MINIMIZATION OF THE GROUND STATE FOR TWO PHASE CONDUCTORS IN LOW CONTRAST REGIME

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ABSTRACT. In this article we consider the problem of the optimal distribution of two conducting materials with given volume inside a fixed domain, in order to minimize the first eigenvalue (the ground state) of a Dirichlet operator. It is known, when the domain is a ball, that the solution is radial, and it was conjectured that the optimal distribution of the materials consists in putting the material with the highest conductivity in a ball around the center. We show that this conjecture is not true in general. For this, we consider the particular case where the two conductivities are close to each other (low contrast regime) and we perform an asymptotic expansion with respect to the difference of conductivities. We find that the optimal solution is the union of a ball and an outer ring when the amount of the material with the higher density is large enough.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^d$ which is to be called the design region. Let m be a given positive number, $0 < m < |\Omega|$, where $|\Omega|$ is the Lebesgue measure of the design region Ω. Two materials with conductivities α and β ($0 < \alpha < \beta$) are distributed in arbitrary disjoint measurable subsets A and B, respectively, of Ω so that $A \cup B = \Omega$ and $|B| = m$. Consider the two-phases eigenvalue problem:

$$-\operatorname{div}(\sigma\nabla u) = \lambda u \ \text{ in } \Omega, \tag{1}$$

$$u = 0 \ \text{ on } \partial\Omega, \tag{2}$$

with the conductivity $\sigma = \alpha\chi_A + \beta\chi_B$. Let λ be the first eigenvalue of (1)-(2) and u the associated eigenvector. The variational formulation for λ is

$$\lambda = \min_{u\in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega \sigma|\nabla u|^2}{\displaystyle\int_\Omega u^2} = \min_{u\in H_0^1(\Omega), \|u\|_2=1} \int_\Omega \sigma|\nabla u|^2, \tag{3}$$

where $\|u\|_2$ denotes the L^2-norm of u. In this paper the set Ω is fixed and we are interested in the dependence of λ on A and B. Since $A = \Omega \setminus B$, λ may be described as a function of B and we write $\lambda = \lambda(B)$. We consider the problem of minimizing $\lambda(B)$ with the constraint that the two phases are to be distributed in fixed proportions:

$$\text{minimize} \quad \lambda(B) \tag{4}$$

$$\text{subject to} \quad B \in \mathcal{B} \tag{5}$$

where

$$\mathcal{B} = \{B \subset \Omega, B \text{ measurable}, |B| = m\} \tag{6}$$

The existence of a solution to the problem (4)-(6) remains an open question. In general, one may evidence microstructural patterns in relation to minimizing sequences and the original problem may have to be relaxed to include microstructural designs. Existence of a solution and optimality conditions in the class of relaxed designs has been discussed in Cox and Lipton [4]. However, the original problem (4)-(6) may still have a solution for particular geometries as is the case when Ω is a ball. When $\Omega = \mathbb{B}(0, R)$ is a ball, the existence of a radially symmetric optimal set has been proved in [1], using rearrangement techniques and a comparison result for Hamilton-Jacobi equations and later, only using rearrangement techniques in [2]. Even, in this case an explicit solution to the problem is not known. It was conjectured in [2, 3], for higher dimensions, that the solution B^* to this problem is a ball $\mathbb{B}(0, R^*)$ like in the one-dimensional case [12], a result known since the 1950's. This conjecture has been recently reinforced by numerical tests in [3] and by the result in [5], where it is shown, using second order shape derivative calculus, that such a configuration is a local minimum for the problem when the volume constraint m is small enough.

In spite of these results, we prove in this paper that the conjecture is not true in general. Indeed, the optimal domain B^* cannot be a ball when α and β are close to each other and m is sufficiently large (cf. Theorem 3). The theoretical base for this result is provided by an asymptotic expansion of the eigenvalue with respect to $\beta - \alpha$ as $\beta \to \alpha$, which allows us to approximate (4)-(6) by a simpler optimization problem (cf. Theorem 2). This is done in Section 2. Through this asymptotic formulation we are not only able to show that the previous conjecture is false but also we are able to compute numerical approximations of the solution in design domains other than balls. The numerical results are presented in Section 4. Another main feature of the paper is the proposal, in Section 3, of a descent algorithm to solve the problem in general. This also permits to establish some necessary optimality conditions and allows us to deduce certain features of the optimal solution.

2. OPTIMAL SETS FOR SMALL CONDUCTIVITY GAP

2.1. **Asymptotic expansion.** In this section we shall look at the problem of minimization of the first eigenvalue in the special case where the conductivities of the two materials, α and β, are close to each other (i.e. are in low contrast regime). Thus, we assume that $\beta = \beta^\varepsilon := \alpha + \varepsilon$ with $\varepsilon > 0$ converging eventually to zero. If the material with conductivity β^ε occupies the sub-domain B of Ω, the conductivity coefficient is, in this case,

$$\sigma = \sigma^\varepsilon(B) := \alpha\chi_A + \beta^\varepsilon \chi_B = \alpha + \varepsilon\chi_B. \tag{7}$$

Let $\lambda^\varepsilon(B)$ be the first eigenvalue in the problem

$$-\operatorname{div}(\sigma^\varepsilon(B)\nabla u^\varepsilon) = \lambda^\varepsilon(B)\, u^\varepsilon \quad \text{in } \Omega, \tag{8}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{9}$$

for the conductivity $\sigma^\varepsilon(B)$. It is well-known, from the Kreĭn-Rutman theorem [13], that the first eigenvalue of a linear elliptic operator is simple and the corresponding eigenfunction is of constant sign (and is the only eigenvalue whose eigenfunction does not change sign). So, we can choose the eigenfunction $u^\varepsilon = u^\varepsilon(B)$ corresponding to $\lambda^\varepsilon(B)$ to be positive and normalize it using the condition

$$\int_\Omega (u^\varepsilon)^2 = 1, \tag{10}$$

In this way, u^ε is uniquely defined. We affirm that, for fixed B, both $\lambda^\varepsilon(B)$ and $u^\varepsilon(B)$ depend analytically on the parameter ε. This result is classical in the perturbation theory of eigenvalues and follows readily, for instance, from Theorem 3, Chapter 2.5 of Rellich [16]. This justifies the ansätze

$$\lambda^\varepsilon(B) = \lambda_0(B) + \varepsilon\lambda_1(B) + \dots \tag{11}$$

$$u^\varepsilon(B) = v_0(B) + \varepsilon v_1(B) + \dots \tag{12}$$

The convergence of the series in (12) holds in the space $H_0^1(\Omega)$. We first make some useful observations about the the terms in ansätze (11)-(12).

Proposition 1. *In ansätze* (11) *and* (12)*, the terms* $\lambda_0(B)$ *and* $v_0(B)$ *are independent of* B*. In fact,* $\lambda_0(B) = \lambda_0$ *is the first eigenvalue in the problem*

$$-\alpha\Delta v_0 = \lambda_0 v_0 \ \ \textit{in}\ \Omega, \tag{13}$$

$$v_0 = 0 \ \ \textit{on}\ \partial\Omega\,. \tag{14}$$

The function v_0 *is the positive eigenfunction corresponding to* λ_0 *and satisfies the normalization condition* $\int_\Omega v_0^2 = 1$*.*

Proof. In view of the analytic dependence of $\lambda^\varepsilon(B)$ and $u^\varepsilon(B)$ on ε, it follows that $\lambda_0(B)$ is the limit, as $\varepsilon \to 0$, of $\lambda^\varepsilon(B)$ and that $v_0(B)$ is the limit of $u^\varepsilon(B)$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$. Recalling that the eigenfunctions $u^\varepsilon(B)$ are positive it follows that $v_0(B)$ is non-negative. Passing to the limit in (8)-(10), as $\varepsilon \to 0$, we obtain that $\lambda_0(B)$ and $v_0(B)$ solve the eigenvalue problem (13)-(14) and $\int_\Omega (v_0(B))^2 = 1$. As $v_0(B)$ is a positive eigenfunction, it follows by the Kreĭn-Rutman theorem that λ_0 is necessarily the first eigenvalue of the eigenvalue problem (13)-(14). Thus, $\lambda_0(B)$ and $v_0(B)$ are independent of B and shall be denoted by λ_0 and v_0, respectively. ☐

Proposition 2. *In the ansatz* (11)*,* $\lambda_1(B)$ *is given explicitly in terms of* v_0 *as follows*

$$\lambda_1(B) = \int_B |\nabla v_0|^2 \, dx\,. \tag{15}$$

The following orthogonality relations hold true

$$\int_\Omega v_0\, v_1(B)\, dx = 0 = \int_\Omega \nabla v_0 \cdot \nabla v_1(B)\, dx\,. \tag{16}$$

Proof. The term $\lambda_1(B)$ in the ansatz (11) is the derivative of $\lambda_\varepsilon(B)$ with respect to ε at $\varepsilon = 0$, whereas the term $v_1(B)$ in the ansatz (12) is the derivative of $u^\varepsilon(B)$ with respect to ε at $\varepsilon = 0$. Differentiating the equations (8)-(10) with respect to ε at $\varepsilon = 0$, we obtain the equations

$$-\operatorname{div}(\alpha\nabla v_1(B)) - \lambda_0 v_1(B) = \operatorname{div}(\chi_B \nabla v_0) + \lambda_1(B)\, v_0 \ \ \text{in}\ \Omega, \tag{17}$$

$$v_1(B) = 0 \ \ \text{on}\ \partial\Omega \tag{18}$$

and the first of the orthogonality relations in (16). We have seen in Proposition 1 that λ_0 is the first eigenvalue of the problem (13)-(14) and is simple, the eigenspace being generated by the eigenfunction v_0. Taking $v_1(B)$ as a test function in (13)-(14) and using the first orthogonality relation in (16), we obtain the second orthogonality relation in (16). Finally, the system (17)-(18) admits a solution, by the Fredholm alternative, if and only if the right hand side is orthogonal to the eigenfunction v_0. This condition leads to the relation

$$\int_\Omega \operatorname{div}(\chi_B \nabla v_0) v_0\, dx + \lambda_1(B) \int_\Omega v_0^2\, dx = 0.$$

As $\int_\Omega v_0^2 = 1$, we obtain

$$\begin{aligned}\lambda_1(B) &= -\int_\Omega \operatorname{div}(\chi_B \nabla v_0) v_0\, dx \\ &= -\int_{\partial\Omega} \chi_B v_0 \nabla v_0 \cdot n\, dS + \int_B |\nabla v_0|^2\, dx = \int_B |\nabla v_0|^2\, dx\,.\end{aligned}$$

☐

Let us denote by

$$\tilde\lambda^\varepsilon(B) = \lambda^\varepsilon(B) - \lambda_0 - \varepsilon\lambda_1(B)$$

the remainder in the ansatz (11). Although $\tilde\lambda^\varepsilon(B)$ is of order ε^2 for fixed B, we need estimates for $\tilde\lambda^\varepsilon(B)$ which are uniform with respect to B. This is given by the following theorem.

Theorem 1. *For $\varepsilon > 0$ sufficiently small, there exists a constant C independent of ε and B such that*

$$|\tilde{\lambda}^\varepsilon(B)| \leq C\,\varepsilon^{\frac{3}{2}} \qquad \forall B \in \mathcal{B}\,. \tag{19}$$

Proof. For the sake of clarity we divide the proof into several steps. Let $\varepsilon > 0$ be a constant which is small compared to 1. In what follows, we use C to denote a generic constant independent of ε and B.

STEP 1: We first show that

$$\lambda^\varepsilon(B) \leq C\,, \qquad \|u^\varepsilon(B)\|_{H^1_0(\Omega)} \leq C\,. \tag{20}$$

The first inequality in (20) follows readily from the variational characterization of $\lambda^\varepsilon(B)$. Indeed, choosing a function $\varphi \in H^1_0(\Omega)$ with $\int_\Omega \varphi^2 = 1$, we have

$$\begin{aligned} \lambda^\varepsilon(B) &= \inf\left\{\int_\Omega \sigma^\varepsilon(B)\nabla u\cdot\nabla u\,dx : u \in H^1_0(\Omega), \int_\Omega u^2 = 1\right\} \\ &\leq \int_\Omega \sigma^\varepsilon(B)\nabla\varphi\cdot\nabla\varphi\,dx \\ &\leq (\alpha+1)\int_\Omega \nabla\varphi\cdot\nabla\varphi\,dx \end{aligned} \tag{21}$$

which proves the first estimate in (20). Now, using the uniform bound for $\lambda^\varepsilon(B)$ and using the fact that the coefficients $\sigma^\varepsilon(B)$ are uniformly elliptic, we have

$$\alpha\|u^\varepsilon(B)\|^2_{H^1_0(\Omega)} \leq \int_\Omega \sigma^\varepsilon(B)\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx = \lambda^\varepsilon(B) \leq C \tag{22}$$

which proves the second estimate in (20).

STEP 2: Next, we show that

$$|\lambda^\varepsilon(B) - \lambda_0| \leq C\,\varepsilon\,. \tag{23}$$

As $\sigma^\varepsilon(B) \geq \alpha$ for all $\varepsilon > 0$, it follows from the variational characterization (21) of $\lambda^\varepsilon(B)$ that

$$\lambda_0 \leq \lambda^\varepsilon(B) \qquad \text{for all } \varepsilon > 0 \text{ and for all measurable } B \subset \Omega\,. \tag{24}$$

On the other hand, using the variational characterization (21) and the fact that

$$\lambda_0 = \int_\Omega \alpha\nabla v_0\cdot\nabla v_0\,dx,$$

we get the following estimate

$$\begin{aligned} \lambda^\varepsilon(B) - \lambda_0 &\leq \int_\Omega \sigma^\varepsilon(B)\nabla v_0\cdot\nabla v_0\,dx - \int_\Omega \alpha\nabla v_0\cdot\nabla v_0\,dx \\ &= \varepsilon\int_\Omega \chi_B\nabla v_0\cdot\nabla v_0\,dx\,. \end{aligned} \tag{25}$$

The claim (23) follows from (24) and (25).

STEP 3: Now, we use the above estimate to show that

$$\|u^\varepsilon(B) - v_0\|_{H^1_0(\Omega)} \leq C\,\sqrt{\varepsilon}\,. \tag{26}$$

To begin with, we have

$$\begin{aligned}\lambda^\varepsilon(B)-\lambda_0 &= \int_\Omega \sigma^\varepsilon(B)\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx - \int_\Omega \alpha\nabla v_0\cdot\nabla v_0\,dx\\ &= \varepsilon\int_\Omega \chi_B\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx\\ &\quad +\alpha\left(\int_\Omega \nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx - \int_\Omega\nabla v_0\cdot\nabla v_0\,dx\right)\\ &= \varepsilon\int_\Omega \chi_B\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx\ + 2\alpha\int_\Omega \nabla v_0\cdot\nabla(u^\varepsilon(B)-v_0)\,dx\\ &\quad +\alpha\int_\Omega \nabla(u^\varepsilon(B)-v_0)\cdot\nabla(u^\varepsilon(B)-v_0)\,dx\,.\end{aligned}$$

Rewriting the previous equality we get

$$\begin{aligned}&\alpha\int_\Omega |\nabla(u^\varepsilon(B)-v_0)|^2\,dx + 2\alpha\int_\Omega\nabla v_0\cdot\nabla(u^\varepsilon(B)-v_0)\,dx\\ &\quad = \lambda^\varepsilon(B)-\lambda_0-\varepsilon\int_\Omega\chi_B\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx\,. \end{aligned}\tag{27}$$

Finally we obtain

$$\begin{aligned}&\left|\alpha\int_\Omega |\nabla(u^\varepsilon(B)-v_0)|^2\,dx + 2\alpha\int_\Omega\nabla v_0\cdot\nabla(u^\varepsilon(B)-v_0)\,dx\right|\\ &\le |\lambda^\varepsilon(B)-\lambda_0| + \varepsilon\left|\int_\Omega\chi_B\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx\right|\\ &\le C\varepsilon\end{aligned}\tag{28}$$

where the last inequality is a consequence of (23) and (20). Dividing by ε and passing to the limit in (28), we have

$$\left|\lim_{\varepsilon\to 0}\alpha\frac{\int_\Omega|\nabla(u^\varepsilon(B)-v_0)|^2\,dx}{\varepsilon} + 2\alpha\int_\Omega\nabla v_0\cdot\nabla v_1(B)\,dx\right| \le C\,. \tag{29}$$

In view of the second orthogonality relation in (16) we conclude that (26) holds.

STEP 4: Finally, we show the estimate for the remainder $\tilde\lambda^\varepsilon(B)$

$$|\tilde\lambda^\varepsilon(B)| = |\lambda^\varepsilon(B)-\lambda_0-\varepsilon\lambda_1(B)| \le C\,\varepsilon^{\frac{3}{2}}\,. \tag{30}$$

On the one hand, we observe that

$$\begin{aligned}\lambda_0+\varepsilon\lambda_1(B)-\lambda^\varepsilon(B) &\ge \int_\Omega\alpha\nabla v_0\cdot\nabla v_0\,dx + \varepsilon\int_\Omega\chi_B\nabla v_0\cdot\nabla v_0\,dx\\ &\quad -\int_\Omega\sigma^\varepsilon(B)\nabla v_0\cdot\nabla v_0\,dx\\ &= 0\,.\end{aligned}\tag{31}$$

On the other hand, we have

$$\begin{aligned}\lambda_0+\varepsilon\lambda_1(B)-\lambda^\varepsilon(B) &\leq \int_\Omega \alpha\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx+\varepsilon\int_\Omega \chi_B\nabla v_0\cdot\nabla v_0\,dx\\ &\qquad -\int_\Omega \sigma^\varepsilon\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx \qquad (32)\\ &\leq \varepsilon\int_\Omega \chi_B\nabla v_0\cdot\nabla v_0\,dx-\varepsilon\int_\Omega \chi_B\nabla u^\varepsilon(B)\cdot\nabla u^\varepsilon(B)\,dx\\ &= \varepsilon\int_\Omega \chi_B\nabla(v_0-u^\varepsilon(B))\cdot\nabla(v_0-u^\varepsilon(B))\,dx\\ &\qquad +2\varepsilon\int_\Omega \chi_B\nabla u^\varepsilon(B)\cdot\nabla(v_0-u^\varepsilon(B))\,dx\,.v \qquad (33)\end{aligned}$$

So, by the result of Step 3, it follows from (31) and (33) that

$$\begin{aligned}|\lambda_0+\varepsilon\lambda_1(B)-\lambda^\varepsilon(B)| &\leq \varepsilon\|v_0-u^\varepsilon(B)\|^2_{H^1_0(\Omega)}+2C\varepsilon\|v_0-u^\varepsilon(B)\|_{H^1_0(\Omega)}\\ &\leq C\varepsilon^2+C\varepsilon^{\frac{3}{2}}\\ &\leq C\varepsilon^{\frac{3}{2}}\,.\end{aligned}$$

This completes the proof of the estimate (19). ☐

Corollary 1. *Let* $\lambda^\varepsilon(B)$ *be the first eigenvalue in* (8)-(9)*,* λ_0 *the first eigenvalue of problem* (13)-(14) *and let* $\lambda_1(B)$ *be as in Proposition 2. Then, we have*

$$\left|\inf_{B\in\mathcal{B}}\lambda^\varepsilon(B)-\lambda_0-\varepsilon\inf_{B\in\mathcal{B}}\lambda_1(B)\right|\leq C\varepsilon^{\frac{3}{2}}\,, \tag{34}$$

where C *is a constant independent of* ε *and* $B\in\mathcal{B}$

Proof. The proof is immediate from the estimate (19). ☐

Corollary 2. *Let* C *be the constant independent of* ε *and* $B\in\mathcal{B}$ *appearing in* (19) *of Theorem 1. If* $B^\star_\varepsilon\in\mathcal{B}$ *is a minimizer of* $\lambda^\varepsilon(\cdot)$ *then we have the following estimate:*

$$\left|\lambda_1(B^\star_\varepsilon)-\inf_{B\in\mathcal{B}}\lambda_1(B)\right|\leq 2\,C\varepsilon^{\frac{1}{2}}. \tag{35}$$

Proof. Let $B^\star_\varepsilon\in\mathcal{B}$ be a minimizer for $\lambda^\varepsilon(B)$. Then, we have

$$\begin{aligned}&\varepsilon\left|\lambda_1(B^\star_\varepsilon)-\inf_{B\in\mathcal{B}}\lambda_1(B)\right|\\ &\quad=\left|(\lambda^\varepsilon(B^\star_\varepsilon)-\lambda_0-\varepsilon\lambda_1(B^\star_\varepsilon))-(\lambda^\varepsilon(B^\star_\varepsilon)-\lambda_0-\varepsilon\inf_{B\in\mathcal{B}}\lambda_1(B))\right|\\ &\quad\leq|\lambda^\varepsilon(B^\star_\varepsilon)-\lambda_0-\varepsilon\lambda_1(B^\star_\varepsilon)|+\left|\inf_{B\in\mathcal{B}}\lambda^\varepsilon(B)-\lambda_0-\varepsilon\inf_{B\in\mathcal{B}}\lambda_1(B)\right|\\ &\quad\leq 2\,C\varepsilon^{\frac{3}{2}} \qquad (36)\end{aligned}$$

where the inequalities follow from (30) and (34). Thus, we have obtained (35). ☐

Remark 1. *The Corollary 1 tells us that, in low contrast regimes, asymptotically the minimum value of* $\lambda^\varepsilon(\cdot)$ *can be calculated approximately by calculating the minimum of* $\lambda_1(\cdot)$ *which is easily achieved using Theorem 2. In addition, the previous corollary gives us to understand that a minimizer for* $\lambda^\varepsilon(\cdot)$ *is approximately a minimizer for* $\lambda_1(\cdot)$ *when* ε *is small. It can be seen, by similar arguments, that a minimizer for* $\lambda_1(\cdot)$ *is approximately a minimizer for* $\lambda^\varepsilon(\cdot)$ *when* ε *is small. This remark will be used later in Section 4 to determine numerical approximations of the set* B *minimizing* $\lambda(B)$ *in low contrast regime.*

We prove the following theorem which provides a characterization of the minimizer of $\lambda_1(\cdot)$, in terms of the level sets of the gradient of v_0.

Theorem 2. *There exists $c^* \geq 0$ such that whenever B is a measurable subset of Ω satisfying*

$$\{x : |\nabla v_0(x)| < c^*\} \subset B \subset \{x : |\nabla v_0(x)| \leq c^*\}$$

and $|B| = m$, then B is an optimal solution for the problem of minimizing $\lambda_1(B)$ over $B \in \mathcal{B}$.

Proof. Let $f(c) := |\{x \in \Omega : |\nabla v_0(x)| \leq c\}|$, f is clearly an increasing function with $0 \leq f(c) \leq |\Omega|$. Let $c^* := \inf\{c : f(c) \geq m\}$. We have $f(c^*) \geq m$, indeed, let $c_k > c^*$ be a decreasing sequence such that $c_k \to c^*$. On one hand $f(c_k) \geq m$ and $\lim_{k\to\infty} f(c_k) \geq m$. On the other hand

$$\begin{aligned}\lim_{k\to\infty} f(c_k) &= \lim_{k\to\infty} |\{x \in \Omega : |\nabla v_0(x)| \leq c_k\}| \\ &= |\cap_{k\in\mathbb{N}} \{x \in \Omega : |\nabla v_0(x)| \leq c_k\}| \\ &= |\{x \in \Omega : |\nabla v_0(x)| \leq c^*\}| = f(c^*),\end{aligned}$$

so that $f(c^*) \geq m$. In a similar way we have $|\{x : |\nabla v_0(x)| < c^*\}| \leq m$, indeed let $c_k < c^*$ be an increasing sequence with $c_k \to c^*$. On one hand $f(c_k) < m$ and $\lim_{k\to\infty} f(c_k) \leq m$. On the other hand

$$\begin{aligned}\lim_{k\to\infty} f(c_k) &= \lim_{k\to\infty} |\{x \in \Omega : |\nabla v_0(x)| \leq c_k\}| \\ &= |\cup_{k\in\mathbb{N}} \{x \in \Omega : |\nabla v_0(x)| \leq c_k\}| \\ &= |\{x \in \Omega : |\nabla v_0(x)| < c^*\}|,\end{aligned}$$

so that $|\{x \in \Omega : |\nabla v_0(x)| < c^*\}| \leq m$.
If B is a measurable set such that $\{x : |\nabla v_0(x)| < c^*\} \subset B \subset \{x : |\nabla v_0(x)| \leq c^*\}$ and $|B| = m$ then B is an optimal solution for the problem of minimization of $\lambda_1(\cdot)$ over $\mathcal{B}$. Indeed if D is any measurable subset of Ω such that $|D| = m$, we shall have $|B \cap D^c| = |B^c \cap D|$. Therefore,

$$\begin{aligned}\int_D |\nabla v_0|^2\, dx &= \int_{D\cap B} |\nabla v_0|^2\, dx + \int_{D\cap B^c} |\nabla v_0|^2\, dx \\ &\geq \int_{D\cap B} |\nabla v_0|^2\, dx + \int_{D^c\cap B} |\nabla v_0|^2\, dx = \int_B |\nabla v_0|^2\, dx\end{aligned}$$

as $|\nabla v_0| \geq c^*$ on $D \cap B^c$ whereas $|\nabla v_0| \leq c^*$ on $D^c \cap B$. ☐

Remark 2. *If $\{x : |\nabla v_0(x)| = c^*\}$ is of measure zero, then the* unique *solution (up to a set of measure zero) is the set $\{x : |\nabla v_0(x)| < c^*\}$ which shall also be open if Ω is a sufficiently smooth domain (as by Theorem 8.14* [6] *it can be concluded that ∇v_0 is of class C^1). In view of Proposition 3 proved below, this will be the case when Ω is a disk.*

2.2. **Disproving the disk conjecture.** In this section we show that, for ε sufficiently small, the distribution of materials which corresponds to placing material β in a ball around the centre of Ω when Ω is a ball is not optimal for the problem (4)-(6).

In particular, let $\Omega = \mathbb{B}(0,1) \subset \mathbb{R}^d$ be the ball of center 0 and radius 1. Then, the solution v_0 of (13)-(14) is radial and smooth. By setting $w_0(|x|) := v_0(x)$, equation (13)-(14) becomes, using the Laplacian in polar (r,θ) or spherical (r,θ,φ) coordinates, for $d = 2,3$

$$r^2 w_0''(r) + (d-1) r w_0'(r) + r^2 \frac{\lambda_0}{\alpha} w_0(r) = 0, \tag{37}$$

$$w_0'(0) = 0,\ w_0(1) = 0. \tag{38}$$

where the boundary conditions (38) correspond to the continuity of the gradient at the origin and the Dirichlet condition on the boundary, respectively. The solution of this equation

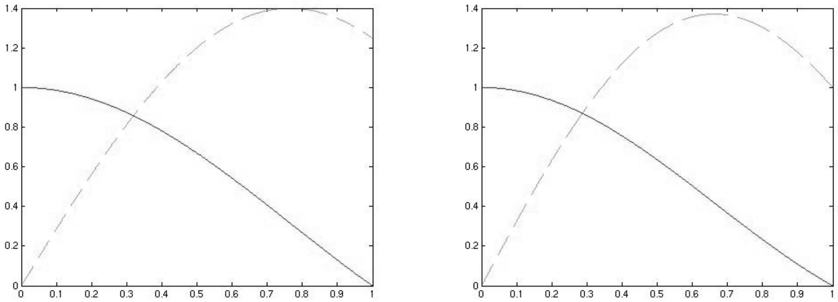


FIGURE 1. Functions $w_0(r)$ (plain), and $w_1(r) = -w_0'(r)$ (dashed) in dimensions $d = 2$ (left) and $d = 3$ (right). r_d^1 is such that w_1 is increasing on $[0, r_d^1]$ and decreasing on $[r_d^1, 1]$, and r_d^0 is such that $w_1(r_d^0) = w_1(1)$.

is

$$w_0(r) = J_0(\eta_d r) \qquad \text{if } d = 2, \tag{39}$$

$$w_0(r) = j_0(\eta_d r) \qquad \text{if } d = 3, \tag{40}$$

where J_0 and j_0 denote Bessel functions of the first and second kind, respectively and η_d (d=2,3) are their respective zeros. We introduce the notation $w_1(r) := -w_0'(r)$. The function w_0 is decreasing and so w_1 is positive. The behaviour of the functions w_0 and w_1 is depicted in Figure 1.

Proposition 3. *For $\Omega = \mathbb{B}(0,1)$, the level set $\{|\nabla v_0| = c\}$ has zero measure for each $c \geq 0$.*

Proof. Since $v_0(x) = w_0(|x|)$, we have $\nabla v_0(x) = w_0'(|x|)\dfrac{x}{|x|}$ and consequently we have

$$|\nabla v_0|(x) = -w_0'(|x|)\,.$$

Thus, the measure of the set $\{|\nabla v_0|(x) = c\}$ is positive if and only if the measure of one of the sets $\{x : w_0'(|x|) = c\}$ or $\{x : w_0'(|x|) = -c\}$ is positive. Thus, if $\{x : w_0'(|x|) = -c\}$ has positive measure, this is the same as saying $S := \{x : \nabla v_0(x) = -c\frac{x}{|x|}\}$ has positive measure. That is, the set $\{x : \nabla\left(v_0(x) + c|x|\right) = 0\}$ has positive measure. By a classical result due to Morrey [15], if $\psi \in W^{1,1}_{\text{loc}}$, then $\nabla\psi = 0$ almost everywhere on the set $\{\psi = 0\}$. Applying this result to the function $\nabla\left(v_0 + c|x|\right)$, we get

$$\nabla^2\left(v_0 + c|x|\right) = 0$$

almost everywhere on the set of positive measure S defined above. So, now taking the trace we obtain

$$-\Delta v_0(x) = \frac{c(d-1)}{|x|}$$

on a set of positive measure. Then, by equation (13), this means that we have

$$w_0(r) = \frac{\alpha c(d-1)}{\lambda_0 r}$$

on a set of positive measure which is clearly in violation of the behaviour of the Bessel function J_0 or j_0. This concludes our proof. □

Let ω_d denote the volume of the unit ball, i.e. we have $\omega_d = \pi$ for $d = 2$ and $\omega_d = 4\pi/3$ for $d = 3$ and let r_d^0 and r_d^1 be the constants mentioned in the caption of Figure 1.

Proposition 4. *When Ω is a ball, in two or three dimensional space, the unique symmetrical optimal domain B^* which is solution of the minimization problem for $\lambda_1(B)$ over $B \in \mathcal{B}$ is of two possible types*

- *Type I: If $m \leq \omega_d (r_d^0)^d$ then $B^* = \mathbb{B}(0, (m/\omega_d)^{1/d})$ or,*
- *Type II: If $m > \omega_d (r_d^0)^d$ then there exists ξ^0 and ξ^1 with*

$$(m/\omega_d)^{1/d} < \xi^0 < \xi^1 < 1$$

and $B^* = \mathbb{B}(0, \xi^0) \cup \Big(\mathbb{B}(0,1) \setminus \overline{\mathbb{B}(0,\xi^1)}\Big).$

Proof. In view of Theorem 2, Proposition 3 and Remark 2, a solution to the problem of minimization of $\lambda_1(\cdot)$ over $\mathcal{B}$ is the set $B^* = \{x \in \Omega : |\nabla v_0(x)| \leq c^*\}$ where c^* is as in Theorem 2. Moreover, it is the unique solution (up to a set of measure zero). We may also write it as $B^* = \{x \in \Omega : w_1(|x|) \leq c^*\}$. Whenever $m \leq \omega_d \ (r_d^0)^d$, the equation $w_1(r) = c^*$ has exactly one solution ξ in $(0,1)$. In this case the solution to our minimization problem is $B^* = \mathbb{B}(0, \xi)$ consisting of a single component in the center. The constraint $|B^*| = m$ implies $\xi = (m/\omega_d)^{1/d}$.

In the case $m > \omega_d \ (r_d^0)^d$, the equation $w_1(r) = c^*$ has two solutions ξ^0 and ξ^1 in $(0,1)$. Then the solution set is

$$B^* = \mathbb{B}(0, \xi^0) \cup \Big(\mathbb{B}(0,1) \setminus \overline{\mathbb{B}(0,\xi^1)}\Big)$$

which completes our proof. ☐

In [3] it was conjectured, based on numerical tests, that the infimum of the first eigenvalue of (4)-(5) is attained when the material with the highest conductivity is placed in a concentric disk in the center of the domain. We prove that this conjecture is *false*, at least in two- or three- dimensional space, for β close to α and when the proportion of the material β exceeds a certain quantity.

Theorem 3. *When $\Omega = B(0,1)$, for $\beta = \alpha + \varepsilon$ sufficiently close to α and given an $m > \omega_d (r_d^0)^d$, the distribution of the materials wherein the material with the higher conductivity β is placed in a concentric disk in the center of the domain is not optimal for the problem* (4)-(5).

Proof. Our proof is based on Corollary 2 and Proposition 4. We shall assume that $m > \omega_d (r_d^0)^d$. Let $B^\star$ be the minimizer of $\lambda_1(\cdot)$ over $B \in \mathcal{B}$, which according to Proposition 4, is the union of a ball $\mathbb{B}(0, \xi_0)$ and a ring or shell $\mathbb{B}(0,1) \setminus \overline{\mathbb{B}(0,\xi_1)}$. Let B_m be the concentric ball in $\mathbb{B}(0,1)$ having the volume m. We show that B_m cannot be the minimizer of $\lambda^\varepsilon(\cdot)$ in the problem (4)-(5) whenever $\beta = \alpha + \varepsilon$ is less than a certain quantity, to be precise, if

$$2C\varepsilon^{\frac{1}{2}} < |\lambda_1(B_m) - \lambda_1(B^*)| \ .$$

For such an $\varepsilon > 0$, if we assume that B_m is a minimizer for $\lambda^\varepsilon(\cdot)$ in the problem (4)-(5), then, by (35) of Corollary 2, we obtain

$$|\lambda_1(B_m) - \lambda_1(B^*)| \leq 2C\varepsilon^{\frac{1}{2}} \ .$$

This is clearly a contradiction. So, for $\varepsilon > 0$ as above, B_m cannot be a minimizer of $\lambda^\varepsilon(\cdot)$ in the problem (4)-(5). ☐

3. DESCENT ALGORITHM AND NECESSARY OPTIMALITY CONDITION

In this section we consider domains Ω with any shape, and not only the case of a disk as in the previous section. We describe an algorithm which, starting from a given initial measurable set B_0, allows to find a new measurable set B_1 such that the first eigenvalue λ_1 is decreased, i.e.

$$\lambda(B_1) \leq \lambda(B_0).$$

Algorithms based on a similar principle have been used successfully to minimize eigenvalues in problems with indefinite weight [9, 10]. It may be used, as we shall see in Subsection 4.2, to obtain numerical results. This algorithm allows to derive necessary optimality conditions; see Corollary 3, which in turn may be used to derive properties of the optimal set B for particular geometries such as symmetric domains or polygonal domains. More details may be found at the end of this section.

Given an initial measurable set B_0, let u_{B_0} and $\lambda(B_0)$ denote the first eigenvector and eigenvalue, respectively, for problem (1)-(2). Introduce the quantity

$$\mathscr{M}(B_0, c) = |\{x : |\nabla u_{B_0}(x)| \leq c\}| \tag{41}$$

Lemma 1. *The function $\mathscr{M}(B_0, c)$ is non-decreasing with respect to c and is such that $\mathscr{M}(B_0, c) \to 0$ as $c \to 0$ and $\mathscr{M}(B_0, c) \to |\Omega|$ as $c \to \infty$. Furthermore, it is a right-continuous function. It is also left continuous at any c if and only if the Lebesgue measure of $\{x : |\nabla u_{B_0}(x)| = c\}$ is zero.*

Proof. The function $\mathscr{M}(B_0, c)$ is monotone non-decreasing due to the set inclusion $\{x : |\nabla u_{B_0}(x)| \leq c\} \subset \{x : |\nabla u_{B_0}(x)| \leq c'\}$ whenever $c \leq c'$. Due to the integrability of $|\nabla u_{B_0}|$, we have $|\{x : |\nabla u_{B_0}(x)| > c\}| \to 0$ as $c \to \infty$ and therefore, we can conclude that $\mathscr{M}(B_0, c) \to |\Omega|$ as $c \to \infty$. Also, $\mathscr{M}(B_0, c) \to 0$ as $c \to 0$ as it is known that the set of critical points for u_{B_0} is of measure zero [14]. The right continuity may be seen from the following

$$\begin{aligned}
\mathscr{M}(B_0, c+) - \mathscr{M}(B_0, c) &= \lim_{c_n \searrow c} \mathscr{M}(B_0, c_n) - \mathscr{M}(B_0, c) \\
&= \lim_{c_n \searrow c} |\{x : c < |\nabla u_{B_0}(x)| \leq c_n\}| \\
&= |\cap_n \{x : c < |\nabla u_{B_0}(x)| \leq c_n\}| \\
&= 0
\end{aligned}$$

as the latter intersection is empty.

On the other hand,

$$\begin{aligned}
\mathscr{M}(B_0, c) - \mathscr{M}(B_0, c-) &= \lim_{c_n \nearrow c} \mathscr{M}(B_0, c) - \mathscr{M}(B_0, c_n) \\
&= \lim_{c_n \nearrow c} |\{x : c_n < |\nabla u_{B_0}(x)| \leq c\}| \\
&= |\cap_n \{x : c_n < |\nabla u_{B_0}(x)| \leq c\}| \\
&= |\{x : |\nabla u_{B_0}(x)| = c\}|,
\end{aligned}$$

which yields the assertion on left continuity. □

Now define

$$c_0 := \inf\{c : \mathscr{M}(B_0, c) \geq m\}. \tag{42}$$

As in the proof of Theorem 2, it may be shown that $|\{x : |\nabla u_{B_0}(x)| \leq c_0\}| \geq m$ and $|\{x : |\nabla u_{B_0}(x)| < c_0\}| \leq m$. Let B_1 be any measurable subset of Ω which satisfies $\{x : |\nabla u_{B_0}(x)| < c_0\} \subset B_1 \subset \{x : |\nabla u_{B_0}(x)| \leq c_0\}$ and $|B_1| = m$.

We have the following result

Theorem 4. *Given an initial measurable set $B_0 \subset \Omega$ with $|B_0| = m$, let c_0 be as defined in* (42) *and let B_1 be a measurable set as above having measure m. Then we have $\lambda(B_1) \leq \lambda(B_0)$. Furthermore, if $\mathscr{M}(B_0, c)$ is continuous, then equality holds if and only if $B_1 = B_0$ almost everywhere.*

Proof. Consider the decompositions

$$\begin{aligned}
B_0 &= (B_0 \cap B_1) \cup (B_0 \cap B_1^c), \\
B_1 &= (B_0 \cap B_1) \cup (B_0^c \cap B_1).
\end{aligned}$$

Since $|B_0| = |B_1| = m$ we have with the above decompositions $|B_0 \cap B_1^c| = |B_0^c \cap B_1|$. Noticing that $|\nabla u_{B_0}| \geq c_0$ on B_1^c and $|\nabla u_{B_0}| \leq c_0$ on B_1, we may write

$$
\begin{aligned}
\int_{B_0} |\nabla u_{B_0}|^2 &= \int_{B_0\cap B_1} |\nabla u_{B_0}|^2 + \int_{B_0\cap B_1^c} |\nabla u_{B_0}|^2 \\
&\geq \int_{B_0\cap B_1} |\nabla u_{B_0}|^2 + c_0^2|B_0 \cap B_1^c| \qquad (43)\\
&= \int_{B_0\cap B_1} |\nabla u_{B_0}|^2 + c_0^2|B_0^c \cap B_1| \\
&\geq \int_{B_0\cap B_1} |\nabla u_{B_0}|^2 + \int_{B_0^c\cap B_1} |\nabla u_{B_0}|^2 = \int_{B_1} |\nabla u_{B_0}|^2, \qquad (44)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lambda(B_0) &= \alpha \int_\Omega |\nabla u_{B_0}|^2 + (\beta-\alpha) \int_{B_0} |\nabla u_{B_0}|^2 \\
&\geq \alpha \int_\Omega |\nabla u_{B_0}|^2 + (\beta-\alpha) \int_{B_1} |\nabla u_{B_0}|^2 \qquad (45)\\
&\geq \min_{u\in H_0^1(\Omega), ||u||_2=1} \left(\alpha \int_\Omega |\nabla u|^2 + (\beta-\alpha) \int_{B_1} |\nabla u|^2 \right) = \lambda(B_1)\,, \qquad (46)
\end{aligned}
$$

and the inequality is proved.

Now, let us assume that $\mathscr{M}(B_0, c)$ is continuous. In view of Lemma 1, the continuity of $\mathscr{M}(B_0, c)$ implies that the set $\{x : |\nabla u_{B_0}(x)| = c_0\}$ has zero measure, so that B_1 can be taken to be $\{x : |\nabla u_{B_0}(x)| \leq c_0\}$. The equality $\lambda(B_1) = \lambda(B_0)$ holds if and only if equality holds in (45), (46). In particular, equality in (45) holds only if equality holds in (43) and (44). Thus, as $|\nabla u_{B_0}(x)| > c_0$ on B_1^c, for equality to hold in (43) we need that $|B_0 \cap B_1^c| = 0$. Consequently, we also have $|B_0^c \cap B_1| = 0$. Thus, we have shown that $B_1 = B_0$ except for a set of measure zero. □

Remark 3. *Using Theorem 4, we may obtain a sequence of domains B_n such that*

$$\lambda(B_n) \leq \lambda(B_{n-1}).$$

However, it is not a priori guaranteed that B_n converges to an admissible set B^* in some topology for which the map $B \mapsto \lambda(B)$ is lower semicontinuous. Even by supposing this, we cannot say that B^* is a global optimum for the problem.

The following corollary derives immediately from Theorem 4.

Corollary 3 (Necessary optimality condition)**.** *If a measurable set B^* is optimal for the problem (4) - (5) and if $\mathscr{M}(B^*, c)$ is continuous at c^* for c^* defined analogously as in* (42) *with B^* replacing B_0, then up to a set of measure zero, B^* is equal to the level set $\{x : |\nabla u_{B^*}(x)| \leq c^*\}$.*

Proof. It is enough to apply the previous theorem taking B^* in place of B_0, c^* instead of c_0 and then to take $B_1 = \{x : |\nabla u_{B^*}(x)| \leq c^*\}$, which is allowed due to the continuity of $\mathscr{M}(B^*, \cdot)$ at c^*. Then, $\lambda(B_1) \leq \lambda(B^*)$ and B^* is optimal, so we have the equality $\lambda(B_1) = \lambda(B^*)$. The conclusion follows, as by Theorem 4, equality holds only if B_1 is almost everywhere equal to B^*. □

Assuming that the hypotheses of Corollary 3 are satisfied we obtain the following results for certain geometries.

The disk case. In the case $\Omega = \mathbb{B}(0, R)$ the optimal set B^* should include the origin. Indeed, in [2], it is shown that the optimal domain is radially symmetric. The regularity and the radial symmetry of the solution imply that the gradient of u vanishes at the origin

0 and therefore, by Corollary 3, it follows that $0 \in B^*$.

The ring or torus case. If one is able to show the radial symmetry of the solution as in the disk case, then due to the Dirichlet condition and the positivity of the solution, it is clear that the gradient of u vanishes at one point along a radius of the domain and by radial symmetry, the gradient of u vanishes on a whole circle whose center is the center of the ring or torus. Using Corollary 3 we obtain that this circle is in B^*. This property may be observed for instance in Figure 5 for different ratios $m/|\Omega|$.

Domains with corners in two dimensions. In this case the optimal set B^* contains a neighbourhood of the corners with angle smaller than π while its complement $A^* = \Omega \setminus B^*$ contains a neighbourhood of the corners with angle greater than π. Indeed, let $P \in \partial\Omega$ be a conical (corner) point of Ω and denote ϑ the associated angle at this corner. The classical theory of solution of elliptic partial differential equations in non-smooth domains [7, 8, 11] establishes that in view of the Dirichlet boundary conditions, u_{B^*} may be written as

$$u_{B^*}(x) = cr^{\pi/\vartheta} \sin\left(\frac{\pi\theta}{\vartheta}\right) + u_s(x),$$

where c is the coefficient of the singularity which depends on the geometry of the domain, θ corresponds to polar coordinates with center P and $\theta = 0$ or $\theta = \vartheta$ on the tangents to $\partial\Omega$ at P, and $u_s \in H^2(\Omega)$ satisfies $u_s(x) = O(r^{\pi/\vartheta+1})$ as $x \to P$. Therefore

$$|\nabla u_{B^*}(x)| = O(r^{\pi/\vartheta-1}) \text{ as } x \to P$$

and

$$|\nabla u_{B^*}(x)| \to 0 \text{ if } \vartheta < \pi, \qquad |\nabla u_{B^*}(x)| \to \infty \text{ if } \vartheta > \pi \qquad \text{as } x \to P.$$

Therefore, in view of Corollary 3, when $\vartheta < \pi$, B^* contains a neighbourhood of P since $|\nabla u_{B^*}(x)| \to 0$ as $x \to P$ and we may find a small enough neighbourhood of P that will be included in $\{x : |\nabla u_{B^*}(x)| \leq c^*\}$. When $\vartheta > \pi$ then $A^* = \Omega \setminus B^*$ contains a neighbourhood of P since $|\nabla u_{B^*}(x)| \to \infty$ as $x \to P$. These properties may be observed in the Figures 2, 3 and 4.

Symmetrical domains. If the domain Ω has symmetries, then it is probably possible to show that the optimal domain has the same symmetries, as in [2]. Thus, if $\Omega \subset \mathbb{R}^d$ has d independent hyperplanes of symmetry, it is expected that the solution set B^* includes the point which is the intersection of these hyperplanes. If Ω is a square for instance, as in Figure 2, the center of this square belongs to B^*.

These properties are corroborated by numerical results shown in Section 4.1 for such domains in the case when α and β are close enough.

4. NUMERICAL RESULTS

The aims of this section are twofold. On the one hand we would like to obtain approximate numerical solutions of problem (4)-(5) by applying Remark 1 and Theorem 2 in the case of nearly equal conductivities. On the other hand, in the general case, we would like to explore the numerical utility of the algorithm described in Section 3.

4.1. **In low contrast regime.**

In this section we compute numerical approximations of solutions of the optimization problem (4)-(5) for general geometries Ω, under the assumption of low contrast regime, i.e. $\beta = \beta^\varepsilon = \alpha + \varepsilon$ for small ε. Following Remark 1, the solution B^* of the auxiliary problem corresponding to the minimization of $\lambda_1(\cdot)$ (see Theorem 2) is then an approximate solution for (4)-(6) for small ε. For the computation of B^*, we compute the function $f(c)$ defined in the proof of Theorem 2 and look for c^* such that $f(c^*) = m$. Numerically, this may be achieved by simple dichotomy for instance.

In Figures 2 to 6, the nearly optimal distribution B^* for various geometries Ω and values m are plotted. We also plot the isolines of $|\nabla v_0|^2$, where v_0 is the solution of (13)-(14), i.e. v_0 is the first eigenvector with constant conductivity α. In all these examples, the features mentioned at the end of Section 3 are seen to hold. For domains with salient angles, such as in Figures 2 to 4, the set B^* contains a neighbourhood of these corners. On the contrary, reentrant corners such as in Figure 4 are always in A^* since the gradient is unbounded at these points. We also observe that the set B^* always contains a point (or more in the case of radial symmetry as in Figure 5) inside the domain. This interior point corresponds to the place where the gradient of v_0 vanishes and thus is expected to be in B^* according to Corollary 3. For domains without corners, the optimal set may or may not touch the boundary as show Figures 5 and 6.

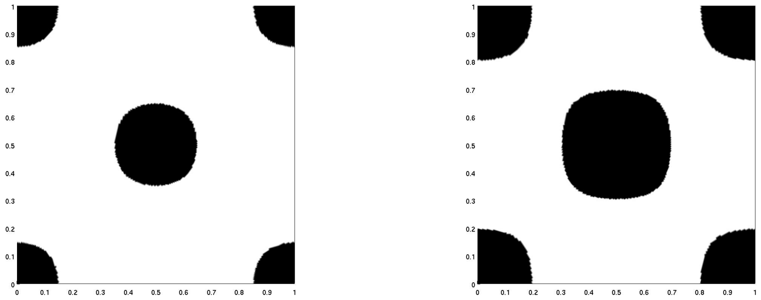
In general the optimal set B^* seem to have a complex structure and a certain regularity, except for some particular values of m as in Figure 2(c).

4.2. **The disk case for general β.** We shall apply the algorithm of Section 3 to get an idea of the optimal solution in the case of a disk for any value of β. As was commented in Remark 3, the convergence of the sequence B_n is not guaranteed and even if the domains B_n do converge to a domain B^*, this domain is not necessarily optimal for problem (4)-(5). So as to make the algorithm more effective, we include a second step, wherein we make local perturbations to the B^* in some descent directions with the aid of the shape derivative obtained in [3] (cf. the same for more details). By repeating these two steps, successively, it is plausible that these iterations lead to a global optimum.

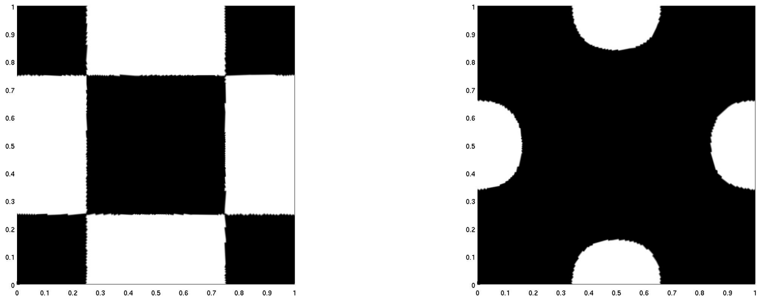
One of the main difficulty when solving the discrete version of (4)-(5) comes from the volume constraint $|B| = m$. Indeed from a numerical point of view it is not possible to satisfy the volume constraint exactly due to the discretization. Given an initial set B_0, the new set B_1 is numerically determined by dichotomy using the gradient of the solution u_{B_0} according to (42). The new set B_1 does not satisfy exactly the constraint $|B_1| = m$, but we may compute an intermediate point I lying between grid points, and such that the constraint $|B_1| = m$ is satisfied. The numerical scheme is modified accordingly using interpolation.

We present results in dimension $d = 2$. For the numerical tests we take $\alpha = 1$, $\Omega = \mathbb{B}(0, 1)$. The initial domain is $B_0 = \mathbb{B}(0, 0.75)$ (cf. Figure 7) and we test different values of $\beta > \alpha$.

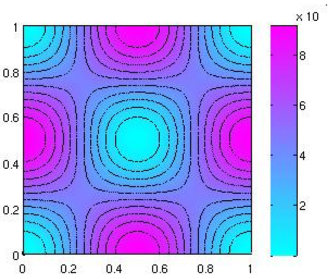
In Figure 8, the optimal sets B^* with conductivity β are depicted in black. These results show that the algorithm is able to perform topological changes, since the initial set B_0 has only one connected component, while the optimal domain B^*, $1 = \alpha < 1.01 < \beta \leq 1.15$, has two connected components (Type II). For $\beta = 1.16$ and higher initial values β, we observe an optimal set B^* of Type I mentioned in Proposition 4. This leads us to think that, for $\beta \geq 1.16$, the proportion of the material β may need to be higher in order to produce solutions of Type II.



(a) $m/|\Omega| = 0.1427$ (b) $m/|\Omega| = 0.2634$

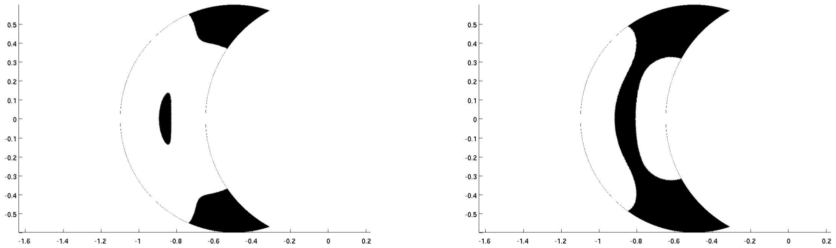


(c) $m/|\Omega| = 0.5$ (d) $m/|\Omega| = 0.8318$

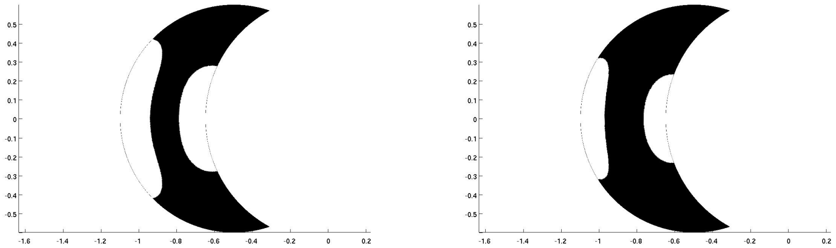


(e) isolines of $|\nabla v_0|^2$

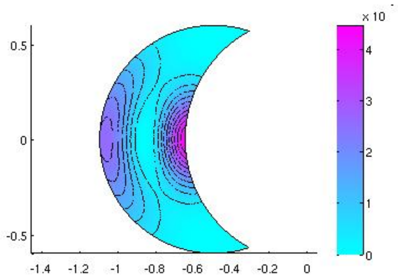
FIGURE 2. Nearly optimal distribution $B^\star$ in the square case. The dark region corresponds to B and the material β, the white region to A and material α. As predicted by Corollary 3, the optimal domain $B^\star$ contains the corners of the square, as well as the center since the gradient of the eigenfunction vanishes at that point.



(a) $m/|\Omega| = 0.2473$ (b) $m/|\Omega| = 0.4861$

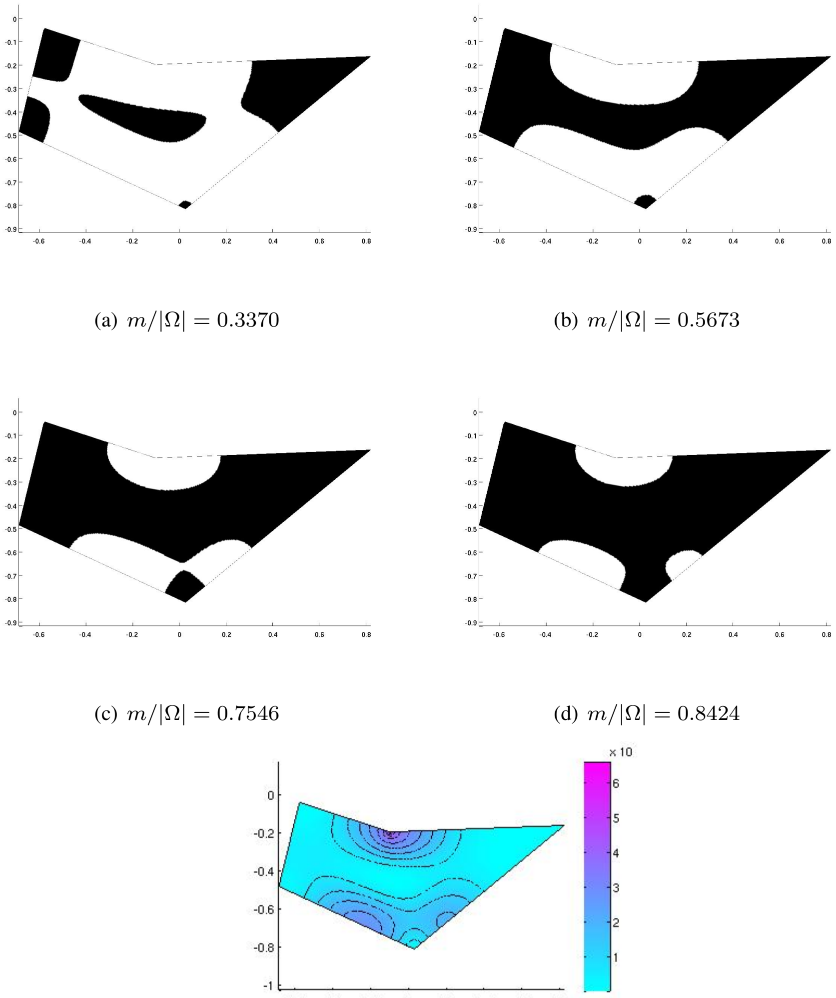


(c) $m/|\Omega| = 0.6377$ (d) $m/|\Omega| = 0.7742$



(e) isolines of $|\nabla v_0|^2$

FIGURE 3. Nearly optimal distributions $B^\star$ in the crescent case for different values of m. The dark region corresponds to B and the material β, the white region to A and material α. As predicted by Corollary 3, the approximately optimal domain $B^\star$ always contains the corners of the crescent and a point in the center where the eigenfunction vanishes.



(a) $m/|\Omega| = 0.3370$

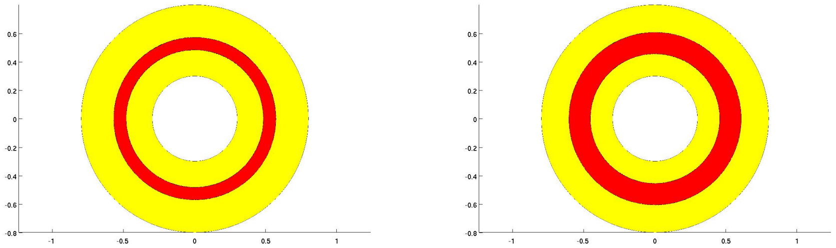
(b) $m/|\Omega| = 0.5673$

(c) $m/|\Omega| = 0.7546$

(d) $m/|\Omega| = 0.8424$

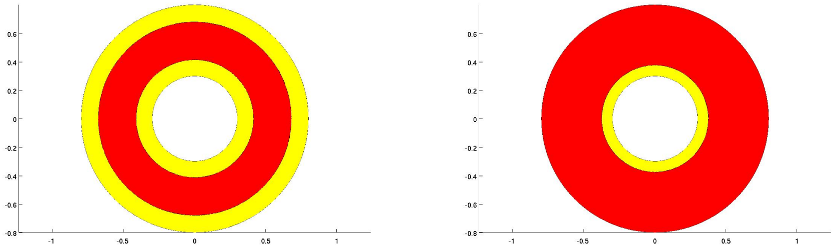
(e) isolines of $|\nabla v_0|^2$

FIGURE 4. Nearly optimal distributions $B^\star$ in the polygon case for different values of m. The dark region corresponds to B and the material β, the white region to A and material α. As predicted by Corollary 3, the approximately optimal domain $B^\star$ contains the corners of the polygon with angle less than π and a point in the center where the gradient of the eigenfunction vanishes. Note that $A^* = \Omega \setminus B^*$ contains a neighbourhood of the point with the reentrant corner, i.e. the corner with angle greater than π. Indeed, in this corner the gradient is unbounded.



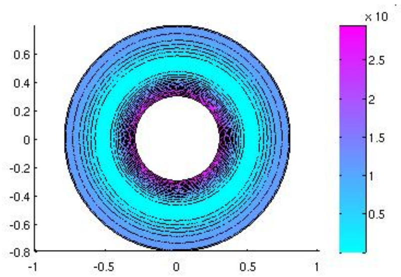
(a) $m/|\Omega| = 0.1702$

(b) $m/|\Omega| = 0.2932$



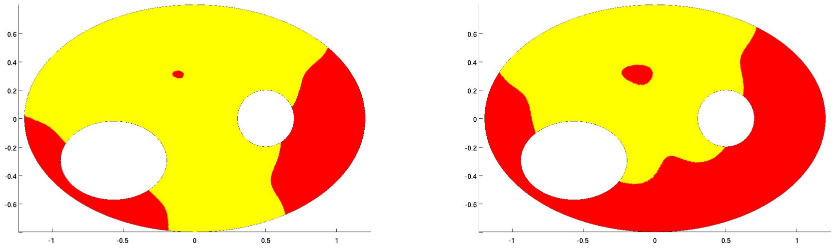
(c) $m/|\Omega| = 0.5308$

(d) $m/|\Omega| = 0.9072$

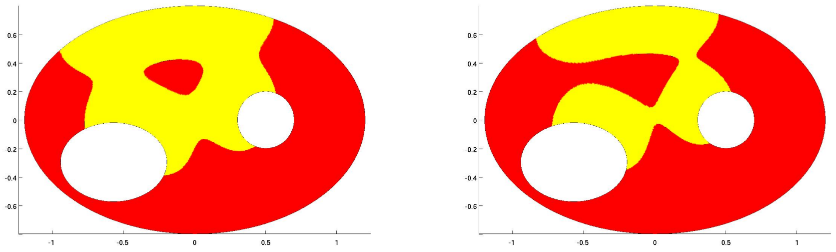


(e) isolines of $|\nabla v_0|^2$

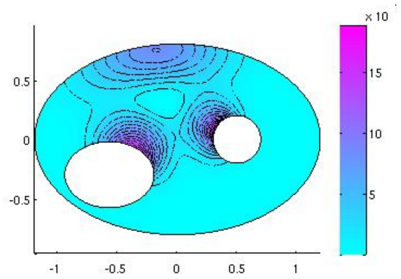
FIGURE 5. Nearly optimal distributions $B^\star$ in the ring case for different values of m. The dark region corresponds to B and the material β, the bright region to A and material α. As predicted by Corollary 3, the approximately optimal domain $B^\star$ contains a ring at mid-distance to the two boundaries. It corresponds to the place where the gradient of the eigenfunction vanishes. For m large enough, the domain B^* touches the outer boundary of the ring, but it never touches the inner boundary, where the gradient attains its maximum (as long as $m < |\Omega|$).



(a) $m/|\Omega| = 0.2551$ (b) $m/|\Omega| = 0.4786$



(c) $m/|\Omega| = 0.5914$ (d) $m/|\Omega| = 0.7011$



(e) isolines of $|\nabla v_0|^2$

FIGURE 6. Nearly optimal distributions $B^\star$ in the case of an ellipse with two holes for different values of m. The dark region corresponds to B and the material β, the bright region to A and material α. In this case we observe that the approximately optimal set B^* has several connected components, except maybe for large m, and touches the boundary. The set B^* always has a connected component in the center of the domain. This corresponds to a point where the gradient of the eigenfunction vanishes.

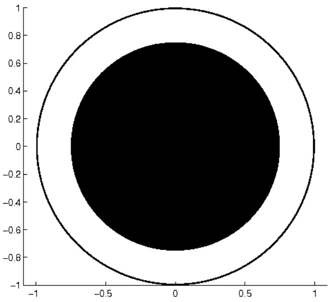
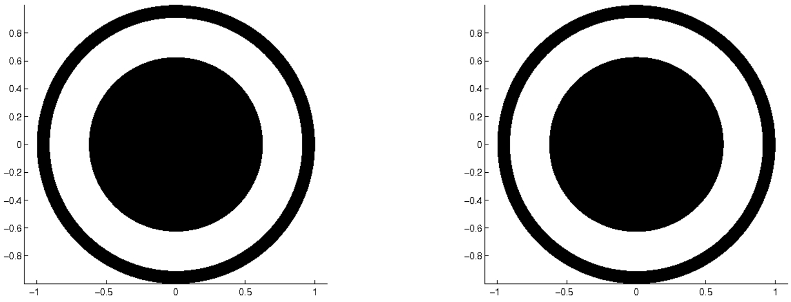
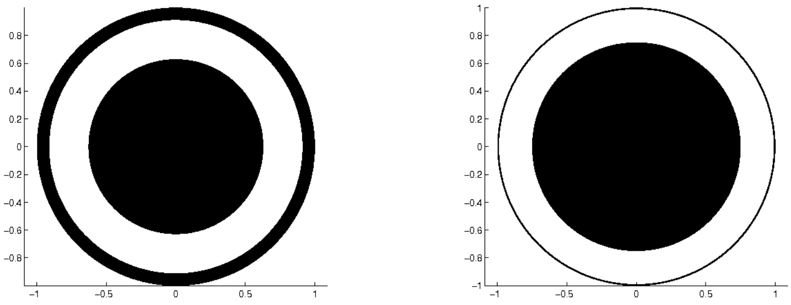


FIGURE 7. Initial domain $B_0 = \mathbb{B}(0, 0.75)$ (black disk) and domain $\Omega = \mathbb{B}(0, 1)$ (black disk and white ring)



(a) $\alpha = 1, \beta = 1.05$

(b) $\alpha = 1, \beta = 1.10$



(c) $\alpha = 1, \beta = 1.15$

(d) $\alpha = 1, \beta = 1.16$

FIGURE 8. Optimal sets A^* and B^* for different values of β close to α. The set B^* is represented in black, the set A^* in white. The optimal region B^* is of type II for $1 = \alpha < 1.01 < \beta \leq 1.15$ and of type I for $\beta = 1.16$.

Acknowledgements. This research project was initiated during the workshop on partial differential equations and optimization at the Centro de Ciencias de Benasque Pedro Pascual, Spain. The authors would like to thanks the organizers for the fruitful conference. The second author acknowledges financial support by the Austrian Ministry of Science and Education and the Austrian Science Foundation FWF under START-grant Y305 "Interfaces and Free Boundaries" and the subproject "Freelevel" of the SFB F32 "Mathematical Optimization and Applications in Biomedical Sciences". The third author acknowledges financial support by CONICYT-FONDECYT project N^o 1090305. The first author thanks the MICDB for partial support through Grant ICM-P05-001-F, Fondap-Basal-Conicyt.

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