Pricing and Hedging Defaultable Claim.

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September 21, 2011

Abstract

We study the pricing and the hedging of claim Ψ which depends of the default times of two firms A and B. In fact, we assume that we can not buy or sell any defaultable bond from the firm B but we can trade a defaultable bond of the firm A. Since the default times of the two firms are correlated, our aim is to find the best price and hedging of Ψ using bond of the firm A. Hence we solve this problem in two cases: first in a Markov framework using indifference price and solving a system of Hamilton Jacobi Bellman equation; and in a secondly in a more general framework (mean-variance tradeoff process non deterministic) using the mean variance hedging approach and solving backward stochastic differential equations.

Keywords Default and Credit risk; Hamilton Jacobi Bellman and Backward Differential equations; Mean variance hedging.

MSC Classification (2010): 60G48 60H10 91G40 49L20

Introduction

Models for pricing and hedging defaultable claim have generated a large debates by academics and practitioners during the last subprime crisis. The challenge is to modelize the expected losses of derivatives portfolio by taking account the counterparties defaults since they have been affected by the crisis and their agreement on the derivatives contracts can potentially vanish. In the literature, models for pricing defaultable securities have been pioneered by Merton [18]. His approach consists of explicitly linking the risk of firm default and the value of the firm. Although this model is a good issue to understand the default risk, it is less useful in practical applications since it is too difficult to capture the dynamics of the firm's value which depends of many macroeconomics factors. In response of these difficulties, Duffie and Singleton [5] introduced the reduced form modeling which

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has been followed by Madan and Unal [17], Jeanblanc and Rutkowski [11] and others. In this approach, the main tool is the ”default intensity process” which describes in short terms the instantaneous probability of default. This process combines with the recovery rate of the firm, represent the main tools necessary to manage the default risk. However, we should manage the default risk considering the financial market as a network where every default can affect another one and the propagation spread as far as the connections exist. In the literature, to deal with this correlation risk, the most popular approach is the copula. This approach consists of defining the joint distribution of the firms on the financial network considered given the marginal distribution of each firm on the network. In static framework [1], Li [16] was the first to develop this approach to modelize the joint distribution of the default times. But since, all computations are done without considering the evolution of the survey probability given available information then we can't describe the dynamics of the derivatives portfolio in this framework. In response of these limits on the static copula approach, El Karoui, Jeanblanc and Ying developped a conditional density approach [6]. An important point, in this framework is that given this density, we can compute explicitly the default intensity processes of firms in the financial market considered. We will follow this approach and work without losing any generality in the explicit case where financial network is defined only with two firms denoted by A and B.

We work on reduced form framework where the intensity process jumps when any default occurs. This jump impacts the default of the firm and makes some correlation between them. We assume that we can not buy or sell any defaultable bond from the firm B but we can trade a defaultable bond of the firm A, we consider two different cases for pricing and hedging a general defaultable claims: the indifference pricing in Markov setting and the Mean-Variance hedging for general cases.

In the first case, our aim is to find, using the correlation between the two firms, the indifference price of any contingent claim given the risk aversion defined by an exponential utility function. We express the indifference pricing as a optimization problem see El Karoui and Rouge [7]and using Delbaen and Schachermayer in [2] approach. Solving the dual problem, we find the solution of the indifference price. The characterization of the optimal probability for the dual optimization problem is solved by Hamilton-Jacobi-Bellman (HJB) equations since the defaultable bond price is assumed to be a Markov process in this framework.

In a second case, we have been interested in hedging in a general framework by Mean-Variance approach. We assume that we work in a general setting (not necessarily Markov), then we can not use the HJB equation to characterize the value function. Mean Variance approach has been introduced by Schweizer in [20] and generalized by many authors ([21], [8], [14], [4], [1], [15], [9]). Most of theses papers use martingales techniques and an important quantity in this context is the **Variance Optimal Martingale Measure** (VOM). The VOM, $\bar{\mathbb{P}}$, is the solution of the dual problem of minimizing the L^2-norm of the density $d\mathbb{Q}/d\mathbb{P}$, over all (signed) local martingale measure $\mathbb{Q}$ for the defaultable bond price of the firm A. If we consider the case of no jump of default, then the bond A price process is continuous; in this case Delbaen and Schachermayer in [3] prove the existence of

[1] The framework where we don't consider the evolution of the survey probability given a filtration

an equivalent VOM $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$. Moreover the price of any contingent claim ψ is given by $\mathbb{E}^{\bar{\mathbb{P}}}(\psi)$. In Laurent and Pham [14], they find explicit characterization of the variance optimal martingale measure in terms of the value function of a suitable stochastic control problem. In discontinuous case, when the so-called Mean-Variance Trade-off process (MVT) is deterministic, Arai [1] prove the same results. Since we work in discontinuous case and since the Variance Trade-off in our case is not deterministic (due to the stochastic default intensity process), we can not apply the standards results. Hence our work is first to characterize the value process of the Mean-Variance problem and secondly make some links with the existence and the characterization of the VOM in some particular cases. However, we really don't need to prove and assume this existence to solve the problem, we solve a system of Backward Differential Equations (BSDE) and we characterize the solution of the problem using BSDE's solutions. The main contribution in this part is the explicit characterization of the BSDE's solutions without using the existence of the VOM, we only use the characterization of the set of solutions via a stochastic control problem studied by Jeanblanc and al. in [10]. Therefore, we deduce that the main BSDE coefficient will follow a quadratic growth and since we work in discontinuous filtration due to defaults events, using Kharroubi and Lim [12] we will split the BSDE's with jumps into many continuous BSDEs with quadratic growth and we conclude the existence of the solution using the standard result of Kobylanski [13].

Hence in a first section, we will give some notations and results relative to credit risk modelling. Secondly, we will study the case of pricing and hedging a defaultable contingent claim in a Markovian framework using indifference pricing. Then in the last section, we will study the pricing and hedging problems in a more general framework (not Markov) using mean variance hedging approach.

1 Notations

Let $T > 0$ be a fixed maturity time and denote by $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{[0,T]}, \mathbb{P})$ an underlying probability space. The filtration $\mathbb{F}$ is generated by a one dimensional Brownian motion W. Let τ^A and τ^B be the two default times of firms A and B. Let define for all $t \in [0,T]$:

$$H_t^A = 1_{\{\tau^A \le t\}} \quad \text{and} \quad H_t^B = 1_{\{\tau^B \le t\}} \tag{1.1}$$

We define now some filtrations

$$\mathcal{G}_t^A = \mathcal{F}_t \vee \mathcal{H}_t^B, \qquad \mathcal{G}_t^B = \mathcal{F}_t \vee \mathcal{H}_t^A \quad and \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^A \vee \mathcal{H}_t^B$$

where $\mathcal{H}^A$ (resp. $\mathcal{H}^B$) is the natural filtration generated by H^A (resp. H^B). And we will denote $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$, $\mathbb{G}^A := \left(\mathcal{G}_t^A\right)_{t\in[0,T]}$ and $\mathbb{G}^B := \left(\mathcal{G}_t^B\right)_{t\in[0,T]}$. We assume that the density assumption holds, then the intensity process exists and is given in Delbean and al. [6]. For $i \in \{A, B\}$, let define by λ^i the $\mathbb{G}^i$-progressively non-negative process which represents the default intensity process of the firm i. For more convenience we work in the enlarged filtered probability space $(\Omega, \mathbb{G} := (\mathbb{G}_t)_{[0,T]}, \mathbb{P})$. We have that

$$M_t^i = H_t^i - \int_0^{t\wedge\tau^i} \lambda_s^i ds$$

are $\mathbb{G}$-martingales. Moreover we have the Kosuoka representation Theorem which is that for any $\mathbb{G}$-martingale M there exist $\mathbb{G}$-predictable processes Z and $U = (U^A, U^B)$ such that

$$M = Z.W + U.M$$

where . denotes the standard scalar product. Using this decomposition, we represent the dynamics of the defaultable claim A in the enlarged filtration $\mathbb{G}$ by

$$\frac{dD_t^A}{D_{t^-}^A} = \mu_t dt + \sigma_t^A dM_t^A + \sigma_t^B dM_t^B + \sigma_t dW_t \tag{1.2}$$

where $\mu, \sigma^A, \sigma^B, \sigma$ are $\mathbb{G}$-predictable bounded processes.

2 Hedging defaultable claim with Markov copula

Let consider $\psi \in \mathcal{G}_T$ a bounded contingent claim which depends on the default times τ^A of the firm A and τ^B of the firm B. Our aim is to find, using the correlation between the two firms, the best hedging and pricing of ψ.

Assumption 2.1. *We assume that $\mu, \sigma^A, \sigma^B, \sigma$ and the intensity processes λ^A, λ^B are deterministic bounded functions of time and D^A, H^A, H^B.*

Remark 2.1. *In this case, (D^A, H^A, H^B) is a Markov process.*

We assume that the risk aversion of investors is given by an exponential utility function with parameter δ which is

$$U(x) = -\exp(-\delta x)$$

Moreover we assume that there is a free bond (without default) which follows the dynamics

$$dD_t^0 = r_t D_t^0 dt$$

Therefore, given an initial wealth $x \geq 0$, if we assume that investors follow an admissible strategies (π^0, π), which are represented by a set $\mathcal{A}$ of predictable processes π such that

$$\int_0^T \pi_s^2 ds < +\infty, \quad \mathbb{P} - a.s,$$

then we can define the dynamics of the wealth $X^{x,\pi}$ by

$$dX_t^{x,\pi} = r_t X_t^{x,\pi} dt + \pi_t \left[(\mu_t - r_t)dt + \sigma_t^A dM_t^A + \sigma_t^B dM_t^B + \sigma_t dW_t \right] \tag{2.3}$$

Therefore, to define the indifference price or the hedging of ψ, we should solve the equation

$$u^\psi(x+p) = u^0(x),$$

where functions u^ψ and u^0 are defined by:

$$u^\psi(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[-\exp(-\delta(X_T^{x,\pi} - \psi)) \right] \quad \text{and} \quad u^0(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[-\exp(-\delta X_T^{x,\pi}) \right] \tag{2.4}$$

2.1 The dual optimization formulation

To deal with the problem (2.4), we use the duality theory developped by Delbaen and Schachermayer in [2]. In fact this theory allow us to find the optimal wealth at the horizon time T and the optimal risk neutral probability $\mathbb{Q}^*$. In the sequel without loose of generality, we will assume that $r_t \equiv 0$. Let recall now some results about the dual theory.

Theorem 2.1. *[Delbaen and Schachermayer] Let U be a utility function which satisfies the standards assumptions and consider the optimization problem $u(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T^{x,\pi})\right]$, then the dual function of u defined by:*

$$v(y) = \sup_{x>0}\{u(x) - xy\}, \quad u(x) = \inf_{y>0}\{v(y) + yx\}$$

is given by

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{M}^e} \mathbb{E}\left(V\left[y\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\right)$$

where V represents the dual function of U and $\mathcal{M}^e$ represents the set of risk neutral probability measure.

Moreover there exists an optimal martingale measure $\mathbb{Q}^*$ which solves the dual problem and we have that the optimal wealth at time T is given by:

$$X_T^{x,\pi^*} = I\left[\nu Z_T^{\mathbb{Q}^*}\right], \text{ where } \nu \text{ is defined s.t. } \quad \mathbb{E}^{\mathbb{Q}^*}\left[X_T^{x,\pi^*}\right] = x.$$

where the function I represents the inverse function of U' and $Z_T^{\mathbb{Q}^*}$ represents the Radon Nikodym on $\mathcal{G}_T$ of $\mathbb{Q}^*$ with respect to $\mathbb{P}$.

Now, we can apply this result to solve our optimization problem (2.4). We will resolve only the case $\psi \neq 0$. Indeed the particular case $\psi = 0$ could be obtain by these results.

Proposition 2.1. *Let $\mathbb{Q}^*$ be the optimal risk neutral probability which solves the dual problem*

$$\inf_{\mathbb{Q}\in\mathcal{M}^e}\left[H(\mathbb{Q}|\mathbb{P}) - \delta\mathbb{E}^{\mathbb{Q}}(\psi)\right] \tag{2.5}$$

then the optimal strategy $\pi^* \in \mathcal{A}$ solution of the optimization problem (2.4) satisfies:

$$-\frac{1}{\delta}\ln\left(Z_T^{\mathbb{Q}^*}\right) + \psi = x + \frac{1}{\delta}\ln\left(\frac{y}{\delta}\right) + \int_0^T \pi_t^* dD_t^A \tag{2.6}$$

where $H(\mathbb{Q}|\mathbb{P})$ represents the entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ $\left(i.e. \quad \mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right)$ and y is a non negative constant.

Proof. The proof is based on the Theorem 2.1. First to match with the assumptions of this Theorem in the case $\psi \neq 0$, we change the historical probability. Let define

$$\frac{d\mathbb{P}^\psi}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \frac{\exp(\delta\psi)}{\mathbb{E}\left[\exp(\delta\psi)\right]} \quad \text{and} \quad \widetilde{u}^\psi(x) = \sup_{\pi\in\mathcal{A}} \mathbb{E}^\psi\left[-\exp(-\delta X_T^{x,\pi})\right],$$

then setting $c = \mathbb{E}\left[\exp(\delta\psi)\right]$ we have

$$\begin{aligned} u^{\psi}(x) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[-\exp(-\delta(X_T^{x,\pi}-\psi))\right] = \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{\psi}}\left[-c\exp(-\delta X_T^{x,\pi})\right] \\ &= \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{\psi}}\left[\exp\left(-\delta(-\frac{1}{\delta}\log(c)+X_T^{x,\pi})\right)\right] = \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{\psi}}\left[\exp\left(-\delta X_T^{x-\frac{1}{\delta}\log(c),\pi}\right)\right] \end{aligned}$$

Hence by the definition of $\widetilde{u}^{\psi}(x)$ we obtain that

$$\widetilde{u}^{\psi}\left(x-\frac{1}{\delta}\ln(c)\right) = u^{\psi}(x)$$

Then using the Theorem 2.1, the dual function of $\widetilde{u}^{\psi}$ is given for all $y>0$ by:

$$\widetilde{v}^{\psi}(y) = \inf_{\mathbb{Q}\in\mathcal{M}^e} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}^{\psi}}\right)\right] \tag{2.7}$$

where

$$V(y) = \sup_{x>0}\{U(x)-xy\} = \sup_{x>0}\{-\exp(-\delta x)-xy\} = \frac{y}{\delta}\left[\ln\left(\frac{y}{\delta}\right)-1\right]$$

Then using this expression of $V(y)$ into (2.7) gives after calculation an explicit expression of the dual function which is

$$\widetilde{v}^{\psi}(y) = V(y) + \frac{y}{\delta}\ln(c) + \frac{y}{\delta}\inf_{\mathbb{Q}\in\mathcal{M}^e}\left[H(\mathbb{Q}|\mathbb{P}) - \delta\mathbb{E}^{\mathbb{Q}}(\psi)\right]$$

Since $\mathbb{Q}^*$ is the optimal risk neutral probability which is solution of (2.5) we deduce that the optimal wealth at time T of the optimization problem (2.4) is given by

$$X_T^{x,\pi^*} = I\left[y\frac{Z_T^{\mathbb{Q}^*}}{Z_T^{\mathbb{Q}^{\psi}}}\right]$$

where y is defined such that $\mathbb{E}^{\mathbb{Q}^*}\left[X_T^{x,\pi^*}\right] = x - \frac{1}{\delta}\ln(c)$ and I is equal to $-V'$. Moreover from Owen [19], we can deduce that there exists an optimal strategy $\pi^*\in\mathcal{A}$ such that:

$$X_T^{x,\pi^*} = I\left[y\frac{Z_T^{\mathbb{Q}^*}}{Z_T^{\mathbb{Q}^{\psi}}}\right] = x - \frac{1}{\delta}\ln(c) + \int_0^T \pi_t^* dD_t^A.$$

In our case, we work under the the case of exponential utility function with parameter δ. So

$$I(y) := -\frac{1}{\delta}\ln\left(\frac{y}{\delta}\right)$$

then we finally get that

$$x - \frac{1}{\delta}\ln(c) + \int_0^T \pi_t^* dD_t^A = -\frac{1}{\delta}\ln\left(\frac{y}{\delta}\right) - \frac{1}{\delta}\log\left(Z_T^{\mathbb{Q}^*}\right) + \psi - \frac{1}{\delta}\ln(c).$$

which conclude the proof of this proposition.

□

2.2 Value function of the dual problem

In this part, we solve the dual problem (2.9) in a Markov framework. In fact, if we consider the same problem with a different set of probability measure like $\mathcal{M}^e = \mathcal{Q}$, where $\mathcal{Q}$ represents the set of all probability measure $\mathbb{Q} \ll \mathbb{P}$, then the value function is given by the entropy of ψ with a parameter δ.

But since we work in a more restricted set of probability $\mathcal{M}^e$ which reprensents the set of all risk neutral probability, the value function is more difficult to precise. To characterize the value function, we first describe the set $\mathcal{M}^e$.

Let $\mathbb{Q} \in \mathcal{M}^e$ and define $Z_T^{\mathbb{Q}}$ be the Radon Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$. Consider the non negative martingale process $Z_t^{\mathbb{Q}} = \mathbb{E}\left[Z_T^{\mathbb{Q}}|\mathcal{G}_t\right]$ and using representation theorem imply that there exists predictable processes ρ^A and ρ^B which take their values in $\mathcal{C} = (-1, +\infty)$ and a predictable process ρ which takes its values in $\mathbb{R}$ such that for all $t \in [0, T]$

$$dZ_t^{\mathbb{Q}} = Z_{t^-}^{\mathbb{Q}} \left(\rho_t^A dM_t^A + \rho_t^B dM_t^B + \rho_t dW_t\right)$$

Since $\mathbb{Q}$ is in $\mathcal{M}^e$, it is a risk neutral probability, then ZD^A is a local martingale. This implies by Ito's calculus the following equation:

$$\mu_t + \rho_t^A \sigma_t^A \lambda_t^A + \rho_t^B \sigma_t^B \lambda_t^B + \rho_t \sigma_t = 0 \tag{2.8}$$

Remark 2.2. *We notice that the process ρ depends explicitly to ρ^A and ρ^B.*

Therefore using equation (2.8), the latter (2.5) can be formulated as find ρ^A and ρ^B which minimize:

$$\inf_{\mathbb{Q} \in \mathcal{M}^e} \mathbb{E}^{\mathbb{Q}}\left[\ln(Z_T^{\mathbb{Q}}) - \delta\psi\right] \tag{2.9}$$

We make now an Assumption.

Assumption 2.2. *The contingent claim $\psi \in \mathcal{G}_T$ is given by*

$$\psi = g(D_T^A)\mathbf{1}_{\{\tau^B > T\}} + f(D_{\tau^B -}^A)\mathbf{1}_{\{\tau^B \leq T\}}$$

where g and f are two bounded continuous functions.

Proposition 2.2. *The value function of the dual problem (2.9) is given by:*

$$V(t, D_t^A, H_t^A, H_t^B) = \inf_{\rho^A, \rho^B \in \mathcal{C}} \mathbb{E}^{\mathbb{Q}}\left[\int_t^T j(s, \rho_s^A, \rho_s^B, D_s^A)ds - \delta g(D_T^A)\mathbf{1}_{\{\tau^B > T\}}\Big| D_t^A, H_t^A, H_t^B\right] \tag{2.10}$$

where the function j is defined by:

$$j(s, \rho_s^A, \rho_s^B, D_s^A) = \sum_{i \in \{A,B\}} \lambda_s^i \left[(1 + \rho_s^i)\ln(1 + \rho_s^i) - \rho_s^i\right] - \delta(1 + \rho_s^B)\lambda_s^B f(D_s^A) + \frac{1}{2}\rho_s^2 \tag{2.11}$$

Proof. The proof is based on the Ito's formula. We write first the dynamics of $\ln(Z^{\mathbb{Q}})$ under $\mathbb{Q}$ which is given by

$$d\ln(Z_t^{\mathbb{Q}}) = \sum_{i\in\{A,B\}} \rho_t^i dM_t^i + \left[\ln(1+\rho_t^i) - \rho_t^i\right] dH_t^i + \rho_t dW_t - \frac{1}{2}\rho_t^2 dt$$

Using Girsanov theorem, the processes defined for all $i \in \{A, B\}$ by

$$\widetilde{M}_t^i = M_t^i - \int_0^t \rho_s^i \lambda_s^i ds \quad and \quad \widetilde{W}_t = W_t - \int_0^t \rho_s ds$$

are $\mathbb{Q}$-martingales. Hence we obtain that

$$\ln(Z_T^{\mathbb{Q}}) - \delta\psi = \int_0^T \sum_{i\in\{A,B\}} \lambda_t^i[(1+\rho_t^i)\ln(1+\rho_t^i) - \rho_t^i]dt - \delta\left[\int_0^T f(D_{t^-}^A)dH_t^B + g(D_T^A)(1-H_T^B)\right] + \int_0^T \frac{1}{2}\rho_t^2 dt + M_T^{\mathbb{Q}}$$

where $M^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale. Then we can rewrite the dual problem using the last expression:

$$\inf_{\mathbb{Q}\in\mathcal{M}^e} \mathbb{E}^{\mathbb{Q}}\left[\ln(Z_T^{\mathbb{Q}}) - \delta\psi\right] = \inf_{\rho^A,\rho^B\in\mathcal{C}} \mathbb{E}^{\mathbb{Q}}\left[\int_0^T j(s,\rho_s^A,\rho_s^B,D_s^A) - \delta(1-H_T^B)g(D_T^A)\right]$$

where j is given in (2.11). Since by Remark 2.1, the process (D^A, H^A, H^B) is a Markov process, then the value function of the dual optimization problem is given by:

$$V(t, D_t^A, H_t^A, H_t^B) = \inf_{\rho^A,\rho^B\in\mathcal{C}} \mathbb{E}^{\mathbb{Q}}\left[\int_t^T j(s,\rho_s^A,\rho_s^B,D_s^A)ds - \delta g(D_T^A)\mathbf{1}_{\{\tau^B>T\}}\middle| D_t^A, H_t^A, H_t^B\right] \tag{2.12}$$

□

We need now to evaluate an explicit form of the value function.

Proposition 2.3. *Let $z = (x, h^A, h^B)$, the value function of the dual optimization problem is solution of the following Hamilton-Jacobi-Bellman equation:*

$$\frac{\partial V}{\partial t}(t,z) + \frac{1}{2}\frac{\partial V}{\partial x^2}(t,z)\sigma^2(t,z) + \inf_{\rho^A,\rho^B\in\mathcal{C}}\{\mathcal{L}_{\rho^A,\rho^B}V(t,z) + j(t,\rho_t^A,\rho_t^B)\},\ V(T,z) = g(x)(1-h^B) \tag{2.13}$$

where

$$\mathcal{L}_{\rho^A,\rho^B}V(t,z) = \sum_{i\in\{A,B\}}\left[-\frac{\partial V}{\partial z}(t,z)\sigma^i(t,z) + \left(V(t,z^i) - V(t,z)\right)\right](1+\rho_t^i)\lambda^i(t,z)$$

$z^i = \left(x(1+\sigma^i(t,z)), h^A + \alpha^i, h^B + 1 - \alpha^i\right)$ *where* $\alpha^A = 1$ *and* $\alpha^B = 0$.
Moreover given the value function, the optimal strategy satisfies:

$$\pi_t^* = -\frac{1}{\delta}\left(\frac{\partial V}{\partial x}(t,z) + \frac{\bar{\rho}_t}{D_{t^-}^A\sigma(t,z)}\right)$$

where the process $\bar{\rho}$ is explicit given with the optimal control $\bar{\rho}^i$, $i \in \{A, B\}$, see the relation (2.8).

Proof. From Proposition 2.2, we find the value function of the dual optimization problem see (2.12). Using Hamilton-Jacobi-Bellman (HJB) equation we get:

$$V(t,D_t^A,H_t^A,H_t^B)=\inf_{\rho^A,\rho^B\in\mathcal{C}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^{t+h}j(s,\rho_s^A,\rho_s^B,D_s^A)ds+V(t+h,H_{t+h}^A,H_{t+h}^B)\big|D_t^A,H_t^A,H_t^B\right]$$

Then the value function solve the HJB equation (2.13). In the next part, we find the optimal strategy given the value function. Let recall that from Theorem 2.1, the optimal risk neutral probability and the value function exist. Let define $\bar\rho^A$, $\bar\rho^B$ and $\bar\rho$ the optimal density parameters. Since $\bar\rho^A$ and ρ^B are optimal for the HJB equation, assuming $\sigma(t,z)\neq 0$, using first order condition we find for $i\in\{A,B\}$:

$$\left[(V(t,z^i)-V(t,z))-x\sigma^i(t,z)\frac{\partial V}{\partial x}(t,z)+\ln(1+\bar\rho_t^i)-\frac{\sigma^i(t,z)}{\sigma(t,z)}\bar\rho_t\right]\lambda^i(t,z)=\delta(1-\alpha^i)f(x)\lambda^i(t,z) \quad (2.14)$$

Then using the HJB equation and the relation (2.14), we find the following relation:

$$-\frac{1}{2}\bar\rho_t^2+\sum_{i\in\{A,B\}}\bar\rho_t^i\lambda^i(t,z)=\sum_{i\in\{A,B\}}(1+\bar\rho_t^i)\frac{\sigma^i(t,z)}{\sigma(t,z)}\bar\rho_t+\frac{1}{2}\frac{\partial^2V}{\partial x^2}(t,z)x^2\sigma^2(t,z)+\frac{\partial V}{\partial t}(t,z) \quad (2.15)$$

Let recall the Ito's decomposition of the process $\ln(Z^{\mathbb{Q}^*})$:

$$\ln(Z_T^{\mathbb{Q}^*})=\int_0^T[\bar\rho_t d\bar W_t+\frac{1}{2}\bar\rho_t^2dt]+\int_0^T\sum_{i\in\{A,B\}}\left[\ln(1+\bar\rho_t^i)dH_t^i-\bar\rho_t^i\lambda^i(t,z)\right]$$

Then using equations (2.15) and (2.14), we find a useful a more explicit decomposition of the process $\ln(Z_T^{\mathbb{Q}^*})$:

$$\begin{aligned}\ln(Z_T^{\mathbb{Q}^*})=&\int_0^T-\frac{1}{2}\frac{\partial^2V}{\partial x^2}(t,z_t)(D_{t^-}^A)^2\sigma^2(t,z_t)dt-\int_0^T\frac{\partial V}{\partial t}(t,z_t)dt+\int_0^T\bar\rho_t d\bar W_t\\&-\sum_{i\in\{A,B\}}\left[(V(t,z_t^i)-V(t,z_t))-D_{t^-}^A\sigma^i(t,z_t)\frac{\partial V}{\partial x}(t,z)\right]dH_t^i\\&+\int_0^T\sum_{i\in\{A,B\}}\frac{\sigma^i(t,z_t)}{\sigma(t,z)}\bar\rho_t[dH_t^i-(1+\bar\rho_t^i)\lambda^i(t,z_t)]+\int_0^T\delta f(D_{t^-}^A)dH_t^B\end{aligned}$$

where $z_t=(D_t^A,H_t^A,H_t^B)$, then using the Ito's decomposition of $V(T,D_T^A,H_T^A,H_T^B)$, we find:

$$\begin{aligned}\ln(Z_T^{\mathbb{Q}^*})=&\int_0^T\frac{\bar\rho_t}{\sigma(t,z)}\left[\sigma(t,z_t)d\bar W_t+\sum_{i\in\{A,B\}}\sigma^i(t,z_t)d\bar M_t^i\right]+\delta f(D_{\tau^B-}^A)\mathbf{1}_{\{\tau^B\leq T\}}\\&-V(T,D_T^A,H_T^A,H_T^B)+V(0,D_0^A,H_0^A,H_0^B)+\int_0^T\frac{\partial V}{\partial x}(t,z_t)dD_t^A\end{aligned}$$

Since

$$V\left(T,D_T^A,H_T^A,H_T^B\right)=-\delta g(D_T^A)(1-H_T^B)$$

and

$$\psi = f(D^A_{\tau^B-})\mathbf{1}_{\{\tau^B\le T\}} + g(D^A_T)(1-H^B_T)$$

then we get:

$$\ln(Z^{\mathbb{Q}^*}_T) - \delta\psi = V(0, D^A_0, H^A_0, H^B_0) + \int_0^T \left[\frac{\bar\rho_t}{D^A_{t^-}\sigma(t,z)} + \frac{\partial V}{\partial x}(t, z_t)\right] dD^A_t$$

From Definition of the value function,

$$V(0, D^A_0, H^A_0, H^B_0) = \mathbb{E}^{\mathbb{Q}^*}\left[\ln(Z^{\mathbb{Q}^*}_T) - \delta\psi\right]$$

using the fact that $\mathbb{E}^{\mathbb{Q}^*}\left[X^{x,\pi^*}_T\right] = x - \frac{1}{\delta}\ln(c)$, where $X^{x,\pi^*}_T = -\frac{1}{\delta}\ln\left(\frac{1}{\delta}\frac{Z^{\mathbb{Q}^*}_T}{Z^{\mathbb{P}^\psi}}\right)$ see Theorem 2.1, we deduce that

$$\mathbb{E}^{\mathbb{Q}^*}\left[-\frac{1}{\delta}\ln(Z^{\mathbb{Q}^*}_T) + \psi - \frac{1}{\delta}\ln(c) - \frac{1}{\delta}\ln\left(\frac{y}{\delta}\right)\right] = x - \frac{1}{\delta}\ln(c)$$

Hence we conclude that

$$V(0, D^A_0, H^A_0, H^B_0) = -\delta x - \ln\left(\frac{y}{\delta}\right)$$

Finally, we find

$$-\frac{1}{\delta}\ln(Z^{\mathbb{Q}^*}_T) + \psi = x + \frac{1}{\delta}\ln\left(\frac{y}{\delta}\right) + \int_0^T -\frac{1}{\delta}\left[\frac{\bar\rho_t}{D^A_{t^-}\sigma(t,z)} + \frac{\partial V}{\partial x}(t, z_t)\right] dD^A_t$$

Therefore from equation (2.6), we obtain the result of the Proposition. □

In conclusion, we have find that since we can characterize the optimal probability for the dual optimization problem using the Delbaen and Schachermayer Theorem, we can characterize the HJB equation and then this allows us to find the optimal strategy for the primal solution for a defaultable contingent claim. Therefore we can find for $\psi = 0$ and $\psi \neq 0$, the optimal strategy in the both cases and deduce the indifference price p of a defaultable contingent claim solving the equation $u^\psi(x+p) = u^0(x)$.

3 Generalization of the hedging in a general framework: Mean-Variance approach

In this part, we assume that we work in a general setting (not necessarily Markov), then we can not use the HJB equation to characterize the value function. The Mean Variance approach is a well-known methodology to manage hedging in general case. It seems to have been introduced in 1992 by Schweizer [20]. An important quantity in this context is the **Variance Optimal Martingale Measure** (VOM). The VOM, $\bar{\mathbb{P}}$, is the solution of the dual problem of minimizing the L^2-norm of the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$, over all (signed) local martingale measure $\mathbb{Q}$ for D^A. Let recall now the Mean-Variance problem:

$$\min_{\pi\in\mathcal{A}} \mathbb{E}\left[(X^{x,\pi}_T - \psi)^2\right]. \tag{3.16}$$

If we assume $\mathbb{G} = \mathbb{F}$ (in the case we do not consider jump of default), the process D^A is continuous; in this case Delbean and Schachermayer [3] prove the existence of an equivalent VOM $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$ and the fact that the price of ψ is given by $\mathbb{E}^{\bar{\mathbb{P}}}(\psi)$. In discontinuous case, when the so-called Mean-Variance Trade-off process (MVT) is deterministic, Arai [1] prove the same results. Since we work in discontinuous case and since the Variance Trade-off is not deterministic (due to the stochastic default intensity process), we cannot apply the standards results. In this part, our work is first to characterize the value process of the Mean-Variance problem and secondly make some links with the existence and the characterization of the VOM in some particular cases. First, let recall some usual spaces:

• $\mathcal{S}^\infty$ is the Banach space of $\mathbb{R}^d$-valued cadlag processes X such that there exists a constant C satisfying

$$\|X\|_{\mathcal{S}^\infty} := \sup_{t\in[0,T]} |X_t| \leq C < +\infty \quad \text{and} \quad \Big(\mathbb{E}\Big[\int_0^T |X_t|^2\, dt\Big]\Big)^{\frac{1}{2}} < +\infty$$

• $\mathcal{H}^2$ is the Hilbert space of $\mathbb{R}$-valued predictable processes Z such that

$$\|Z\|_{\mathcal{H}^2} := \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2\, dt\Big]\Big)^{\frac{1}{2}} < +\infty$$

• $\mathcal{H}^2_\lambda$ is the Hilbert space of $\mathbb{R}^2$-valued predictable processes such that

$$\|V\|_{\mathcal{H}^2_\lambda} := \Big(\mathbb{E}\Big[\sum_{j\in\{A,B\}} \int_0^T |V_t^j|^2 \lambda_t^j dt\Big]\Big)^{\frac{1}{2}} = \Big(\mathbb{E}\Big[\int_0^T |V|_t^2 dt\Big]\Big)^{\frac{1}{2}} < +\infty$$

where

$$|v|_t := \Big[\sum_{j\in\{A,B\}} |v_j^2|\lambda_t^j\Big]^{\frac{1}{2}} \in \mathbb{R}_+ \cup \{+\infty\} \tag{3.17}$$

for $v \in R^2$ and $t \in [0,T]$.

• BMO is the space of $\mathbb{G}$-adapted matingale such that for any stopping times $0 \leq \sigma \leq \tau \leq T$, there exists a non negative constant $c > 0$ such that:

$$\mathbb{E}\left[[M]_\tau - [M]_{\sigma^-}|\mathcal{G}_\sigma\right] \leq c.$$

when $M = Z.W \in$ BMO, to simplify notation we write $Z \in$ BMO.

Definition 3.1 ($R_2(\mathbb{P})$ condition)**.** *Let Z be a uniformtly integrable martingale with $Z_0 = 1$ and $Z_T > 0$, we say that Z satisfies reverse Hölder $R_2(\mathbb{P})$ under $\mathbb{P}$ if there exist a constant $c > 0$ such that for every stopping times σ, we have:*

$$\mathbb{E}\left[\left(\frac{Z_T^2}{Z_\sigma^2}\right)^2 |\mathcal{G}_\sigma\right] \leq c.$$

3.1 Characterization of the optimal cost via BSDE

In this part, we characterize the functional cost of the Mean-Variance problem using dynamic programming via BSDE. Given an admissible strategy $\pi \in \mathcal{A}$, let define $J_t(\pi)$ the functional cost of the Mean-Variance problem at time $t \in [0,T]$. From dynamic programming, we find for any $\pi \in \mathcal{A}$ that $J(\pi)$ is a submartingale and for the optimal strategy $\pi^* \in \mathcal{A}$ that $J(\pi^*)$ is a martingale. Let recall the usual assumption:

$$J_t(\pi) = \Theta_t (X_t^{x,\pi} - Y_t)^2 + \xi_t \tag{3.18}$$

where Θ is a non-negative $\mathbb{G}$-adapted process and Y, ξ are two $\mathbb{G}$-adapted processes. Therefore using the properties of the functional cost, we find the following characterization of the triplet (Θ, Y, ξ) using BSDE. First let give some characterizations of the process Θ.

Proposition 3.4. *The process Θ is a submartingale, moreover there exists a non negative constant $\delta > 0$ such that $\Theta_t \geq \delta > 0$.*

Proof. From Theorem 1.4 of Jeanblanc and al. [10], we get that Θ is a positive submartingale. To prove that there exists a non-negative constant δ such that $\Theta \geq \delta > 0$, first we should prove that there exists a martingale measure $\mathbb{Q} \in R_2(\mathbb{P})$ and then we can deduce from Lemma 2.1 of [10] that the assertion is satisfied. Let given $\mathbb{Q}$ a martingale measure and denote by Z_T its Radon Nikodym density with respect to $\mathbb{P}$ on $\mathcal{G}_t$. We define $Z_t = \mathbb{E}[Z_T|\mathcal{G}_t]$ for $t \leq T$. Therefore from representation theorem, there exist three $\mathbb{G}$-predictable processes ρ^A, ρ^B and ρ such that $dZ_t/Z_{t^-} = \rho_t^A dM_t^A + \rho_t^B dM_t^B + \rho_t dW_t$. Moreover we get $\mu_t^A + \sigma^A \rho_t^A \lambda_t^A + \sigma_t^B \rho_t^B \lambda_t^B = 0$; then for $i \in \{A, B\}$, we can choose $\rho^i > -1$ such that there are bounded, then from the boundness condition of $\mu, \sigma^A, \sigma^B, \lambda^A$ and λ^B, we deduce ρ is also bound. Therefore, we deduce the stochastic logaritm $\mathcal{L}(Z)$ of Z is BMO moreover there exists a non negative constant h, such that $1 + \Delta\mathcal{L}(Z) > h$ then from Theorem 2.14 of Delbean and al. [4], the chosen martingale measure $\mathbb{Q} \in R_2(\mathbb{P})$. □

Proposition 3.5. *There exists an optimal strategy $\pi^* \in \mathcal{A}$ such that*

$$J_t(\pi^*) = \Theta_t (X_t^{x,\pi} - Y_t)^2 + \xi_t$$

is a martingale, where the triplet (θ, Y, ξ) follows the BSDEs:

$$\begin{aligned}
\frac{d\Theta_t}{\Theta_{t^-}} &= -g_t^1(\Theta_t, \theta_t, \beta_t)dt + \theta_t.dM_t + \beta_t dW_t, && \Theta_t = 1 \\
dY_t &= -g_t^2(Y_t, U_t, Z_t)dt + U_t.dM_t \ + Z_t dW_t, && Y_T = \psi \\
d\xi_t &= -g_t^3(\xi_t, \epsilon_t, R_t)dt + \ \epsilon_t.dM_t + R_t dW_t, && \xi_T = 0.
\end{aligned} \tag{3.19}$$

where $\theta = (\theta^A, \theta^B)$, $U = (U^A, U^B)$ and $\epsilon = (\epsilon^A, \epsilon^B)$ are $\mathbb{G}$-predictable processes and the martingale of defaults $M = (M^A, M^B)$.

The different coefficients g^j, $j = \{1,2,3\}$ are defined by:

$$g^1_t(\Theta_t,\theta_t,\beta_t) = -\frac{\left[\mu_t + \sum_{i\in\{A,B\}}\theta^i_t\sigma^i_t\lambda^i_t + \sigma_t\beta_t\right]^2}{\sigma^2_t + \sum_{i\in\{A,B\}}(1+\theta^i_t)\left(\sigma^i_t\right)^2\lambda^i_t}$$

$$g^2_t(Y_t,U_t,Z_t) = -\frac{\left[\mu_t + \sum_{i\in\{A,B\}}\theta^i_t\sigma^i_t\lambda^i_t + \sigma_t\beta_t\right]\left[\sigma_t Z_t + \sum_{i\in\{A,B\}}(1+\theta^i_t)\sigma^i_t U^i_t\lambda^i_t\right]}{\sigma^2_t + \sum_{i\in\{A,B\}}(1+\theta^i_t)\left(\sigma^i_t\right)^2\lambda^i_t} + \sum_{i\in\{A,B\}}\theta^i_t U^i_t\lambda^i_t + \beta_t Z_t$$

$$g^3_t(\xi_t,\epsilon_t,R_t) = \Theta_{t^-}\left[Z^2_t + \sum_{i\in\{A,B\}}\left(U^i_t\right)^2(1+\theta^i_t)\lambda^i_t - \frac{\left(Z_t\sigma_t + \sum_{i\in\{A,B\}}\sigma^i_t U^i_t(1+\theta^i_t)\lambda^i_t\right)^2}{\sigma^2_t + \sum_{i\in\{A,B\}}(1+\theta^i_t)\left(\sigma^i_t\right)^2\lambda^i_t}\right]$$

Proof. Let π be an admissible strategy, let find the triplet (Θ,Y,ξ) such that for all $t \le T$

$$J_t(\pi) = \Theta_t\left(X^{x,\pi}_t - Y_t\right)^2 + \xi_t$$

is a submartingale. Since we know the dynamics of the wealth and the representation of the triplet (Θ,Y,ξ) (3.19); from Ito's formula and integration by part formula for jump process, we find the decomposition of $J(\pi)$. Let recall that for any S,L semimartingale, we have that

$$d(S_tL_t) = S_{t^-}dL_t + L_{t^-}dS_t + d[S,L]_t$$

In our framework since jump comes from defaults events we get

$$d[S,L]_t = \langle S^c,L^c\rangle_t + \sum_{i\in\{A,B\}}\Delta S^i_t\Delta L^i_t dH^i_t$$

Applying this results for $S = L = (X^{x,\pi} - Y)$ gives:

$$d(X^{x,\pi}-Y)^2_t = 2(X^{x,\pi}_{t^-} - Y_{t^-})\left[(\pi_t\mu_t + g^2_t)dt + \sum_{i\in\{A,B\}}(\pi_t\sigma^i_t - U^i_t)dM^i_t + (\pi_t\sigma_t - Z_t)dW_t\right] + (\sigma_t\pi_t - Z_t)^2dt + \sum_{i\in\{A,B\}}\left(\pi_t\sigma^i_t - U^i_t\right)^2dH^i_t$$

Secondly take $S = \Theta$ and $L = (X^{x,\pi} - Y)^2$, let define $K := (X^{x,\pi} - Y)$, we find:

$$\begin{aligned} d\left(\Theta K^2\right)_t &= 2K_{t^-}\Theta_{t^-}\left[(\pi_t\mu_t + g_t^2)dt + \sum_{i\in\{A,B\}}(\pi_t\sigma_t^i - U_t^i)dM_t^i + (\pi_t\sigma_t - Z_t)dW_t\right] \\ &\quad + \Theta_{t^-}(\sigma_t\pi_t - Z_t)^2dt + \sum_{i\in\{A,B\}}\Theta_{t^-}(\pi_t\sigma_t^i - U_t^i)^2dH_t^i - \Theta_{t^-}K_{t^-}^2g_t^1dt \\ &\quad + \Theta_{t^-}K_{t^-}^2\left[\sum_{i\in\{A,B\}}\theta_t^idM_t^i + \beta_tdW_t\right] + 2K_{t^-}\Theta_{t^-}(\pi_t\sigma_t - Z_t)\beta_tdt \\ &\quad + \sum_{i\in\{A,B\}}\left[(\pi_t\sigma_t^i - U_t^i)^2 + 2K_{t^-}(\pi_t\sigma_t^i - U_t^i)\right]\theta_t^i\Theta_{t^-}dH_t^i \end{aligned}$$

Using this decomposition, we can write explicitly the dynamics of the functional cost $J(\pi)$ for any $\pi \in \mathcal{A}$, $dJ_t(\pi) = dM_t^\pi + dV_t^\pi$:

$$dJ_t(\pi) = dM_t^\pi + \Theta_{t^-}\left[\pi_t^2a_t + 2\pi_t(b_tK_t + c_t) + 2K_t(g_t^2 - u_t) - K_t^2g_t^1 + v_t\right]dt - g_t^3dt \quad (3.20)$$

where processes are defined respectively by:

$$\begin{aligned} a_t &= \sigma_t^2 + \sum_{i\in\{A,B\}}(\sigma_t^i)^2(1+\theta_t^i)\lambda_t^i > 0, \quad b_t = \mu_t + \sigma_t\beta_t + \sum_{i\in\{A,B\}}\sigma_t^i\theta_t^i\lambda_t^i \\ c_t &= -\sigma_tZ_t - \sum_{i\in\{A,B\}}\sigma_t^iU_t^i(1+\theta_t^i)\lambda_t^i, \quad v_t = Z_t^2 + \sum_{i\in\{A,B\}}(U_t^i)^2(1+\theta_t^i)\lambda_t^i \\ u_t &= \beta_tZ_t + \sum_{i\in\{A,B\}}U_t^i\theta_t^i\lambda_t^i \end{aligned}$$

The fact that, for any π, the process $J(\pi)$ is a submartingale and the fact that there exists π^* such that $J(\pi^*)$ is a martingale imply that we should find π^* such that the finite variation part of $J(\pi^*)$ vanishes. Then for any π , $V^\pi \geq V^{\pi^*} = 0$.

Therefore the strategy π^* minimize the finite variation of V^π. Since the coefficients g^1, g^2 and g^3 do not depend on π, using the first order condition (for any $t \in [0,T]$, $\pi_t \to V_t^\pi$ is convex since $a > 0$ using Proposition 3.4), we find that

$$\pi_t^* = -\frac{b_tK_t + c_t}{a_t}, \quad t \leq T \quad (3.21)$$

where $K_t := X_t^{x,\pi} - Y_t$. Therefore using the explicit expression of the optimal strategy on (3.20), we find:

$$\begin{aligned} dJ_t(\pi) &= dM_t^{\pi^*} + \Theta_{t^-}\left[-\frac{(b_tK_t + c_t)^2}{a_t} + 2K_t(g_t^2 - u_t) - K_t^2g_t^1 + v_t\right]dt - g_t^3dt \\ &= dM_t^{\pi^*} + \Theta_{t^-}\left[-K_t^2\left(g_t^1 + \frac{b_t^2}{a_t}\right) + 2K_t\left(g_t^2 - u_t - \frac{b_tc_t}{a_t}\right)\right]dt + \left((v_t - \frac{c_t^2}{a_t})\Theta_{t^-} - g_t^3\right)dt \end{aligned}$$

then setting $g_t^1 + \frac{b_t^2}{a_t} = 0$, $g_t^2 - u_t - \frac{b_tc_t}{a_t} = 0$ and $(v_t - \frac{c_t^2}{a_t})\Theta_{t^-} - g_t^3 = 0$. Finally we find the expected results. □

Remark 3.3. *If we find the solution of the both first BSDEs then the solution of the third is given explicitly using representation theorem. Moreover we find:*

$$\xi_t = \mathbb{E}\left[\int_t^T \left((v_s - \frac{c_s^2}{a_s})\Theta_{s^-}\right) ds \Big| \mathcal{G}_t\right], \quad t \leq T.$$

In the complete market case, we have that the tracking error $\xi \equiv 0$ since the hedging is perfect.

3.2 Characterization of the VOM using BSDEs

We show in Proposition 3.8 that the Proposition 3.4 is equivalent to consider the triplet $(\frac{1}{\Theta}, \theta, \beta)$ in $\mathcal{S}^\infty \times \mathcal{S}^\infty \times \mathrm{BMO}$. This assertion leads us to construct the VOM in some complete and incomplete markets. We find also that the price of the defaultable contingent claim ψ via the VOM. We consider three different cases:

i. Complete market (where we assume $\mathbb{G} = \mathbb{F}$ and $\mathbb{G} = \mathbb{H}^A$)

ii. Incomplete market (where we consider only the case $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$).

iii. Incomplete market (where we consider the case $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$).

We have explicit solution of the VOM with respect to the process Θ in the first two cases.

3.2.1 Complete market

If we assume that $\mathbb{G} = \mathbb{F}$ (we do not consider the default impact of A and B on the asset dynamics of the firm A) or $\mathbb{G} = \mathbb{H}^A$ (we don't consider the spread of the market) then with the usual assumptions on the assets parameters, the financial market is complete. Hence, the VOM is the unique risk neutral probability and its dynamics can be found explicitly. Our goal in this part is to find the solution of the triple BSDEs given the VOM $\bar{\mathbb{P}}$.

Proposition 3.6. *Let $\bar{\mathbb{P}}$ be the VOM (the unique risk neutral probability) and let define $\bar{Z}_T$ be the Radon Nikodym density of $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$. We denote $\bar{Z}_t = \mathbb{E}\left[\bar{Z}_T|\mathcal{G}_t\right]$, then for all $t \leq T$, we have that*

$$\Theta_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}$$

Moreover for all $t \in [0, T]$ we have that

$$Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$$

Proof. We will consider the two cases

(i) $\mathbb{G} = \mathbb{F}$

(ii) $\mathbb{G} = \mathbb{H}^A$

First case: Let consider the case where $\mathbb{G}$ is equal to $\mathbb{F}$ and let the process L defined by the stochastic differential equation given by

$$dL_t = L_{t^-}\rho_t dW_t$$

where $\rho.W \in \mathrm{BMO}$, using Ito's formula we find:

$$\begin{aligned} d\left(\frac{L_t^2}{\Theta_t}\right) &= \frac{L_t^2}{\Theta_t}\left[(2\rho_t - \beta_t)dW_t + (\beta_t^2 + g_t^1 - 2\beta_t\rho_t + \rho_t^2)dt\right] \\ &= \frac{L_t^2}{\Theta_t}\left[(2\rho_t - \beta_t)dW_t + \left((\beta_t - \rho_t)^2 - (\frac{\mu_t}{\sigma_t} + \beta_t)^2\right)dt\right] \\ &= \frac{L_t^2}{\Theta_t}\left[(2\rho_t - \beta_t)dW_t + \left((-\rho_t - \frac{\mu_t}{\sigma_t})(2\beta_t + \rho_t + \frac{\mu_t}{\sigma_t})\right)dt\right] \end{aligned}$$

Then if we set for all $t \leq T$ that

$$\rho_t := -\frac{\mu_t}{\sigma_t}$$

then the process $\frac{L^2}{\Theta}$ is a true martingale using the bound condition of $\left(\frac{1}{\Theta}, \mu, \sigma\right)$ and the BMO property of β. Therefore we get:

$$\mathbb{E}\left(\frac{L_T^2}{\Theta_T}\Big|\mathcal{G}_t\right) = \frac{L_t^2}{\Theta_t}, \quad t \leq T$$

Since $\Theta_T = 1$, we find the expected result.

Moreover we obtain that $L = \bar{Z}$ which is the Radon-Nikodym of the unique risk neutral probability; $g_t^2 = -\frac{\mu_t}{\sigma_t}Z_t$, $g_t^3 = 0$, then $Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$ and $\xi_t = 0, t \leq T$.

Second case: Let now consider the case where $\mathbb{G}$ is equal to $\mathbb{H}$ and let the process L define by the stochastic differential equation given by

$$dL_t = L_{t^-}\rho_t^A dM_t^A$$

where $\rho^A.M^A \in \mathrm{BMO}$, using Ito's formula we find:

$$\begin{aligned} d\left(\frac{L_t^2}{\Theta_t}\right) &= \frac{L_{t^-}^2}{\Theta_{t^-}}\left[\left(\frac{(1+\rho_t^A)^2}{1+\theta_t^A} - 1\right)dM_t^A + \left(\frac{((\theta_t^A)^2 + (\rho_t^A)^2 - 2\rho_t^A\theta_t^A)\lambda_t^A}{1+\theta_t^A} + g_t^1\right)dt\right] \\ &= \frac{L_{t^-}^2}{\Theta_{t^-}}\left[\left(\frac{(1+\rho_t^A)^2}{1+\theta_t^A} - 1\right)dM_t^A + \frac{1}{1+\theta_t^A}\left((\rho_t^A - \theta_t^A)^2 - (\frac{\mu_t}{\sigma_t^A\lambda_t^A} + \theta_t^A)^2\right)\lambda_t^A dt\right] \\ &= \frac{L_{t^-}^2}{\Theta_{t^-}}\left[\left(\frac{(1+\rho_t^A)^2}{1+\theta_t^A} - 1\right)dM_t^A + \frac{1}{1+\theta_t^A}\left((\rho_t^A + \frac{\mu_t}{\sigma_t^A\lambda_t^A})(-2\theta_t^A + \rho_t^A - \frac{\mu_t}{\sigma_t^A\lambda_t^A})\right)\lambda_t^A dt\right] \end{aligned}$$

then if we set for all $t \leq T$

$$\rho_t^A := -\frac{\mu_t}{\lambda_t^A\sigma_t^A}$$

then the process $\frac{L^2}{\Theta}$ is a true martingale using the bound condition of $\Theta, \mu, \sigma^A, \theta^A$. Hence we get:

$$\mathbb{E}\left(\frac{L_T^2}{\Theta_T}\Big|\mathcal{G}_t\right) = \frac{L_t^2}{\Theta_t}, \quad t \leq T$$

Since $\Theta_T = 1$, we find the expected result. Moreover $L = \bar{Z}$ the Radon-Nikodym of the unique risk neutral probability and $g_t^2 = -\frac{\mu_t}{\lambda_t^A} U_t^A$, $g_t^3 = 0$, then $Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$ and $\xi_t = 0, t \le T$. □

Remark 3.4. *We can find the existence of solution of the triple BSDEs using only the explicitly given VOM.*

3.2.2 Incomplete market

In the incomplete market case the remark 3.4 doesn't hold true. The VOM depends on the dynamics of the triplet (Θ, θ, β). In the particular case where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$, we can find that the Proposition 3.6 holds true. But in the case $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$, we can not prove the existence of the VOM but we still characterize the process Θ with some martingale measure.

Proposition 3.7. *Let consider the incomplete market $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$, then the VOM (defiines the local martingale measure $\mathbb{Q}$ which minimizes the L^2-norm of $Z^{\mathbb{Q}}$), $\bar{Z}_T$ represents the Radon Nikodym density of $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$ and $\bar{Z}_t = \mathbb{E}\left[\bar{Z}_T|\mathcal{G}_t\right]$. We find for all $t \le T$*

$$\Theta_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}$$

Moreover

$$Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$$

In more general case, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$, we can only prove there exists a martingale measure $\bar{\mathbb{P}}$ such that:

$$\Theta_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}, \quad t \le T.$$

and

$$Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right], \quad t \le T.$$

Proof. **First step:**
We consider the case where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$. Let consider $\mathbb{Q}$ is a martingale measure for D^A and let define $Z_T^{\mathbb{Q}}$ its Radon Nikodym density with respect to $\mathbb{P}$ on $\mathcal{G}_T$. we define the process $Z_t^{\mathbb{Q}} = \mathbb{E}\left[Z_T^{\mathbb{Q}}|\mathcal{G}_t\right]$. Using martingale theorem representation there exists two $\mathbb{G}$-predictable processes ρ^A and ρ such that

$$dZ_t = Z_{t^-}\left[\rho_t^A dM_t^A + \rho_t dW_t\right]$$

Using Ito's formula, we find:

$$d\left(\frac{(Z_t^{\mathbb{Q}})^2}{\Theta_t}\right) = \frac{\left(Z_{t^-}^{\mathbb{Q}}\right)^2}{\Theta_{t^-}}\left[\left(\frac{(1+\rho_t^A)^2}{1+\theta_t^A} - 1\right) dM_t^A + (2\rho_t - \beta_t)dW_t + j_t dt\right] \tag{3.22}$$

where $j_t = (\rho_t - \beta_t)^2 + \frac{(\rho_t^A - \theta_t^A)^2}{1+\theta_t^A}\lambda_t^A + g_t^1$. Since $\mathbb{Q}$ is a martingale measure for D^A we get using (2.8) that

$$\mu_t^A + \rho_t^A \sigma_t^A \lambda_t^A + \rho_t \sigma_t = 0$$

Hence using this equation we can find ρ^A using ρ and plotting this result on the expression of j. We find:

$$j_t = (\rho_t - \beta_t)^2 + \frac{(\mu_t + \sigma_t\rho_t + \sigma_t^A\theta_t^A\lambda_t^A)^2}{(1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A} - \frac{(\mu_t + \beta_t\sigma_t + \theta_t^A\sigma_t^A\lambda_t^A)^2}{(1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A + \sigma_t^2}$$

Let now define

$$\bar{\rho}_t = \rho_t - \beta_t, \quad \bar{a}_t = \sigma_t^2 + (1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A$$

and

$$\bar{b}_t = \mu_t + \sigma_t\beta_t + \sigma_t^A\theta_t^A\lambda_t^A$$

then we get:

$$j_t = \frac{1}{(1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A}\left[\bar{a}_t\bar{\rho}_t + 2\bar{\rho}_t\bar{b}_t\sigma_t + \frac{\bar{b}_t^2\sigma_t^2}{\bar{a}_t}\right] = \frac{\bar{a}_t}{(1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A}\left(\bar{\rho}_t + \frac{\bar{b}_t\sigma_t}{\bar{a}_t}\right)^2$$

Then with (3.22), $j \geq 0$ and the process $\frac{(Z^{\mathbb{Q}})^2}{\Theta}$ is a submartingale (since $Z^{\mathbb{Q}}$ is a martingale and $\frac{1}{\Theta} \in \mathcal{S}^\infty$). We deduce $\mathbb{E}\left[\frac{(Z_T^{\mathbb{Q}})^2}{\Theta_T}\right] \geq \frac{(Z_0^{\mathbb{Q}})^2}{\Theta_0}$, since $\Theta_T = 1$ and $Z_0^{\mathbb{Q}} = 1$.

Finally we get for any martingale measure for D^A that $\mathbb{E}\left[(Z_T^{\mathbb{Q}})^2\right] \geq \frac{1}{\Theta_0}$. Moreover if we set $\bar{\rho}_t = -\frac{\bar{b}_t\sigma_t}{\bar{a}_t}$, then $\bar{Z}$ is a true martingale measure since $(\Theta, \theta, \beta) \in \mathcal{S}^\infty \times \mathcal{S}^\infty \times \text{BMO}$ and μ, σ^A, σ^B are bounded (the process b,a, ρ and ρ^A are bounded). We call $\bar{\mathbb{P}}$ the martingale measure under this condition then $\mathbb{E}\left[\bar{Z}_T^2\right] = \frac{1}{\Theta_0}$. We deduce $\bar{\mathbb{P}}$ is the martingale measure which minimizes the L^2-norm of Z and $\bar{\Theta}_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}, t \leq T$. Using the explicit expression of ρ we find:

$$\rho_t = -\frac{\sigma_t\bar{b}_t}{\bar{a}_t} + \beta_t$$
$$\rho_t^A = -\frac{(1+\theta_t^A)\sigma_t^A\bar{b}_t}{\bar{a}_t} + \theta_t^A$$

Moreover since

$$\begin{aligned} g_t^2 &= \frac{-\bar{b}_t(\sigma_t Z_t + (1+\theta_t^A)U_t^A\sigma_t^A\lambda_t^A)}{\bar{a}_t} + \beta_t Z_t + U_t^A\lambda_t^A \\ &= Z_t\left(-\frac{\bar{b}_t\sigma_t}{\bar{a}_t} + \beta_t\right) + U_t^A\left(-\frac{(1+\theta_t^A)\sigma_t^A\bar{b}_t}{\bar{a}_t} + \theta_t^A\right)\lambda_t^A \\ &= Z_t\rho_t + U_t^A\rho_t^A\lambda_t^A \end{aligned}$$

then we conclude that $Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$. Therefore the characterization of the price of ψ (using Mean-Variance approach) and the VOM in this incomplete case is well defined using the triplet (Θ, θ, β) associated to the first BSDE.

Second step:

We consider the general case where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$. Let consider $\mathbb{Q}$ is a martingale measure for D^A and let define $Z_T^{\mathbb{Q}}$ its Radon Nikodym density with respect to $\mathbb{P}$ on $\mathcal{G}_T$. We define the process $Z_t^{\mathbb{Q}} = \mathbb{E}\left[Z_T^{\mathbb{Q}}|\mathcal{G}_t\right]$. Using martingale theorem representation there exists two $\mathbb{G}-$ predictable processes ρ^A, ρ^B and ρ such that

$$dZ_t = Z_{t^-}\left[\rho_t^A dM_t^A + \rho_t^B dM_t^B + \rho_t dW_t\right]$$

Using Ito's formula, we find:

$$d\left(\frac{(Z_t^{\mathbb{Q}})^2}{\Theta_t}\right) = \frac{(Z_{t^-}^{\mathbb{Q}})^2}{\Theta_{t^-}}\left[\sum_{i\in\{A,B\}}\left(\frac{(1+\rho_t^i)^2}{1+\theta_t^i}-1\right)dM_t^i + (2\rho_t-\beta_t)dW_t + j_t dt\right]$$

where $j_t = (\rho_t-\beta_t)^2 + \sum_{i\in\{A,B\}}\frac{(\rho_t^i-\theta_t^i)^2}{1+\theta_t^i}\lambda_t^i + g_t^1$. Since $\mathbb{Q}$ a martingale measure for D^A we get by (2.8)

$$\mu_t^A + \sum_{i\in\{A,B\}}\rho_t^i\sigma_t^i\lambda_t^i + \rho_t\sigma_t = 0$$

Hence using this equation, we can find ρ^A using ρ and ρ^B and plotting this result on the expression of j. Let first recall a notation:

$$a_t = \sigma_t^2 + \sum_{i\in\{A,B\}}(1+\theta_t^i)(\sigma_t^i)^2\lambda_t^i \quad and \quad b_t = \mu_t + \sigma_t\beta_t + \sum_{i\in\{A,B\}}\theta_t^i\sigma_t^i\lambda_t^i$$

then we find:

$$\begin{aligned}
C_t :=&(1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A j_t \\
&= (\rho_t-\beta_t)^2[\sigma_t^2 + (1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A] + \frac{(\rho_t^B-\theta_t^B)^2}{1+\theta_t^B}\lambda_t^B\left[\sum_{i\in\{A,B\}}(1+\theta_t^i)(\sigma_t^i)^2\lambda_t^i\right] \\
&+ \frac{b_t^2}{a_t}\left[\sigma_t^2 + (1+\theta_t^B)(\sigma_t^B)^2\lambda_t^B\right] + 2(\rho_t^B-\theta_t^B)(\rho_t-\beta_t)\sigma_t\sigma_t^B\lambda_t^B \\
&+ 2b_t\left[(\rho_t-\beta_t)\sigma_t + (\rho_t^B-\theta_t^B)\sigma_t^B\lambda_t^B\right]
\end{aligned}$$

Then from the two first term we add and move an additional process to find the process a, we get:

$$\begin{aligned}
C_t \;=\; & \left[(\rho_t-\beta_t)^2 a_t + \frac{b_t^2}{a_t}\sigma_t^2 + 2b_t(\rho_t-\beta_t)\sigma_t\right] + (1+\theta_t^B)\lambda_t^B\left[\frac{(\rho_t^B-\theta_t^B)^2}{(1+\theta_t^B)^2}a_t + 2b_t\sigma_t^B\frac{\rho_t^B-\theta_t^B}{1+\theta_t^B}\right. \\
&\left.+\frac{b_t^2}{a_t^2}(\sigma_t^B)^2\right] + (1+\theta_t^B)\lambda_t^B\left[2(\rho_t-\beta_t)\frac{(\rho_t^B-\theta_t^B)}{(1+\theta_t^B)}\sigma_t\sigma_t^B - (\rho_t-\beta_t)^2(\sigma_t^B)^2 - \frac{(\rho_t^B-\theta_t^B)^2}{(1+\theta_t^B)^2}\sigma_t^2\right]
\end{aligned}$$

Finally we find a more explicit expression of the coefficient j:

$$\begin{aligned}
C_t \;=\; & a_t\left[\left((\rho_t-\beta_t) + \frac{b_t}{a_t}\sigma_t\right)^2 + (1+\theta_t^B)\lambda_t^B\left(\frac{(\rho_t^B-\theta_t^B)}{1+\theta_t^B} + \frac{b_t\sigma_t^B}{a_t}\right)^2\right] \\
& -(1+\theta_t^B)\lambda_t^B(\sigma_t^B)^2\left((\rho_t-\beta_t) - \frac{\sigma_t}{\sigma_t^B}\frac{\rho_t^B-\theta_t^B}{1+\theta_t^B}\right)^2
\end{aligned}$$

It follows that if we set $\rho_t - \beta_t := -\frac{b_t}{a_t}\sigma_t$ and $\rho_t^B - \theta_t^B := -(1+\theta_t^B)\sigma_t^B\frac{b_t}{a_t}$, then we find $j = 0$ and $\rho_t^A - \theta_t^A = -\sigma_t^A\frac{b_t}{a_t}$.

Since $(\frac{1}{\Theta}, \theta, \beta) \in \mathcal{S}^\infty \times \mathcal{S}^\infty \times \mathrm{BMO}$ and μ, σ^A, σ^B and σ are bounded then the processes b, a, ρ, ρ^A and ρ^B are bounded. Therefore, we deduce there exists a martingale measure $\bar{\mathbb{P}}$ such that

$$\delta \leq \Theta_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}, \quad t \leq T.$$

Moreover we find for all $t \leq T$ that

$$g_t^2 = Z_t\rho_t + \sum_{i\in\{A,B\}} U_t^i\rho_t^i\lambda_t^i$$

then $Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$.

To identify that $\bar{\mathbb{P}}$ is the VOM in this case, we should prove that $j \geq 0$ as in the first case; but from the last expression of j, we can't prove that this condition holds true. We remark that the assertion of VOM will be justify if one of the following equality is satisfied:

$$\sigma_t^B(\rho_t - \beta_t) = \sigma_t\frac{\rho_t^B - \theta_t^B}{1+\theta_t^B}, \quad \sigma_t^A\frac{\rho_t^B - \theta_t^B}{1+\theta_t^B} = \sigma_t^B\frac{\rho_t^A - \theta_t^A}{1+\theta_t^A} \quad \text{and} \quad \sigma_t^A(\rho_t - \beta_t) = \sigma_t\frac{\rho_t^A - \theta_t^A}{1+\theta_t^A}.$$

□

3.3 Existence of BSDE's solution

We prove in this part the existence of the triplet (Θ, θ, β). Note that to prove the existence of the triple, we dont need the assumption that the VOM exists and should satisfied the $\mathbb{R}_2(\mathbb{P})$ condition (this assumption implies the Radon-Nikodym of the VOM $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$ is non-negative). Proving that the triple exists justify from Proposition 3.7 that the VOM exists. Moreover if the triple is defined such that $\bar{Z}$ is non negative implies that $\bar{\mathbb{P}}$ satisfies the $R_2(\mathbb{P})$ condition.

Proposition 3.8. *The process* $(\frac{1}{\Theta}, \theta, \beta) \in \mathcal{S}^\infty \times \mathcal{S}^\infty \times \mathrm{BMO}$. *Moreover there exists* δ *such that:*

$$\delta \leq \Theta_t \leq 1, \quad t \in [0, T].$$

Proof. The proof is a consequence of Proposition 3.4 and Proposition 3.7. Indeed from Proposition 3.7 , we deduce $\frac{1}{\Theta} \in S^\infty$. Moreover since $\Theta > 0$ and from the equation:

$$\Theta_t = \Theta_T + \int_t^T \Theta_{s-}g_s^1(\Theta_s, \theta_s, \beta_s)ds - \int_t^T \Theta_{s-}\theta_s.dM_s - \int_t^T \Theta_{s-}\beta_s dW_s.$$

we conclude $\delta \leq \Theta_t \leq \mathbb{E}\left[\Theta_T|\mathcal{G}_t\right] \leq 1$. We can remark that we find the same conclusion $(\Theta \leq 1)$ as Jeanblanc and al. [10]. Actually, since the process Θ is bounded, from Lemma 3.1 in Appendix we conclude that the process $(\Theta_{t-}\theta_t)_{t\leq T}$ is bounded. Therefore since $\Theta \geq \delta > 0$, we find that the process θ is bounded too. To prove that $\beta \in \mathrm{BMO}$, we use Ito formula to:

$$\int_\tau^T \Theta_t^2\beta_t^2 dt + \int_\tau^T \Theta_t^2 \sum_{i\in\{A,B\}} (\theta_t^i)^2\lambda_t^i dt \leq \Theta_T^2 - \theta_\tau^2 + \int_\tau^T 2\Theta_{t-}g_t^1 dt + M_T - M_\tau.$$

where τ is a $\mathbb{G}$-stopping times and M is a martingale (from Doob's decomposition of the submartinagle Θ), since $g^1 < 0$ and $0 < \delta \leq \Theta \leq 1$ then taking the conditional expectation we find:

$$\mathbb{E}\left[\int_\tau^T \beta_t^2 dt \Big| \mathcal{G}_\tau\right] \leq \frac{1}{\delta^2}(1-\delta^2)$$

We conclude $\beta.W \in \mathrm{BMO}$ (for simplify notation $\beta \in \mathrm{BMO}$). □

Actually we can prove the existence of the triple BSDEs since we have more informations on the space of solution.

Theorem 3.2. *There exists a triplet* $(\Theta, \theta, \beta) \in \mathcal{S}^\infty \times \mathcal{S}^\infty \times \mathrm{BMO}$ *solution of the BSDE*

$$\frac{d\Theta_t}{\Theta_{t^-}} = -g_t^1(\Theta_t, \theta_t, \beta_t)dt + \theta_t.dM_t + \beta_t dW_t, \qquad \Theta_T = 1.$$

Moreover given the triplet (Θ, θ, β)*, we can prove the existence of solutions of the two BSDEs* (g^2, ψ) *and* $(g^3, 0)$*.*

Proof. The proof is divided in two parts. In the first part, we give a general bound for the coefficient of the first BSDE and in the second part we deal with splitting technics to deal with the existence of the BSDE.

First step: Let define $\bar{g}_t = \Theta_{t^-} g_t^1$, $\bar{\theta}_t^i = \Theta_{t^-}\theta_t^i$ for $i \in \{A, B\}$ and $\bar{\beta}_t = \Theta_{t^-}\beta_t$, $t \leq T$ and let define the BSDE $(\bar{g}, \Theta_T)$:

$$d\Theta_t = -\bar{g}_t dt + \bar{\theta}_t.dM_t + \bar{\beta}_t dW_t.$$

Let define some properties of $\bar{g}$:

$$\begin{aligned}
|\bar{g}_t(\Theta_t, \theta_t, \beta_t)| &= \Theta_{t^-} \frac{\left[\mu_t + \sum_{i\in\{A,B\}} \theta_t^i \sigma_t^i \lambda_t^i + \sigma_t \beta_t\right]^2}{\sigma_t^2 + \sum_{i\in\{A,B\}}(1+\theta_t^i)(\sigma_t^i)^2 \lambda_t^i} \\
&\leq 3\Theta_{t^-}\left[\frac{\mu_t^2}{\sigma_t^2} + \frac{\left(\sum_{i\in\{A,B\}} \theta_t^i \sigma_t^i \lambda_t^i\right)^2}{\sum_{i\in\{A,B\}}(1+\theta_t^i)(\sigma_t^i)^2\lambda_t^i} + \beta_t^2\right] \\
&\leq 6\Theta_{t^-}\left[\frac{\mu_t^2}{\sigma_t^2} + \beta_t^2 + (1+\theta_t^A)\lambda_t^A + (1+\theta_t^B)\lambda_t^B\right]
\end{aligned}$$

since the process μ, σ and Θ are bounded then there exists a non negative constant $C > 0$ such that:

$$|\bar{g}_t(\Theta_t, \theta_t, \beta_t)| \leq C\left[1 + \Theta_{t^-}\beta_t^2 + \sum_{i\in\{A,B\}}(1+\Theta_{t^-}\theta_t^i)\lambda_t^i\right]$$

Since the processes $(\Theta_{t^-}\theta_t^i)_{t\leq T}$ and λ^i, $i \in \{A, B\}$ are bounded then there exists a non-negative constant C such that:

$$|\bar{g}_t(\Theta_t, \bar{\theta}_t, \bar{\beta}_t)| \leq C\left(1 + \frac{|\bar{\beta}_t|^2}{\delta}\right). \tag{3.23}$$

We can split the BSDE on different continuous BSDE (see Kharroubi and Lim [12]) and since the bound of the coefficient does not depend on the jump process then the different coefficients of the splitting BSDEs will satisfy a quadratic growth.

Second step: (The splitting of the jump BSDE):

First we define $\tau^1 = \tau^A \wedge \tau^B$, $\tau^2 = \tau^A \vee \tau^B$ and

$$\Delta_k = \{(l_1, \cdots l_k) \in (\mathbb{R}^+)^k : l_1 \leq \cdots l_k\}, \quad 1 \leq k \leq 2.$$

Since we work with the same assumption (density assumption) and notation as in [12] then we can decomposed Θ_T and $\bar{g}$ such that:

$$\Theta_T \;=\; \gamma^0 1_{\{\{0\leq T\leq \tau_1\}\}} + \gamma^1(\{\tau^1\})1_{\{\tau^1\leq T\leq \tau^2\}} + \gamma^2(\tau^1,\tau^2)1_{\{\tau^2>T\}}$$

and

$$\bar{g}_t(\Theta_t, \bar{\theta}_t, \bar{\beta}) \;=\; \bar{g}^0_t(\Theta, \bar{\theta}, \bar{\beta})1_{\{0\leq T\leq \tau_1\}} + \bar{g}^1_t(\Theta, \bar{\theta}, \bar{\beta}, \tau^1)1_{\{\tau^1\leq T\leq \tau^2\}} + \bar{g}^2_t(\Theta, \bar{\theta}, \bar{\beta}, \tau^1, \tau^2)1_{\{T>\tau^2\}}$$

where γ^0 is $\mathcal{F}_T$-measurable, γ^k is $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k)$-measurable for $k = 1, 2$ and $\bar{g}^0$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable, $\bar{g}^k$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\Delta^k)$. Moreover the variables $\gamma^k \in (0,1)$ for $k = \{0,1,2\}$ since $\Theta_T \in (0,1)$ (see proposition 3.1 in Kharroubi and Lim [12]).

Let now give the main result of splitting BSDE which is a first step to prove the existence of the triplet $(\Theta, \bar{\theta}, \bar{\beta})$. Let $l \in \Delta_2$ and let assume the following BSDEs:

$$d\Theta^2_t(l) = -\bar{g}^2_t(\Theta^2_t(l), 0, \beta_t(l))dt + \beta_t(l)dW_t, \quad \Theta^2_T(l) = \gamma^2(l). \tag{3.24}$$

admits a solution $(\Theta^2_t(l), \beta_t(l)) \in S^\infty([l_2 \wedge T, T]) \times \mathcal{H}^2[l_2 \wedge T, T]$ and for $k = \{0, 1\}$

$$d\Theta^k_t(l_{(k)}) = -\bar{g}^k_t\left[\Theta^{k+1}_t(l_{(k)}), (\Theta^k_t - \Theta^{k-1}_t)(l_{(k)}), \bar{\beta}_t(l_{(k)})\right]dt + \bar{\beta}_t(l_{(k)})dW_t, \quad \Theta^k_T(l_{(k)}) = \gamma^k(l_{(k)}). \tag{3.25}$$

admits a solution $\left(\Theta^k(l_{(k)}, \bar{\beta}^k(l_{(k)})\right) \in \mathcal{S}^\infty([l_k \wedge T, T]) \times \mathcal{H}^2[l_k \wedge T, T]$. where $l_{(k)} = (l_1, \cdots l_k)$

then the triple $(\Theta, \bar{\theta}, \bar{\beta})$ is given by:

$$\begin{aligned} \Theta_t &= \Theta^0_t 1_{\{t<\tau^1\}} + \Theta^1_t(\tau^1)1_{\{\tau^1\leq t\leq \tau^2\}} + \Theta^2_t(\tau^1,\tau^2)1_{\{\tau^2<t\}} \\ \bar{\beta}_t &= \bar{\beta}^0_t 1_{\{t<\tau^1\}} + \bar{\beta}^1_t(\tau^1)1_{\{\tau^1\leq t\leq \tau^2\}} + \bar{\beta}^2_t(\tau^1,\tau^2)1_{\{\tau^2<t\}} \\ \bar{\theta}_t &= \bar{\theta}^0_t 1_{\{t<\tau^1\}} + \bar{\theta}^1_t(\tau^1)1_{\{\tau^1\leq t\leq \tau^2\}} \end{aligned} \tag{3.26}$$

where $\bar{\theta}_t(\tau_{(k)}) = \Theta^{k+1}_t(\tau_{(k)}, t) - \Theta^k_t(\tau_{(k)})$ and $\tau_{(k)} = (\tau_1, \cdots \tau_k)$, for $k = \{0, 1\}$.

Therefore, to prove the existence of the triple (Θ, θ, β) we should prove the existence of the BSDEs (3.24) and (3.25). Firstly, let remark for $k = \{0, 1, 2\}$, $\gamma^k \in (0, 1)$ and $0 \leq \Theta^k(l_{(k)}) \leq 1$. Secondly, from the boundness of $\bar{g}$ from (3.23) we deduce using the same arguments of [12](proof of Proposition 3.1 step 2) that there exists a non negative constant such that:

$$|\bar{g}^k_t(y, z, u, l_{(k)})| \leq C\left(1 + |z|^2\right).$$

To resume, to prove the existence of the BSDEs (3.24) and (3.25) it is sufficient to prove the existence of the BSDE:

$$dx_t = -f_t(x_t, z_t) + z_t dW_t, \quad x_T = \gamma \in (0,1) \tag{3.27}$$

where $|f_t(x,z)| \le C\left(1+|z|^2\right)$. From Kobylanski result see [13], there exist a pair $(X,Z) \in \mathcal{S}^\infty \times$ BMO maximal solution, then for $k = \{0,1,2\}$, there exists a solution (Θ^k, Z^k) associated to (g^k, γ^k). Therefore we conclude the existence of the triple $(\Theta, \bar{\theta}, \bar{\beta}) \in S^\infty \times S^\infty \times$ BMO using (3.26). □

3.4 Special case and explicit solution of the BSDE

We assume $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$ and that the parameters of the asset's dynamics are constant before and after the default time τ^A. Moreover, we assume that the intensity process is given by $\lambda_t = \lambda(1 - H_t^A)$. Using theses assumptions, we find an explicit solution of the BSDE associated to $(\bar{g}, \Theta_T)$ using the splitting approach.

Assumption 3.3. *The processes $\mu, \sigma, \sigma^A, \lambda$ in (1.2) satisfy the following sassumptions:*

$$\begin{aligned}
\mu_t &= \mu(H_t) = \mu(0)1_{\{\tau^A > t\}} + \mu(1)1_{\{\tau^A \le t\}}, \\
\sigma_t &= \sigma(H_t) = \sigma(0)1_{\{\tau^A > t\}} + \sigma(1)1_{\{\tau^A \le t\}}, \\
\sigma_t^A &= \sigma^A(H_t) = \kappa 1_{\{\tau^A > t\}}, \\
\lambda_t &= \lambda(H_t) = \lambda 1_{\{\tau^A > t\}}.
\end{aligned}$$

such that $\mu(0)\kappa = \sigma(0)^2 + \kappa^2\lambda$.

Proposition 3.9. *Under Assumption 3.3, there exists a solution of the BSDE associated to $(\bar{g}, \Theta)$ given by:*

$$\Theta_t = \exp\left[-\frac{\mu(0)}{\kappa}(T-t)\right]1_{\{\tau^A > t\}} + \exp\left[-\left(\frac{\mu(1)}{\sigma(1)}\right)^2 (T-t)\right]1_{\{\tau^A \le t\}}, \quad t \le T.$$

Proof. To prove Proposition 3.9, let first recall that using the splitting approach developped by [12], we can write the BSDE before and after the default. Let recall that in our case:

$$\begin{aligned}
\Theta_t &= \Theta_t^0 1_{\{t < \tau^A\}} + \Theta_t^1(\tau^A)1_{\{\tau^A \le t\}} \\
g_t^1 &= g_t^{1,0}1_{\{t < \tau^A\}} + g_t^{1,1}1_{\{\tau^A \le t\}}
\end{aligned}$$

where Θ^0 and Θ^1 satisfy the following dynamics:

$$\begin{aligned}
-\frac{d\Theta_t^0}{\Theta_t^0} &= g_t^{1,0}(\Theta_t^0, \theta_t^A, \beta_t^0)dt - \beta_t^0 dW_t + \lambda\theta_t^A dt, \quad \Theta_T^0 = 1, \\
-\frac{d\Theta_t^1(l)}{\Theta_t^1(l)} &= g_t^{1,1}(\Theta_t^1(l), 0, \beta_t^1(l))dt - \beta_t^1(l)dW_t, \quad \Theta_T^1(l) = 1
\end{aligned}$$

with

$$g_t^{1,0}(\Theta_t^0,\theta_t^A,\beta_t^0) = -\frac{\left[\mu(0)+\theta_t^A\kappa\lambda+\sigma(0)\beta_t^0\right]^2}{\sigma(0)^2+(1+\theta_t^A)\kappa^2\lambda} \quad \text{and} \quad g_t^{1,1}(\Theta_t^1(l),0,\beta_t^1(l)) = -\frac{\left[\mu(1)+\sigma(1)\beta_t^1(l)\right]^2}{\sigma(1)^2}$$

where $l\in\Delta_1$ and $\Theta_t^1(t)-\Theta_t^0=\theta_t^A\Theta_t^0$, see proof of Theorem 3.2 for more details. Using Assumption 3.3, we set $\beta^1(l)=0$ and we find $g_t^{1,1}(\Theta_t^1,0,\beta_t^1(l)) = -\left(\frac{\mu(1)}{\sigma(1)}\right)^2$, since $\Theta_T^1(l)=1$, then $\Theta^1(l)=\Theta^1$, we get:

$$\Theta_t^1 = \exp\left[-\left(\frac{\mu(1)}{\sigma(1)}\right)^2(T-t)\right], \quad t\le T.$$

To find the solution of the first one, we set $\beta^0=0$ and from Assumption 3.3

$$\mu(0)\kappa = \sigma(0)^2+\kappa^2\lambda$$

we deduce $g_t^{1,0}(\Theta_t^0,\theta_t^A,\beta_t^0) = -\frac{\mu(0)}{\kappa}-\theta_t^A\lambda$. Therefore we find that Θ^0 satisfies the dynamics:

$$-\frac{d\Theta_t^0}{\Theta_t^0} = -\frac{\mu(0)}{\kappa}dt, \quad \Theta_T^0=1.$$

we get $\Theta_t^0=\exp\left[-\frac{\mu(0)}{\kappa}(T-t)\right]$ and we find the expected result. □

Appendix

Lemma 3.1. *Let consider X and Y two $\mathbb{G}$-predictable processes such that for $i\in\{A,B\}$, $Y_{\tau_i}=X_{\tau_i}$. Then, $X_t=Y_t$ on $(\tau_i\ge t)$ a.s. Moreover, if $X_{\tau_i}\le Y_{\tau_i}$, then $X_t\le Y_t$ a.s on $(\tau_i\ge t)$.*

Proof. Assume that X and Y are bounded. If $X_{\tau_i}=Y_{\tau_i}$, then $\int_0^\infty |X_t-Y_t|dH_t^i=0$ and

$$0=\mathbb{E}\left(\int_0^\infty |X_t-Y_t|dH_t^i\right)=\mathbb{E}\left[\int_0^\infty |X_t-Y_t|\lambda_t^i dt\right].$$

Therefore, we have $X_t=Y_t$ on $(\tau^i\ge t)$. Moreover, if $X_{\tau_i}\le Y_{\tau_i}$, we consider the predictable process V defined as $V_t=Y_t 1_{\{X_t\le Y_t\}}$. Then $V_{\tau^i}=Y_{\tau^i}$ and by using the first part of the proof, we obtain $V_t=Y_t$ on $(\tau^i\ge t)$. The general case follows. □

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