On separated Carleson sequences in the ball of $\mathbb{C}^n$.

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Abstract

A. Hartmann proved recently that just one bounded holomorphic function in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, was enough to characterize interpolating sequences for $H^\infty(\mathbb{D})$. In this work we generalize this to the case of the unit ball in $\mathbb{C}^n$. Precisely to a sequence S of points in $\mathbb{B}$ we associate canonically a pair $S_1,\ S_2$ of points in $\mathbb{B}$ such that : if there is a bounded holomorphic function f on $\mathbb{B}$ and a $\delta < 1$ with $|f| \leq \delta$ on S_1 and $|f| \geq 1$ on S_2, then S is a separated Carleson sequence.

In one variable we prove that this condition is also necessary. Because in one variable separated Carleson sequence is equivalent to interpolating sequence, we recover a characterization of interpolating sequences just by one function.

Contents

1 Introduction.

Let $\mathbb{D}$ the unit disc in $\mathbb{C}$ and S a sequence of points in $\mathbb{D}$.

We consider the hyperbolic distance in $\mathbb{D}$:

$$d_H(a,\ b) := \text{arccos}\ (1 + \frac{2\left|a-b\right|^2}{(1-\left|a\right|^2)(1-\left|b\right|^2)}).$$

To say that the sequence S is separated means that there is a $\delta > 0$ such that

$$\forall a,b \in S,\ a \neq b,\ d_H(a,b) \geq \delta \iff \left|\frac{a-b}{1-\overline{a}b}\right| \geq \delta' \iff |a-b| \leq$$

To say that the sequence S is Blaschke means that $\displaystyle\sum_{a\in S}(1-\left|a\right|)<\infty$ and this implies that the Blaschke product $B_S(z):=\displaystyle\prod_{a\in S}\frac{a-z}{1-\bar{a}z}\frac{\left|a\right|}{a}$ converges in $\mathbb{D}$ to a function in $H^{\infty}(\mathbb{D})$, i.e. B_S is holomorphic and bounded in $\mathbb{D}$ and is zero exactly on S.

We shall also need the notion of Carleson measure. Let $(\zeta,\ h)\in\mathbb{T}\times(0,1)$, and note

$W(\zeta,h):=\{z\in\mathbb{D}::\left|1-\bar{\zeta}z\right|<h\}$ the associated Carleson window.

If ν is a borelian measure in $\mathbb{D}$, we shall say that ν is Carleson if there is a constant $C>0$ such that

$$\forall\zeta\in\mathbb{T},\ \forall h\in(0,1),\ \left|\nu\right|(W(\zeta,h))\leq Ch.$$

If ν is Carleson, we have the embedding Carleson theorem :

$$\forall p\geq 1,\ \forall g\in H^p,\ \int_{\mathbb{D}}\left|g\right|^p\,d\left|\nu\right|\leq C\left\|g\right\|_{H^p}^p,$$

i.e. $H^p\subset L^p(\left|\nu\right|)$ in a continuous way. A sequence S will be a Carleson sequence if the canonical measure associated to it :

$\mu_S:=\sum_{a\in S}(1-\left|a\right|)\delta_a$ is a Carleson measure.

We shall use the following theorem, basis of the duality H^1-BMOA,

$f\in H^{\infty}(\mathbb{D})\Rightarrow(1-\left|z\right|)\left|\partial f(z)\right|^2\,dm(z)$ is a Carleson measure.

(This implication is still true for $f\in BMOA$.)

Definition 1 *We shall say that S is interpolating (for H^{∞}) if*

$$\forall\lambda\in\ell^{\infty}(S),\ \exists f\in H^{\infty}::\forall a\in S,\ f(a)=\lambda_a.$$

L. Carleson characterized these sequences for $H^{\infty}(\mathbb{D})$ by the condition :

$$\inf_{a\in S}\prod_{b\in S\backslash\{a\}}\left|\frac{a-b}{1-\bar{b}a}\right|>0.$$

in particular this sequence S is Blaschke and separated.

One can see easily that this condition is equivalent to the fact that S is *dual bounded* in $H^{\infty}(\mathbb{D})$, which means :

$$\exists C>0,\ \forall a\in S,\ \exists\rho_a\in H^{\infty}(\mathbb{D}),\ \left\|\rho_a\right\|_{\infty}\leq C::\forall b\in S,\rho_a(b)=\delta_{ab}.$$

We just take $\rho_b(z):=\dfrac{B_b(z)}{B_b(b)}$ with $B_b(z):=\prod_{a\in S\backslash\{b\}}\frac{a-z}{1-\bar{a}z}\frac{\left|a\right|}{a}$.

So the *metric condition* which caracterises the interpolation is equivalent to the existence of an infinity of functions verifying the above conditions.

A. Hartmann [3] showed that this can be reduced to a condition on **only one** function :

Theorem 1 *Let S be a separated Blaschke sequence in the unit disc $\mathbb{D}$ of $\mathbb{C}$. There is a partition (S_1,S_2) of S such that if there is a function $f\in H^{\infty}(\mathbb{D})$ with*

$f=0$ on S_1 and $f=1$ on S_2,

then S is interpolating for $H^{\infty}(\mathbb{D})$.

The aim of this work is to show that we can weakened the condition on the function and suppress the two conditions on the sequence, these conditions will be automatically fulfilled.

We shall extend this in the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. In this setting N. Varopoulos [5] proved that if the sequence S is $H^\infty(\mathbb{B})$ interpolating, then S is a separated Carleson sequence. P. Thomas [4] proved that if S is $H^p(\mathbb{B})$ interpolating, which is weaker than $H^\infty(\mathbb{B})$, then S is a separated Carleson sequence. A better result was obtained in [2] : if the sequence S is just dual bounded in $H^p(\mathbb{B})$, then it is separated and Carleson. Again in the ball dual boundedness involves an infinity of functions and in this work we shall show that just one function is enough.

In the sequel $d_H(a,b)$ will be the hyperbolic distance in $\mathbb{B}$.

Definition 2 *We shall say that the partition (S_1,S_2) of S is good if there is $\varphi : S_1 \to S_2$ such that $\forall a \in S_1,\ d_H(a,\varphi(a)) = \inf_{c\in S\setminus\{a\}} d_H(a,c)$ and if there is $\psi : S_2 \to S_1$ such that $\forall b \in S_2,\ d_H(b,\psi(b)) = \inf_{c\in S\setminus\{b\}} d_H(b,c)$.*
The partition (S_1,S_2) is "very good" if moreover

$$M_1 := \sup_{a\in S_1} d_H(a,\varphi(a)) < \infty,\ M_2 := \sup_{b\in S_2} d_H(b,\psi(b)) < \infty.$$

We shall show that any discrete sequence in $\mathbb{B}$ (in fact in any metric space) admits a good partition (S_1,S_2) and if the partition is not very good then we shall add points to it $S'_j \supset S_j,\ j=1,2$ to make the new pair (S'_1,S'_2) a very good partition of $S' = S'_1 \uplus S'_2$.

Definition 3 *The sequence $S \subset \mathbb{B}$ is ultra-separated if there is a function f in $H^\infty(\mathbb{B})$ such that $|f| \leq \delta 1$ on S'_1 and $|f| \geq 1$ on S'_2 for any very good partition (S'_1,S'_2) associated to the initial sequence S.*

Now we can state the main theorem.

Theorem 2 *If the sequence S is ultra-separated in the ball $\mathbb{B}$, then it is Carleson and separated. In the unit disc of $\mathbb{C}$ the converse is also true.*

2 Proofs.

The following lemma says that there are always good partitions for a discrete sequence.

Lemma 1 *Let S be a discrete sequence in the metric space $(X,\ d)$. There is a good partition (S_1,S_2) of S.*

Proof
Take a point $O \in X$ and $a_1 \in S$ such that $d(a_1,\ O)$ is minimal, then $b_1 \in S$ a nearest neighbour for the distance d of a_1 and define $\varphi(a_1) = b_1$. Take a_2 a nearest neighbour of $b_1$and define $\psi(b_1) := a_2$; if $a_2 = a_1$ we stop at this "perfect" pair (a_1,b_1) with $\psi(b_1) := a_1$. If not we continue with b_2 nearest neighbour of a_2 etc... We stop at a perfect pair. This way we get a branch B_1 finite or infinite. We put all the "a" in S_1 and all the "b" in S_2.
If it remains points in S we have that the points in $S\setminus B_1$ are far from the points in B_1 by construction. We take a point c in $S\setminus B_1$ the nearest from O.

A) If all the nearest points from c are in B_1, which may happen, we take one of them, d, now if d is in S_1, we put c in S_2 and we set $\psi(c) := d$. If d is in S_2, we put c in S_1 and we set $\varphi(c) := d$. This completes B_1 and we start all again.

B) If c has a nearest neighbour which is not in B_1, we start a new branch B_2 etc...
A new point may have its nearest neighbour in B_1 or in B_2, etc... Then we put it in B_1 or in B_2, ... as in the step A).

We continue this way in order to exhaust S.

The S_1 part is all the "a" and S_2 is all the "b".
Then S is a bipartite graph with components $S_j,\ j = 1, 2$ on which the two applications $\varphi,\ \psi$ are well defined. ■

2.1 Completion of the partition.

If the hyperbolic distance between a and $\varphi(a)$ or between b and $\psi(b)$ can be arbitrarily big, we shall have to add points to S_1 or to S_2.

Let $M_1 := \sup_{a\in S_1} d(a, \varphi(a)),\ M_2 := \sup_{b\in S_2} d(b, \psi(b))$; if $M := \max (M_1,\ M_2) < \infty$ then we have nothing to do. If $M = \infty$, then we choose an integer m, for instance $m = 6$, and we shall add points the following way.

For $a \in S_1$ such that $d(a, \varphi(a)) > m$ we add a point b to S_2 such that $b \in (a,\ \varphi(a))$ and $d(a, b) = 1$; of course we set $\varphi(a) = b$ and we have $\psi(b) = a$, because there is no point of S nearest b than a.
We do the same for $b \in S_2$ if $d(b, \psi(b)) > m$, we add a point a in S_1 on the segment $(b, \psi(b))$ such that $d(a, b) = 1$ and we set $\psi(b) := a$ and of for the same reason than above we have $\varphi(a) = b$.

By adding these points we have new S_1 and S_2 such that the new M is bounded by m.

Let us show now the lemma.

Lemma 2 *If the sequence S is ultra-separated in $H^\infty(\mathbb{B})$ then it is separated.*

Proof.
The existence of the ultra-separating f gives easily that the sequence S is separated. Let $a,\ b$ two points of S. Suppose that $a \in S_1$; then if b is in S_2, the ultra-separating function f is such that

$$|f(a)| < \delta \text{ and } |f(b)| \geq 1,$$

hence a and b are separated.

If $b \in S_1$, let $c = \varphi(a) \in S_2$, because c is the nearest neighbour of a, we have $d(a, b) \geq d(a, c)$, hence, because a and c are separated, this is the same for a and b. ■

On the other hand, taking powers of f, we can manage to have δ as small as we wish.

2.2 Proof of the sufficiency.

We have $\forall a \in S_1,\ |f(a)| \leq \delta$ hence because $f \in H^\infty(\mathbb{B})$ we have $|f| \leq 2\delta$ in a hyperbolic neighbourhood of a, for instance in $B_a(\epsilon) := \{z \in \mathbb{C}^n :: |\Phi_a(z)| < \epsilon\}$, where Φ_a is the automorphism of the ball exchanging a and 0. Take ϵ small enough in order that these pseudo-balls are disjoint, this is possible because the sequence is separated.

Let $b=\varphi(a)\in S_2$ the neighbour of a, then $|f(b)|\geq 1$ hence because $f\in H^{\infty}$, $|f|\geq 1-\delta$ in a hyperbolic neighbourhood of b, for instance in $B_b(\epsilon)$.

Hence the derivative of f is big in a "tube" linking $B_a(\epsilon)$ to $B_b(\epsilon)$. Precisely let $\gamma(t),\ t\in[0,1]$ be a smooth curve in $\mathbb{B}$ such that $\gamma(0)=\alpha\in B_a(\epsilon)$ and $\gamma(1)=\beta\in B_b(\epsilon)$, we have

$$1-3\delta\leq|f(\beta)-f(\alpha)|=\left|\int_{\gamma}\partial f(z)\,dz\right|=\left|\int_0^1\partial f\circ\gamma(t)\cdot\gamma'(t)dt\right|\leq\int_{\alpha}^{\beta}|\partial f(z)|\,d\,|z|.$$

Let $d\mu(z):=c_{\alpha\beta}\dfrac{d\,|z|}{\rho(z)}$ be the measure on the curve γ, with $\dfrac{1}{c_{\alpha\beta}}=\displaystyle\int_{\alpha}^{\beta}d\mu(z)$.

This measure μ is a probability measure and we have by Hölder :

$$c_{\alpha\beta}^2(1-3\delta)^2\leq\left(\int_{\gamma}\rho(z)\,|\partial f(z)|\,d\mu(z)\right)^2\leq\int_{\gamma}\rho(z)^2\,|\partial f(z)|^2\,d\mu(z)=c_{\alpha\beta}\int_{\gamma}\rho(z)\,|\partial f(z)|^2\,d\,|z|.$$

So

$$(1-3\delta)^2\leq\frac{1}{c_{\alpha\beta}}\int_{\gamma}\rho(z)\,|\partial f(z)|^2\,d\,|z|.$$

We get, with our parametrisation,

$$\textstyle\int_{\gamma}d\mu(z)=\int_0^1\frac{\gamma'(t)}{\rho\circ\gamma(t)}dt.$$

Let

$$A=\inf_{t\in[0,1]}\frac{1}{\rho\circ\gamma(t)},\ \ B=\sup_{t\in[0,1]}\frac{1}{\rho\circ\gamma(t)},$$

then

$$A\times|\gamma|\leq\ \textstyle\int_{\gamma}d\mu(z)\leq B\times|\gamma|\,,$$

where $|\gamma|$ denotes the euclidean length of the curve γ.

Finally

$$(1-3\delta)^2\leq\frac{1}{c_{\alpha\beta}}\int_{\gamma}\rho(z)\,|\partial f(z)|^2\,d\,|z|\lesssim\frac{1}{A\times|\gamma|}\int_{\gamma}\rho(z)\,|\partial f(z)|^2\,d\,|z|.$$

3 On the geometry of the unit ball in $\mathbb{C}^n$.

Let $a\in\mathbb{B}$ and define the pseudo-ball associated to it:

$$B_a(\epsilon):=\{z\in\mathbb{B}::|\Phi_a(z)|<\epsilon\}.$$

We have, by the non-isotropy of the geometry in the ball, the following lema.

Lemma 1 *Let S_a be the euclidean sphere of center 0 and radius $|a|$. The area of $S_a\cap B_a(\epsilon)$ is equivalent to $\epsilon^n(1-|a|)^n$.*

Proof.

Well known. ∎

3.1 A construction of pseudo-geodesics.

Let $a,b\in\mathbb{B}$ be two points in the ball ; we shall connect them by a smooth curve $\mathcal{G}(a,b))$ the following way :

• consider the 2-real plane P containing $(0,a,b)$; it cut the ball as a real disc, then see it as a complex disc and take the geodesic, in the hyperbolic metric of this disc, passing through a,b. It is the arc of a circle orthogonal to the boundary passing through a,b.

Take $\mathcal{G}(a,b)$ to be the part of this arc between a and b.
We have two clear facts:

• the minimum of the distance $\rho(z)$ to the boundary of the ball, for $z\in\mathcal{G}(a,b)$, is attained at a or b ;

• the euclidean length of $\mathcal{G}(a,b)$ is smaller than π times the length of the straight line between a and b.

Integrating in the cone C_b linking the vertex b to the points in $B_a(\epsilon)$ by use of these pseudo-geodesics, we get because of the anisotropic geometry of these pseudo-balls

$$\int_{C_b}\rho(z)\left|\partial f(z)\right|^2 dm(z)\gtrsim \epsilon^{2n-1}(1-|a|)^n\min(1-|a|\,,\ 1-|b|)\,\frac{(1-3\delta)^2}{|a-b|}.$$

The same in the cone C_a of vertex b and containing $B_a(\epsilon)$,

$$\int_{C_a}\rho(z)\left|\partial f(z)\right|^2 dm(z)\gtrsim \epsilon^{2n-1}(1-|b|)^n\min(1-|a|\,,\ 1-|b|)\,\frac{(1-3\delta)^2}{|a-b|}.$$

And the integral in the tube is bigger than the maximum of these two integrals :

$$\int_{tube}\rho(z)\left|\partial f(z)\right|^2 dm(z)\gtrsim \epsilon^{2n-1}\max\left((1-|a|)^n,\ (1-|b|)^n\right)\times\min\left((1-|a|),(1-|b|)\right)\times\tfrac{1}{|a-b|}.$$

Because M is finite we have that $d_H(a,b)<M$ which means that there is a $0<\gamma_M$ such that $|a-b|\leq\gamma_M\min\,(1-|a|\,,\ 1-|b|)$.
Hence the pseudo ball $Q_a:=B(a,\varphi(a))$ "centered" at a and containing the "tube" verifies

$$\int_{Q_a}\rho(z)\left|\partial f(z)\right|^2 dm(z)\gtrsim\frac{\epsilon^{2n-1}}{\gamma_M}\times\max\left((1-|a|)^n,\ (1-|b|)^n\right).$$

Revoir d'ici These pseudo-balls Q_a are disjoint because $\varphi(a)$ is the nearest point to a in S.

Now let us see that the measure $\nu:=\sum_{a\in S_1'}(1-|a|)^n\delta_a$is a Carleson one.

Let $W(\zeta,h)$ a Carleson window in the ball ; we have

$$\int_{W(\zeta,h)}d\nu:=\sum_{a\in S_1}(1-|a|)^n\lesssim\frac{\gamma_M}{\epsilon^{2n-1}}\sum_{a\in W(\zeta,h)\cap S_1}\int_{Q_a}\rho(z)\left|\partial f(z)\right|^2 dm(z).$$

But the pseudo-balls Q_a are disjoint and if $a\in W(\zeta,h)$ then $Q_a\subset W(\zeta,c_Mh)$ with a uniform c_M, because of the hyperbolic geometry. So we have

$$\int_{W(\zeta,h)}d\nu\leq\frac{\gamma_M}{\epsilon}\int_{W(\zeta,c_Mh)}\rho(z)\left|\partial f(z)\right|^2 dm(z)\lesssim\frac{\gamma_M}{\epsilon}c_Mh^n,$$

because $\rho\left|\partial f\right|^2 dm$ is a Carleson measure since $f\in H^\infty(\mathbb{B})$. So ν is also Carleson.

Hence we have that S_1 is a separated Carleson sequence. The same for S_2 and, because $S=S_1\cup S_2$ is still separated, we have that S is also a separated Carleson sequence.

3.2 Proof of the necessity in the case of the unit disc.

Suppose that the sequence S is interpolating for $H^\infty(\mathbb{D})$ then take a good partition $(S_1,\ S_2)$ we have that S_j is still interpolating.

If M_1,M_2 are finite, then we can choose a function $f\in H^\infty(\mathbb{D})$ such that $f=0$ on S_1 and 1 on S_2 and we are done.

If M_1 or M_2 is infinite, we add points as it was done above. The sequence A of added points to S_1 for instance is associated to a subsequence B of S_1, B is then interpolating and hence the way that the points are added, just a "translation" of the points of B makes also of A an interpolating sequence. Because $S_1' := S \cup A$ is separated, S_1' is still an interpolating sequence. The same for S_2, hence $S' := S_1' \cup S_2'$ is an interpolating sequence.

Now we can choose $f \in H^\infty(\mathbb{D})$ such that $f = 0$ on S_1 and $f = 1$ on S_2' to be done. ■

Remark 1 *In the ball of* $\mathbb{C}^n,\ n \geq 2$, *there is a big difference between interpolating sequences and separated Carleson ones. For instance we can take a maximum net* S *of separated points in the unit disc and consider this sequence* S *in the unit ball* $\mathbb{B}$ *of* $\mathbb{C}^2$ *:* S *is Carleson in* $\mathbb{B}$ *and* S *is separated [1] but* S *is not* $H^\infty(\mathbb{B})$ *interpolating because if so,* S *would be interpolating also for* $H^\infty(\mathbb{D})$*and this is false by the characterization of Carleson.*

Remark 2 *This result is not completely satisfactory because we have to add points to the good partition to make it a very good one in case where there are points arbitrarily far from there nearest neighbour.*

So the question : is this result optimal ? I.e. is there a separated sequence S *where there are points arbitrarily far from there nearest neighbour,* S *not interpolating and such that there is a function* $f \in H^\infty(\mathbb{D}),\ |f|_{S_1} \leq \delta < 1$ *and* $|f|_{S_2} \geq 1$ *with* (S_1, S_2) *a good partition of* S *?*

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