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TRUNCATIONS OF HAAR DISTRIBUTED MATRICES, TRACES AND BIVARIATE BROWNIAN BRIDGE

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Abstract. Let U be a Haar distributed matrix in $\mathbb{U}(n)$ or $\mathbb{O}(n)$. We show that after centering the two-parameter process

$$W^{(n)}(s,t) = \sum_{i\le \lfloor ns\rfloor, j\le \lfloor nt\rfloor} |U_{ij}|^2$$

converges in distribution to the bivariate tied-down Brownian bridge.

1. Introduction

Let σ be a random permutation uniformly distributed on the symmetric group $\mathcal{S}_n$. Define for $p,q\le n$

$$X_{p,q}^{(n)} = \#\{1\le i\le p\ ,\ \sigma(i)\le q\}.$$

In [7], G. Chapuy proved that a suitable normalization of $X_{p,q}^{(n)}$ converges in distribution to the bivariate tied-down Brownian bridge. Note that $X_{p,q}^{(n)} = \mathrm{Tr}(\Sigma_{p,q}\Sigma_{p,q}^\star)$ where $\Sigma_{p,q}$ is the truncated matrix of size $p\times q$ of Σ, the permutation matrix associated with σ, and * means adjoint. In this paper, we prove a similar result when the symmetric group is replaced by the unitary group or the orthogonal group, equipped with the Haar measure.

Let U be a Haar unitary, resp. orthogonal, matrix in $\mathbb{U}(n)$, resp. in $\mathbb{O}(n)$. We consider, for $p\le n$ and $q\le n$, the upper-left $p\times q$ submatrix $V_{p,q}$ and the $p\times p$ Hermitian matrix

$$H_{p,q} = V_{p,q}V_{p,q}^\star\,.$$

We are interested in the asymptotic behavior of

$$T_{p,q} = \mathrm{Tr}\, H_{p,q} = \sum_{i\le p, j\le q} |U_{i,j}|^2\,. \tag{1.1}$$

Setting

$$Y_{p,q}^{(n)} = T_{p,q} - \mathbb{E}T_{p,q}\,,$$

we define a sequence of two-parameter processes $W^{(n)}$ by

$$W^{(n)} := \left(Y^{(n)}_{\lfloor ns\rfloor,\lfloor nt\rfloor}, s,t\in[0,1]\right)$$

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Chapuy used the space $C([0,1]^2)$ completing its process in such a way that it is continuous and affine on each closed "lattice triangle". We prefer using the multidimensional generalization of Skorokhod space $D([0,1]^2)$ given by [5]. It consists of functions from $[0,1]^2$ to $\mathbb{R}$ which are at each point right continuous (with respect to the natural partial order of $[0,1]^2$) and admit limits in all "orthants". The space $D([0,1]^2)$ is endowed with the topology of Skorokhod (see [5] for the definition).
Our main result is the following

Theorem 1.1. *The process $W^{(n)}$ converges in distribution in $D([0,1]^2)$ to a tied-down Brownian bridge $\sqrt{\frac{2}{\beta}}W^{(\infty)}$ where $W^{(\infty)}$ is a centered continuous Gaussian process on $[0,1]^2$ of covariance*

$$\mathbb{E}[W^{(\infty)}(s,t)W^{(\infty)}(s',t')] = (s \wedge s' - ss')(t \wedge t' - tt'),$$

$\beta = 2$ in the unitary case and $\beta = 1$ in the orthogonal case.

Previous works are related to our problem. First, Borel in 1906 shows that for a uniformly distributed point on the $(n-1)$-dimensional (real) sphere, the scaled first coordinate converges in distribution to the standard normal. Since that time, many authors studied the entries and partial traces of matrices from the orthogonal and unitary group. In particular Diaconis and d'Aristotile ([12, 13]) proved that the sequence of one-parameter processes

$$\left\{ \sum_{i=1}^{\lfloor ns \rfloor} U_{ii} \ , \ s \in [0,1] \right\}_n$$

converges in distribution to the complex Brownian motion. Besides, Silverstein [26] proved that for q fixed, the sequence of one-parameter processes

$$\left\{ n^{1/2}\Big(\sum_{i=1}^{\lfloor ns \rfloor} |U_{iq}|^2 - s\Big) \ , \ s \in [0,1] \right\}_n \tag{1.2}$$

converges in distribution to the Brownian bridge.

In the paper [27], Silverstein discussed the similarity between the matrix of eigenvectors of a (real) sample covariance matrix and a Haar distributed orthogonal matrix, with a one-parameter parameter process analogous to (1.2). We also refer to [2, Chapter 10] for the behavior of eigenvectors of sample covariance matrices and universality conjectures. To extend this study and after reading a first draft of our result, Djalil Chafai conjectured that if M is a $n \times n$ matrix with i.i.d. entries having the same four first moments as the complex Gaussian standard and if $\mathcal{U}$ denotes the matrix of eigenvectors of $N = MM^*$, then the sequence $\mathcal{W}^{(n)}$ obtained by changing U_{ij} into $\mathcal{U}_{ij}$ converges to the tied-down Brownian bridge as in Theorem 1.1. At the moment of posting the present version, we are aware that Florent Benaych-Georges ([4]) answered positively to the conjecture when N is a Wigner matrix, under a fourth moment hypothesis.

The rest of the paper is organized as follows. In section 2, we introduce the basic notions on permutations, partitions, classical cumulants. The paper is then split into two parts. Section 3 deals with the unitary case and Section 4 is devoted to the orthogonal case. In each part, we develop the combinatorics associated with the group[1] (the Weingarten function and the associated cumulants).

In particular, we state a formula for the cumulants of variables of the form $X = \mathrm{Tr}(AUBU^\star)$ for deterministic matrices A, B of size n. In the unitary case, the formula follows from the results of [21]. We then apply the above formula to the computation of the second and fourth cumulants of $T_{p,q}$. The proof of Theorem 1.1 is then divided in two parts: tightness of the family of distributions of $W^{(n)}$ and convergence of the finite dimensional laws. To prove the tightness, we use a criterion of Bickel and Wichura for two-parameter processes, with the help of the estimates obtained in section 3.2, resp 4.3. The convergence of the finite dimensional distributions to Gaussian distributions relies on the computations of their cumulants and their asymptotics. The expression of the cumulants follows from the previous section and their limit follows from the asymptotics of cumulants of unitary, resp. orthogonal, Weingarten functions, obtained in [8], resp. [11]. Let us mention that simultaneously and independently of the present paper, precise computations in the orthogonal case are presented in [23] and [24]. In section 5, we give complementary remarks and connections with other problems.

2. Preliminaries

For n a positive integer we set $[n] := \{1, 2, \cdots, n\}$.

2.1. **Partitions, permutations.** We call $A = \{A_1, \cdots, A_s\}$ a partition of $[k]$ if the A_i $(1 \le i \le s)$ are pairly disjoint, non-void subsets of $[k]$ such that $A_1 \cup \cdots \cup A_s = [k]$. We call $A_1, \cdots, A_s$ the blocks of A. The number of blocks of A is denoted by $\#(A)$. The set of all partitions of $[k]$ is denoted by $\mathcal{P}(k)$. One partition A is said to refine another B, denoted $A \le B$ provided every block of A is contained in some block of B. Given two partitions A and B, $A \wedge B$ (resp. $A \vee B$) is the largest (resp. smallest) partition which refines (resp. is refined by) both A and B. Under these operations, the partially ordered set $\mathcal{P}(k)$ is a lattice. We denote by 1_k the largest partition of $[k]$ (one-block partition), and by 0_k the smallest one (k-blocks partition). For $A, B \in \mathcal{P}(k)$ with $A \le B$ we denote by $[A, B]$ the interval

$$[A, B] = \{C \in \mathcal{P}(k) \mid A \le C \le B\}.$$

Since we will make some use of the Möbius function for partitions, let us just recall ([28] Sect 3.6) that for two real functions f, g defined on $\{(A, B) \in$

[1]We will deal with the symplectic case in a forthcoming paper.

$\mathcal{P}(k) \times \mathcal{P}(k); A \leq B\}$, we have:

$$f(A,B) = \sum_{C \in [A,B]} g(A,C) \tag{2.1}$$

if and only if

$$g(A,B) = \sum_{C \in [A,B]} \text{Möb}(C,B) f(A,C) \tag{2.2}$$

with

$$\text{Möb}(C,B) := \prod_i \left((-1)^{i-1}(i-1)!\right)^{p_i}$$

where p_i is the number of blocks of B that contain exactly i blocks of C.

Let $\mathcal{S}_k$ be the set of permutations on k elements. With $\sigma \in \mathcal{S}_k$, we associate the set $\mathcal{C}(\sigma)$ of its cycles, whose number is denoted by $\#(\sigma)$. We denote by 0_σ the partition whose blocks are the cycles of σ, or when the context is clear, just by σ. For $\pi \in \mathcal{S}_k$, a partition $A = (A_1, \dots A_l)$ of $[k]$ is called π-invariant if π leaves invariant each block A_i that is $0_\pi \leq A$ (which we just write $\pi \leq A$).

Finally, we define $\mathcal{M}_{2k}$ as the set of pairings of $[2k]$, i.e. of partitions where each block consists of exactly two elements. It is then convenient to encode the set $[2k]$ by

$$[2k] \cong \{1, \dots, k, \bar{1}, \dots, \bar{k}\}\,.$$

Given two pairings p_1, p_2, we define the graph $\Gamma(p_1, p_2)$ as follows. The vertex set is $[2k]$ and the edge set consists of the pairs of p_1 and p_2. Let $\text{loop}(p_1, p_2)$ the number of connected components of $\Gamma(p_1, p_2)$.

2.2. **Cumulants.** For $r \geq 1$, κ_r denotes the classical cumulant of order r (see [25], [21] p.215). It is a multilinear function of r variables defined as follows: if $a_1, \dots, a_r$ are random variables,

$$\kappa_r(a_1, \dots, a_r) = \sum_{C \in \mathcal{P}(r)} \text{Möb}(C, 1_r)\, \mathbb{E}_C(a_1, \dots a_r)$$

where for $C = \{C_1, \dots, C_k\} \in \mathcal{P}(r)$,

$$\mathbb{E}_C(a_1, \dots, a_r) = \prod_{i=1}^r \mathbb{E}(\prod_{j \in C_i} a_j)\,. \tag{2.3}$$

More generally, relative cumulants are defined, for $A \leq B \in \mathcal{P}(r)$ as

$$\kappa_{A,B}(a_1, \cdots, a_r) = \sum_{C \in [A,B]} \mathbb{E}_C(a_1, \cdots, a_r)\, \text{Möb}(C,B)\,. \tag{2.4}$$

From the equivalence between (2.1) and (2.2) we have, for any $A \leq B \in \mathcal{P}(r)$

$$\mathbb{E}_B(a_1, \cdots, a_r) = \sum_{C \in [A,B]} \kappa_{A,C}(a_1, \cdots, a_r)\,. \tag{2.5}$$

2.3. **Matrices.** For a matrix $M = (M_{ij})_{i,j\le n}$, we denote by Tr the trace and by tr the normalized trace

$$\mathrm{Tr}(M) = \sum_{i=1}^{n} M_{ii}, \qquad \mathrm{tr}(M) = \frac{1}{n}\sum_{i=1}^{n} M_{ii}.$$

For $\pi \in \mathcal{S}_k$ and $M = (M_1, \dots M_k)$ a k-tuple of $n \times n$ matrices, we set

$$\mathrm{Tr}_\pi(M) = \mathrm{Tr}_\pi(M_1, \dots, M_k) = \prod_{C\in\mathcal{C}(\pi)} \mathrm{Tr}(\prod_{j\in C} M_j).$$

Let s be a fixed integer and let $\{M_1, \dots, M_s\}_n$ be a sequence of $n \times n$ deterministic matrices. We say that $\{M_1, \dots M_s\}_n$ has a limit distribution if there exists a non commutative probability space $(\mathcal{A}, \varphi)$ and $a_1, \dots a_s \in \mathcal{A}$ such that for any polynomial p in s non commuting variables,

$$\lim_{n\to\infty} \mathrm{tr}(p(M_1, \dots, M_s)) = \varphi(p(a_1, \dots, a_s)).$$

3. The unitary group

3.1. **Preliminary remarks: Some moments.** Let U be a Haar distributed matrix on $\mathbb{U}(n)$ the unitary group of size n. We have the following important relations (see [17], Proposition 4.2.3 and [18]). If $U_{i,j}$ is the generic element of U, then the random variable $|U_{i,j}|^2$ follows the beta distribution on $[0, 1]$ with parameter $(1, n-1)$ of density $(n-1)(1-x)^{n-2}$. Thus,

$$\mathbb{E}|U_{i,j}|^2 = \frac{1}{n}\ , \ \mathbb{E}|U_{i,j}|^4 = \frac{2}{n(n+1)}\ , \ \mathrm{Var}\,|U_{i,j}|^2 = \frac{n-1}{n^2(n+1)}\,, \tag{3.1}$$

and more generally

$$\mathbb{E}|U_{ij}|^{2k} = \frac{(n-1)!k!}{(n-1+k)!}\,. \tag{3.2}$$

If $X = |U_{i,j}|^2$ and $Y = |U_{i,k}|^2$ with $k \neq j$, then (X, Y) follows the Dirichlet distribution on $\{0 \le x, y, x + y \le 1\}$ with parameters $(1, 1, n-2)$ of density $(n-1)(n-2)(1-x-y)^{n-3}$. Thus

$$\mathbb{E}\left(|U_{i,j}|^2|U_{i,k}|^2\right) = \frac{1}{n(n+1)}\,. \tag{3.3}$$

Besides, if $i \neq k, j \neq \ell$,

$$\mathbb{E}\left(|U_{i,j}|^2|U_{k,\ell}|^2\right) = \frac{1}{n^2-1}\,. \tag{3.4}$$

From these relations, we can compute the first moments of $T_{p,q}$, defined in (1.1).

Proposition 3.1. *The mean and the variance of $T_{p,q}$ are given by:*

$$\mathbb{E}T_{p,q} = \sum_{i\le p, j\le q} \mathbb{E}|U_{ij}|^2 = pq\mathbb{E}|U_{11}|^2 = \frac{pq}{n}\,. \tag{3.5}$$

and

$$\operatorname{Var} T_{p,q} = pq\frac{n^2 - n(p+q) + pq}{n^2(n^2-1)}. \tag{3.6}$$

Assume that $p/n \to s$*,* $q/n \to t$*, then,*

$$\lim_n \frac{1}{n}\mathbb{E}T_{p,q} = st \ , \ \lim_n \operatorname{Var} T_{p,q} = st(1-(s+t)+st) = st(1-s)(1-t).$$

Proof:

$$\begin{aligned}
\mathbb{E}T_{p,q}^2 &= \sum_{i,k\le p, j,l\le q} \mathbb{E}|U_{ij}|^2|U_{kl}|^2 \\
&= \sum_{i\le p, j\le q} \mathbb{E}|U_{ij}|^4 + \sum_{i\le p, j\ne l\le q} \mathbb{E}|U_{ij}|^2|U_{il}|^2 \\
&\quad + \sum_{i\ne k\le p, j\le q} \mathbb{E}|U_{ij}|^2|U_{kj}|^2 + \sum_{i\ne k\le p, j\ne l\le q} \mathbb{E}|U_{ij}|^2|U_{kl}|^2 \\
&= pq\frac{2}{n(n+1)} + pq(q-1)\frac{1}{n(n+1)} \\
&\quad + p(p-1)q\frac{1}{n(n+1)} + p(p-1)q(q-1)\frac{1}{n^2-1} \\
&= pq\left(\frac{p+q}{n(n+1)} + \frac{(p-1)(q-1)}{n^2-1}\right).
\end{aligned}$$

This yields (3.6).

Remark 3.2. *An easy consequence of the above Proposition is*

$$\lim_n \frac{1}{n} T_{\lfloor ns\rfloor, \lfloor nt\rfloor} = st$$

in probability. Actually the convergence is uniform in $s, t \in [0,1]$ *(see section 5).*

3.2. **Combinatorics for the unitary group.** Let $\mathbb{U}(n)$ denote the unitary group of size n endowed with the Haar probability measure. The generic element will be denoted by U and its (i,j) coefficient by U_{ij}. In [11], Collins and Sniady proved the following integration formula on $\mathbb{U}(n)$, see also [8]. Let $\mathcal{M}_{2k}^U$ denote the set of pairings of $[2k]$, pairing each element of $[k]$ with an element of $[\bar{k}]$. Let $G^{\mathbb{U}(n)}$ be the Gram matrix[2]

$$G^{\mathbb{U}(n)} = (G^{\mathbb{U}(n)}(p_1,p_2))_{p_1,p_2\in\mathcal{M}_{2k}^U} := (n^{\operatorname{loop}(p_1,p_2)})_{p_1,p_2\in\mathcal{M}_{2k}^U}.$$

Then the unitary Weingarten matrix $\mathrm{Wg}^{\mathbb{U}(n)}$ is the pseudo inverse of $G^{\mathbb{U}(n)}$.

[2]The term of Gram matrix comes from the theory of representations of groups and algebras, see [10].

Proposition 3.3. *For every choice of indices* $\mathbf{i} = (i_1, \dots, i_k, i_{\bar 1}, \dots, i_{\bar k})$ *and* $\mathbf{j} = (j_1, \dots, j_k, j_{\bar 1}, \dots, j_{\bar k})$,

$$\mathbb{E}\left(U_{i_1 j_1} \dots U_{i_k j_k} \bar U_{i_{\bar 1} j_{\bar 1}} \dots \bar U_{i_{\bar k}, j_{\bar k}}\right) = \sum_{p_1, p_2 \in \mathcal{M}^U_{2k}} \delta^{p_1}_{\mathbf{i}} \delta^{p_2}_{\mathbf{j}} \operatorname{Wg}^{\mathbb{U}(n)}(p_1, p_2) \tag{3.7}$$

where $\delta^{p_1}_{\mathbf{i}}$ *(resp.* $\delta^{p_2}_{\mathbf{j}}$*) is equal to* 1 *or* 0 *if* $\mathbf{i}$ *(resp.* $\mathbf{j}$*) is constant on each pair of* p_1 *(resp.* p_2*) or not.*

It is clear that with each pairing $p \in \mathcal{M}^U_{2k}$ we can associate a unique $\sigma \in \mathcal{S}_k$ such that $p = \prod_{i=1}^k (i, \overline{\sigma(i)})$. It is known that if p_1 is associated with α and p_2 with β, then $\operatorname{Wg}^{\mathbb{U}(n)}(p_1, p_2)$ is a function of $\beta\alpha^{-1}$ denoted by $\operatorname{Wg}(n, \beta\alpha^{-1})$, so that (3.7) becomes

$$\mathbb{E}\left(U_{i_1 j_1} \dots U_{i_k, j_k} \bar U_{i_{\bar 1} j_{\bar 1}} \dots \bar U_{i_{\bar k}, j_{\bar k}}\right) = \sum_{\alpha, \beta \in \mathcal{S}_k} \tilde\delta^{\alpha}_{\mathbf{i}} \tilde\delta^{\beta}_{\mathbf{j}} \operatorname{Wg}(n, \beta\alpha^{-1}) \tag{3.8}$$

where $\tilde\delta^\alpha_{\mathbf{i}} = 1$ if $i(s) = i(\overline{\alpha(s)})$ for every $s \le k$ and 0 otherwise. In particular, if $\pi \in \mathcal{S}_k$, we have

$$\operatorname{Wg}(n, \pi) = \mathbb{E}(U_{11} \dots U_{kk} \bar U_{1\pi(1)} \dots \bar U_{k\pi(k)})\,. \tag{3.9}$$

The Weingarten functions for $k = 1, 2$ are given by (see [8]):

$$\begin{aligned} \operatorname{Wg}(n, (1)) &= \frac{1}{n} \\ \operatorname{Wg}(n, (1)(2)) &= \frac{1}{n^2 - 1}\ , \ \operatorname{Wg}(n, (12)) = -\frac{1}{n(n^2-1)}\,. \end{aligned} \tag{3.10}$$

From these equations, we can recover (3.3), (3.4).
We can now state a proposition which is a particular case of [21, Theorem 3.10].

Proposition 3.4. *Let* U *be Haar distributed on* $\mathbb{U}(n)$*. Let* $D = (D_1, \dots D_k)$ *and* $\bar D = (D_{\bar 1}, \dots D_{\bar k})$ *be two families of deterministic matrices of size* n*. We set, for* $1 \le i \le r$,

$$X_i = \operatorname{Tr}(D_i U D_{\bar i} U^\star)\,.$$

Then,

$$\kappa_r(X_1, \dots, X_r) = \sum_{\alpha, \beta \in \mathcal{S}_r} \sum_A C_{\beta\alpha^{-1}, A} \operatorname{Tr}_\alpha(\bar D) \operatorname{Tr}_{\beta^{-1}}(D) \tag{3.11}$$

where in the second sum $A \in \mathcal{P}(r)$ *is such that*

$$\beta\alpha^{-1} \le A \text{ and } A \vee \beta \vee \alpha = 1_r\,, \tag{3.12}$$

and $C_{\sigma, A}$ *are the relative cumulants of the unitary Weingarten function (see* [11]*). Moreover, if the sequence* $\{D, \bar D\}_n$ *has a limit distribution, then for* $r \ge 3$,

$$\lim_{n\to\infty} \kappa_r(X_1, \dots, X_r) = 0.$$

Remark 3.5. *When writing* $\mathrm{Tr}_\alpha(\bar{D})$ *in* (3.11)*, we consider* α *as a permutation acting on* $[\bar{k}]$*. The formula* (3.11) *is not given exactly on this form in* [21] *but in our particular case where* $X_i = \mathrm{Tr}(D_i U D_{\bar{i}} U^\star)$ *with the* D_i *deterministic, the formulas are equivalent. We shall prove a similar formula in the orthogonal case.*

For the sake of completeness, let us give the meaning of $C_{\pi,A}$. Let $\pi \in \mathcal{S}_r$ and $a_i = U_{ii}\bar{U}_{i\pi(i)}$. We denote for a π invariant partition $\Pi \in \mathcal{P}(r)$

$$E_\Pi(\pi) := \mathbb{E}_\Pi(a_1, \dots, a_r)\,, \tag{3.13}$$

i.e., owing to (3.9)

$$E_\Pi(\pi) = \prod_{k=1}^{s} \mathrm{Wg}(\pi|_{V_k})\,, \tag{3.14}$$

where $\Pi = \{V_1, \dots, V_s\}$. Then, if we set

$$C_{\pi,A} := \kappa_{\pi,A}(a_1, \dots, a_r)\,,$$

the summation formula (2.4) yields

$$C_{\pi,A} = \sum_{\pi \le C \le A} \mathrm{Wg}(\pi|_{V_1}) \cdots \mathrm{Wg}(\pi|_{V_1})\, \mathrm{M\ddot{o}b}(C, A) \tag{3.15}$$

and the reverse one (2.5)

$$E_C(\pi) = \sum_{A \in [\pi, C]} C_{\pi,A}\,. \tag{3.16}$$

We will use these formulas in the orthogonal case.

3.3. Computations of the second and fourth cumulants of $T_{p,q}$.

3.3.1. *The covariance of* $T_{p,q}$. The fundamental remark is that

$$H_{p,q} = D_1 U D_{\bar{1}} U^\star$$

with $D_1 = I_p, D_{\bar{1}} = I_q$, where I_k is the matrix of projection on the k first coordinates. Note that D_1 and $D_{\bar{1}}$ are commuting projectors and that if $p/n \to s, q/n \to t$, $\{D_1, D_{\bar{1}}\}_n$ has a limit distribution with a_1, a_2 commuting projectors on $(\mathcal{A}, \varphi)$ such that $a_1 a_2 = a_1$ if $s < t$ and $= a_2$ if $t < s$, and $\varphi(a_1) = s, \varphi(a_2) = t$.
Let $p, p', q, q' \le n$. We now give an application of Proposition 3.4 to the computation of $\mathrm{cov}(T_{p,q}, T_{p',q'}) = \kappa_2(T_{p,q}, T_{p',q'})$. This can also be done, using the computations of Section 3.1.
We set $D_2 = I_{p'}, D_{\bar{2}} = I_{q'}$ and apply formula (3.11) to $X_1 = T_{p,q}$, $X_2 = T_{p',q'}$, $r = 2$. The different possibilities for α, β and A satisfying (3.12) are gathered in the following table, where 0 and 1 stand for the convenient permutation or partitions on [2]:

α	β	$\beta\alpha^{-1}$	A
0	0	0	1
1	0	1	1
0	1	1	1
1	1	0	0 or 1

The relative cumulants are given by (see (3.10), (2.4))

$$C_{1,1} = -\frac{1}{n(n^2-1)}\ ,\ C_{0,1} = -\frac{1}{n^2} + \frac{1}{n^2-1} = \frac{1}{n^2(n^2-1)}\ ,\ C_{0,0} = \frac{1}{n^2}$$

The different products of traces are quite obvious, so that plugging into (3.11), we get

$$\kappa_2(T_{p,q}, T_{p',q'}) = \frac{(p\wedge p')(q'\wedge q')}{n^2-1} - \frac{(p\wedge p')qq'}{n(n^2-1)} - \frac{pp'(q\wedge q')}{n(n^2-1)} + \frac{pp'q'q'}{n^2(n^2-1)}. \tag{3.17}$$

In the limit $p/n \to s, q/n \to t, p'/n \to s', q'/n \to t'$, we get

$$\begin{aligned}\lim_n \kappa_2(T_{p,q}, T_{p',q'}) &= (s\wedge s')(t\wedge t') - (s\wedge s')tt' - ss'(t\wedge t') + ss'tt' \\ &= (s\wedge s' - ss')(t\wedge t - tt'). \end{aligned}\tag{3.18}$$

3.3.2. *The fourth cumulant.* We now give an estimate for $\kappa_4(T_{p,q})$. From (3.11) with $r=4$,

$$\kappa_4 = \sum_{\alpha,\beta\in\mathcal{S}_4}\sum_A C_{\beta\alpha^{-1},A}\ \mathrm{Tr}_\alpha(\bar{D})\,\mathrm{Tr}_{\beta^{-1}}(D) \tag{3.19}$$

where A runs over the partitions of [4] satisfying condition (3.12), and finally

$$D_i = I_p\ ,\ D_{\bar{i}} = I_q\quad (i\le 4)\,.$$

We have now

$$\mathrm{Tr}_{\beta^{-1}}(D) = p^{\#(\beta)}\ ,\ \mathrm{Tr}_\alpha(\bar{D}) = q^{\#(\alpha)}\,.$$

In [8, Cor. 2.9], Collins proved that the order of $C_{\beta\alpha^{-1},A}$ is at most $n^{-8-\#(\beta\alpha^{-1})+2\#(A)}$. Finally,

$$C_{\beta\alpha^{-1},A}\,\mathrm{Tr}_{\beta^{-1}}(D)\,\mathrm{Tr}_\alpha(\bar{D}) = O\left(n^{-8-\#(\beta\alpha^{-1})+2\#(A)}p^{\#(\beta)}q^{\#(\alpha)}\right)\,. \tag{3.20}$$

From equation (20) in [21], we see that

$$2\#(A) + \#(\alpha) + \#(\beta) - \#(\beta\alpha^{-1}) \le 6$$

and the expression in (3.20) is of order

$$\begin{aligned} n^{-8-\#(\pi)+2\#(A)}p^{\#(\beta)}q^{\#(\alpha)} &\le p^{\#(\beta)}q^{\#(\alpha)}n^{-2-\#(\alpha)-\#(\beta)} \\ &\le (p/n)^{\#(\beta)-1}(q/n)^{\#(\alpha)-1}pqn^{-4} \\ &\le p^2q^2n^{-4}\,. \end{aligned}$$

We conclude that

$$\kappa_4 = O\left(p^2q^2n^{-4}\right)\,. \tag{3.21}$$

3.4. **Proof of Theorem 1.1.**

3.4.1. *Tightness.* According to Bickel and Wichura [5, Theorem 3], since our processes are null on the axes, the tightness of the family of distributions of $W^{(n)}$ is in force as soon as the condition $\mathcal{C}(\beta,\gamma)$ with $\beta > 1$ is satisfied (see (2), (3) in [5]):

$$\mathbb{E}(|W^{(n)}(B)|^{\gamma_1}|W^{(n)}(C)|^{\gamma_2}) \le (\mu(B))^{\beta_1}(\mu(C))^{\beta_2} \tag{3.22}$$

where $\gamma = \gamma_1+\gamma_2 > 0$ and $\beta = \beta_1+\beta_2 > 1$, B and C are two adjacent blocks in $[0,1]^2$ and $W^{(n)}(B)$ denotes the increment of $W^{(n)}$ around B, given by

$$W^{(n)}(B) = W^{(n)}_{s',t'} - W^{(n)}_{s',t} - W^{(n)}_{s,t'} + W^{(n)}_{s,t}$$

for $B =]s,s']\times]t,t']$, μ is a finite positive measure on $[0;1]^2$ with continuous marginals.

From Cauchy-Schwarz inequality, (3.22) is implied by

$$\mathbb{E}(|W^{(n)}(B)|^{2\gamma_1}) \le (\mu(B))^{2\beta_1}. \tag{3.23}$$

Moreover, it is enough to consider blocks whose corner points are in $T^n = \{\frac{p}{n}, 0\le p\le n\}\times\{\frac{q}{n}, 0\le q\le n\}$ (see [5], p. 1665.)

Let $p \le p' \le n$ and $q \le q' \le n$ and $B =]\frac{p}{n},\frac{p'}{n}]\times]\frac{q}{n},\frac{q'}{n}]$

$$\begin{aligned} W^{(n)}(B) := \Delta^{(n)}_{p,q}(p',q') &= Y^{(n)}_{p',q'} - Y^{(n)}_{p',q} - Y^{(n)}_{p,q'} + Y^{(n)}_{p,q} \\ &= \sum_{p+1\le i\le p'}\ \sum_{q+1\le i\le q'} |U_{i,j}|^2 - \mathbb{E}(|U_{i,j}|^2). \end{aligned}$$

If we show that there exists a constant C, such that for all n,

$$\mathbb{E}\left[\left(\Delta^{(n)}_{p,q}(p',q')\right)^4\right] \le C\frac{(p'-p)^2(q'-q)^2}{n^4}, \tag{3.24}$$

then (3.23) is satisfied with $\gamma_1 = 2$, $\beta_1 = 1$ and μ is the Lebesgue measure. Since $\Delta^{(n)}_{p,q}(p',q')$ has the same distribution as $Y^{(n)}_{p'-p,q'-q}$, it is enough to show

$$\mathbb{E}\left[\left(Y^{(n)}_{p,q}\right)^4\right] = O(p^2q^2n^{-4})\,. \tag{3.25}$$

If X is a real random variable, an elementary computation gives

$$\mathbb{E}(X-\mathbb{E}X)^4 = \kappa_4 + 3\kappa_2^2\,, \tag{3.26}$$

where κ_r is the r-th cumulant of X. Taking $X = T_{p,q} = \operatorname{Tr} D_1UD_{\bar 1}U^\star$, we saw above in (3.6) that

$$\kappa_2 = \operatorname{Var} T_{p,q} \le 2\frac{pq}{n^2}\,. \tag{3.27}$$

Gathering (3.26) , (3.27) and (3.21) we get that (3.25) is checked, which proves the tightness.

3.4.2. *Finite-dimensional laws.* Let $(a_i)_{i\le k}\in\mathbb{R}$, $(s_i,t_i)_{i\le k}\in[0,1]^2$. We must prove the convergence in distribution of $X^{(n)} := \sum_{i=1}^k a_i W^{(n)}_{s_i,t_i}$ to a Gaussian distribution.
Let us denote $p_i=\lfloor ns_i\rfloor$, $q_i=\lfloor nt_i\rfloor$. Then

$$X^{(n)} = \sum_{i=1}^k a_i Y^{(n)}_{p_i,q_i} = \sum_{i=1}^k a_i[\mathrm{Tr}(D_iUD_{\bar{i}}U^\star) - \mathbb{E}(\mathrm{Tr}(D_iUD_{\bar{i}}U^\star))]$$

where $D_i = I_{p_i}$, $D_{\bar{i}} = I_{q_i}$.
$\{D_i, D_{\bar{i}}, i=1,\dots k\}$ are commuting projectors with a limit distribution $\{q_i,q_{\bar{i}}, i=1,\dots k\}$ on a probability space $(\mathcal{A},\phi_1)$ with $\phi_1(q_i)=s_i$, $\phi_1(q_{\bar{i}})=t_i$ and $q_iq_j=q_i$ if $u_i\le u_j$ (and $=q_j$ otherwise) where $u_i=s_i$ for i odd and $u_i=t_i$ for i even.
Let $r\ge 3$, then

$$\begin{aligned}\kappa_r(X^{(n)},\dots,X^{(n)}) &= \sum_{i_1,\dots,i_r=1}^k a_{i_1}\dots a_{i_r}\kappa_r(Y^{(n)}_{p_{i_1},q_{i_1}},\dots,Y^{(n)}_{p_{i_r},q_{i_r}})\\ &= \sum_{i_1,\dots,i_r=1}^k a_{i_1}\dots a_{i_r}\kappa_r(X_{i_1},\dots,X_{i_r})\end{aligned}$$

where $X_{i_p}=\mathrm{Tr}(D_{i_p}UD_{\bar{i}_p}U^\star)$. From Proposition 3.4

$$\lim_{n\to\infty}\kappa_r(X_{i_1},\dots,X_{i_r}) = 0. \qquad (3.28)$$

Now, the second cumulant is given by

$$\kappa_2(X^{(n)},X^{(n)}) = \sum_{i,j=1}^k a_ia_j\kappa_2(\mathrm{Tr}(D_iUD_{\bar{i}}U^\star),\mathrm{Tr}(D_jUD_{\bar{j}}U^\star)).$$

From (3.18)

$$\begin{aligned}&\lim_n \kappa_2(\mathrm{Tr}(D_iUD_{\bar{i}}U^\star),\mathrm{Tr}(D_jUD_{\bar{j}}U^\star))\\ &= (s_i\wedge s_j - s_is_j)(t_i\wedge t_j - t_it_j).\end{aligned}$$

Thus, we get the convergence of $X^{(n)}$ to a centered Gaussian distribution with variance

$$\sum_{i,j=1}^k a_ia_j(s_i\wedge s_j - s_is_j)(t_i\wedge t_j - t_it_j).$$

It follows that the finite-dimensional laws of the process $W^{(n)}$ converge to the finite-dimensional laws of the tied-down Brownian bridge. ■

4. The orthogonal case

4.1. **Combinatorics for the orthogonal group.** Let $\mathbb{O}(n)$ denote the orthogonal group of size n endowed with the Haar probability measure. The generic element will be denoted by O and its (i,j) coefficient by O_{ij}. In

[11], Collins and Sniady proved the following integration formula on $\mathbb{O}(n)$, see also [10, Theorem 2.1] for the following formulation. Let $G^{\mathbb{O}(n)}$ be the Gram matrix[3]

$$G^{\mathbb{O}(n)} = (G^{\mathbb{O}(n)}(p_1, p_2))_{p_1,p_2\in\mathcal{M}_{2k}} := (n^{\text{loop}(p_1,p_2)})_{p_1,p_2\in\mathcal{M}_{2k}} \,,$$

where $\mathcal{M}_{2k}$ is the set of pairings of $[2k]$ defined in Section 2.1. Then, the orthogonal Weingarten matrix $\mathrm{Wg}^{\mathbb{O}(n)}$ is the pseudo inverse of $G^{\mathbb{O}(n)}$.

Proposition 4.1. *For every choice of indices* $\mathbf{i} = (i_1, \dots, i_k, i_{\bar{1}}, \dots, i_{\bar{k}})$ *and* $\mathbf{j} = (j_1, \dots, j_k, j_{\bar{1}}, \dots, j_{\bar{k}})$,

$$\mathbb{E}\left(O_{i_1j_1}\dots O_{i_kj_k}O_{i_{\bar{1}}j_{\bar{1}}}\dots O_{i_{\bar{k}}j_{\bar{k}}}\right) = \sum_{p_1,p_2\in\mathcal{M}_{2k}} \delta_{\mathbf{i}}^{p_1}\delta_{\mathbf{j}}^{p_2}\,\mathrm{Wg}^{\mathbb{O}(n)}(p_1,p_2) \qquad (4.1)$$

where $\delta_{\mathbf{i}}^{p_1}$ *(resp.* $\delta_{\mathbf{j}}^{p_2}$*) is equal to* 1 *or* 0 *if* $\mathbf{i}$ *(resp.* $\mathbf{j}$*) is constant on each pair of* p_1 *(resp.* p_2*) or not.*

We now identify $\mathcal{M}_{2k}$ as the quotient set $\mathcal{S}_{2k}/H_k$ where H_k is a subgroup of $\mathcal{S}_{2k}$ known as the hyperoctahedral group and defined as follows (see [15] and [6]). With each $g \in \mathcal{S}_{2k}$, we associate the product of disjoint transpositions:

$$\eta(g) = \prod_{i=1}^{k}(g(i)\ g(\bar{i}))\,,$$

which can be identified as an element of $\mathcal{M}_{2k}$.

We set $\gamma = \prod_{i=1}^{k}(i\ \bar{i})$ and define $H_k = \{g \in \mathcal{S}_{2k}, \gamma g = g\gamma\}$, the centralizer of γ. We have the following equivalence

$$\eta(g) = \eta(g') \Longleftrightarrow \exists h \in H_k, g = g'h\,,$$

implying $\mathcal{M}_{2k} \cong \mathcal{S}_{2k}/H_k$.

According to Proposition 3.3 in [11], see also [10], $\mathrm{Wg}^{\mathbb{O}(n)}(\eta(g_1), \eta(g_2))$ depends only on the conjugacy class of $\eta(g_1)\eta(g_2)$ and we can define the orthogonal Weingarten function of $\mathcal{S}_{2k}$, denoted by $\mathrm{W}\Lambda^{\mathbb{O}(n)}$ (see [6, p. 511]) by

$$W\Lambda^{\mathbb{O}(n)}(g) = \mathrm{Wg}^{\mathbb{O}(n)}(\eta(Id), \eta(g))$$

and we have

$$\mathrm{Wg}^{\mathbb{O}(n)}(\eta(g_1), \eta(g_2)) = \mathrm{W}\Lambda^{\mathbb{O}(n)}(g_1^{-1}g_2).$$

In the sequel, we shall drop the superscript $\mathbb{O}(n)$ keeping in mind for the asymptotics that $W\Lambda$ depends on the size n.

It is clear from the definitions that $\mathrm{W}\Lambda$ is invariant on the classes of the

[3] See footnote 1.

double coset space $H_k \backslash \mathcal{S}_{2k}/H_k$.
Then, formula (4.1) can be written as

$$\mathbb{E}\left(O_{i_1 j_1}\dots O_{i_k j_k} O_{i_{\bar 1} j_{\bar 1}}\dots O_{i_{\bar k} j_{\bar k}}\right) = \frac{1}{|H_k|^2}\sum_{g_1, g_2 \in \mathcal{S}_{2k}} \delta_{\mathbf{i}}^{\eta(g_1)}\delta_{\mathbf{j}}^{\eta(g_2)}\, \mathrm{W}\Lambda(g_1^{-1}g_2). \tag{4.2}$$

We now describe the generators of $\mathcal{S}_{2k}/H_k$ following the presentation in [15], (see also [6] with a slightly different definition for particular permutations). For $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \{-1, 1\}^k$, we define τ_ϵ by

$$\tau_\epsilon = \prod_{i, \epsilon_i = -1} (i\bar{i}) \in H_k.$$

For $\pi \in \mathcal{S}_k$, we define $t_\pi \in \mathcal{S}_{2k}$ by

$$t_\pi(i) = i;\ t_\pi(\bar{i}) = \overline{\pi(i)}\,.$$

We shall now parametrize $\mathcal{S}_{2k}/H_k$ using special permutations.

Definition 4.2. A pair $(\epsilon, \pi) \in \{-1, 1\}^k \times \mathcal{S}_k$ is particular if, for any cycle c of π, we have $\epsilon_i = 1$ where i is the smallest element of c. We say that the corresponding permutation $g_{\epsilon,\pi} := \tau_\epsilon t_\pi$ in $\mathcal{S}_{2k}$ is particular.

Proposition 4.3. *(see Theorem 8 in* [15]*) The class $\mathcal{S}_{2k}/H_k$ containing $\tau_\epsilon t_\pi$ has exactely $2^{\#(\pi)}$ elements of the form $\tau_{\epsilon'} t_{\pi'}$. Every class of $\mathcal{S}_{2k}/H_k$ contains exactely one particular permutation of the form $\tau_\epsilon t_\pi$. There are $\frac{(2k)!}{2^k k!}$ particular permutations $g_l = \tau_{\epsilon_l} t_{\pi_l}$, $l \le \frac{(2k)!}{2^k k!}$.*

Let $\Sigma \in \mathcal{S}_{2k}$. From the above Proposition, there exists a particular pair (ϵ, σ) such that $\Sigma = \tau_\epsilon t_\sigma h$ with $h \in H_k$. In particular,

$$\Sigma \sim t_\sigma \text{ in } H_k \backslash \mathcal{S}_{2k}/H_k \text{ and } \mathrm{W}\Lambda(\Sigma) = \mathrm{W}\Lambda(\mathrm{t}_\sigma).$$

We now recall how to find σ from Σ (see [15, Proposition 18]). First consider the pairing $\eta(\Sigma)$ and the 2-regular graph $\Gamma(\Sigma)$ (i.e. with all the cycles having even length) defined by

$$\Gamma(\Sigma) = \eta(\Sigma) \cup \eta(Id).$$

Then, any cycle Γ_j of length $2q_j$ of $\Gamma(\Sigma)$ is of the form

$$\Gamma_j = (i_{j1}^{\pm}, i_{j1}^{\mp}, \dots, i_{jq_j}^{\pm}, i_{jq_j}^{\mp})$$

where the ordered couple $i^{\pm}i^{\mp}$ is such that $i^{\pm}i^{\mp} \in \{i, \bar{i}\}$ with $i^{\pm} = i$ implies $i^{\mp} = \bar{i}$ and $i^{\pm} = \bar{i}$ implies $i^{\mp} = i$. By convention, the starting point of Γ_j is $i_{j1}^{\pm} = \min\{\bar{i}_{jl}, l \le q_j\}$. Set $\sigma_j = (i_{j1}, \dots, i_{jq_j})$. Then $\sigma = \prod \sigma_j$.

4.2. **The variance of $T_{p,q}$.** From Proposition 4.1 and the value of the Weingarten function for $k = 2$ ([11], [10, Examples 2.1-3.1], see also [18]), we have: For $i \ne k$, $j \ne \ell$,

$$\mathbb{E}\left(O_{ij}^2 O_{k\ell}^2\right) = \frac{n+1}{n(n+2)(n-1)}.$$

For $j \neq k$

$$\mathbb{E}\left(O_{ij}^2 O_{ik}^2\right) = \frac{1}{n(n+2)}$$

and

$$\mathbb{E} O_{ij}^4 = \frac{3}{n(n+2)}.$$

From these 3 relations, we easily get

Proposition 4.4. *The mean and the variance of $T_{p,q}$ are given by:*

$$\mathbb{E} T_{p,q} = \sum_{i \le p, j \le q} \mathbb{E} O_{ij}^2 = pq\, \mathbb{E} O_{11}^2 = \frac{pq}{n}\,. \tag{4.3}$$

and

$$\operatorname{Var} T_{p,q} = 2pq \frac{n^2 - n(p+q) + pq}{n^2(n+2)(n-1)}. \tag{4.4}$$

Assume that $p/n \to s$, $q/n \to t$, then,

$$\lim_n \frac{1}{n} \mathbb{E} T_{p,q} = st\ ,\ \lim_n \operatorname{Var} T_{p,q} = 2st(1-(s+t)+st) = 2st(1-s)(1-t).$$

4.3. **Mixed moments of the variables T_{p_i,q_i}.** As in the unitary case, we need to compute cumulants $\kappa_r(X_1, \dots X_r)$ where the variables X_i are of the form

$$X_i = \operatorname{Tr}(D_i O D_{\bar{i}} O^{-1}).$$

We shall first establish an analogue of Proposition 3.4. We assume in the following that the matrices D_i are deterministic and symmetric. Even if we shall deal later only with diagonal matrices, the computations are the same for general matrices.

Proposition 4.5. *Let O be Haar distributed on $\mathbb{O}(n)$. Let $D = (D_1, \dots D_k)$ and $\bar{D} = (D_{\bar{1}}, \dots D_{\bar{k}})$ be two families of deterministic and symmetric matrices of size n. We set, for $1 \le i \le r$,*

$$X_i = \operatorname{Tr}(D_i O D_{\bar{i}} O^{-1}).$$

Then,

$$\kappa_r(X_1, \dots, X_r) = \sum_{(\alpha,\beta,\epsilon) \in \mathcal{S}_r \times \mathcal{S}_r \times \{\pm 1\}^r} \lambda_{\alpha,\beta,r} \sum_A C_{\sigma,A} \operatorname{Tr}_\alpha(D) \operatorname{Tr}_{\beta^{-1}}(\bar{D}) \tag{4.5}$$

where in the second sum

- *$\sigma \in \mathcal{S}_r$ is a function of α, β, ϵ satisfying $t_{\alpha^{-1}} \tau_\epsilon t_\beta \sim t_\sigma$* in $H_k \backslash S_{2k} / H_k$ *see* (4.8)*,*
- *$A \in \mathcal{P}(r)$ is such that $\sigma \le A$ and $A \vee \alpha \vee \beta = 1_r$,*
- *$C_{\sigma,A}$ are the relative cumulants of the orthogonal Weingarten function (see* [11]*)*
- *the combinatorial coefficient $\lambda_{\alpha,\beta,\epsilon}$ is $2^{r - \#(\alpha) - \#(\beta)}$.*

Proof: We first give a formula for the mixed moments, following [6, Eq (20)]:

$$\mathbb{E}[\prod_{i=1}^{k} \mathrm{Tr}(D_i O D_{\bar{i}} O^{-1})] = \frac{1}{|H_k|^2} \sum_{g,g' \in \mathcal{S}_{2k}} W\Lambda(g^{-1}g') M^+_D(g) M^-_{\bar{D}}((g')^{-1}) \quad (4.6)$$

where the $\pm$ generalized moments M are functions on $\mathcal{S}_{2k}$, resp. H_k-right and H_k-left invariant: for Y a k-tuple of random real matrices,

$$M^+_Y(g) = \mathbb{E}[\mathrm{Tr}_{\pi_l}(Y_1^{\epsilon_1(l)}, \dots, Y_k^{\epsilon_k(l)})] \text{ if } g \in g_l H_k$$

$$M^-_Y(g) = \mathbb{E}[\mathrm{Tr}_{\pi_l}(Y_1^{\epsilon_1(l)}, \dots, Y_k^{\epsilon_k(l)})] \text{ if } g \in H_k g_l$$

with $Y^{-1} = Y^t$ the transpose of Y.
Formula (4.6) follows from (4.2) and straightforward computations. Since the matrices D are symmetric, the moments do not depend of the sequence ϵ. Using the parametrisation $g = \tau_{\epsilon_1} t_\alpha h$ and $g' = \tau_{\epsilon_2} t_\beta h'$ with $h, h' \in H_k$, we can rewrite (4.6) as

$$\mathbb{E}[\prod_{i=1}^{k} \mathrm{Tr}(D_i O D_{\bar{i}} O^{-1})] = \sum_{\alpha,\beta \in \mathcal{S}_k, \epsilon \in \{\pm 1\}^k} \lambda_{\alpha,\beta,k} W\Lambda(t_{\alpha^{-1}} \tau_\epsilon t_\beta) \mathrm{Tr}_\alpha(D) \mathrm{Tr}_{\beta^{-1}}(\bar{D}) \quad (4.7)$$

The coefficient $\lambda_{\alpha,\beta,k}$ comes from the fact that we do not impose the sequences ϵ_1 and ϵ_2 associated with α, resp. β to be particular and $\epsilon = \epsilon_1 \epsilon_2$. From Proposition 4.3, $\lambda_{\alpha,\beta,k} = 2^{k - \#(\alpha) - \#(\beta)}$
As recalled before, with the triplet $(\alpha, \beta, \epsilon)$, we can associate a particular pair (σ, ϵ') such that:

$$t_{\alpha^{-1}} \tau_\epsilon t_\beta = \tau_{\epsilon'} t_\sigma h \quad (4.8)$$

with $h \in H$. Then, $\mathrm{W}\Lambda(t_{\alpha^{-1}} \tau_\epsilon t_\beta) = \mathrm{W}\Lambda(t_\sigma)$, denoted below by $\mathrm{W}\Lambda(\sigma)$. In the following, we will denote by $\sigma := \sigma(\alpha, \beta, \epsilon)$ the permutation constructed above.

We now compute the cumulant, following the same scheme as in [21]. Let $C = \{V_1, \dots V_k\} \in \mathcal{P}(r)$. We denote by S_{V_i} the permutations on V_i.

$$\begin{aligned}
\mathbb{E}_C(X_1, \dots, X_r) &= \prod_{i=1}^{k} \mathbb{E}\left(\prod_{j \in V_i} X_j\right) \\
&= \sum_{(\alpha_i,\beta_i,\epsilon_i) \in S_{V_i} \times S_{V_i} \times \{\pm 1\}^{|V_i|}; i \le k} (\prod_{i=1}^{k} \lambda_{\alpha_i,\beta_i,|V_i|}) W\Lambda(\sigma_1) \dots W\Lambda(\sigma_k) \dots \\
&\qquad \dots \mathrm{Tr}_{\alpha_1}(D) \mathrm{Tr}_{\beta_1^{-1}}(\bar{D}) \dots \mathrm{Tr}_{\alpha_k}(D) \mathrm{Tr}_{\beta_k^{-1}}(\bar{D}) \\
&= \sum_{\substack{(\alpha, \beta, \epsilon) \in \mathcal{S}_r \times \mathcal{S}_r \times \{\pm 1\}^r \\ \alpha, \beta \le C}} \lambda_{\alpha,\beta,r} \mathrm{W}\Lambda_C(\sigma) \mathrm{Tr}_\alpha(D) \mathrm{Tr}_{\beta^{-1}}(\bar{D})
\end{aligned}$$

where $\sigma \in \mathcal{S}_r$ is such that $\sigma \leq C$ and (see [11, section 3.4])

$$\mathrm{W}\Lambda_C(\sigma) = \prod_{i=1}^{k} \mathrm{W}\Lambda(\sigma_{|V_i}).$$

Collins and Sniady defined the cumulants of orthogonal Weingarten functions that we will denote by $C_{\sigma,A}$ for A a partition σ-invariant. They satisfy like (3.15) and (3.16):

$$\mathrm{W}\Lambda_C(\sigma) = \sum_{A \in [\sigma, C]} C_{\sigma,A}\,.$$

Thus,

$$\mathbb{E}_C(X_1, \dots X_r) = \sum_{\substack{\alpha, \beta \leq C \\ \epsilon \in \{\pm 1\}^r}} \lambda_{\alpha,\beta,r} \sum_{A \in [\sigma, C]} C_{\sigma,A} \operatorname{Tr}_\alpha(D) \operatorname{Tr}_{\beta^{-1}}(\bar{D}).$$

Now,

$$\begin{aligned}
&\kappa_r(X_1, \dots, X_r) \\
&\quad = \sum_C \text{Möb}(C, 1_r)\, \mathbb{E}_C(X_1, \dots X_r) \\
&\quad = \sum_C \text{Möb}(C, 1_r) \sum_{\substack{(\alpha,\beta,\epsilon) \in \mathcal{S}_r \times \mathcal{S}_r \times \{\pm 1\}^r \\ \alpha, \beta \leq C}} \lambda_{\alpha,\beta,r} \sum_{A \in [\sigma, C]} C_{\sigma,A} \operatorname{Tr}_\alpha(D) \operatorname{Tr}_{\beta^{-1}}(\bar{D}) \\
&\quad = \sum_{(\alpha,\beta,\epsilon) \in \mathcal{S}_r \times \mathcal{S}_r \times \{\pm 1\}^r} \lambda_{\alpha,\beta,r} \sum_{\sigma \leq A} \sum_{\{C; A, \alpha, \beta \leq C\}} \text{Möb}(C, 1_r) C_{\sigma,A} \operatorname{Tr}_\alpha(D) \operatorname{Tr}_{\beta^{-1}}(\bar{D}) \\
&\quad = \sum_{(\alpha,\beta,\epsilon) \in \mathcal{S}_r \times \mathcal{S}_r \times \{\pm 1\}^r} \lambda_{\alpha,\beta,r} \sum_{\sigma \leq A; A \vee \alpha \vee \beta = 1_r} C_{\sigma,A} \operatorname{Tr}_\alpha(D) \operatorname{Tr}_{\beta^{-1}}(\bar{D})
\end{aligned}$$

where the last equality follows from

$$\sum_{C; A, \alpha, \beta \leq C} \text{Möb}(C, 1_r) = \begin{cases} 1 & \text{if } A \vee \alpha \vee \beta = 1_r, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

We now study the asymptotic of (4.5) when $n \to \infty$. We assume that the family of $n \times n$ matrices (D_i) has a limit distribution. It is known (see [11, Theorem 3.16]) that the order of $C_{\sigma,A}$ is $n^{-2r-\#(\sigma)+2\#(A)}$. In [11], the asymptotic of the cumulant is given in terms of a metric l on pairings. We use that $l(p_\sigma, p_{Id}) = |\sigma| := r - \#(\sigma)$ where $p_\sigma = \prod(i, \overline{\sigma(i)})$ is the pairing associated with σ and $p_{Id} = \gamma$.
Now,

$$\operatorname{Tr}_\alpha(D) \operatorname{Tr}_{\beta^{-1}}(\bar{D}) = n^{\#(\alpha)+\#(\beta)} \operatorname{tr}_\alpha(D) \operatorname{tr}_{\beta^{-1}}(\bar{D}) = O(n^{\#(\alpha)+\#(\beta)}).$$

Therefore, for given α, β, ϵ, the corresponding term in the cumulant is of order

$$n^{-2r-\#(\sigma)+2\#(A)+\#(\alpha)+\#(\beta)},$$

where σ and A satify the conditions quoted in Proposition 4.5, i.e.

$$\begin{cases} \sigma \text{ is the permutation defined by (4.8)}, \\ \sigma \leq A \text{ and } A \vee \alpha \vee \beta = 1_r. \end{cases} \tag{4.9}$$

Proposition 4.6. *Under the conditions* (4.9)*, for* $r \geq 3$*,*

$$-2r - \#(\sigma) + 2\#(A) + \#(\alpha) + \#(\beta) + 1 \leq 0\,. \tag{4.10}$$

Corollary 4.7. *Let* $\{D_i, i \in [2r]\}_n$ *be a sequence of deterministic and symmetric matrices of size n which has a limit distribution and* $X_i = \mathrm{Tr}(D_i O D_{\bar{i}} O^{-1})$*. For* $r \geq 3$*,*

$$\lim_{n\to\infty} \kappa_r(X_1, \dots, X_r) = 0 \tag{4.11}$$

We first recall the following lemma (see [21] and the proof therein for the second assertion below).

Lemma 4.8. *For* $A, B \in \mathcal{P}(k)$ *we have*

$$\#(A) + \#(B) \leq k + \#(A \vee B)\,.$$

Moreover, if there exists a block A_i *of* A *and* B_j *of* B *such that* $\#(A_i \cap B_j) = l$*, then,*

$$\#(A) + \#(B) \leq k - l + 1 + \#(A \vee B)\,.$$

Proof of Proposition 4.6 From the above lemma and Condition (4.9), we have:

$$\#(\alpha) + \#(\beta) \leq r + \#(\alpha \vee \beta)\,, \tag{4.12}$$

$$\#(A) + \#(\alpha \vee \beta) \leq r + 1\,, \tag{4.13}$$

$$\#(A) \leq \#(\sigma)\,. \tag{4.14}$$

The proof relies on the following property:

Lemma 4.9. *If* α *or* β *has a fixed point, then there is a strict inequality in* (4.13) *or in* (4.14)

Proof of Lemma 4.9 We denote by Σ the permutation of $\mathcal{S}_{2r}$ given by $t_{\alpha^{-1}} \tau_\epsilon t_\beta$.

1) Assume that i is a fixed point of α and β. Then $(\Sigma(i), \Sigma(\bar{i})) = (i, \bar{i})$ or $(\bar{i}, i)$ depending on the sign of $\epsilon(i)$ and therefore, i is a fixed point of σ. In this case, $\sigma \vee \alpha \vee \beta \neq 1_r$ and therefore $\#(A) < \#(\sigma)$.

2) Assume that i is a fixed point of β and $\alpha(i) \neq i$. Then $\{\Sigma(i), \Sigma(\bar{i})\} = \{i, \overline{\alpha^{-1}(i)}\}$ and therefore the elements $\{i, \alpha^{-1}(i)\}$ are in the same cycle of σ (thus in the same block of A), thery are obviously in the same block of α. From Lemma 4.8, the inequality is strict in (4.13).

3) Assume that i is a fixed point of α and $\beta(i) \neq i$.
$\Sigma(\overline{\beta^{-1}(i)}) = i$ or $\bar{i}$ according to the sign of ϵ_i and

$$\Sigma(\beta^{-1}(i)) = \beta^{-1}(i) \text{ or } \overline{\alpha^{-1}\beta^{-1}(\mathrm{i})}.$$

Thus, we find two distinct elements $(i, \beta^{-1}(i))$ (or $(i, \alpha^{-1}\beta^{-1}(i))$) belonging to a cycle of σ (thus to the same block of A) and belonging to a block of $\alpha \vee \beta$ (which is β^{-1}-invariant and $\alpha^{-1}\beta^{-1}$-invariant). Therefore, the inequality (4.13) is strict. □

Let α and β two permutations in $\mathcal{S}_r$. If $\#(\alpha) + \#(\beta) \leq r - 2 + \#(\alpha \vee \beta)$, then, (4.10) is satisfied, using (4.13) and (4.14). From (4.12), it remains to study the two cases:

1) $\#(\alpha) + \#(\beta) = r + \#(\alpha \vee \beta)$,

2) $\#(\alpha) + \#(\beta) = r - 1 + \#(\alpha \vee \beta)$.

Case 1) It is not difficult to see that in this case, there exist two different fixed points for α or β. For example, a unique fixed point for both α and β would imply:

$$\#(\alpha) + \#(\beta) \leq 1 + \frac{r-1}{2} + 1 + \frac{r-1}{2} = r + 1.$$

This is not possible since in this case, $\alpha \vee \beta$ has at least two blocks. So, let $i \neq j$ the two fixed points. Several situations can occur:

a) i, j are fixed points of both α and β. Then, as seen above, i, j are fixed points of σ. In this case, $\#(A) \leq \#(\sigma) - 2$, leading to (4.10).

b) i is a fixed point of α and β and j is a fixed point of one of them. From the previous Lemma,

$$\#(A) \leq \#(\sigma) - 1 \text{ and } \#(A) + \#(\alpha \vee \beta) \leq r - 1 + 1 = r$$

leading to (4.10).

c) i, j are fixed points of one of the two permutations. For example, i, j are fixed points of β. If $\#\{i, j, \alpha^{-1}(i), \alpha^{-1}(j)\} \geq 3$, then, from Lemmas 4.8 and 4.9, it is not difficult to see that $\#(A) + \#(\alpha \vee \beta) \leq r - 2 + 1$ (either a block of σ and a block of α have 3 common points or two blocks of σ and two blocks of α have two common points, etc.), leading to (4.10).
If $\#\{i, j, \alpha^{-1}(i), \alpha^{-1}(j)\} = 2$, in this case, α and σ have a 2-cycle (i, j). This yields $\#(A) + \#(\alpha \vee \beta) \leq r - 1 + 1$ but we also have $\#(A) < \#(\sigma)$ since $\sigma \vee (\alpha \vee \beta) \neq 1_r$. The other cases are similar, leading to

$$-\#(\sigma) + 2\#(A) + \#(\alpha \vee \beta) \leq r - 2 + 1 = r - 1$$

proving (4.10).

Case 2): $\#(\alpha) + \#(\beta) = r - 1 + \#(\alpha \vee \beta)$. If there exists a fixed point, then one of the inequality in (4.13) or (4.14) is strict and we are done. If there is no fixed point, this implies that r is even, $\#(\alpha) = \#(\beta) = \frac{r}{2}$ and $\alpha \vee \beta = 1_r$. An equality in (4.13) is true only for $A = 0_r$, the partition in singletons. This would imply that $\sigma = Id$. Let $i \leq r$,

$$\Sigma(i) = i \,\text{or}\, \overline{\alpha^{-1}(i)} \text{ and } \Sigma(\bar{i}) = \beta(i) \,\text{or}\, \overline{\alpha^{-1}\beta(\text{i})}.$$

Therefore, $\sigma = Id$ implies that $\alpha^{-1}\beta(i) = i$ or $\alpha^{-1}(i) = \beta(i)$ and α and β have a common 2-cycle. This is not possible ($\alpha \vee \beta = 1_r$) except if $r = 2$. □.

4.4. **The fourth cumulant.** We give an estimate for $\kappa_4(T_{p,q})$. From the previous section,

$$\kappa_4(T_{p,q}) = \sum_{(\alpha,\beta,\epsilon)\in S_4\times S_4\times\{\pm 1\}^4} \lambda_{\alpha,\beta,4} \sum_{A;A\vee\alpha\vee\beta=1_4} C_{\sigma,A} \operatorname{Tr}_\alpha(D) \operatorname{Tr}_{\beta^{-1}}(\bar{D})$$

where $D = (I_p, I_p, I_p, I_p)$ and $\bar{D} = (I_q, I_q, I_q, I_q)$. From the asymptotic behavior of the cumulant, each term is of order

$$n^{-8-\#(\sigma)+2\#(A)} p^{\#(\alpha)} q^{\#(\beta)} \tag{4.15}$$

where σ, A, α, β satisfy condition (4.9).
First assume that $\#(\alpha) = \#(\beta) = 1$. Then the order of (4.15) is at most:

$$\frac{pq}{n^4} \leq \frac{p^2q^2}{n^4}.$$

Now assume that α or β has at least two blocks. From Proposition 4.6, the order of (4.15) is at most

$$n^{-1-\#(\alpha)-\#(\beta)} p^{\#(\alpha)} q^{\#(\beta)}.$$

It is easy to see that this is at most of order $\frac{p^2q^2}{n^4}$. For example, α and β has at least two blocks,

$$n^{-1-\#(\alpha)-\#(\beta)} p^{\#(\alpha)} q^{\#(\beta)} \leq \frac{p^2q^2}{n^5} \left(\frac{p}{n}\right)^{\#(\alpha)-2} \left(\frac{q}{n}\right)^{\#(\beta)-2}.$$

In the remaining cases, we find a majoration by $\frac{pq^2}{n^4}$ or $\frac{p^2q}{n^4}$. Therefore, we obtain

$$\kappa_4(T_{p,q}) = O\left(\frac{p^2q^2}{n^4}\right). \tag{4.16}$$

4.5. **Proof of Theorem 1.1.** The proof is similar to the proof in the unitary case, using the asymptotic vanishing cumulants of order $r \geq 3$ (Corollary 4.7) and the estimate (4.16) for the fourth cumulant, ensuring the tightness of the family of distributions.

5. Complementary remarks

1) Since the sup norm is continuous for the Skorokhod topology, Theorem 1.1 implies that

$$\sup_{s,t\in[0,1]} |W^{(n)}(s,t)| \to \sup_{s,t\in[0,1]} |W^{(\infty)}(s,t)|$$

in distribution, which implies that

$$\sup_{s,t\in[0,1]} |\frac{1}{n} T_{\lfloor ns\rfloor,\lfloor nt\rfloor} - st| \to 0 \tag{5.1}$$

in probability.

2) The definition of our process focusses on the trace of a random matrix $H_{p,q}$. This trace is a linear statistic of its empirical spectral distribution, i.e.

$$T_{p,q} = \operatorname{Tr} H_{p,q} = p \int x d\mu(x)\,,$$

where

$$\mu = \frac{1}{p}\sum_{k=1}^{p} \delta_{\lambda_k}\,,$$

and the λ_k's are the eigenvalues of $H_{p,q}$. If we are interested only in marginals ($p = \lfloor ns \rfloor, q = \lfloor nt \rfloor$, with $s, t \in (0,1)$ fixed), we can look directly at the asymptotic behavior of μ when $n \to \infty$ and then deduce results from the continuity of the mapping $\mu \mapsto \int x d\mu(x)$ on $\mathcal{M}_1([0,1])$. It is known that, if $p \le q$ and $p + q \le n$, the random matrix $H_{p,q}$ belongs to the Jacobi unitary/orthogonal ensemble ([9] Theorem 2.2, [1] Prop. 4.1.4), which entails that the joint distribution of eigenvalues has a density proportional to

$$\prod_{k=1}^{p} \lambda_i^{a-1}(1-\lambda_i)^{b-1} \prod_{1\le i<j\le p} |\lambda_i - \lambda_j|^{\beta} \tag{5.2}$$

where $a = (q-p+1)\frac{\beta}{2}$ and $b = (n-p-q+1)\frac{\beta}{2}$. The sequence of empirical spectral distributions converges to the generalized Kesten-McKay distribution. When $s \le \min(t, 1-t)$, this distribution has a density which can be parametrized by s, t or by the endpoints of its support (u_-, u_+) with $0 \le u_- < u_+ \le 1$:

$$\pi_{u_-,u_+}(x) = C_{u_-,u_+} \frac{\sqrt{(x-u_-)(u_+-x)}}{2\pi x(1-x)} \tag{5.3}$$

where

$$C_{u_-,u_+}^{-1} := \frac{1}{2}\left[1 - \sqrt{u_-u_+} - \sqrt{(1-u_-)(1-u_+)}\right].$$

The relation between (s,t) and $u_\pm$ is

$$u_\pm = \left[\sqrt{s(1-t)} \pm \sqrt{(1-s)t}\right]^2 .$$

By continuity, in all cases, we recover a weak form of (5.1), i.e.

$$\lim_n \frac{1}{n} T_{\lfloor ns \rfloor, \lfloor nt \rfloor} = s \int x \pi_{u_-,u_+}(x) dx = st\,,$$

in probability.

It could also be possible to recover the limiting fluctuations of the marginal distribution of $T_{\lfloor ns \rfloor, \lfloor nt \rfloor}$ with s, t fixed, i.e.

$$T_{\lfloor ns \rfloor, \lfloor nt \rfloor} - \mathbb{E} T_{\lfloor ns \rfloor, \lfloor nt \rfloor} \xrightarrow{\text{law}} \mathcal{N}(0, \frac{2}{\beta} s(1-s)t(1-t))$$

from the known results on the fluctuations of linear statistics of μ. Actually, the result of Johansson [19] is not specific of the Jacobi ensemble, but uses a model of random matrices invariant by conjugation, with polynomial external field. Here, the ensemble is invariant but the potential is logarithmic (see (5.2)).

The result is a Gaussian limit with the good variance.

At another level, in the same asymptotics as above, Hiai and Petz [16] proved that the family of empirical spectral distributions satisfies the Large Deviation Principle in $\mathcal{M}_1([0,1])$ with speed $\beta n^2/2$ and good rate function, which in the case $s < t < 1/2$ is

$$\begin{aligned}\mathcal{I}(\nu) &= -s^2 \int\int \log|x-y| d\nu(x) d\nu(y) \\ &\quad -s\int ((1-s-t)\log(1-x) + (t-s)\log x)\, d\nu(x) + I_0(s,t)\,.\end{aligned}$$

where $I_0(s,t)$ is some constant (the limiting free energy). Appealing again to the continuity of the mean, we deduce from the contraction principle that $n^{-1}T_{\lfloor ns\rfloor,\lfloor nt\rfloor}$ satisfies the LDP at scale n^{-2} with good rate function

$$I(c) = \inf\{\mathcal{I}(\nu); \nu\in\mathcal{M}_1([0,1]), \int_0^1 x d\nu(x) = c\}\,.$$

3) In multivariate (real) analysis of variance, the random variable $T_{p,q}$ is known as the Bartlett-Nanda-Pillai statistics. The exact distribution of $T_{p,q}$ is known by its Laplace transform which is an hypergeometric function of matrix argument ([22] p.479). Various asymptotic studies have been performed, essentially p,q fixed, $n\to\infty$ (large sample framework), or high-dimensional framework with q fixed, $n,p\to\infty$ and $p/n\to s<1$ (see for instance [14]). The asymptotic regime of the present paper ($p/n\to s, q/n\to t$) is considered in Section 4.4 of the book [2] and a CLT for the statistic $T_{p,q}$ may be deduced from Theorem 2.2 of [3].

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