

A Tight Bound on the Set Chromatic Number*

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Abstract

The purpose of this note is to provide a tight bound on the set chromatic number of a graph in terms of its chromatic number. Namely, for all graphs G , we show that $\chi_s(G) \geq \lceil \log_2 \chi(G) \rceil + 1$, where $\chi_s(G)$ and $\chi(G)$ are the set chromatic number and the chromatic number of G , respectively. This answers in the affirmative a conjecture of Gera, Okamoto, Rasmussen and Zhang.

1 Introduction

There is a plethora of work devoted to neighbor-distinguishing vertex (or edge) colorings in graphs. Essentially, given a function f defined on the set of vertices of a graph the goal is to obtain a vertex coloring (or an edge coloring) such that $f(u) \neq f(v)$ whenever u and v are two adjacent vertices. (Obviously, the values taken by f depends on the coloring used.) This approach to graph coloring permits to gather in a synthetic framework several variants of colorings such as set colorings, metric colorings and sigma colorings. An interesting line of research is to estimate how these notions relate to each others and, in particular, how they behave with respect to the (usual) chromatic number of a graph. We refer the reader to the survey by Chartrand, Okamoto and Zhang [2] for further information.

The notion of a *set coloring* was first introduced by Chartrand, Okamoto, Rasmussen and Zhang [1]. Given a graph $G = (V, E)$ and a (not necessarily proper) k -coloring $c : V \rightarrow \{1, 2, \dots, k\}$ of its vertices, let

$$NC(v) := \{c(u) \mid (u, v) \in E\}$$

be the *neighborhood color set* of a vertex $v \in V$. The coloring c is *set neighbor-distinguishing*, or simply a *set coloring*, if $NC(u) \neq NC(v)$ for every pair (u, v) of adjacent vertices in G .

The minimum number of colors, k , required for such a coloring is the *set chromatic number* $\chi_s(G)$ of G . Note that $\chi_s(G) \leq \chi(G)$ for every graph G . Moreover, the set chromatic number is bounded from below by the logarithm of the clique number as follows.

Theorem 1 ([1]). *For every graph G ,*

$$\chi_s(G) \geq 1 + \lceil \log_2 \omega(G) \rceil. \quad (1)$$

A natural strengthening of (1) would be to replace the clique number with the chromatic number. In particular, if $\chi(G) = \omega(G)$ then

$$\chi_s(G) \geq 1 + \lceil \log_2 \chi(G) \rceil, \quad (2)$$

and Gera, Okamoto, Rasmussen and Zhang [3] showed that (2) is tight, in a strong sense.

Theorem 2 ([3]). *For each pair (a, b) of integers such that $2 \leq a \leq b \leq 2^{a-1}$, there exists a perfect graph G with $\chi_s(G) = a$ and $\chi(G) = b$.*

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Furthermore, they conjectured that (2) holds for all graphs G . The purpose of this note is to confirm this conjecture.

Theorem 3. *For all graphs G ,*

$$\chi(G) \leq 2^{\chi_s(G)-1}.$$

2 Proof of Theorem 3

Before starting the proof, notice that for every complete graph K_n with $n \geq 2$,

$$\chi_s(K_n) = \chi(K_n) = n.$$

This fact allows us to proceed by double induction: an ascending induction on the number of vertices of the graph and, for a fixed number of vertices, a descending induction on the number of edges.

Let $G = (V, E)$ be a graph with $n \geq 2$ vertices and m edges. The assertion of Theorem 3 holds if $n = 2$ or $m = \binom{n}{2}$ (that is, G is complete). So we assume now that $n > 2$, $m < \binom{n}{2}$ and $\chi(H) \leq 2^{\chi_s(H)-1}$ for all graphs H with either less than n vertices or exactly n vertices and more than m edges.

We shall study the structure of G under the additional assumption that $\chi(G) > 2^{\chi_s(G)-1}$. This will allow us to exhibit a proper coloring of G using at most $2^{\chi_s(G)-1}$, thereby obtaining a contradiction and concluding the proof.

First, notice that for any two non-adjacent vertices u and v of G ,

$$\chi_s(G) < \chi_s(G + uv). \quad (3)$$

Indeed, if there exists a pair (u, v) of vertices violating (3), then

$$\chi(G) \leq \chi(G + uv) \leq 2^{\chi_s(G+uv)-1} \leq 2^{\chi_s(G)-1},$$

which contradicts our assumption on G .

Now, set $a := \chi_s(G)$ and let $c : V \rightarrow \{1, \dots, a\}$ be a set coloring. We consider the a color classes V_1, \dots, V_a where V_i is the set of vertices assigned color i .

We observe that no two vertices u and v in a same color class can have the same neighborhood color set. This follows from the definition of a set coloring if u and v are adjacent, so suppose that u and v are not adjacent. Consider the graph G' that results from identifying the vertices u and v of G into a new vertex z . Note that the vertex coloring of G' naturally induced by c (that is, with z being assigned color i) is a valid set coloring of G' , so $\chi_s(G) \geq \chi_s(G')$. However, as $\chi(G) \leq \chi(G')$, the induction hypothesis applied to G' yields that $\chi(G) \leq 2^{\chi_s(G')-1} \leq 2^{\chi_s(G)-1}$; a contradiction.

We are now in a position to describe the structure of the color classes. We assert that for each $i \in \{1, \dots, a\}$, the subgraph H_i of G induced by the vertices in V_i is either a clique or a clique and an isolated vertex.

We prove the assertion in two steps. Fix $i \in \{1, \dots, a\}$. First, we show that for every two non-adjacent vertices u and v in V_i , it holds that $i \notin NC(u) \cap NC(v)$, that is one of u and v has no neighbors in V_i . Indeed, as reported earlier the graph $G' := G + uv$ has set chromatic number at least $a + 1$, so that c is not a set coloring of G' . Consequently, $NC_{G'}(u) \neq NC_G(u)$ or $NC_{G'}(v) \neq NC_G(v)$ hence $i \notin NC(u) \cap NC(v)$, as wanted.

Now, to prove the assertion, assume that V_i does not induce a clique in G . What precedes implies that there is a vertex $v \in V_i$ that has no neighbors in V_i . If two vertices u and w in $V_i \setminus \{v\}$ are not adjacent then, similarly, we may assume that u has no neighbors in V_i . We now prove that every neighbor of u in G is also a neighbor of v . By symmetry of the roles played by u and v , this would imply that u and v have the same neighborhood, hence the same neighborhood color set; a contradiction. So let x be a neighbor of u in G and assume that x is not adjacent to v . Note that $x \in V_j$ for some $j \neq i$ and $i \in NC(x) \setminus NC(v)$. Let G' be the graph obtained from G by adding the edge xv . We know that c cannot be a set coloring of G' . Moreover, $NC_{G'}(x) = NC_G(x)$ and $i \notin NC_{G'}(v) = NC_G(v) \cup \{j\}$. Consequently, we infer that v has a neighbor y in G such that $NC_G(y) = NC_{G'}(v)$. However, $i \in NC_G(y)$ yet $i \notin NC_{G'}(v)$. This contradiction finishes the proof of the assertion.

We can now exhibit a proper coloring of G using at most 2^{a-1} colors, which will complete the proof of Theorem 3.

We color the vertices of G using the alphabet $\{0, 1\}^{a-1}$. Let $\mathbf{0}$ be the zero vector and let $e_j := 1 \cdots 101 \cdots 1$ where the zero is in position j . Call a vertex of *Type 1* if $v \in V_i$ but $i \notin NC(v)$, and of *Type 2*, otherwise. Now, we define a coloring χ , which is related to the characteristic vector of the neighborhood color set. Let v be a vertex of G , so $v \in V_i$ for some $i \in \{1, \dots, a\}$. If v is of Type 1, let

$$\chi(v) := \begin{cases} \mathbf{0} & \text{if } i = a \\ e_i & \text{if } i \neq a. \end{cases}$$

Suppose that v is of Type 2. If $v \in V_a$ then for each $j \in \{1, \dots, a-1\}$ let

$$\chi(v)_j := \begin{cases} 1 & \text{if } j \in NC(v) \\ 0 & \text{if } j \notin NC(v). \end{cases}$$

Otherwise, that is, if v is of Type 2 and $v \in V_i$ with $i < a$, then for each $j \in \{1, \dots, a-1\}$ let

$$\chi(v)_j := \begin{cases} 1 & \text{if } j \neq i \text{ and } j \in NC(v) \\ 0 & \text{if } j \neq i \text{ and } j \notin NC(v) \\ 1 & \text{if } j = i \text{ and } a \in NC(v) \\ 0 & \text{if } j = i \text{ and } a \notin NC(v). \end{cases}$$

We now show that χ is a proper coloring of G .

First note that the coloring χ is proper on each part V_i , since two (adjacent) vertices in V_i have different neighborhood color sets (both containing i). Furthermore, any two vertices of Type 1 are assigned distinct vectors by χ . Now, let $u \in V_i$ and $v \in V_j$ be two adjacent vertices with $i \neq j$. Suppose first that u is of Type 1 but v is of Type 2. Then $\chi(v)_i = 1$ and $\chi(u)_i = 0$ if $i \neq a$, whereas $\chi(u)_j = 1$ and $\chi(v)_j = 0$ if $i = a$. Finally, assume that both u and v are of Type 2. As c is a set coloring of G , there must exist $\ell \in NC(u) \Delta NC(v)$. If $\ell \neq a$, then $\chi(u)_\ell \neq \chi(v)_\ell$ since $\ell \notin \{i, j\}$. If $\ell = a$, we may assume without loss of generality that $a \notin NC(u)$. Then $\chi(u)_i = 0$ while $\chi(v)_i = 1$ since u and v are adjacent. Thus, the coloring χ is proper, which concludes the proof of Theorem 3. ■

References

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