ *with* $\lambda_{\tau_k} = \lambda$ *and* $X_{\tau_{k+1}} = j \in \mathcal{S}^d$ *is given by*

$$\Phi_{\tau_k,\tau_{k+1},j} := \mathbb{E}\left[\exp\left(-u\int_{\tau_k}^{\tau_{k+1}} r_s ds\right)|\lambda_{\tau_k} = \lambda, X_{\tau_{k+1}} = j\right] = \exp\left\{-A(\Delta_{t_k}, j)\lambda - B(\Delta_{t_k}, j)\right\} \tag{2.4}$$

where

$$\Delta_{t_k} = \tau_{k+1} - \tau_k$$

and

$$A(\Delta_{t_k}, j) = \frac{2}{\gamma_j + \kappa_j} - \frac{4\gamma_j}{\gamma_j + \kappa_j}\frac{1}{(\gamma_j + \kappa_j)\exp(\gamma_j\Delta_{t_k}) + \gamma_j - \kappa_j} \tag{2.5}$$

$$B(\Delta_{t_k}, j) = -\frac{\kappa_j\theta_j(\gamma_j + \kappa_j)}{\sigma_j^2}\Delta_{t_k} + 2\frac{\kappa_j\theta_j}{\sigma_j^2}\ln\left((\gamma_j + \kappa_j)\exp(\gamma_j\Delta_{t_k}) + \gamma_j - \kappa_j\right) - 2\frac{\kappa_j\theta_j}{\sigma_j^2}\ln(2\gamma_j) \tag{2.6}$$

$$\gamma_j = \sqrt{\kappa_j^2 + 2\sigma_j^2} \tag{2.7}$$

Proof. We recall that the constant parameter CIR process defined in (1.1) is an affine process (see Duffie and al. [8]). So as in each step of time $[\tau_k, \tau_{k+1}[$, the stochastic process X is constant. So the process λ is a classical CIR with constant parameter on this each step. So on each step of time $[\tau_k, \tau_{k+1}[$, the process λ is affine, hence we can assume that the expression of $\Phi_{\tau_k,\tau_{k+1},j}$ is given by the form

$$\exp\left\{-A(\Delta_{t_k}, j)\lambda_{\tau_k} - B(\Delta_{t_k}, j)\right\} \tag{2.8}$$

for some functions $A(\Delta_{t_k}, j)$ and $B(\Delta_{t_k}, j)$. Hence we proceed similarly as the proof of Proposition 1 of Gourieroux [15]. By iterating expectation and taking a small time step dt, we have that

$$\begin{aligned}
\Phi_{\tau_k,\tau_{k+1},j}(1) &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds\right)|\lambda_{\tau_k+dt} = \tilde{\lambda}, X_{\tau_{k+1}} = j\right]|\lambda_{\tau_k} = \lambda, X_{\tau_{k+1}} = j\right] \\
&= \mathbb{E}_{\tau_k,j}\left[\mathbb{E}_{\tau_k+dt,j}\left[\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds\right)\right]\right]
\end{aligned}$$

We cut the integral in $\int_{\tau_k}^{\tau_{k+1}} = \int_{\tau_k}^{\tau_k+dt} + \int_{\tau_k+dt}^{\tau_{k+1}}$

$$\begin{aligned}
\Phi_{\tau_k,\tau_{k+1},j}(1) &= \mathbb{E}_{\tau_k,j}\left[\exp\left(-\int_{\tau_k}^{\tau_k+dt}\lambda_s ds\right)\mathbb{E}_{\tau_k+dt,j}\left[\exp\left(-\int_{\tau_k+dt}^{\tau_{k+1}}\lambda_s ds\right)\right]\right] \\
&= \mathbb{E}_{\tau_k,j}\left[\exp\left(-\lambda_{\tau_k}dt\right)\Phi_{\tau_k+dt,\tau_{k+1},j}(1)\right]
\end{aligned}$$

We use now the Assumption of the form of the conditional Laplace transform Φ

$$\Phi_{\tau_k,\tau_{k+1},j}(1) = \mathbb{E}_{\tau_k,j}\left[\exp\left(-\lambda_{\tau_k}dt\right)\exp\left(-A(\Delta_{t_k}-dt,j)\lambda_{\tau_k+dt}-B(\Delta_{t_k}-dt,j)\right)\right]$$

Since we take a small time step dt we obtain

$$\begin{aligned}
&= \mathbb{E}_{\tau_k,j}\left[\exp\left\{-\lambda_{\tau_k}dt - A(\Delta_{t_k}-dt,j)\left(\lambda_{\tau_k}+\kappa_j(\theta_j-\lambda_{\tau_k})dt+\sigma_j\sqrt{\lambda_{\tau_k}}dW_t\right)-B(\Delta_{t_k}-dt,j)\right\}\right] \\
&= \mathbb{E}_{\tau_k,i,j}[\exp\{-\lambda_{\tau_k}dt - A(\Delta_{t_k}-dt,j)\lambda_{\tau_k} - A(\Delta_{t_k}-dt,j)\kappa_j\theta_j dt \\
&\quad +A(\Delta_{t_k}-dt,j)\kappa_j\lambda_{\tau_k}dt - A(\Delta_{t_k}-dt,j)\sigma_j\sqrt{\lambda_{\tau_k}}dW_t - B(\Delta_{t_k}-dt,j)\}] \\
&= \exp\{-\lambda_{\tau_k}dt - A(\Delta_{t_k}-dt,j)\lambda_{\tau_k} - A(\Delta_{t_k},j)\kappa_j\theta_j + A(\Delta_{t_k},j)\kappa_j\lambda_{\tau_k} + \frac{1}{2}A(\Delta_{t_k},j)^2\sigma_j^2\lambda_{\tau_k}dt - B(\Delta_{t_k}-dt,j)\}
\end{aligned}$$

We regroup the λ_{τ_k} in factor

$$\begin{aligned}
\Phi_{\tau_k,\tau_{k+1},j}(1) &= \exp\left\{\left(-dt - A(\Delta_{t_k}-dt,j) + A(\Delta_{t_k},j)\kappa_j + \frac{1}{2}A(\Delta_{t_k},j)^2\sigma_j^2 dt\right)\lambda_{\tau_k}\right\} \\
&\quad \exp\left\{-A(\Delta_{t_k},j)\kappa_j\theta_j - B(\Delta_{t_k}-dt,j)\right\}
\end{aligned}$$

By identification this expression of the Laplace transform and our Assumption of the form of this expression (2.8). We get the following system with $\lambda_{\tau_k} = \lambda$,

$$\begin{aligned}
A(\Delta_{t_k},j) &= dt + A(\Delta_{t_k}-dt,j) - A(\Delta_{t_k},j)\kappa_j - \frac{1}{2}A(\Delta_{t_k},j)^2\sigma_j^2 dt \\
B(\Delta_{t_k},j) &= A(\Delta_{t_k},j)\kappa_j\theta_j + B(\Delta_{t_k}-dt,j)
\end{aligned}$$

Taking $dt \to 0$, we obtain the two functions as solution of the system

$$\begin{aligned}
A(\Delta_{t_k},j) &= dt + A(\Delta_{t_k}-dt,j) - A(\Delta_{t_k},j)\kappa_j - \frac{1}{2}A(\Delta_{t_k},j)^2\sigma_j^2 dt \\
B(\Delta_{t_k},j) &= A(\Delta_{t_k},j)\kappa_j\theta_j + B(\Delta_{t_k}-dt,j)
\end{aligned}$$

$$\frac{\partial A(\Delta_{t_k},j)}{\partial \Delta_{t_k}} = 1 - \kappa_j A(\Delta_{t_k},j) - \frac{1}{2}\sigma_j^2 A(\Delta_{t_k},j)^2 \quad (2.9)$$

$$\frac{\partial B(\Delta_{t_k},j)}{\partial \Delta_{t_k}} = \kappa_j\theta_j A(\Delta_{t_k},j) \quad (2.10)$$

With initial condition equal to $A(0,j) = 0$ and $B(0,j) = 0$, for all $j \in \mathcal{S}^d$ since $\Phi_{\tau_k,\tau_k,j} = \mathbb{E}_{\tau_k}\left[\exp\left(-\int_{\tau_k}^{\tau_k}\lambda_s ds\right)\right] = 1$ The calculus of the solution of this system of Riccati equation is straightforward and can be find for instance in Cox and al. [7]. □

We would like now to give an explicit form of the conditional Laplace transform of the CIR process between time 0 and T. This is done by the following Theorem.

Theorem 2.1. *Assume that for all $t \in [0,T]$, the process (λ_t) follows a regime switching CIR, then we have for all $\lambda_0 = \lambda > 0$ and $X_{\tau_1} = i_0 \in \mathcal{S}^d$ that*

$$\begin{aligned}
\Phi_{0,T,\lambda,X}(1) = \mathbb{E}\left[\exp\left(-\int_0^T \lambda_s ds\right) | \lambda_0 = \lambda, \mathcal{F}_T^X\right] &:= \mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] \\
&= \exp\left\{-\sum_{j=1}^{M} B_{M-j}(\Delta_{t_{j-1}})\right\} \exp\left(-A_0(\Delta_{t_0}, i_0)\lambda\right)
\end{aligned}$$

where

$$A_0(\Delta_{t_0}) = \frac{2}{\gamma^1+\kappa^1} - \frac{4\gamma^1}{\gamma^1+\kappa^1}\frac{1}{(\gamma^1+\kappa^1)\exp(\gamma^1\Delta_{t_0}) + \gamma^1 - \kappa^1} \tag{2.11}$$

$$\begin{aligned}
B_{M-k}(\Delta_{t_{j-1}}) = & -\frac{\kappa^{M-k+1}\theta^{M-k+1}(\gamma^{M-k+1}+\kappa^{M-k+1})}{(\sigma^{M-k+1})^2}\Delta_{t_{j-1}} \\
& +2\frac{\kappa^{M-k+1}\theta^{M-k+1}}{(\sigma^{M-k+1})^2}\ln\left((\gamma^{M-k+1}+\kappa^{M-k+1})\exp(\gamma^{M-k+1}\Delta_{t_{j-1}}) + \gamma^{M-k+1} - \kappa^{M-k+1}\right) \\
& -2\frac{\kappa^{M-k+1}\theta^{M-k+1}}{(\sigma^{M-k+1})^2}\ln\left(2\gamma^{M-k+1}\right)
\end{aligned} \tag{2.12}$$

$$\gamma^{M-k+1} = \sqrt{(\kappa^{M-k+1})^2 + 2(\sigma^{M-k+1})^2} \tag{2.13}$$

where we denote for simplicity $\kappa^j = \kappa(X_{t_j})$, $\theta^j = \theta(X_{t_j})$ and $\sigma^j = \sigma(X_{t_j})$

Proof. We have a sequence of increasing time $0 = \tau_0 < \tau_1 < \cdots < \tau_M = T$. Hence

$$\begin{aligned}
\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] &= \mathbb{E}_{\lambda,X}\left[\exp\left(-\sum_{k=0}^{M-1}\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)\right] = E_{\lambda,X}\left[\exp\left(-\sum_{k=0}^{M-1}\int_{\tau_k}^{\tau_k+\Delta_{t_k}} \lambda_s ds\right)\right] \\
&= \mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-1}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)\right]
\end{aligned}$$

By Assumption 1.1, conditioning with respect to $\overline{\mathcal{F}}_{\tau_{M-1}} := \overline{\mathcal{F}}_{M-1}$, we obtain

$$\begin{aligned}
\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] &= \mathbb{E}_{\lambda,X}\left[\mathbb{E}\left[\prod_{k=0}^{M-1}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right) |\overline{\mathcal{F}}_{M-1}\right]\right] \\
&= \mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}} \lambda_s ds\right)\mathbb{E}\left[\exp\left(-\int_{\tau_{M-1}}^{\tau_M} \lambda_s ds\right) |\overline{\mathcal{F}}_{M-1}\right]\right]
\end{aligned}$$

We know that $\mathbb{E}\left[\exp\left(-\int_{\tau_{M-1}}^{\tau_M} \lambda_s ds\right) |\overline{\mathcal{F}}_{M-1}\right]$ is equal to $\Phi(\tau_{M-1}, \tau_M, X_M)$, where X_M is equal to X_{τ_M}. So applying Proposition 2.1, we have that

$$\mathbb{E}\left[\exp\left(-\int_{\tau_{M-1}}^{\tau_M} \lambda_s ds\right) |\overline{\mathcal{F}}_{M-1}\right] = \exp\left\{-A_{M-1}(\Delta_{t_{M-1}}, X_M)\lambda_{\tau_{M-1}} - B_{M-1}(\Delta_{t_{M-1}}, X_M)\right\}$$

We recall that the quantities $A_{M-1}(\Delta_{t_{M-1}}, X_M)$ and $B_{M-1}(\Delta_{t_{M-1}}, X_M)$ are constants. Hence replacing this result in the expectation gives

$$\begin{aligned}\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] &= \mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds\right)\exp\left\{-A_{M-1}(\Delta_{t_{M-1}}, X_M)\lambda_{\tau_{M-1}} - B_{M-1}(\Delta_{t_{M-1}}, X_M)\right\}\right] \\ &= \exp\left\{-B_{M-1}(\Delta_{t_{M-1}}, X_M)\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-1}(\Delta_{t_{M-1}}, X_M)\lambda_{\tau_{M-1}}\right)\right]\end{aligned}$$

For the visibility of the calculus we will denote by A_{k-1} (resp. B_{k-1}) the quantity $A_{k-1}(\Delta_{t_{k-1}}, X_k)$ (resp. $B_{k-1}(\Delta_{t_{k-1}}, X_k)$) for all $k \in \{0, \ldots, M-1\}$. Hence

$$\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] = \exp\left\{-B_{M-1}\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)\right]$$

We conditioning again with respect to $\overline{\mathcal{F}}_{M-2}$ to obtain

$$\begin{aligned}\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] &= \exp\left\{-B_{M-1}\right\}\mathbb{E}_{\lambda,X}\left[\mathbb{E}\left[\prod_{k=0}^{M-2}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)|\overline{\mathcal{F}}_{M-2}\right]\right] \\ &= \exp\left\{-B_{M-1}\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-3}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds\right)\mathbb{E}\left[\exp\left(-\int_{\tau_{M-2}}^{\tau_{M-1}}\lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)|\overline{\mathcal{F}}_{M-2}\right]\right]\end{aligned}$$

To continue, we need to evaluate the conditional expectation

$$\mathbb{E}\left[\exp\left(-\int_{\tau_{M-2}}^{\tau_{M-1}}\lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)|\overline{\mathcal{F}}_{M-2}\right]$$

For this we will denote by $\varphi_{\tau_{M-2},\Delta_{t_{M-2}}}$ this conditional expectation and we will use the Lemma below to give an explicit form of this quantity.

Lemma 2.1. *Assume that for all $k \in \{1, \ldots, M\}$ that the conditional expectation $\varphi_{\tau_{M-k},\Delta_{t_{M-k}}}$ has a exponential affine structure form given by*

$$\varphi_{\tau_{M-k},\Delta_{t_{M-k}}} = \exp\left(-A_{M-k}(\Delta_{t_{M-k}}, X_{M-k+1})\lambda_{\tau_{M-k}} - B_{M-k}(\Delta_{t_{M-k}}, X_{M-k+1})\right) \quad (2.14)$$

Then we can find explicits forms for functions $A_{M-k}(\Delta_{t_{M-k}}, X_{M-k+1})$ and $B_{M-k}(\Delta_{t_{M-k}}, X_{M-k+1})$ which are given by equations (2.5) and (2.6) with the recursive condition that $A_{M-k}(0) = A_{M-k+1}$ and initial condition $B_{M-k}(0) = 0$.

Proof. We proceed similarly as the proof of Proposition 2.1. Indeed

$$\begin{aligned}\varphi_{\tau_{M-k},\Delta_{t_{M-k}}} &:= \mathbb{E}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}}\lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k+1}}\right)|\overline{\mathcal{F}}_{M-k}\right] \\ &= \mathbb{E}_{M-k}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}}\lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k}+\Delta_{t_{M-k}}}\right)\right]\end{aligned}$$

We take again a small time interval $dt \ll \Delta_{t_{M-2}}$ to obtain

$$\mathbb{E}_{M-k}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}} \lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k}+\Delta_{t_{M-k}}}\right)\right]$$
$$= \mathbb{E}_{M-k}\left[\mathbb{E}_{M-k+dt}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}} \lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k}+\Delta_{t_{M-k}}}\right)\right]\right]$$

Then

$$\mathbb{E}_{M-k+dt}\left[\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k+1}} \lambda_s ds - A_{M-k+1}\lambda_{\tau_{M-k+1}}\right)\right]$$
$$= \mathbb{E}_{M-k}\left[\varphi(\tau_{M-k}+dt, \Delta_{t_{M-k}} - dt)\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k}+dt} \lambda_s ds\right)\right]$$

We use the Assumption of the form of φ, this is equal to

$$\mathbb{E}_{M-k}\left[\exp\left(-A_{M-k}(\Delta_{t_{M-k}} - dt, X_{M-k+1})\lambda_{\tau_{M-k}+d_t} - B_{M-k}(\Delta_{t_{M-k}} - dt, X_{M-k+1})\right)\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k}+dt} \lambda_s ds\right)\right]$$

We simplify the notation and obtain

$$= \quad \mathbb{E}_{M-k}\left[\exp\left(-A_{M-k}(\Delta_{t_{M-k}} - dt)\lambda_{\tau_{M-k}+d_t} - B_{M-k}(\Delta_{t_{M-k}} - dt)\right)\exp\left(-\int_{\tau_{M-k}}^{\tau_{M-k}+dt} \lambda_s ds\right)\right]$$

For small dt and using the stochastic differential equation of $(\lambda)_{t\in[0,T]}$, we obtain

$$\begin{aligned} = \quad & \mathbb{E}_{M-k}[\exp\{-A_{M-k}(\Delta_{t_{M-k}} - dt)\left[\lambda_{\tau_{M-k}} + \kappa^{M-k+1}\left(\theta^{M-k+1} - \lambda_{\tau_{M-k}}\right)dt + \sigma^{M-k+1}\sqrt{\lambda_{\tau_{M-k}}}dW_t\right] \\ & -B_{M-k}(\Delta_{t_{M-k}} - dt) - \lambda_{\tau_{M-k}}dt\}] \end{aligned}$$

where $\kappa^{M-k+1} = \kappa(X_{\tau_{M-k+1}})$, $\theta^{M-k+1} = \theta(X_{\tau_{M-k+1}})$ and $\sigma^{M-k+1} = \delta(X_{\tau_{M-k+1}})$.

$$\begin{aligned} = \quad & \exp\left\{-A_{M-k}(\Delta_{t_{M-k}} - dt)\lambda_{\tau_{M-k}} - A_{M-k}(\Delta_{t_{M-k}} - dt)\kappa^{M-k+1}\left(\theta^{M-k+1} - \lambda_{\tau_{M-k}}\right)dt\right\} \\ & \times\exp\left\{-B_{M-k}(\Delta_{t_{M-k}} - dt) - \lambda_{\tau_{M-k}}d_t\right\}\mathbb{E}_{M-k}\left[\exp\left(-A_{M-k}(\Delta_{t_{M-k}} - dt)\sigma^{M-k+1}\sqrt{\lambda_{\tau_{M-k}}}dW_t\right)\right] \\ = \quad & \exp\left\{-A_{M-k}(\Delta_{t_{M-k}} - dt)\lambda_{\tau_{M-k}} - A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1}\left(\theta^{M-k+1} - \lambda_{\tau_{M-k}}\right)dt\right\} \\ & \times\exp\left\{-B_{M-k}(\Delta_{t_{M-k}} - dt) - \lambda_{\tau_{M-k}}dt\right\}\exp\left(\frac{1}{2}A^2_{M-k}(\Delta_{t_{M-k}})(\sigma^{M-k+1})^2\lambda_{\tau_{M-k}}dt\right) \end{aligned}$$

By identifying with the assumed expression of φ in (2.14), we get

$$\begin{cases} A_{M-k}(\Delta_{t_{M-k}}) = A_{M-k}(\Delta_{t_{M-k}} - dt) - A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1}dt - \frac{1}{2}A^2_{M-k}(\Delta_{t_{M-k}})(\sigma^{M-k+1})^2dt + dt \\ B_{M-k}(\Delta_{t_{M-k}}) = B_{M-k}(\Delta_{t_{M-k}} - dt) + A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1}\theta^{M-k+1}dt \end{cases}$$

Taking d_t close to zero, we get the two functions as solutions of:

$$\begin{cases} \frac{\partial A_{M-k}(\Delta_{t_{M-k}})}{\partial \Delta_{t_{M-k}}} &= -A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1} - \frac{1}{2}A^2_{M-k}(\Delta_{t_{M-k}})(\sigma^{M-k+1})^2 + 1 \\ \frac{\partial B_{M-k}(\Delta_{t_{M-k}})}{\partial \Delta_{t_{M-k}}} &= A_{M-k}(\Delta_{t_{M-k}})\kappa^{M-k+1}\theta^{M-k+1} \end{cases}$$

with condition for $\Delta_{t_{M-k}} \equiv 0$, $A_{M-k}(0) = A_{M-k+1}$ and $B_{M-k}(0) = 0$.

Hence by the proof of Proposition 2.1, we know the explicit form of $A_{M-k}(\Delta_{t_{M-k}})$ and $B_{M-k}(\Delta_{t_{M-k}})$ which are given by equations (2.5), (2.6) with the new recursive condition that $A_{M-k}(0) = A_{M-k+1}$ and initial condition $B_{M-k}(0) = 0$ □

We come back to the proof of the Theorem 2.1, by applying the Lemma 2.1 with $k = 2$. We obtain

$$\mathbb{E}\left[\exp\left(-\int_{\tau_{M-2}}^{\tau_{M-1}} \lambda_s ds - A_{M-1}\lambda_{\tau_{M-1}}\right)|\overline{\mathcal{F}}_{M-2}\right] = \exp\left(-A_{M-2}(\Delta_{t_{M-2}})\lambda_{\tau_{M-2}} - B_{M-2}(\Delta_{t_{M-2}})\right)$$

with deterministic function $A_{M-2}(\Delta_{t_{M-2}})$ and $B_{M-2}(\Delta_{t_{M-2}})$. Hence

$$\begin{aligned}\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T r_s ds\right)\right] &= \exp\{-B_{M-1}\}\,\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-3}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds\right)\exp\left(-A_{M-2}(\Delta_{t_{M-2}})\lambda_{t_{M-2}} - B_{M-2}(\Delta_{t_{M-2}})\right)\right] \\ &= \exp\{-B_{M-1} - B_{M-2}\}\,\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-3}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-2}(\Delta_{t_{M-2}})\lambda_{\tau_{M-2}}\right)\right]\end{aligned}$$

Conditioning an other time with respect to now $\overline{\mathcal{F}}_{M-3}$. We obtain

$$\begin{aligned}\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] &= \exp\{-B_{M-1} - B_{M-2}\}\,\mathbb{E}_{\lambda,X}\left[\mathbb{E}\left[\prod_{k=0}^{M-3}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-2}\lambda_{\tau_{M-2}}\right)|\overline{\mathcal{F}}_{M-3}\right]\right] \\ &= \exp\left\{-\sum_{j=1}^{2}B_{M-j}\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-4}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds\right)\mathbb{E}\left[\exp\left(-\int_{\tau_{M-3}}^{\tau_{M-2}}\lambda_s ds - A_{M-2}\lambda_{\tau_{M-2}}\right)|\overline{\mathcal{F}}_{M-3}\right]\right]\end{aligned}$$

We can now again apply Lemma 2.1 with $k = 3$. We obtain again that

$$\mathbb{E}\left[\exp\left(-\int_{\tau_{M-3}}^{\tau_{M-2}} \lambda_s ds - A_{M-2}\lambda_{\tau_{M-2}}\right)|\overline{\mathcal{F}}_{M-3}\right] = \exp\left(-A_{M-3}(\Delta_{t_{M-3}})\lambda_{\tau_{M-3}} - B_{M-3}(\Delta_{t_{M-3}})\right)$$

And so

$$\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] = \exp\left\{-\sum_{j=1}^{3}B_{M-j}\right\}\mathbb{E}_{\lambda,X}\left[\prod_{k=0}^{M-4}\exp\left(-\int_{\tau_k}^{\tau_{k+1}}\lambda_s ds - A_{M-3}(\Delta_{t_{M-3}})\lambda_{\tau_{M-3}}\right)\right]$$

By iterating the conditioning with respect to $\overline{\mathcal{F}}_{M-k}$, k going to 4 to M and applying the Lemma 2.1 we finally obtain

$$\mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T \lambda_s ds\right)\right] = \exp\left\{-\sum_{j=1}^{M}B_{M-j}\right\}\exp\left(-A_0(\Delta_{t_0})\lambda_{\tau_0}\right)$$

with by hypothesis $\lambda_{\tau_0} = \lambda$ and $A_0(\Delta_{t_0}) = A_0(\Delta_{t_0}, X_{\tau_1})$ with $X_{\tau_1} = i_0 \in \mathcal{S}^d$. □

We can now obtain the general expression of the conditional Laplace transform of the regime switching CIR process by Theorem 2.1.

Corollary 2.1. *For all $u \in \mathbb{C}$, we have that the conditional Laplace transform of the regime switching CIR process conditioned on $\lambda_0 = \lambda$ and $X_{\tau_1} = i_0 \in \mathcal{S}^d$ is given by*

$$\begin{aligned}\Phi_{0,T,\lambda,X}(u) &:= \mathbb{E}_{\lambda,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right] \\ &= \exp\left\{-\sum_{j=1}^{M} \tilde{B}_{M-j}\left(\Delta_{t_{M-j}}\right)\right\}\exp\left(-\tilde{A}_0\left(\Delta_{t_0}, i_0\right)\lambda\right) \end{aligned} \tag{2.15}$$

where the function $\tilde{B}_{M-j}$ for $j = \{1,\dots,M\}$ and $\tilde{A}_0$ are given by equations (2.11) and (2.12) taking parameters $\tilde{\kappa}^j := \kappa(\tilde{X}_{\tau_j}) = \kappa^j$, $\tilde{\theta}^j := \theta(\tilde{X}_{\tau_j}) = u\theta^j$ and $\tilde{\sigma}^j := \sigma(\tilde{X}_{\tau_j}) = \sqrt{u}\sigma^j$.

Proof. Since $\mathbb{E}\left[\exp\left(-u\int_0^T \lambda_s ds\right)|\lambda_0 = \lambda, \mathcal{F}_T^X\right] = \mathbb{E}_{\lambda,X}\left[\exp\left(-\int_0^T (u\lambda_s) ds\right)\right]$. This is the conditional Laplace transform of a process $(u\lambda)_t$ which is still a CIR process with new parameters $\tilde{\kappa}_t = \kappa_t$, $\tilde{\theta}_t = u\theta_t$ and $\tilde{\sigma}_t = \sqrt{u}\sigma_t$, for all $t \in [0,T]$. Hence applying Theorem 2.1 with this set of parameters gives the result as expected. □

2.2 Analytic approximation

We now give an analytical approximation to evaluate the conditional Laplace transform of a regime switching CIR. For this, we extend the result obtained in Choi and Wirjanto [6]. They obtained an analytic approximation of a zero coupon bond price in the particular case of constants parameters model and with constant time step discretization model.

2.2.1 Construction of the times grid

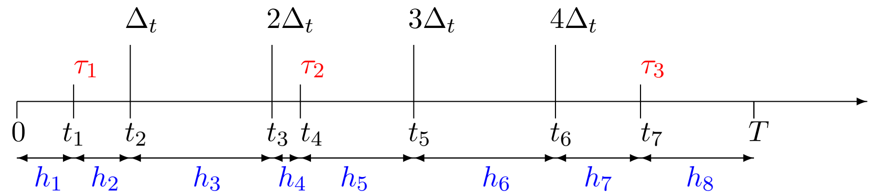
Let Δ_t be a fixed time step, then starting in time 0 we partition the time interval $[0,T]$ in time step of

- size Δ_t if there is no jump of the Markov process between time 0 to Δ_t.
- size τ_1 if there is the first jump of the Markov process at stopping time τ_1 less tan Δ_t.

Hence we denote by h_1 the first time step of size Δ_t or τ_1. Then we will proceed as the following: at time t_k, corresponding of the time after the step h_k, we construct the step h_{k+1} of size

- Δ_t if there is no jump of the Markov process between time t_k to $t_k + \Delta_t$.
- τ_i if there is the i jumps of the Markov process at stopping time τ_i less tan $t_k + \Delta_t$.

As an example of the construction



This construction imply that $h_k = t_k - t_{k-1} \leq \Delta_t$ and that the parameter of the regime switching CIR are constants (and bounded) in this each time interval $[t_k, t_{k+1}[$, $k \in \{0, 1, \ldots, n-1\}$. It follows as an application of the tree property of conditional expectation that the conditional Laplace transform of λ which follows a regime switching CIR is given by

$$\begin{aligned}\Phi_{0,T,\lambda,X}(u) &:= \mathbb{E}\left[\exp\left(-u\int_0^T \lambda_s ds\right)|\lambda_0, \mathcal{F}_T^X\right] \\ &= \mathbb{E}_{\lambda_0,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right] \\ &= \mathbb{E}^{t_0}_{\lambda_0,X}\mathbb{E}^{t_1}_{\lambda,X}\ldots\mathbb{E}^{t_{n-1}}_{\lambda,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right] \qquad (2.16)\end{aligned}$$

Proposition 2.2. *Let for all $k \in \{1, \ldots, n-1\}$,*

$$F_k = \exp\left(\frac{h^3_{n-k+1}}{8}u^2\sigma^2_{n-k}a^2_{n-k+1}\lambda_{n-k}\right) \qquad (2.17)$$

Then we have

$$\mathbb{E}^{t_0}_{\lambda_0,X}\left[\frac{\exp\left(-u\int_0^T \lambda_s ds\right)}{\prod_{i=1}^{n-1}F_i}\right] = \exp\left(-\frac{u}{2}\sum_{k=1}^n h_k^2 a_k\kappa_{k-1}\theta_{k-1} - \frac{u}{2}h_1\lambda_0\left[1 + a_1\left(1-\kappa_0 h_1\right)\right]\right)F_n \qquad (2.18)$$

where

$$a_{n-1} = 1 + \frac{h_n}{h_{n-1}} + \frac{h_n}{h_{n-1}}a_n\left(1 - h_n\kappa_{n-1}\right) \quad \textit{and} \quad a_n = 1 \qquad (2.19)$$

Proof. Using trapezoidal rule we obtain that the expectation at time t_{n-1} is given by

$$\begin{aligned}\mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right] &= \mathbb{E}^{t_{n-1}}_{\lambda_0,X}\left[\exp\left(-u\sum_{i=1}^n\left(\frac{\lambda_i+\lambda_{i-1}}{2}h_i\right)\right)\right] \\ = \exp\left(-u\sum_{i=1}^{n-2}\left(\frac{\lambda_i+\lambda_{i-1}}{2}h_i\right) - u\frac{\lambda_{n-2}}{2}h_{n-1}\right)&\mathbb{E}^{t_{n-1}}_{\lambda,X}\left[\exp\left(-\frac{u}{2}\left[h_n\lambda_n + h_n\lambda_{n-1} + h_{n-1}\lambda_{n-1}\right]\right)\right]\end{aligned}$$

Using the approximation

$$\lambda_n \simeq \lambda_{n-1} + \kappa_{n-1}\left(\theta_{n-1} - \lambda_{n-1}\right)h_n + \sigma_{n-1}\sqrt{\lambda_{n-1}}\Delta W_{n-1}$$

where $\Delta W_{n-1} = W_n - W_{n-1}$ and denote by G_{n-2} the quantity

$$\exp\left(-u\sum_{i=1}^{n-2}\left(\frac{\lambda_i+\lambda_{i-1}}{2}h_i\right)-u\frac{\lambda_{n-2}}{2}h_{n-1}\right)$$

We obtain that $\mathbb{E}_{\lambda_0,X}^{t_{n-1}}\left[\exp\left(-u\int_0^T\lambda_s ds\right)\right]$ is equal to

$$\begin{aligned}
& G_{n-2}\mathbb{E}_{\lambda_0,X}^{t_{n-1}}\left[\exp\left(-\frac{u}{2}\left[h_n\left(\lambda_{n-1}+\kappa_{n-1}\left(\theta_{n-1}-\lambda_{n-1}\right)h_n+\sigma_{n-1}\sqrt{\lambda_{n-1}}\Delta W_{n-1}\right)+h_n\lambda_{n-1}+h_{n-1}\lambda_{n-1}\right]\right)\right] \\
= \; & G_{n-2}\exp\left(-\frac{u}{2}\left[h_n\lambda_{n-1}+h_n^2\kappa_{n-1}\theta_{n-1}-h_n^2\kappa_{n-1}\lambda_{n-1}+h_n\lambda_{n-1}+h_{n-1}\lambda_{n-1}\right]\right) \\
& \times\mathbb{E}_{\lambda_0,X}^{t_{n-1}}\left[\exp\left(-\frac{u}{2}h_n\sigma_{n-1}\sqrt{\lambda_{n-1}}\Delta W_{n-1}\right)\right]
\end{aligned}$$

Moreover we have that if $\epsilon\sim\mathcal{N}(0,1)$ then for a constant K we have

$$\mathbb{E}\left[\exp\left(K\sqrt{T}\epsilon\right)\right]=\exp\left(\frac{K^2T}{2}\right) \tag{2.20}$$

Applying (2.20) and factorize by $-\frac{u\lambda_{n-1}h_{n-1}}{2}$ give

$$\begin{aligned}
&= G_{n-2}\exp\left(\frac{u^2}{8}h_n^3\sigma_{n-1}^2\lambda_{n-1}a_n^2\right)\exp\left(-\frac{u\lambda_{n-1}h_{n-1}}{2}\left[1+\frac{h_n}{h_{n-1}}+\frac{h_n}{h_{n-1}}a_n\left(1-h_n\kappa_{n-1}\right)\right]-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}\right) \\
&= G_{n-2}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}\right)\exp\left(-\frac{u\lambda_{n-1}h_{n-1}}{2}a_{n-1}\right)F_1
\end{aligned}$$

Hence

$$\mathbb{E}_{\lambda_0,X}^{t_{n-1}}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{F_1}\right] = G_{n-2}\exp\left(\frac{u^2}{8}h_n^3\sigma_{n-1}^2\lambda_{n-1}\right)\exp\left(-\frac{u\lambda_{n-1}h_{n-1}}{2}a_{n-1}\right)$$

Then we can obtain the conditional expectation based on the information until t_{n-2} with G_{n-3} the quantity

$$\exp\left(-u\sum_{i=1}^{n-3}\left(\frac{\lambda_i+\lambda_{i-1}}{2}h_i\right)-u\frac{\lambda_{n-3}}{2}h_{n-2}\right)$$

$$\mathbb{E}_{\lambda_0,X}^{t_{n-2}}\left[\mathbb{E}_{\lambda,X}^{t_{n-1}}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{F_1}\right]\right] = G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n^2\kappa_{n-1}\theta_{n-1}\right)$$

$$\times\mathbb{E}^{t_{n-2}}_{\lambda_0,X}\left[\exp\left(-\frac{u}{2}\left[\lambda_{n-1}h_{n-1}a_{n-1}+h_{n-2}\lambda_{n-2}+h_{n-1}\lambda_{n-2}\right]\right)\right]$$
$$= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}\right)\exp\left(\frac{u^2}{8}h_{n-1}^3\sigma_{n-2}^2\lambda_{n-2}a_{n-1}^2\right)$$
$$\exp\left(-\frac{u}{2}\lambda_{n-2}h_{n-1}a_{n-1}-\frac{u}{2}\kappa_{n-2}\left(\theta_{n-2}-\lambda_{n-2}\right)h_{n-1}^2a_{n-1}-h_{n-2}\lambda_{n-2}-h_{n-1}\lambda_{n-2}\right)$$
$$= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}\right)\exp\left(\frac{u^2}{8}h_{n-1}^3\sigma_{n-2}^2\lambda_{n-2}a_{n-1}^2\right)\exp\left(-\frac{u}{2}h_{n-1}^2a_{n-1}\kappa_{n-2}\theta_{n-2}\right)$$
$$\times\exp\left(-\frac{u}{2}\lambda_{n-2}h_{n-1}a_{n-1}+\frac{u}{2}\kappa_{n-2}\lambda_{n-2}h_{n-1}^2a_{n-1}-h_{n-2}\lambda_{n-2}-h_{n-1}\lambda_{n-2}\right)$$
$$= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}-\frac{u}{2}h_{n-1}^2a_{n-1}\kappa_{n-2}\theta_{n-2}\right)F_2$$
$$\times\exp\left(-\frac{u}{2}\lambda_{n-2}h_{n-2}\left[1+\frac{h_{n-1}}{h_{n-2}}+\frac{h_{n-1}}{h_{n-2}}a_{n-1}\left(1-\kappa_{n-2}h_{n-1}\right)\right]\right)$$
$$= G_{n-3}\exp\left(-\frac{u}{2}h_n^2a_n\kappa_{n-1}\theta_{n-1}-\frac{u}{2}h_{n-1}^2a_{n-1}\kappa_{n-2}\theta_{n-2}\right)\exp\left(-\frac{u}{2}\lambda_{n-2}h_{n-2}a_{n-2}\right)F_2$$

Hence repeating iterating calculus gives

$$\mathbb{E}^{t_{n-k}}_{\lambda_0,X}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{\prod_{i=1}^{k-1}F_i}\right]=G_{n-k-1}\exp\left(-\frac{u}{2}\sum_{i=1}^{k}h_{n-k+i}^2a_{n-k+i}\kappa_{n-k+i-1}\theta_{n-k+i-1}\right)$$
$$\times\exp\left(-\frac{u}{2}\lambda_{n-k}h_{n-k}a_{n-k}\right)F_k$$

Then until time t_0, we obtain finally the expected result.

$$\mathbb{E}^{t_0}_{\lambda_0,X}\left[\frac{\exp\left(-u\int_0^T\lambda_s ds\right)}{\prod_{i=1}^{n-1}F_i}\right]=\exp\left(-\frac{u}{2}\sum_{k=1}^{n}h_k^2a_k\kappa_{k-1}\theta_{k-1}-\frac{u}{2}h_1\lambda_0\left[1+a_1\left(1-\kappa_0h_1\right)\right]\right)F_n$$

□

Theorem 2.2. *For all $u\in\mathbb{C}$, the conditional Laplace transform Φ of the regime switching CIR process is given by*

$$\ln\left(\Phi_{0,T,\lambda,X}(u)\right) = \ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\exp\left(-u\int_0^T\lambda_s ds\right)\right]\right)$$
$$= -\frac{u}{2}\sum_{k=1}^{n}h_k^2a_k\kappa_{k-1}\theta_{k-1}-\frac{u}{2}h_1\lambda_0\left[1+a_1\left(1-\kappa_0h_1\right)\right]$$
$$+\sum_{k=1}^{n}\ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\exp\left\{\frac{h_{n-k+1}^3}{8}u^2\sigma_{n-k}^2a_{n-k+1}^2\left[\lambda_0+\sum_{i=0}^{n-k}\kappa_i(\theta_i-\lambda_i)h_{i+1}+\sum_{i=0}^{n-k}\sigma_i\sqrt{\lambda_i}\Delta W_i\right]\right\}\right]\right)\quad(2.21)$$

where the sequence a is defined in Proposition 2.2.

Proof. As in [6], we see that it would be difficult to compute the expression $\mathbb{E}^{t_{n-k-1}}_{\lambda_0,X}\left[F_k\right]$ explicitly. What why we simply approximate the expression F_k at time t_{n-k} by $\mathbb{E}^{0}_{\lambda_0,X}\left[F_k\right]$. First we can use the following approximation

$$\lambda_{n-k} \simeq \lambda_0+\sum_{i=0}^{n-k}\kappa_i(\theta_i-\lambda_i)h_{i+1}+\sum_{i=0}^{n-k}\sigma_i\sqrt{\lambda_i}\Delta W_i$$

Then

$$
\begin{aligned}
F_k &= \exp\left(\frac{h_{n-k+1}^3}{8}u^2\sigma_{n-k}^2 a_{n-k+1}^2 \lambda_{n-k}\right) \\
&= \exp\left(\frac{h_{n-k+1}^3}{8}u^2\sigma_{n-k}^2 a_{n-k+1}^2 \left[\lambda_0 + \sum_{i=0}^{n-k}\kappa_i(\theta_i-\lambda_i)h_{i+1} + \sum_{i=0}^{n-k}\sigma_i\sqrt{\lambda_i}\Delta W_i\right]\right) \quad (2.22)
\end{aligned}
$$

Hence approximate the expression of F_k at time t_{n-k} by the expectation at time 0, we obtain

$$
\begin{aligned}
\ln\left(\mathbb{E}_{\lambda_0,X}^{t_0}\left[\frac{\exp\left(-u\int_0^T \lambda_s ds\right)}{\prod_{i=1}^{n-1} F_i}\right]\right) &\simeq \ln\left(\mathbb{E}_{\lambda_0,X}^{t_0}\left[\frac{\exp\left(-u\int_0^T \lambda_s ds\right)}{\prod_{k=1}^{n} \mathbb{E}_{\lambda_0,X}^{t_0}\left[F_k\right]}\right]\right) \\
&= \ln\left(\frac{\mathbb{E}_{\lambda_0,X}^{t_0}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right]}{\prod_{k=1}^{n} \mathbb{E}_{\lambda_0,X}^{t_0}\left[F_k\right]}\right) \\
&= \ln\left(\mathbb{E}_{\lambda_0,X}^{t_0}\left[\exp\left(-u\int_0^T \lambda_s ds\right)\right]\right) - \ln\left(\prod_{k=1}^{n}\mathbb{E}_{\lambda_0,X}^{t_0}\left[F_k\right]\right) \\
&= \ln\left(\Phi_{0,T,\lambda,X}(u)\right) - \sum_{k=1}^{n}\ln\left(\mathbb{E}_{\lambda_0,X}^{t_0}\left[F_k\right]\right)
\end{aligned}
$$

Then using (2.22) gives the result.

□

Remark 2.2. *In the no switching regime case, we find the same expression as in [6]. Indeed we have in this case that* $h_k \equiv h = \frac{T}{N}$ *for an* $N \in \mathbb{N}$ *and* $(\kappa_k, \theta_k, \sigma_k) \equiv (\kappa, \theta, \sigma)$.

3 Application to price defaultable bond

3.1 Credit migration model

Let $T > 0$ be a fixed maturity time and denote by $(\Omega, \overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{[0,T]}, \mathbb{P})$ an underlying probability space.

Definition 3.2. *A* **Notation** *is a label given by an entity which measure the viability of a firm. This graduate notation goes to 1 to K, which means default of the firm. We will call a* **Indicator of notation** *a continuous time homogeneous Markov chain* $(X)_{t\in[0,T]}$ *on the finite space* $\mathcal{S} = \{1, \ldots, K\}$.

As in Section 1, we will denote by $\mathcal{F}_t^X := \{\sigma(X_s); 0 \le s \le t\}$, the natural filtration generated by the continuous Markov chain X. The generator matrix of X will be denoted by Π^X and it is given by

$$
\Pi_{ij}^X \ge 0 \quad \text{if } i \ne j \text{ for all } i, j \in \mathcal{S} \quad \text{and} \quad \Pi_{ii}^X = -\sum_{j\ne i}\Pi_{ij} \quad \text{otherwise.} \qquad (3.23)
$$

We can remark that Π^X_{ij} represents the intensity of the jump from state i to state j. In this part, we work in hazard rate framework and we use the credit rating migration modelization. We find conditions to work in Markov setting. We will assume that the intensity of default of each firms will depend on the Indicator of notation of each firms. For simplicity and without restriction we will work we two firms denoted by A and B.

3.1.1 Markov Copula

Let A and B be two firms with respectively: Indicator of notation $(X^A)_{t\in[0,T]}$ and $(X^B)_{t\in[0,T]}$. Generator matrix Π^A and Π^B and natural filtration $(\mathcal{F}^A)_{t\in[0,T]}$ and $(\mathcal{F}^B)_{t\in[0,T]}$. Let us denote by X the bivariate processes $X = (X^A, X^B)$, which is a finite continuous time Markov chain with respect to its natural filtration $\mathcal{F}^X = \mathcal{F}^{A,B}$. We recall now the Corollary 5.1 of Bielecki and al. [2], applied on our case, which gives the condition that the components of the bivariate processes X are themselves Markov chain with respect to their natural filtration.

Corollary 3.2. *Consider two Markov chain X^A and X^B, with respect to their own filtrations $\mathcal{F}^A$ and $\mathcal{F}^B$, and with values in $\mathcal{S}$. Suppose that their respective generators are Π^A_{ij} and Π^B_{hk} with i, j, h and k are in $\mathcal{S}$. Consider the system of equations in the unknowns $\Pi^X_{ij,hk}$ where $i, j, h, k \in \mathcal{S}$ and $(i, h) \neq (j, k)$:*

$$\sum_{k\in\mathcal{S}} \Pi^X_{ij,hk} = \Pi^A_{ij} \quad \forall h, i, j \in \mathcal{S}, i \neq j \quad \textit{and} \quad \sum_{j\in\mathcal{S}} \Pi^X_{ij,hk} = \Pi^B_{hk} \quad \forall i, h, k \in \mathcal{S}, h \neq k \tag{3.24}$$

Suppose that the above system admits a solution such that the matrix $\Pi^Z := \left(\Pi^Z_{ij,hk}\right)_{i,j,h,k\in\mathcal{S}}$ with

$$\Pi^X_{ii,hh} = - \sum_{(j,k)\in\mathcal{S}\times\mathcal{S},(j,k)\neq(i,h)} \Pi^X_{ij,hk} \tag{3.25}$$

properly defines an infinitesimal generator of a Markov chain with values in $\mathcal{S} \times \mathcal{S}$. Consider, the bivariate Markov chain $X = (X^A, X^B)$ on $\mathcal{S} \times \mathcal{S}$ with generator matrix Π^X. Then, the components X^A and X^B are Markov chain with respect to their own filtrations, their generators are Π^A and Π^B.

Hence we can formulate a Definition of a Markov copula.

Definition 3.3. *A* **Markov copula** *between the Markov chains X^A and X^B is any solution to system (3.24) such that the matrix Π^X, with $\Pi^X_{ii,hh}$ given in (3.25), properly defines an infinitesimal generator of a Markov chain with values in $\mathcal{S} \times \mathcal{S}$.*

3.1.2 Markov copula in the hazard rate framework

Assume that there is only two state $\mathcal{S} = \{0, 1\}$, before and after the default time. We denote by $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ such that $\mathcal{F}_t = \overline{\mathcal{F}}_t \vee \mathcal{F}^X_t$. Let τ^A and τ^B be the two default times of firms A and B. Let define for all $t \in [0, T]$:

$$H^A_t = 1_{\{\tau^A\leq t\}} \quad \text{and} \quad H^B_t = 1_{\{\tau^B\leq t\}} \tag{3.26}$$

We define now some filtrations

$$\mathcal{G}_t^A = \mathcal{F}_t \vee \mathcal{H}_t^B, \quad , \quad \mathcal{G}_t^B = \mathcal{F}_t \vee \mathcal{H}_t^A \quad and \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^A \vee \mathcal{H}_t^B$$

where $\mathcal{H}^A$ (resp. $\mathcal{H}^B$) is the natural filtration generated by H^A (resp. H^B). And we will denote $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$, $\mathbb{G}^A := \left(\mathcal{G}_t^A\right)_{t\in[0,T]}$ and $\mathbb{G}^B := \left(\mathcal{G}_t^B\right)_{t\in[0,T]}$. Let now consider $\lambda^i := \lambda^i(X)$, for $i \in \{A, B\}$ two $\mathbb{F}$-progressively non-negative processes defined on $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with the filtration $\mathbb{F}$. We assume that $\int_0^\infty \lambda^i(X_s)ds = +\infty$ and we set:

$$\tau^i = \inf \left\{ t \in \mathbb{R}^+, \int_0^t \lambda^i(X_s)ds \geq -\ln(U^i) \right\}, \quad i \in \{A, B\}.$$

where U^i are mutually independent uniform random variables defined on $(\Omega, \mathcal{G}, \mathbb{P})$ which are independent of λ^i. The stopping times τ^A and τ^B are totally inaccessible and conditionally independent given the filtration $\mathbb{F}$, moreover the $(\mathcal{H})$-hypothesis is satisfied (i.e. that every local $\mathbb{F}$-martingale is a local $\mathbb{G}$-martingale too). The process λ^i is called the $\mathbb{F}$-intensity of the firm i and we have that

$$M_t^i = H_t^i - \int_0^{t\wedge\tau^i} \lambda^i(X_s)_s ds$$

are $\mathbb{G}$-martingales. In general case, processes λ^i are $\mathbb{F} \vee \mathbb{G}^{(i)}$-adapted which jump when any default occurs. This jump impacts the default of the firm and makes some correlation between the firms. In our case, the correlation is constructed using the $\mathbb{F}$-Markov chain $X = (X^A, X^B)$. Since from the explicit formula of the intensity given the survey probability for each $i \in \{A, B\}$:

$$\lambda_t^i = -\frac{1}{\mathbb{P}(\tau^i \geq t|\mathcal{G}_t^i)} \frac{d\mathbb{P}(\tau^i \geq \theta|\mathcal{G}_t^i)}{d\theta}\Big|_{\theta=t}$$

We can find, from Bielecki and al. [3] (Example 4.5.1 p 94), that the formula of the conditional survey probability with respect to $\mathbb{G}$ is given by:

$$\mathbb{P}(\tau^i \geq \theta|\mathcal{G}_t) = 1_{\tau^i\geq t}\mathbb{E}\left[e^{-\int_t^\theta \lambda^i(X_s)ds}|\mathcal{F}_t\right] \tag{3.27}$$

for $i \in \{A, B\}$. The Markov chain X will explain how the curve of the default bond moves with different states (regimes) of the financial market.

3.1.3 Construction of the Markov chain

We are now going to present the canonical construction of a conditional Markov chain X, based on a given filtration $\mathbb{F}$ and a stochastic infinitesimal generator Π. This construction can be found in Bielecki and Rukowski [4] or Eberlein and Ozkan [10], which we follow closely in the exposition. Each component $\Pi_{ij}^X : \Omega \times [0, T] \to \mathbb{R}^+$ are bounded, $\mathbb{F}$-progressively measurable stochastic processes. We recall that for every $i, j \in \mathcal{S}, i \neq j$, processes Π_{ij} are non-negative and $\Pi_{ii}(t) = -\sum_{j\neq i} \Pi_{ij(t)}$. The process X is constructed from an initial distribution μ and the $\overline{\mathbb{F}}$-conditional adapted infinitesimal generator Π by enlarging the underlying probability space $(\Omega, \overline{\mathcal{F}}, \mathbb{P}_T)$ to a probability space denoted in the sequel by $(\Omega, \mathcal{F}, \mathbb{Q}_T)$. The new probability space is obtained as a product space of the underlying one with a probability space supporting the initial distribution μ of X and a probability space supporting a sequence of uniformly distributed random variables, which control, together with the entries of the infinitesimal generator Π, the laws of jump times $(\tau_k)_{k\in\mathbb{N}}$ of X and jump heights. We denote by $\mathbb{F}$ its trivial extension from the original probability space $(\Omega, \overline{\mathcal{F}}, \mathbb{P}_T)$ to $(\Omega, \mathcal{F}, \mathbb{Q}_T)$. We refer to [4] or Grbac [14] for detail of this construction. However an important step of this construction is that they construct a discrete time process $(\overline{X}_k)_{k\in\mathbb{N}}$ which allow us to construct the migration process X as

$$X_t := \overline{X}_{k-1} \quad \text{for all} \quad t \in [\tau_{k-1}, \tau_k[, \quad k \geq 1 \tag{3.28}$$

where τ_k are the jump time. An important result is that the progressive enlargement of filtration $\mathcal{F}_t := \overline{\mathcal{F}}_t \vee \mathcal{F}_t^X$, $t \in [0, T]$ satisfies the $(\mathcal{H})$-hypothesis which is that every local $\overline{\mathbb{F}}$-martingale is a local $\mathbb{F}$-martingale too. For the next of this paper we will work under the enlarging probability space $(\Omega, \mathcal{F}, \mathbb{Q}_T)$ where $\mathcal{F}_t := \overline{\mathcal{F}}_t \vee \mathcal{F}_T^X$. The expectation will be take under the probability measure $\mathbb{Q}_T$ but for simplicity we will note $\mathbb{E}^{\mathbb{Q}_T}$ by $\mathbb{E}$.

3.2 Pricing defaultable bond with Markov copula

3.2.1 Defaultable Model

Let $W = (W^1, W^2)$ be a bi-dimensionnal real Brownian motion defined as

$$\overline{\mathcal{F}_t} = \sigma\{W_s^1; 0 \leq s \leq t\} \quad and \quad \mathcal{F}_t^X = \sigma\{W_s^2; 0 \leq s \leq t\}$$

We suppose the existence of two firms A and B with our own indicator of notation X^A and X^B which are $\mathcal{F}^X$-adapted. Then there is two infinitesimal generator Π^A and Π^B generated by the Brownian motion W^2. We construct the credit migration process X as the Markov copula of the bivariate process (X^A, X^B). We will denote by Π^X the infinitesimal generator process of X which is a matrix with K^2 rows and columns since the cardinal of the state of notation is K. Denote by $N = K^2$, we can write Π^X as

$$\Pi^X = \begin{pmatrix} \pi_{(1,1)} & \cdots & \pi_{(1,N)} \\ \pi_{(2,1} & \cdots & \pi_{(2,N)} \\ \vdots & & \vdots \\ \pi_{(N,1)} & \cdots & \pi_{(N,N)} \end{pmatrix}$$

The possible state are N couples which are given by

$$\mathcal{E} := \{(1,1), (1,2), \ldots, (1,K), (2,1), (2,2), \ldots, (2,K), \ldots (K,1), (K,2), \ldots, (K,K)\}$$

With this construction, we obtain two sequence of increasing $\mathbb{F}^X$-stopping time in interval $[0, T]$, where the notation of each firm change. We denote by Γ^A this subdivision

$$0 \leq \tau_1^A < \tau_2^A < \cdots < \tau_n^A \leq T$$

Respectively by Γ^B

$$0 \leq \tau_1^B < \tau_2^B < \cdots < \tau_m^B \leq T$$

Remark 3.3. *This is no reason that the number of change of credit notation n and m of the two firms A and B to be the equal.*

So in each time intervals $[\tau_k^i, \tau_{k+1}^i[$, with $i = A$ and $k \in \{1, \dots n\}$ or $i = B$ and $k \in \{1, \dots m\}$ the credit migration process X is constant.

Assumption 3.2. *We assume that the both intensities processes λ^A and λ^B follow a regime switching CIR given for $i = \{A, B\}$ by*

$$d\lambda_t^i = \kappa(X_t)(\theta(X_t) - \lambda_t^i)dt + \sigma(X_t)\sqrt{\lambda_t^i}dW_t^1 \tag{3.29}$$

We will assume in the sequel that the risk-free interest rate (r_t) is deterministic for all $t \in [0, T]$.

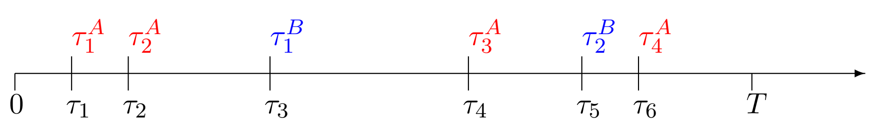
Remark 3.4. *We have that the intensity process (λ_t^i) depends on the value of the credit migration process $X = (X^A, X^B)$. Hence each firm A and B have an increasing sequence of stopping times given by:*

– *for the firm A it is $0 \leq \tau_1^A < \tau_2^A < \cdots < \tau_n^A \leq T$*

– *for the firm B it is $0 \leq \tau_1^B < \tau_2^B < \cdots < \tau_m^B \leq T$*

Hence with this two sequence, we construct another sequence by reorganizing this two sequence in once where we put all the stopping time τ_i^A, $i \in \{1, \dots n\}$ and τ_j^B, $j \in \{1, \dots, m\}$ in increasing order. We obtain a new increasing sequence of stopping time of size $M \in \mathbb{N}$ given by $0 \leq \tau_1 < \tau_2 < \cdots < \tau_M \leq T$

As an example of the construction



Let $(X_t)_{t \in [0,T]}$ be a credit migration process which take value in a finite state space $\mathcal{S} \times \mathcal{S}$ such that for all $k \in \{0, \dots, M-1\}$ and all $t \in [\tau_k, \tau_{k+1}[$, X_t is constant.

Remark 3.5. *By this construction, we have that on each interval $t \in [\tau_k, \tau_{k+1}[$ that the regime switching CIR process λ^i defined in (3.29) is a classical CIR with constant parameter.*

Example 3.1. *We can take a sequence of increasing stopping time which is an equidistant subdivision of the time interval $[0, T]$ on M interval $\tau_k = k\Delta_t = k\frac{T}{M}$. Then we can take $(X_t)_{t \in [0,T]}$ as continuous time bivariate homogeneous Markov chain on finite state space $\mathcal{S}$ which is constant on each time interval $[\tau_k, \tau_{k+1}[= [\tau_k, \tau_k + \Delta_t[$.*

3.2.2 Explicit zero coupon bond price formula

Definition 3.4. *We will denote by* $\left(D^i_{t,T}\right)_{t\in[0,T]}$, $i=\{A,B\}$ *the price of a defaultable discounted bond price which pays \$1 at the maturity T.*

Using the partitioning time and the notation defined as the previous subsection, then by the general asset pricing theory in Harrison and Pliska [17] and [18], the defaultable discounted bond price $D_{t,T}$ at time t given $\lambda=\lambda_0$ and $X_{\tau_1}=i_0\in\mathcal{S}\times\mathcal{S}$ is given by

Proposition 3.3. *For* $i=\{A,B\}$, *we have for all* $t\in[0,T]$ *that*

$$D^i_{t,T}=(1-H^i_t)\mathbb{E}\left[\exp\left(-\int_t^T(r_s+\lambda^i_s)ds\right)|\mathcal{F}^X_t,\lambda_0\right] \tag{3.30}$$

Remark 3.6. *The quantity* $\left(r_t+\lambda^i_t\right)_{t\in[0,T]}$ *can be see as a default-adjusted interest rate process. The part* $\left(\lambda^i_t\right)_{t\in[0,T]}$ *is the risk-neutral mean loss rate of the instrument due to the default of the firm* $i\in\{A,B\}$. *The quantity* $\left(r_t+\lambda^i_t\right)_{t\in[0,T]}$ *therefore represents the probability and the timing of default, as well as for the effect of losses on default. This model allows us to capture a economic health of each firm since for each firm* $i\in\{A,B\}$, *the stochastic process* (λ^i_t) *has parameters whose values depend on the credit notation of the firm. And by the construction of the migration process* X, *we have correlation between each firm notation. This allows the model to capture financial health correlation between each firm, like the impact of the default of one firm against the others.*

Proposition 3.4. *Under Assumptions 1.1 and 3.2, we have for* $i\in\{A,B\}$ *that the defaultable bond price can be obtained by two formula*

1. *Riccati Approach:*

$$D^i_{0,T}=(1-H^i_0)\exp\left(-\int_0^T r_s ds\right)\exp\left\{-\sum_{j=1}^{M}B_{M-j}(\Delta_{t_{j-1}})\right\}\exp\left(-A_0(\Delta_{t_0},i_0)\lambda\right) \tag{3.31}$$

where quantity A and B are the same as given in Theorem 2.1.

2. *Analytic Approximation:*

$$D^i_{0,T}=(1-H^i_0)\exp\left(-\int_0^T r_s ds\right)\exp\left\{-\frac{u}{2}\sum_{k=1}^{n}h^2_{n-k+1}a_{n-k+1}\kappa_{n-k}\theta_{n-k}-\frac{u}{2}h_1\lambda_0\left[1+a_1\left(1-\kappa_0h_1\right)\right]\right\}$$
$$\times\exp\left\{\sum_{k=1}^{n}\ln\left(\mathbb{E}^{t_0}_{\lambda_0,X}\left[\exp\left(\frac{h^3_{n-k+1}}{8}u^2\sigma^2_{n-k}a^2_{n-k+1}\left[\lambda_0+\sum_{i=0}^{n-k}\kappa_i(\theta_i-\lambda_i)h_{i+1}+\sum_{i=0}^{n-k}\sigma_i\sqrt{\lambda_i}\Delta W_i\right]\right)\right]\right)\right\} \tag{3.32}$$

where the sequence a *is defined in Proposition 2.2.*

Proof. For point 1. apply Theorem 2.1 and Proposition 3.3 to the particular case $t=0$ and $\theta=T$. And for point 2. apply Theorem 2.2 with $u=1$. □

4 Simulation: Pricing zero coupon Bond in the two firms case with two regime.

We work on simulated data. We fix the time maturity of zero coupon bond at $T = 10$ (i.e. a ten years ahead maturity). We chose that the deterministic interest rate equals to zero.

4.1 The model parameters and heuristic

In the one regime case, which is called "standard economic" regime, we take for value of the CIR default intensity the value in Table 1.

Parameters	κ	θ	σ
Values	0.1	0.15	0.15

Table 1: Parameters values of the CIR default intensity.

This set of parameters gives a value of the Zero coupon Bond equals to 0.6086. We now do some simulation of our paper model. First we begin with the heuristic of the calculus of the defaultable Bond price. We will proceed by a Monte Carlo approach with $MC \in \mathbb{N}$ steps:

1. We know the value of the infinitesimal generator Π^X of the credit migration process X.
2. We generate a sequence of increasing stopping time and the time correspondent trajectory of X.
3. (a) We apply the formula (3.31) to calculate the price of this defaultable Bond price for the firm A or B.
   (b) We apply the construction of the time grid studied in subsection 2.2.1. Then we applied the formula (3.32).
4. We come back to step 1. until we will have do MC times this methods.

Hence assume that we have 2 regimes which represents a "normal" economic regime (regime 0) and a "crisis" regime (regime 1). The Credit migration process X is then done in a set of four state

$$\{(0,0);(1,0);(0,1);(1,1)\}$$

We fix the infinitesimal generator Π^X of the credit migration process X such that

$$\Pi^X = \begin{pmatrix} -0.1083 & 0.0455 & 0.0455 & 0.0174 \\ 0.0542 & -0.1644 & 0.0082 & 0.1004 \\ 0.0542 & 0.01 & -0.1644 & 0.1003 \\ 0.0542 & 0.01 & 0.01 & -0.0741 \end{pmatrix}$$

which correspond to a transition matrix of

$$P^X = \begin{pmatrix} 0.90 & 0.04 & 0.04 & 0.02 \\ 0.05 & 0.85 & 0.01 & 0.09 \\ 0.05 & 0.01 & 0.85 & 0.09 \\ 0.05 & 0.01 & 0.01 & 0.93 \end{pmatrix}$$

In other words, if we are in a state where only the firm A is on ”crisis” (i.e. state (1,0)) the probability that the firm B go into ”crisis” in the next time step is 0.01.

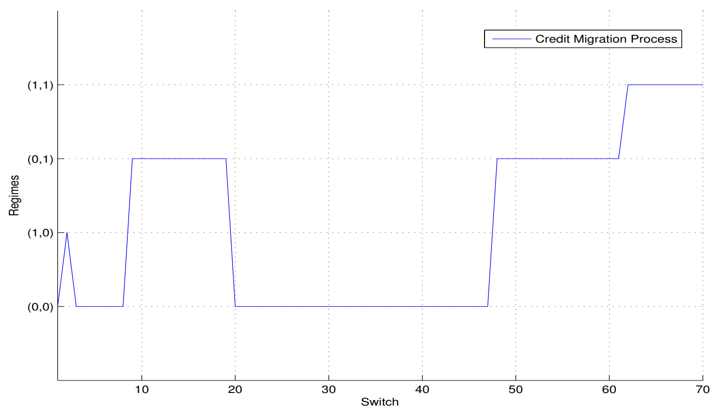


Figure 1: Example of trajectory of the credit migration process X.

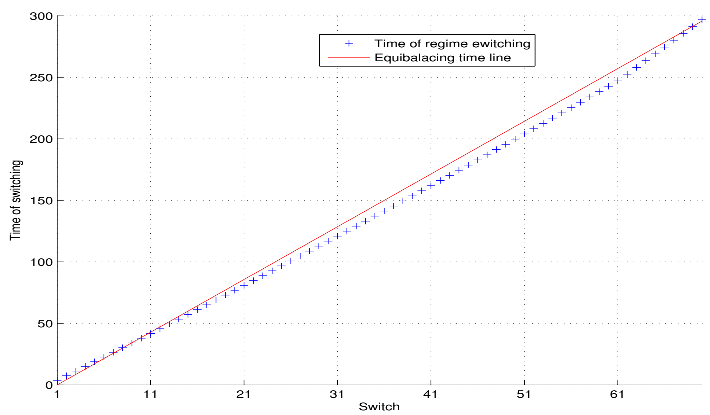


Figure 2: Example of instant of regime switching of the credit migration process X.

Figure 1 and 2 give an example of the trajectory of the credit migration process X and of the sequence of stopping time τ where the credit migration process jumps.

We need to have four set of CIR default intensity parameters. Let for $i \in \{A, B\}$, ν^i, ξ^i and ρ^i be real valued such that the set of parameters are given by Table 2

Parameters	κ_X	θ_X	σ_X
(0,0)	0.1	0.15	0.15
(1,0)	$0.1 + \nu^A$	$0.15 + \xi^A$	$0.15 + \rho^A$
(0,1)	$0.1 + \nu^B$	$0.15 + \xi^B$	$0.15 + \rho^B$
(1,1)	$0.1 + \nu^A + \nu^B$	$0.15 + \xi^A + \xi^B$	$0.15 + \rho^A + \rho^B$

Table 2: Parameters values of the CIR default intensity in the 2 regimes case.

Remark 4.7. *For $i \in \{A, B\}$, the constant ν^i, ξ^i and ρ^i are choose such that the CIR condition holds, i.e. $2\kappa_X\theta_X \geq \sigma_X^2$*

4.2 Comparison of the different formula to evaluate defaultable bond price.

4.2.1 Convergence

We know that the formula of the conditional survey probability with respect to $\mathbb{G}$ is given by equation (3.27). We would like now to compare the different formula to pricing defaultable zero coupon bond (i.e. formulas (3.27), (3.31) and (3.32)).

In tables 3, 4 and 5, we resume the convergence results in the case of a four state regime parameters defined as in Table 2.

Parameters:	ν^A	ν^B	ξ^A	ξ^B	ρ^A	ρ^B
Values:	0.2	0	0	0.3	0	0.1

Table 3: Values of the constant parameters defined in Table 2.

Regimes:	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
Bond price values:	0.6086	0.3777	0.2740	0.0668

Table 4: Values of the Bond price standard formula in $t = 0$ in each regime with a maturity $T = 10$ years.

Remark 4.8. *We take for time step parameter Δ_t appearing in subsection 2.2.1 for the calculus of (3.32) the value 0.01. Indeed, we obtain similar price for different value of Δ_t.*

With this set of parameters, we obtain what we expected. Indedd, we can see in Table 4 that in "crisis" regime (i.e. $(1,0)$, $(0,1)$ and $(1,1)$) the probability of default of the firm is bigger than in "standard economic" regime (i.e. $(0,0)$). In Table 5, we can see that all formuals converge then the number of Monte Carlo simulations increase. Whereas the bond price value given by formula (3.31) based on Riccati approach or formula (3.32) based

Bond Price	(3.31) (std)	C.T.(sec.)	(3.32) (Std)	C.T.	(3.27)	C.T.
$MC = 100$	0.5619 (0.1110)	1.94	0.5585 (0.1699)	15.98	0.6500	1.95
$MC = 300$	0.5692 (0.1015)	5.34	0.5587 (0.1602)	52.52	0.6233	6.61
$MC = 400$	0.5736 (0.0949)	6.87	0.5649 (0.1505)	60.48	0.6400	9.58
$MC = 500$	0.5748 (0.0927)	7.97	0.5658 (0.1511)	78.32	0.6360	12.44
$MC = 1000$	0.5738 (0.0961)	16.31	0.5654 (0.1533)	146.51	0.6220	33.23
$MC = 2000$	0.5727 (0.0995)	27.33	0.5646 (0.1533)	221.22	0.5770	96.78

Table 5: Results for the formulas convergence in $t = 0$ with initial regime the regime $(0, 0)$ and maturity $T = 10$ years.

Bond Price	(3.32)
$\Delta_t = 1$	0.5612
$\Delta_t = 0.1$	0.5648
$\Delta_t = 0.01$	0.5649

on analytic approach converge quicker than the value given by formula (3.27). Indeed, it is sufficient to take 400 Monte Carlo simulation to converge while it is necessary to take at least 2000 Monte Carlo simulations with formula (3.27). The difference of 10^{-1} on the value given by (3.31) and (3.32) could be due to the error approximation of the conditional expectation at time t_{n-k} of F_k (see. proof of Theorem 2.2). Hence our two formulas need less simulation than formula (3.27) to converge. Moreover we observe that the Riccati approach formula (3.31) need a smaller computation time. Only 6.87 sec while formula (3.32) needs 60.48 sec and formula (3.27) needs 96.78 sec. Hence formula based on Riccati approach needs ten times less times than Analytic approach to converge.

4.2.2 Bond price with respect to the maturity T.

Bond Price	(3.31) ($MC = 400$)	(3.32) ($MC = 400$)	(3.27) ($MC = 2000$)
$T = 1$	0,9926	0,9923	0,9940
$T = 2$	0,9709	0,9696	0,9770
$T = 5$	0,8458	0,8405	0,8480
$T = 7$	0,7376	0,7261	0,7365
$T = 10$	0,5736	0,5649	0,5770
$T = 15$	0,3579	0,3948	0,3505

Table 6: Value of the Defaultable zero coupon bond price at time 0 with respect to the maturity time T with $\Delta_t = 0.01$.

We observe in Table 6 and Figure 3 that the three formulas give similar result. Whereas, first, we made this simulation taking 2000 Monte Carlo simulations for the Probabilistic

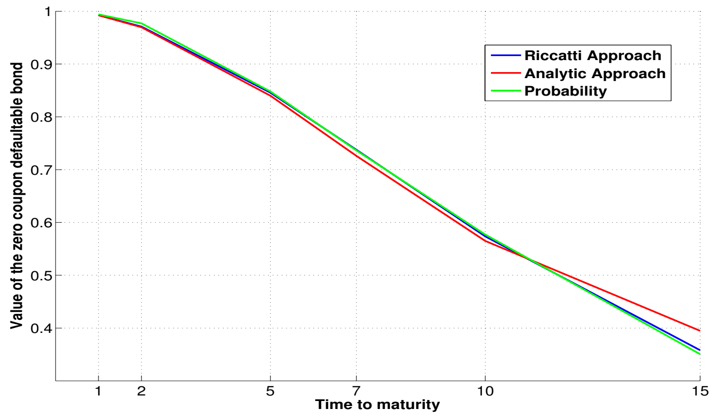


Figure 3: Graphs of the value of the Defaultable zero coupon bond price at time 0 with respect to the maturity time T with $\Delta_t = 0.01$ (MC=400 for the two first formulas and 2000 for the Probability approach).

approach (formula (3.27)). Secondly, we remark, when the maturity T is greater than 10, that the result given by the analytic approximation is no more better than the other. This relative mispricing was observe in the non regime switching case and uniform step time model discretization in [6] as soon as the maturity T is greater than 10.

4.3 Other simulations with Riccati approach formula.

4.3.1 Bond Price all over time $t \in [0, T]$

Taking parameters as in Table 3, we can draw the value of a defaultable zero coupon bond price over time $t \in [0, T]$ using formula (3.31). An example is given in Figure 4.

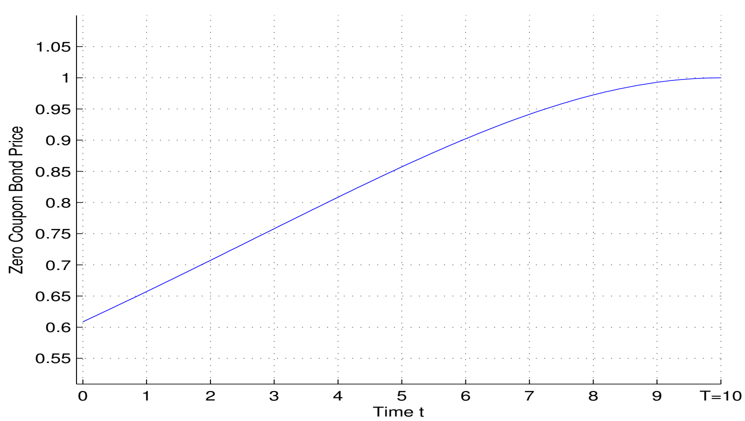


Figure 4: Price of a defaultable zero coupon bond price in each time t between time 0 to maturity T.

4.3.2 Bond Price in function of probability that B goes to crisis

Taking parameters as in Table 3, we evaluate the price of a defaultable zero coupon bond in functon of the probability $P\left(X_{t+\Delta_t} = (0,1)|X_t = (0,0)\right)$ and $P\left(X_{t+\Delta_t} = (1,1)|X_t = (0,0)\right)$. This is $p^X_{1,3}$ and $p^X_{1,4}$. Hence we take a parametric transition matrix of the form: a transition matrix of

$$P^X = \begin{pmatrix} 1-a-3b & a & 2b & b \\ 0.05 & 0.85 & 0.01 & 0.09 \\ 0.05 & 0.01 & 0.85 & 0.09 \\ 0.05 & 0.01 & 0.01 & 0.93 \end{pmatrix}$$

where $a, b \in [0,1]$. We obtain the following result:

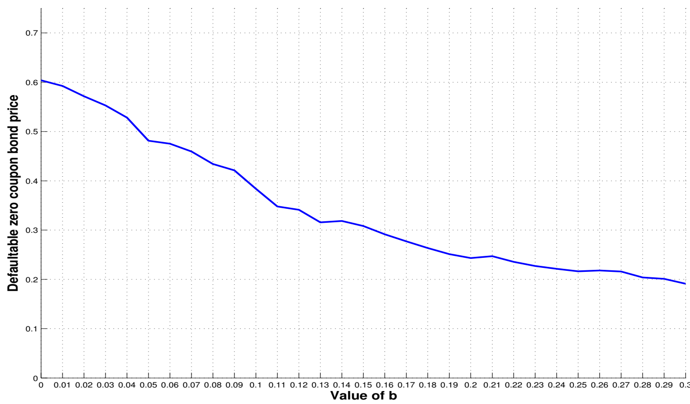


Figure 5: Price of a defaultable zero coupon bond price in $t = 0$ for maturity $T = 10$ and values of $a = 0.04$ in function of b.

Hence we observe in Figure 5 that when b grows up (i.e. the probability $P\left(X_{t+\Delta_t} = (1,1)|X_t = (0,0)\right)$), the price of the defaultable zero coupon bond price of the firm A decreases. This means that the economic status of the firm B (the probability to go in crisis) impact the value of the defaultable zero coupon bond of the firm A.

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