The eccentricity sequences of Fibonacci and Lucas cubes

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Abstract

The Fibonacci cube Γ_n is the subgraph of the hypercube induced by the binary strings that contain no two consecutive 1's. The Lucas cube Λ_n is obtained from Γ_n by removing vertices that start and end with 1. The eccentricity of a vertex u, denoted $e_G(u)$ is the greatest distance between u and any other vertex v in the graph G. For a given vertex u of Γ_n we characterize the vertices v such that $d_{\Gamma_n}(u,v) = e_{\Gamma_n}(u)$. We then obtain the generating functions of the eccentricity sequences of Γ_n and Λ_n. As a corollary we deduce the number of vertices of a given eccentricity.

Key words: Fibonacci cubes, Lucas cubes, Median Graph, Hypercube

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1. Introduction

An interconnection topology can be represented by a graph $G = (V, E)$, where V denotes the processors and E the communication links. The hypercube Q_n is a popular interconnection network because of its structural properties.

The Fibonacci cube was introduced in [Hsu93] as a new interconnection network. This graph is an isometric subgraph of the hypercube which is inspired in the Fibonacci numbers. It has attractive recurrent structures such as its decomposition into two subgraphs which are also Fibonacci cubes by themselves. Structural properties of these graphs were more extensively studied afterwards [Kla05, MS02, DTS02, Gre06, KP05, TV07, EMPZ06, KM11, CKMR11, KMP11, Kla11].

Lucas cubes, introduced in [MCS01], have attracted the attention as well due to the fact that these cubes are closely related to the Fibonacci cubes. They have also been widely studied [DTS02, CKMR11, KMP11, IKR10].

We will next define some concepts needed in this paper. A *Fibonacci*

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string of length n is a binary string $b_1 b_2 \cdots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words, a Fibonacci string does not contain two consecutive 1's. *The Fibonacci cube* Γ_n is the subgraph of Q_n induced by the Fibonacci strings of length n. Adjacent vertices in Γ_n differ in one bit. Because of the empty string, $\Gamma_0 = K_1$. A Fibonacci string of length n is a *Lucas string* if $b_1 \cdot b_n \neq 1$. That is, a Lucas string has no two consecutive 1's including the first and the last elements of the string. The *Lucas cube* Λ_n is the subgraph of Q_n induced by the Lucas strings of length n. We have $\Lambda_0 = \Lambda_1 = K_1$.

The usual notation $d_G(u, v)$ for the shortest path distance between vertices u and v in a connected graph G will be used through this paper. The *eccentricity* of a vertex u, denoted $e_G(u)$ is the greatest distance between u and any other vertex v in the graph. When no confusion is possible we will shorten these notations to $d(u, v)$ and $e(u)$. Clearly, not all the vertices of Γ_n or Λ_n have the same eccentricity as it happens in Q_n. We say that v *satisfies* the eccentricity of u when $d(u, v) = e(u)$. The *radius* of a graph G, denoted $rad(G)$, is the minimum eccentricity among the vertices of G, while the *diameter* of G, denoted $diam(G)$ is the maximum eccentricity among the vertices of the graph.

The radius, $rad(\Gamma_n) = \left\lceil \frac{n}{2} \right\rceil$ and diameter, $diam(\Gamma_n) = n$ of the Fibonacci cubes are obtained in [MS02]. Similarly $rad(\Lambda_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and $diam(\Lambda_n) = 2 \left\lfloor \frac{n}{2} \right\rfloor$ are determined in [MCS01].

We define the *eccentricity sequence* of G as the sequence $\{a_k\}_{k=0}^{diam(G)}$ of nonnegative integers, where a_k is the number of vertices of eccentricity k in G.

In the next table, we show the number of vertices of eccentricity k in Γ_n and in Λ_n for $n = 1$ to 10 which can be computed by hand.

n	0	1		2			3				4					5						6						
k	0	0	1	0	1	2	0	1	2	3	0	1	2	3	4	0	1	2	3	4	5	0	1	2	3	4	5	6
$\Gamma:$	1	0	2	0	1	2	0	0	3	2	0	0	1	5	2	0	0	0	4	7	2	0	0	0	1	9	9	2
$\Lambda:$	1	1	0	0	1	2	0	1	3	0	0	0	1	4	2	0	0	1	5	5	0	0	0	0	1	9	6	2

n	7								8									9										10										
k	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	10
$\Gamma:$	0	0	0	0	5	16	11	2	0	0	0	0	1	14	25	13	2	0	0	0	0	0	6	30	36	15	2	0	0	0	0	0	1	20	55	49	17	2
$\Lambda:$	0	0	0	1	7	14	7	0	0	0	0	0	1	16	20	8	2	0	0	0	0	1	9	30	27	9	0	0	0	0	0	0	1	25	50	35	10	2

Table 1: Number of vertices of eccentricity k in Γ_n and Λ_n.

The purpose of this work is to determine the eccentricity sequences of the Fibonacci and Lucas cubes, Γ_n and Λ_n, for any value of n.

This paper is organized as follows: In the next section, we state the notation required for the Fibonacci cubes. In section three, we characterize the vertices of Γ_n that satisfy the eccentricity of a given vertex (Theorem 3.7). In section four, we obtain the eccentricity sequence of the Fibonacci cubes (Corollary 4.4). Finally, in section five, the eccentricity sequence of the Lucas cube is given (Corollary 5.17).

2. Notation of Fibonacci cubes

Let $\mathcal{F}_n$ be the set of strings of Γ_n.
Let $\mathcal{F}_n^{od\,\cdot}$ be the set of strings of Γ_n that begin with an odd number of 0's,
$\mathcal{F}_n^{ev\,\cdot}$ the set of strings of Γ_n that begin with an even number (eventually null) of 0's,
$\mathcal{F}_n^{ev^*\,\cdot}$ the set of strings of Γ_n that begin with an even number, not null of 0's and
$\mathcal{F}_n^{\varnothing\,\cdot}$ the set of strings of Γ_n that do not begin with a 0.
We have thus $\mathcal{F}_n = \mathcal{F}_n^{od\,\cdot} \uplus \mathcal{F}_n^{ev\,\cdot} = \mathcal{F}^{od\,\cdot} \uplus \mathcal{F}^{ev^*\,\cdot} \uplus \mathcal{F}_n^{\varnothing\,\cdot}$, where $\uplus$ is the disjoint union of sets.
Let $\mathcal{F}_n^{\cdot\,od}$ be the set of strings in Γ_n that end with an odd number of 0's.
Similarly, we define $\mathcal{F}_n^{\cdot\,b}$ where $b \in \{ev, ev^*, \varnothing\}$.
Let $\mathcal{F}_n^{od\,od}$ be the set of strings in Γ_n that begin and end with an odd number of 0's.
In the same way, we define $\mathcal{F}_n^{ab}$ where $a, b \in \{od, ev, ev^*, \varnothing, \cdot\}$.
Note that $\mathcal{F}_n^{\cdot\,\cdot} = \mathcal{F}_n$. Let $\mathcal{F}_{n,k}$ the set of strings of Γ_n with eccentricity k.
For any $a, b \in \{od, ev, ev^*, \varnothing, \cdot\}$, let $\mathcal{F}_{n,k}^{a\,b} = \mathcal{F}_n^{a\,b} \cap \mathcal{F}_{n,k}$ and $f_{n,k}^{a\,b}$ be $|\mathcal{F}_{n,k}^{a\,b}|$.
We will denote by $f^{a\,b}$ the generating function

$$f^{a\,b}(x, y) = \sum_{n,k \geq 0} f_{n,k}^{a\,b}\; x^n y^k$$

3. Eccentricity of a vertex of $\boldsymbol{\Gamma_n}$.

In this section, we show that a vertex x in Γ_n can be written uniquely as the concatenation of particular strings. We give some results concerning the eccentricity of these substrings. These results lead us to compute $e(x)$ and to characterize the vertices y in Γ_n that satisfy $e(x)$. Finally, we determine the last character of the strings y at distance $e(x)$ (Corollary 3.8). This latter result will be very useful through this paper.

Let us recall that Γ_n is an isometric subgraph of Q_n, i.e.:

Proposition 3.1. *The distance $d_{\Gamma_n}(a, b)$ between a and b in Γ_n is $d_{Q_n}(a, b)$, the number of positions in which the two strings a and b differ.*

Proof. Let $a = (a_1a_2 \cdots a_n)$, $b = (b_1b_2 \cdots b_n) \in \Gamma_n$ and let $z = (z_1z_2 \cdots z_n) \in Q_n$ be defined as

$$z_i = \begin{cases} a_i \text{ if } a_i = b_i \\ 0 \text{ if } a_i \neq b_i, \end{cases}$$

Note first that z is a Fibonacci string. Indeed $z_i = z_{i+1} = 1$ would imply $a_i = a_{i+1} = 1$. Consider now a shortest path in Q_n from a to b, $s = (a = s_0, s_1, \cdots, z, \cdots, s_j = b)$, obtained by concatenation of a shortest path from a to z and a shortest path from z to b. It is easy to see that all the vertices of s belong to Γ_n as well thus s is also a path in Γ_n. Furthermore s is a shortest path in Γ_n because, as a subgraph, $d_{\Gamma_n}(a, b) \geq d_{Q_n}(a, b)$. $\square$

We will thus shorten the notation $d_{\Gamma_n}(a, b)$ to $d(a, b)$ in this section. Let us denote by $x = (ab)$ the concatenation of two strings a and b.

Proposition 3.2. *Let* $z \in \mathcal{F}_n$ *such that* $z = (xy)$ *with* $x \in \mathcal{F}_{n_1}$, $y \in \mathcal{F}_{n_2}$ *and* $n_1 + n_2 = n$, *then*

$$e(z) \leq e(x) + e(y)$$

Proof. Let $c \in \mathcal{F}_n$ such that $d(z, c) = e(z)$. Then $c = (ab)$ with $a \in \mathcal{F}_{n_1}$ and $b \in \mathcal{F}_{n_2}$.
By the definition of eccentricity, $d(x, a) \leq e(x)$ and $d(y, b) \leq e(y)$.
Then $e(xy) = d(xy, ab) = d(x, a) + d(y, b) \leq e(x) + e(y)$. $\square$

Proposition 3.3. *Let* $z \in \mathcal{F}_n$ *such that* $z = (xy)$ *with* $x \in \mathcal{F}_{n_1}$, $y \in \mathcal{F}_{n_2}$ *and* $n_1 + n_2 = n$. *If* $e(xy) = e(x) + e(y)$, *then any string* $u \in \mathcal{F}_n$ *that satisfies* $d(u, z) = e(z)$, *can be decomposed in* $u = (vw)$ *with* $v \in \mathcal{F}_{n_1}$, $w \in \mathcal{F}_{n_2}$ *such that* $d(v, x) = e(x)$ *and* $d(w, y) = e(y)$.

Proof. Consider a string $u \in \mathcal{F}_n$ that verifies the eccentricity of z, then $u = (vw)$ with $v \in \mathcal{F}_{n_1}$, $w \in \mathcal{F}_{n_2}$ and $e(xy) = d(vw, xy) = d(v, x) + d(w, y)$.
But $d(v, x) \leq e(x)$ and $d(w, y) \leq e(y)$.
Thus, we must have $d(v, x) = e(x)$ and $d(w, y) = e(y)$. $\square$

Because a Fibonacci string of length n is a binary string with no consecutive 1's, the next proposition is clear

Proposition 3.4. *The strings of* $\mathcal{F}_n$ *with* $n \geq 0$, *can be uniquely written as*

$$x = 0^{l_0}10^{l_1}10^{l_2} \cdots 10^{l_p}$$

with $p \geq 0$, $l_0, l_p \geq 0$ *and* $l_1, \cdots, l_{p-1} \geq 1$.

Proposition 3.5. *For* $l \geq 0$,

$$e(10^l) = e(0^l) + 1$$

Proof. Again, by Proposition 3.2, $e(10^l) \leq 1 + e(0^l)$. Assume that $y \in \mathcal{F}_l$ is a string that satisfy the eccentricity of 0^l, then $(0y) \in \mathcal{F}_{l+1}$ is at distance $e(0^l) + 1$ of 10^l. ☐

We associate next, to every string $0^l \in \mathcal{F}_l$, a set of strings $W(0^l)$ of $\mathcal{F}_l$ in the following way:

$$W(0^l) = \begin{cases} \{1(01)^{\lfloor \frac{l-1}{2} \rfloor}\} \text{ if } l \text{ is odd} \\ \{(10)^a(01)^b / 2a + 2b = l, a, b \geq 0\} \text{ if } l \text{ is even} \end{cases}$$

Proposition 3.6. *For* $l \geq 0$,

$$e(0^l) = \lfloor \frac{l+1}{2} \rfloor$$

Furthermore, the strings of $W(0^l)$ *are the only strings that satisfy the eccentricity of* 0^l.

Proof. The eccentricity of 0^l is the maximum number of 1 in a string of $\mathcal{F}_l$. This number is $\frac{l}{2}$ if l is even and $\frac{l+1}{2}$ if l is odd. It is immediate to verify that in both cases the Fibonacci strings having a maximum number of 1's are those of $W(0^l)$. ☐

Theorem 3.7. *For every* $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p}$ *in* $\mathcal{F}_n$, *with* $p, l_0, l_p \geq 0; l_1, \cdots, l_{p-1} \geq 1$,

$$e(x) = p + \sum_{i=0}^{p} \lfloor \frac{l_i+1}{2} \rfloor$$

Furthermore, the strings that verify the eccentricity of x *are the strings*

$$y = w_0 0 w_1 0 \cdots w_{p-1} 0 w_p$$

where $w_i \in W(0^{l_i})$ *for* $i = 0, 1, \cdots, p$.

Proof. Let $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p} \in \mathcal{F}_n$, with $p, l_0, l_p \geq 0; l_1, \cdots, l_{p-1} \geq 1$. Then, from Proposition 3.2, $e(x) \leq e(0^{l_0}) + e(10^{l_1}) + e(10^{l_2}) + \cdots + e(10^{l_p})$. Combining Propositions 3.5 and 3.6, $e(x) \leq \lfloor \frac{l_0+1}{2} \rfloor + \sum_{i=1}^{p}(\lfloor \frac{l_i+1}{2} \rfloor + 1)$.
Hence $e(x) \leq p + \sum_{i=0}^{p} \lfloor \frac{l_i+1}{2} \rfloor$.
Furthermore, any string $y = w_0 0 w_1 0 \cdots w_{p-1} 0 w_p$ with $w_i \in W(0^{l_i})$ satisfies $d(x, y) = p + \sum_{i=0}^{p} \lfloor \frac{l_i+1}{2} \rfloor$, then we have the equality for the eccentricity.
Given that the strings of $W(0^{l_i})$ are the only ones that verify the eccentricity of 0^{l_i}, by Proposition 3.3, the only strings $z \in \mathcal{F}_n$ that satisfy $d(x, z) = e(x)$ are those of the form of y. ☐

We will use frequently the following consequence:

Corollary 3.8. *For every* $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p} \in \mathcal{F}_n$*, with* $p,\, l_0,\, l_p \geq 0$*;* $l_1, \cdots, l_{p-1} \geq 1$*,* $n \geq 1$*, the following are true:*

(i) *if* l_p *is an odd number and* $y \in \mathcal{F}_n$ *satisfies the eccentricity of* x*, then* $y = (y'1)$ *with* $y' \in \mathcal{F}_{n-1}$*,*

(ii) *if* l_p *is a not null even number, then there exist* $y',\ y'' \in \mathcal{F}_{n-1}$*, such that* $y = (y'0)$ *and* $y = (y''1)$*, both satisfy* $e(x)$*,*

(iii) *if* $l_p = 0$ *and* $y \in \mathcal{F}_n$ *satisfy the eccentricity of* x*, then* $y = (y'0)$ *with* $y' \in \mathcal{F}_{n-1}$*.*

Proof. Consider $y \in \mathcal{F}_n$ such that $d(x, y) = e(x)$.

(i) Since l_p is odd, the only string of $W(0^{l_p})$ is $1(01)^{\lfloor \frac{l_i-1}{2} \rfloor}$. Thus, $y = (y'1)$.

(ii) Because l_p is a not null even number, $W(0^{l_p}) = \{(10)^a(01)^b / 2a + 2b = l_p,\ a, b \geq 0\}$. When $b = 0$ then $a \geq 1$ and y takes the form $y = (y'0)$. When $b \geq 1$ then $y = (y''1)$. The two cases are possible since l_p is not null.

(iii) Given that $l_p = 0$, it follows from Theorem 3.7 that $y = (y'0)$.

□

Notice that if we consider the beginning of a word $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p} \in \mathcal{F}_n$ rather than the end, then the symmetrical of Corollary 3.8 occurs. In this case $(i), (ii)$ and (iii) will be satisfied according to the parity of l_0.

4. Eccentricity sequence of Fibonacci cubes

Considering two subsets, namely, $\mathcal{F}^{\cdot\, od}_{n,k}$ and $\mathcal{F}^{\cdot\, ev}_{n,k}$, we will compute $f(x, y)$, the generating function of the eccentricity sequence of the Fibonacci cube's strings. As a corollary, the value of $f_{n,k}$ is also determined.

Proposition 4.1. *For* $n \geq 1$*,* $k \geq 1$*,*

$$f^{\cdot\, od}_{n,k} = f^{\cdot\, ev}_{n-1,k-1}$$

Proof. Let $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p} \in \mathcal{F}^{\cdot\, od}_{n,k}$, thus $p,\, l_0 \geq 0;\, l_1, \cdots, l_{p-1},\, l_p \geq 1;\ n \geq 1,\, k \geq 1$ and assume that l_p is an odd number. Notice that $l_p - 1$ is a possibly null even number. Then $x = (\theta(x)0)$ with $\theta(x) \in \mathcal{F}^{\cdot\, ev}_{n-1}$ such that

$$\theta(x) = \begin{cases} 0^{l_0}10^{l_1}\cdots 10^{l_p-1} \text{ if } l_p \geq 3 \\ 0^{l_0}10^{l_1}\cdots 10^{l_{p-1}}1 \text{ if } l_p = 1. \end{cases}$$

We have $e(x) \leq e(\theta(x)) + 1$. Furthermore, by Corollary 3.8, (ii) and (iii), there exists a vertex $y = (y'0)$ with $d(y, \theta(x)) = e(\theta(x))$. Since $d((y'01), x) = e(\theta(x)) + 1$, we have $e(x) = e(\theta(x)) + 1$, and θ is a 1 to 1 mapping between $\mathcal{F}^{\cdot\, od}_{n,k}$ and $\mathcal{F}^{\cdot\, ev}_{n-1,k-1}$. □

Proposition 4.2. *For* $n \geq 3,\ k \geq 2$*,*

$$f^{\cdot\, ev}_{n,k} = f^{\cdot\, ev}_{n-2,k-1} + f^{\cdot\, ev}_{n-2,k-2} + f^{\cdot\, ev}_{n-3,k-2}.$$

Proof. Let $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p} \in \mathcal{F}^{\cdot\, ev}_{n,k}$, hence p, l_0, $l_p \geq 0$; $l_1, \cdots, l_{p-1} \geq 1$; $n \geq 3$, $k \geq 2$. As l_p is an even number, we will distinguish two cases:

(i) If $l_p \geq 2$, then $x = (x'00)$ with $x' \in \mathcal{F}^{\cdot\, ev}_{n-2}$. Furthermore, by theorem 3.7, $e(x') = e(x) - 1 = k - 1$ thus $x' \in \mathcal{F}^{\cdot\, ev}_{n-2,k-1}$.

(ii) If $l_p = 0$ then let us consider l_{p-1}.
If l_{p-1} is odd, then $x = (x'1)$ with $x' \in \mathcal{F}^{\cdot\, od}_{n-1}$. If y satisfies $e(x')$, then $d((y0), x) = e(x') + 1$. Therefore, $e(x') = e(x) - 1$ and $x' \in \mathcal{F}^{\cdot\, od}_{n-1,k-1}$.
If l_{p-1} is even, then since l_{p-1} cannot be null, $x = (x'001)$ with $x' \in \mathcal{F}^{\cdot ev}_{n-3}$. Because $e(001) = 2$, then $e(x) \leq e(x') + 2$. The equality is reached because if y is such that $d(x', y) = e(y)$, then $d((y010), x) = e(y) + 2$. Then $x' \in \mathcal{F}^{\cdot\, ev}_{n-3,k-2}$.

Then $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}^{\cdot\, ev}_{n,k}$ and $\mathcal{F}^{\cdot\, ev}_{n-2,k-1} \cup \mathcal{F}^{\cdot\, od}_{n-1,k-1} \cup \mathcal{F}^{\cdot\, ev}_{n-3,k-2}$. By the previous proposition, $f^{\cdot\, od}_{n-1,k-1} = f^{\cdot\, ev}_{n-2,k-2}$ and we are done. ☐

Theorem 4.3.

$$f^{\cdot\, ev}(x,y) = f^{ev\,\cdot}(x,y) = \frac{1}{1 - x(x+1)y}, \tag{4.1}$$

$$f^{\cdot\, od}(x,y) = f^{od\,\cdot}(x,y) = \frac{xy}{1 - x(x+1)y}, \tag{4.2}$$

thus the generating function for the eccentricity sequence is

$$\sum_{n,k\geq 0} f_{n,k}\ x^n y^k = \frac{1 + xy}{1 - x(x+1)y}.$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{F}^{\cdot\, ev}$, thus $p \geq 0$; $l_0, l_p \geq 0$; $l_1, \cdots, l_{p-1} \geq 1$ and p is even.
Let $r(x) = 0^{l_p}10^{l_{p-1}}\cdots 10^{l_0}$ in $\mathcal{F}^{ev\,\cdot}$. Then r is a 1 to 1 mapping between $\mathcal{F}^{\cdot\, ev}$ and $\mathcal{F}^{ev\,\cdot}$.
Hence for any $n,\ k \geq 0$, $f^{\cdot\, ev}_{n,k} = f^{ev\,\cdot}_{n,k}$ and $f^{\cdot\, ev}(x,y) = f^{ev\,\cdot}(x,y)$.
The same applies for $x \in \mathcal{F}^{\cdot\, od}$, therefore $f^{\cdot\, od}(x,y) = f^{od\,\cdot}(x,y)$.

We will first demonstrate the equality (4.1), considering the linear recurrence given by Proposition 4.2, and the following initial values:

$$f^{\cdot\, ev}_{0,0} = f^{\cdot\, ev}_{1,1} = f^{\cdot\, ev}_{2,1} = f^{\cdot\, ev}_{2,2} = 1 \text{ and}$$

$$f^{\cdot\,ev}_{n,0} = 0 \text{ for } n \geq 1,\ f^{\cdot\,ev}_{n,1} = 0 \text{ for } n \geq 3,$$
$$f^{\cdot\,ev}_{n,k} = 0 \text{ for } k > n.$$

The generating function

$$f^{\cdot\,ev}(x,y) = \sum_{n,k\geq 0} f^{\cdot\,ev}_{n,k} x^n y^k$$

satisfies the equation

$$f^{\cdot\,ev}(x,y) = 1 + xy + x^2y + x^2y^2 + \sum_{n\geq 3,\ k\geq 2} f^{\cdot\,ev}_{n,k} x^n y^k.$$

Then

$$\begin{aligned}
f^{\cdot\,ev}(x,y) &= 1 + xy + x^2y + x^2y^2 + \sum_{n\geq 3,\ k\geq 2} (f^{\cdot\,ev}_{n-2,k-1} + f^{\cdot\,ev}_{n-2,k-2} + f^{\cdot\,ev}_{n-3,k-2}) x^n y^k \\
&= 1 + xy + x^2y + x^2y^2 + \sum_{n\geq 3,\ k\geq 2} (f^{\cdot\,ev}_{n-2,k-1} x^{n-2} y^{k-1}) x^2 y \\
&\qquad + \sum_{n\geq 3,\ k\geq 2} (f^{\cdot\,ev}_{n-2,k-2} x^{n-2} y^{k-2}) x^2 y^2 \\
&\qquad + \sum_{n\geq 3,\ k\geq 2} (f^{\cdot\,ev}_{n-3,k-2} x^{n-3} y^{k-2}) x^3 y^2 \\
&= 1 + xy + x^2y + x^2y^2 + (f^{\cdot\,ev}(x,y) - 1)x^2y + (f^{\cdot\,ev}(x,y) - 1)x^2y^2 + f^{\cdot\,ev}(x,y)x^3y^2.
\end{aligned}$$

Hence

$$f^{\cdot\,ev}(x,y) = \frac{1}{1 - x(x+1)y}.$$

For the equality (4.2), we will use the relation given by Proposition 4.1 and the initial values

$$f^{\cdot\,od}_{0,k} = f^{\cdot\,od}_{n,0} = 0 \text{ for } n,\ k \geq 0.$$

Thus

$$\begin{aligned}
f^{\cdot\,od}(x,y) = \sum_{n,k\geq 0} f^{\cdot\,od}_{n,k} x^n y^k &= \sum_{n,k\geq 1} f^{\cdot\,od}_{n,k} x^n y^k \\
&= xy \sum_{n,k\geq 1} f^{\cdot\,ev}_{n-1,k-1} x^{n-1} y^{k-1} = xy f^{\cdot\,ev}(x,y).
\end{aligned}$$

Therefore,

$$f^{\cdot\,od}(x,y) = \frac{xy}{1 - x(x+1)y}.$$

□

Corollary 4.4. *For all n, k such that $n \geq k \geq 1$,*

$$f_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k}$$

Furthermore, $f_{0,0} = 1$ and $f_{n,0} = 0$ for $n > 0$.

Proof.

$$\begin{aligned} f^{\cdot\, ev}(x,y) &= \frac{1}{1-x(x+1)y} = \sum_{b\geq 0}(xy(1+x))^b \\ &= \sum_{b\geq 0}\left[x^b y^b \sum_{a=0}^{b} x^a \binom{b}{a}\right] = \sum_{b\geq 0}\sum_{a=0} x^{a+b} y^b \binom{b}{a} \\ &= \sum_{n\geq 0}\sum_{k=0}^{n} x^n y^k \binom{k}{n-k}. \end{aligned}$$

Therefore,

$$f^{\cdot\, ev}_{n,k} = \binom{k}{n-k}.$$

The proof for $f^{\cdot\, od}_{n,k}$ is similar to the proof of $f^{\cdot\, ev}_{n,k}$ since $f^{\cdot\, od}(x,y)$ is xy times $f^{\cdot\, ev}(x,y)$. Hence

$$\begin{aligned} f^{\cdot\, od}(x,y) &= \frac{xy}{1-x(x+1)y} = xy\sum_{b\geq 0}(xy(1+x))^b \\ &= xy\sum_{b\geq 0}\sum_{a=0}^{b} x^{a+b} y^b \binom{b}{a} = \sum_{b\geq 0}\sum_{a=0}^{b} x^{a+b+1} y^{b+1} \binom{b}{a} \\ &= \sum_{n\geq 1}\sum_{k=1}^{n} x^n y^k \binom{k-1}{n-k}. \end{aligned}$$

Thus $f^{\cdot\, od}_{n,k} = \binom{k-1}{n-k}$ when $n \geq k \geq 1$, and $f^{\cdot\, od}_{n,0} = 0$ for $n \geq 0$. In conclusion $f_{n,k} = f^{\cdot\, ev}_{n,k} + f^{\cdot\, od}_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k}$ □

Using the precedent corollary, it is immediate to deduce the value of $rad(\Gamma_n)$ determined in [MS02]:

Corollary 4.5. *The value of $k \geq 0$ that satisfies $\min_k\{f_{n,k} \mid f_{n,k} > 0\}$ is $k = rad(\Gamma_n) = \lceil \frac{n}{2} \rceil$.*

Notice that using

$$\sum_{i=0}^{m} \binom{m-i}{i} = F_{m+1}$$

(see [GKP94], pg. 289, equation 6.130), we obtain

$$\sum_{i=0}^{n} f_{n,k} = \sum_{k=1}^{n}\left(\binom{k}{n-k}+\binom{k-1}{n-k}\right)$$
$$= \sum_{i=0}^{n}\binom{n-i}{i}+\sum_{i=0}^{n-1}\binom{n-1-i}{i} = F_{n+1}+F_n = F_{n+2}$$

which is consistent with

$$|V(\Gamma_n)| = F_{n+2}.$$

5. Eccentricity sequence of Lucas cubes

We will use the same notation for the strings in the Fibonacci cube to define the strings in the Lucas cube. In all the previous sections, when we referred to Fibonacci sets, we used the letter $\mathcal{F}$. For the Lucas sets, we will use the letter $\mathcal{L}$.

Accordingly, the functions for the Lucas cube will be defined in the same way as in the Fibonacci cube, but with a different letter, ℓ.

In this section, we will compute the generating function of the eccentricity sequence of the Lucas cube's strings, $\ell(x, y)$. For this aim, we will prove that the sets $\mathcal{L}^{a\,b}_{n,k}$ and $\mathcal{F}^{a\,b}_{n,k}$ are the same for all (a, b) excluding two sets, namely, $\mathcal{L}^{od\,od}$ and $\mathcal{L}^{\varnothing\,\varnothing}$. We proceed to compute the values of $\ell^{od\,od}_{n,k}$ and $\ell^{\varnothing\,\varnothing}_{n,k}$ as well as the values of $f^{od\,od}_{n,k}$ and $f^{\varnothing\,\varnothing}_{n,k}$. These results and Theorem 4.3 will give us the eccentricity sequence that we search. As a corollary we obtain the value of $\ell_{n,k}$.

Note further that Λ_n is an isometric subgraph of Γ_n and Q_n, i.e.:

Proposition 5.1. *For all* $x, y \in \mathcal{L}_n, n \geq 1$,

$$d_{\Lambda_n}(x, y) = d_{\Gamma_n}(x, y) = d_{Q_n}(x, y)$$

Proof. We will prove this proposition in the same way that we proved that $d_{\Gamma_n}(x, y) = d_{Q_n}(x, y)$ at the beginning of Section 3.
We have $d_{\Lambda_n}(x, y) \geq d_{Q_n}(x, y)$. Assume $x = (x_1x_2\cdots x_n)$, $y = (y_1y_2\cdots y_n)$ and let $z = (z_1z_2\cdots z_n) \in Q_n$ be defined as

$$z_i = \begin{cases} x_i \text{ if } x_i = y_i \\ 0 \text{ if } x_i \neq y_i, \end{cases}$$

then the path $s = (x = s_0, s_1, \cdots, z, \cdots, s_j = y)$ considered in proposition 3.1 is a shortest path in Q_n from x to y using only vertices of Λ_n, thus the equality is obtained. □

Proposition 5.2. *For* $x \in \mathcal{L}_n, n \geq 1$,

$$e_{\Lambda_n}(x) \leq e_{\Gamma_n}(x)$$

Proof. Let $x \in \mathcal{L}_n$. Then using proposition 5.1 and the fact that $\mathcal{L}_n \subset \mathcal{F}_n$, we have

$$e_{\Lambda_n}(x) = \max_{z \in \mathcal{L}_n}\{d_{\Lambda_n}(x,z)\} = \max_{z \in \mathcal{L}_n}\{d_{\Gamma_n}(x,z)\} \leq \max_{y \in \mathcal{F}_n}\{d_{\Gamma_n}(x,y)\} = e_{\Gamma_n}(x).$$

□

Proposition 5.3. *For* $x \in \mathcal{L}_n \setminus \mathcal{L}_n^{od\,od}, n \geq 1$,

$$e_{\Lambda_n}(x) = e_{\Gamma_n}(x).$$

Proof. Let $x \in \mathcal{L}_n \setminus \mathcal{L}_n^{od\,od}$ and without loss of generality, let us assume that x ends with an even (eventually null) number of 0's. By Corollary 3.8 (ii) and (iii), there exists $y \in \mathcal{F}_n$ such that $d_{\Gamma_n}(x,y) = e_{\Gamma_n}(x)$ and y ends with a 0. Therefore, $y \in \mathcal{L}_n$ and

$$d_{\Lambda_n}(x,y) = d_{\Gamma_n}(x,y) = e_{\Gamma_n}(x).$$

□

Let us observe that $\ell_{n,k}$ can be decomposed as follows:

$$\ell_{n,k} = \ell_{n,k}^{od\,od} + \ell_{n,k}^{od\,ev^*} + \ell_{n,k}^{od\,\varnothing} + \ell_{n,k}^{ev^*od} + \ell_{n,k}^{ev^*ev^*} + \ell_{n,k}^{ev^*\varnothing} + \ell_{n,k}^{\varnothing\,od} + \ell_{n,k}^{\varnothing\,ev^*} + \ell_{n,k}^{\varnothing\,\varnothing}.$$

Corollary 5.4. *For* $n \geq 0$, $k \geq 0$,

$$\begin{aligned}
\ell_{n,k}^{od\,ev^*} &= \ell_{n,k}^{ev^*od} = f_{n,k}^{od\,ev^*},\\
\ell_{n,k}^{od\,\varnothing} &= \ell_{n,k}^{\varnothing\,od} = f_{n,k}^{od\,\varnothing},\\
\ell_{n,k}^{ev^*ev^*} &= f_{n,k}^{ev^*ev^*},\\
\ell_{n,k}^{ev^*\varnothing} &= \ell_{n,k}^{\varnothing\,ev^*} = f_{n,k}^{\varnothing\,ev^*}.
\end{aligned}$$

Proof. When $n = 0$ all these numbers are null. Assume $n \geq 1$ and let $x \in \mathcal{F}_n^{a\,b}$ with $(a,b) \neq (\varnothing, \varnothing)$ then $x \in \mathcal{L}_n^{a\,b}$.
Furthermore if $(a,b) \neq (od, od)$ we have, by Proposition 5.3, $e_{\Lambda_n}(x) = e_{\Gamma_n}(x)$ and

$$\mathcal{L}_{n,k}^{a\,b} = \mathcal{F}_{n,k}^{a\,b}.$$

□

In order to obtain $\ell_{n,k}$, we will compute the values of the functions $\ell_{n,k}^{a\,b}$ in terms of $f_{n,k}^{a\,b}$. For this reason, we will come again to the Fibonacci cube in this part of the section.

Proposition 5.5. *For $n, k \geq 2$,*

$$f^{od\,od}_{n,k} = f^{od\,od}_{n-2,k-2} + f^{od\,ev^*}_{n-2,k-2} + f^{od\,od}_{n-2,k-1}$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{F}^{od\,od}_{n,k}$, $n, k \geq 2$, thus p, l_0, $l_p \geq 0$; $l_1, \cdots, l_{p-1} \geq 1$ and l_0, l_p are odd numbers. Let us consider l_p. We distinguish 2 cases:

(i) If $l_p = 1$, then $p \neq 0$ and $x = (x'10)$ where x' is either in $\mathcal{F}^{od\,ev^*}_{n-2}$ or in $\mathcal{F}^{od\,od}_{n-2}$.
Let $y \in \mathcal{F}_{n-2}$ such that $d(x', y) = e(x')$, then $d(y01, x'10) = e(x') + 2$ and since $e(10) = 2$ then $e(x) \leq e(x') + 2$.
Therefore $e(x) = e(x') + 2$ and $x' \in \mathcal{F}^{od\,ev^*}_{n-2,k-2}$ or $x' \in \mathcal{F}^{od\,od}_{n-2,k-2}$.

(ii) If $l_p \geq 3$, then $x = (x'00)$ with $x' \in \mathcal{F}^{od\,od}_{n-2}$. There exists $y \in \mathcal{F}_{n-2}$ such that $d(y, x') = e(x')$ then $d(y01, x'00) = e(x') + 1$ and $e(x) \leq e(x') + 1$. Therefore $e(x) = e(x') + 1$ and $x' \in \mathcal{F}^{od\,od}_{n-2,k-1}$.

Then $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}^{od\,od}_{n,k}$ and $\mathcal{F}^{od\,od}_{n-2,k-2} \cup \mathcal{F}^{od\,ev^*}_{n-2,k-2} \cup \mathcal{F}^{od\,od}_{n-2,k-1}$. □

Consider a string $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{F}^{od\,ev^*}_{n,k}$. We will demonstrate next, that when we remove a 0 from 0^{l_p}, we obtain a string that belongs to $\mathcal{F}^{od\,od}_{n-1,k} \setminus \{$ words composed by an odd number $(n-1)$ of 0's $\}$.
For this purpose, for even n and eccentricity k, let $g^{even}_{n,k}$ be the number of strings in $\mathcal{F}_n$ composed only by 0's. Notice that by Proposition 3.6, $n = 2k$, then

$$g^{even}_{n,k} = \begin{cases} 1 & \text{if } n = 2k \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.6. *For $n \geq 1$, $k \geq 0$,*

$$f^{od\,ev^*}_{n,k} = f^{od\,od}_{n-1,k} - g^{even}_{n,k}.$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{F}^{od\,ev^*}_{n,k}$, $n \geq 1$, $k \geq 0$, thus $p \geq 1$; $l_0, l_1, \cdots, l_{p-1} \geq 1$ and $l_p \geq 2$. Then $x = (x'0)$ with $x' \in \mathcal{F}^{od\,od}_{n-1}$ such that

$$x' = 0^{l_0}10^{l_1}1\cdots 10^{l_p-1}.$$

Then by Corollary 3.8 (i), all the strings of $\mathcal{F}_{n-1}$ that satisfy the eccentricity of x' have the form $y = (y'1)$. Thus $e(x') = e(x)$, and $x' \in \mathcal{F}^{od\,od}_{n-1,k}$. Conversely, for any string $z \in \mathcal{F}^{od\,od}_{n-1}$ that is not composed only by 0's, the string $(z0) \in \mathcal{F}^{od\,ev^*}_{n}$.
Therefore, $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}^{od\,ev^*}_{n,k}$ and $\mathcal{F}^{od\,od}_{n-1,k} \setminus \{$ words composed by an odd number $(n-1)$ of 0's $\}$. □

Proposition 5.5 can be rewritten in terms of $f^{od\,od}$ using the result of Proposition 5.6, which gives us the next

Proposition 5.7. *For* $n \geq 3,\ k \geq 2,$

$$f^{od\,od}_{n,k} = f^{od\,od}_{n-2,k-2} + f^{od\,od}_{n-2,k-1} + f^{od\,od}_{n-3,k-2} - g^{even}_{n-2,k-2}.$$

Notice that

$$\begin{aligned} g^{even}(x,y) &= \sum_{n,k\geq 0} g^{even}_{n,k} x^n y^k \\ &= \sum_{n,k\geq 0} x^{2k} y^k = 1 + x^2y + x^4y^2 + x^6y^3 + \cdots \\ &= \frac{1}{1-x^2y} \end{aligned} \tag{5.1}$$

Proposition 5.8.

$$f^{od\,od}(x,y) = \frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)}, \tag{5.2}$$

$$f^{od\,ev^*}(x,y) = \frac{x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)}. \tag{5.3}$$

Proof. Considering that

$$f^{od\,od}_{1,1} = 1 \text{ and } f^{od\,od}_{n,k} = 0 \text{ for other values } n \leq 2 \text{ or } k \leq 1, \text{ then}$$

$$\begin{aligned} f^{od\,od}(x,y) &= \sum_{n,k\geq 0} f^{od\,od}_{n,k} x^n y^k \\ &= xy + \sum_{n\geq 3,\ k\geq 2} f^{od\,od}_{n,k} x^n y^k \end{aligned}$$

Therefore by Proposition 5.7

$$\begin{aligned} f^{od\,od}(x,y) - xy &= \sum_{n\geq 3,\ k\geq 2} (f^{od\,od}_{n-2,k-2} + f^{od\,od}_{n-2,k-1} + f^{od\,od}_{n-3,k-2} - g^{even}_{n-2,k-2}) x^n y^k \\ &= x^2y^2 \sum_{n\geq 1,\ k\geq 0} f^{od\,od}_{n,k} x^n y^k + x^2y \sum_{n,k\geq 1} f^{od\,od}_{n,k} x^n y^k \\ &\quad + x^3y^2 \sum_{n,k\geq 0} f^{od\,od}_{n,k} x^n y^k - x^2y^2 \sum_{n\geq 1,\ k\geq 0} g^{even}_{n,k} x^n y^k \end{aligned}$$

thus

$$f^{od\,od}(x,y) - xy = x^2y^2 f^{od\,od}(x,y) + x^2y f^{od\,od}(x,y) + x^3y^2 f^{od\,od}(x,y) - x^2y^2(g^{even}(x,y) - 1)$$

and by relation (5.1),

$$f^{od\,od}(x,y)(1-x^2y^2-x^2y-x^3y^2)=xy+x^2y^2+\frac{x^2y^2}{1-x^2y},$$

thus

$$f^{od\,od}(x,y)=\frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)}.$$

Now we will prove equation (5.3). First we observe that $f_{0,0}^{od\,ev^*}=0$ then

$$f^{od\,ev^*}(x,y)=\sum_{n\geq 1,\ k\geq 0} f_{n,k}^{od\,ev^*}x^ny^k$$

and by Proposition 5.6,

$$\begin{aligned}f^{od\,ev^*}(x,y)&=\sum_{n\geq 1,\ k\geq 0}(f_{n-1,k}^{od\,od}-g_{n,k}^{even})x^ny^k=\sum_{n,k\geq 0}f_{n,k}^{od\,od}x^{n+1}y^k-\sum_{n\geq 1,\ k\geq 0}g_{n,k}^{even}x^ny^k\\&=xf^{od\,od}(x,y)-(g^{even}(x,y)-1).\end{aligned}$$

Therefore, by relation (5.1),

$$\begin{aligned}f^{od\,ev^*}(x,y)&=\frac{x^2y(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)}-\frac{x^2y}{1-x^2y}\\&=\frac{x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)}.\end{aligned}$$

□

Proposition 5.9. *For* $n,k\geq 1$,

$$f_{n,k}^{od\,\varnothing}=f_{n-1,k-1}^{od\,ev^*}+f_{n-1,k-1}^{od\,od}.$$

Proof. Let $x=0^{l_0}10^{l_1}1\cdots 10^{l_p}\in\mathcal{F}_{n,k}^{od\,\varnothing}$; $n,\ k\geq 1$. Thus $p\geq 1$; $l_0,l_1,\cdots,l_{p-1}\geq 1$ and $l_p=0$. We have therefore, $x=(x'1)$ with x' either in $\mathcal{F}_{n-1}^{od\,od}$ or in $\mathcal{F}_{n-1}^{od\,ev^*}$.
Let $y\in\mathcal{F}_{n-1}$ such that $d(x',y)=e(x')$.
Then $d((x'1),(y0))=e(x')+1$ and $e(x)\leq e(x')+1$, thus $e(x)=e(x')+1$.
Therefore, x' belongs to $\mathcal{F}_{n-1,k-1}^{od\,od}$ or to $\mathcal{F}_{n-1,k-1}^{od\,ev^*}$.
Then $x\to x'$ is a 1 to 1 mapping between $\mathcal{F}_{n,k}^{od\,\varnothing}$ and $\mathcal{F}_{n-1,k-1}^{od\,od}\cup\mathcal{F}_{n-1,k-1}^{od\,ev^*}$.
□

Proposition 5.10.

$$f^{od\,\varnothing}(x,y)=\frac{x^2y^2}{(1+xy)(1-xy-x^2y)}.$$

Proof. Considering that

$$f^{od\,\varnothing}_{n,0} = f^{od\,\varnothing}_{0,k} = 0 \text{ for } n,\, k \geq 0, \text{ we have}$$

$$f^{od\,\varnothing}(x,y) = \sum_{n,k\geq 1} f^{od\,\varnothing}_{n,k} x^n y^k.$$

Then from Proposition 5.9,

$$\begin{aligned} f^{od\,\varnothing}(x,y) &= \sum_{n,k\geq 1} (f^{od\,ev^*}_{n-1,k-1} + f^{od\,od}_{n-1,k-1}) x^n y^k \\ &= \sum_{n,k\geq 1} (f^{od\,ev^*}_{n-1,k-1} x^{n-1} y^{k-1}) xy + \sum_{n,k\geq 1} (f^{od\,od}_{n-1,k-1} x^{n-1} y^{k-1}) xy \\ &= f^{od\,ev^*}(x,y)xy + f^{od\,od}(x,y)xy. \end{aligned}$$

Thus by Proposition 5.8,

$$\begin{aligned} f^{od\,\varnothing}(x,y) &= \frac{x^4y^3xy}{(1+xy)(1-x^2y)(1-xy-x^2y)} + \frac{xy(1-x^2y-x^3y^2)xy}{(1+xy)(1-x^2y)(1-xy-x^2y)} \\ &= \frac{x^2y^2(1-x^2y)}{(1+xy)(1-x^2y)(1-xy-x^2y)} \\ &= \frac{x^2y^2}{(1+xy)(1-xy-x^2y)}. \end{aligned}$$

□

Proposition 5.11. *For* $n \geq 1,\ k \geq 0$,

$$f^{ev^*\varnothing}_{n,k} = f^{od\,\varnothing}_{n-1,k},$$

thus

$$f^{ev^*\varnothing}(x,y) = x f^{od\,\varnothing}(x,y).$$

Proof. The equality is true when $n = 1$ or $n = 2$. Then let $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p} \in \mathcal{F}^{ev^*\varnothing}_{n,k}$, with $n \geq 3$ and $k \geq 0$. Thus $p \geq 1$; $l_0 \geq 2$; $l_1, \cdots, l_{p-1} \geq 1$; $l_p = 0$.
As $l_0 > 0$, then $x = (0x')$ with $x' \in \mathcal{F}^{od\varnothing}_{n-1}$.
By Proposition 3.2,

$$e(x) \leq e(x') + 1.$$

Let's suppose that $e(x) = e(x') + 1$, then there exists $y = (1y')$ such that $d(y', x') = e(x')$. By a symmetry argument and Corollary 3.8, y' must begin with 1 which leads us to a contradiction.
Therefore, $e(x) = e(x')$. Thus $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}^{ev^*\varnothing}_{n,k}$

and $\mathcal{F}^{od\varnothing}_{n-1,k}$.

Considering the fact that $f^{ev^*\varnothing}_{0,k} = 0$ for $k \geq 0$, we have:

$$\begin{aligned} f^{ev^*\varnothing}(x,y) &= \sum_{n,k\geq 0} f^{ev^*\varnothing}_{n,k} x^n y^k = \sum_{n\geq 1,k\geq 0} f^{ev^*\varnothing}_{n,k} x^n y^k \\ &= \sum_{n\geq 1,k\geq 0} x f^{od\varnothing}_{n-1,k} x^{n-1} y^k = x f^{od\varnothing}(x,y). \end{aligned}$$

$\square$

Proposition 5.12. *For* $n \geq 3,\ k \geq 1,$

$$f^{\varnothing\,\varnothing}_{n,k} = f^{\varnothing\, ev^*}_{n-1,k-1} + f^{\varnothing\, od}_{n-1,k-1}.$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{F}^{\varnothing\,\varnothing}_{n,k}$ with $n \geq 3$, $k \geq 1$. Thus $p \geq 2$; $l_1, \cdots, l_{p-1} \geq 1$ and $l_0 = l_p = 0$.

Then $x = (x'1)$ with $x' \in \mathcal{F}^{\varnothing\, ev^*}_{n-1}$ if l_{p-1} is an even number and $x' \in \mathcal{F}^{\varnothing\, od}_{n-1}$ if l_{p-1} is odd.

By Proposition 3.2, $e(x) \leq e(x') + 1$.

Let $y' \in \mathcal{F}_{n-1}$ such that $d(x', y') = e(x')$, then $d((y'0), (x'1)) = e(x') + 1$.

Hence $e(x) = e(x') + 1$. Thus $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}^{\varnothing\,\varnothing}_{n,k}$ and $\mathcal{F}^{\varnothing\, ev^*}_{n-1,k-1} \cup \mathcal{F}^{\varnothing\, od}_{n-1,k-1}$. $\square$

Proposition 5.13.

$$f^{\varnothing\,\varnothing}(x,y) = 1 + xy + \frac{xy(x^3y^2 + x^2y^2)}{(1+xy)(1-xy-x^2y)}.$$

Proof. Let us consider the next initial values:

$$f^{\varnothing\,\varnothing}_{0,0} = f^{\varnothing\,\varnothing}_{1,1} = 1 \text{ and } f^{\varnothing\,\varnothing}_{n,k} = 0 \text{ for other values } n \leq 2 \text{ or } k = 0.$$

Then

$$\begin{aligned} f^{\varnothing\,\varnothing}(x,y) &= \sum_{n,k\geq 0} f^{\varnothing\,\varnothing}_{n,k} x^n y^k \\ &= 1 + xy + \sum_{n\geq 3,k\geq 1} f^{\varnothing\,\varnothing}_{n,k} x^n y^k. \end{aligned}$$

Then by Proposition 5.12,

$$\begin{aligned} f^{\varnothing\,\varnothing}(x,y) - 1 - xy &= \sum_{n\geq 3,k\geq 1} (f^{\varnothing\, ev*}_{n-1,k-1} + f^{od\,\varnothing}_{n-1,k-1}) x^n y^k \\ &= xy \sum_{n\geq 2,k\geq 0} (f^{\varnothing\, ev*}_{n,k} + f^{od\,\varnothing}_{n,k}) x^n y^k \end{aligned}$$

Observe that when $n \leq 1$

$$f_{n,k}^{\varnothing\, ev*} + f_{n,k}^{od\, \varnothing} = 0.$$

Hence

$$f^{\varnothing\, \varnothing}(x,y) - 1 - xy = xy(f^{\varnothing\, ev*}(x,y) + f^{od\, \varnothing}(x,y)).$$

From Proposition 5.11,

$$f^{\varnothing\, \varnothing}(x,y) = 1+xy+xy(xf^{od\, \varnothing}(x,y)+f^{od\, \varnothing}(x,y)) = 1+xy+xy(1+x)f^{od\, \varnothing}(x,y).$$

Substituting $f^{od\, \varnothing}(x,y)$ from Proposition 5.10, we obtain the desired result.
□

Proposition 5.14. *For $n \geq 3$, $k \geq 1$,*

$$\ell_{n,k}^{od\, od} = f_{n,k+1}^{od\, od}$$

thus

$$\ell^{od\, od}(x,y) = y^{-1} f^{od\, od}(x,y).$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{L}_{n,k}^{od\, od}$, $n \geq 3$, $k \geq 1$. Thus $p \geq 0$; $l_0, l_1 \cdots, l_{p-1}, l_p \geq 1$
By Corollary 3.8 (i) and by symmetry, every y such that $d(x,y) = e_{\Gamma_n}(x)$ has the form $y = (1y'1)$, with $y' \in \mathcal{F}_{n-2}$. Then, $y \notin \mathcal{L}_n$ and $e_{\Lambda_n}(x) < e_{\Gamma_n}(x)$. Furthermore, note that the string $(1y'0) \in \mathcal{L}_n$. Thus $d((1y'0),x) = e_{\Gamma_n}(x) - 1$. Hence $e_{\Lambda_n}(x) = e_{\Gamma_n}(x) - 1$.
Thus, $x \rightarrow x$ maps $\mathcal{L}_{n,k}^{od\, od}$ into $\mathcal{F}_{n,k+1}^{od\, od}$.
For the second part of the Proposition, consider the initial values

$$\ell_{1,0}^{od\, od} = 1 \text{ and } \ell_{n,k}^{od\, od} = 0 \text{ for other values } n \leq 2 \text{ or } k = 0.$$

Thus

$$\begin{aligned} \ell^{od\, od}(x,y) &= \sum_{n,k\geq 0} \ell_{n,k}^{od\, od} x^n y^k = x + \sum_{n\geq 3, k\geq 1} f_{n,k+1}^{od\, od} x^n y^k \\ &= x + y^{-1} \sum_{n\geq 3, k\geq 1} f_{n,k+1}^{od\, od} x^n y^{k+1}. \end{aligned}$$

But

$$f^{od\, od}(x,y) = xy + \sum_{n\geq 3, k\geq 2} f_{n,k}^{od\, od} x^n y^k,$$

thus

$$\ell^{od\, od}(x,y) = x + y^{-1}(f^{od\, od}(x,y) - xy) = y^{-1} f^{od\, od}(x,y).$$

□

Proposition 5.15.

$$\ell^{\varnothing\,\varnothing}(x,y) = 1.$$

Proof. The empty word is the only string that belongs to some $\mathcal{L}_n$ that neither begins nor ends with a 0. Thus $\ell^{\varnothing\,\varnothing}_{n,k} = 0$ for $n \geq 1$. $\square$

Theorem 5.16. *The generating function for the eccentricity sequence of Lucas cube is*

$$\ell(x,y) = \sum_{n,k\geq 0} \ell_{n,k} x^n y^k = \frac{1+x^2y}{1-xy-x^2y} + \frac{1}{1+xy} - \frac{1-x}{1-x^2y}.$$

Proof. Recall that

$$\ell_{n,k} = \ell^{od\,od}_{n,k} + \ell^{od\,ev^*}_{n,k} + \ell^{od\,\varnothing}_{n,k} + \ell^{ev^*od}_{n,k} + \ell^{ev^*ev^*}_{n,k} + \ell^{ev^*\varnothing}_{n,k} + \ell^{\varnothing\,od}_{n,k} + \ell^{\varnothing\,ev^*}_{n,k} + \ell^{\varnothing\,\varnothing}_{n,k}.$$

and we have the same decomposition for $f_{n,k}$.

From Corollary 5.4, when $(a,b) \neq (od,od)$ and $(a,b) \neq (\varnothing,\varnothing)$, then $\ell^{a\,b}_{n,k} = f^{a\,b}_{n,k}$. Thus

$$\ell_{n,k} = f_{n,k} - f^{od\,od}_{n,k} - f^{\varnothing\,\varnothing}_{n,k} + \ell^{od\,od}_{n,k} + \ell^{\varnothing\,\varnothing}_{n,k}.$$

Thus, the generating function

$$\ell(x,y) = \sum_{n,k\geq 0} \ell_{n,k} x^n y^k$$

satisfies the equation

$$\ell(x,y) = \sum_{n,k\geq 0} (f_{n,k} - f^{od\,od}_{n,k} - f^{\varnothing\,\varnothing}_{n,k} + \ell^{od\,od}_{n,k} + \ell^{\varnothing\,\varnothing}_{n,k}).$$

By Theorem 4.3 and Propositions 5.8, 5.13, 5.14 and 5.15, we conclude that

$$\begin{aligned}
\ell(x,y) = {} & \frac{1+xy}{1-xy-x^2y} - \frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)} \\
& - \left(1+xy+\frac{xy(x+1)x^2y^2}{(1+xy)(1-xy-x^2y)}\right) \\
& + y^{-1}\left(\frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)}\right) + 1 \\
= {} & \frac{1+x-x^2y+x^2y^2-x^3y+x^3y^2-x^4y^2-x^5y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)} \\
= {} & \frac{1}{1+xy} - \frac{1-x}{1-x^2y} + \frac{1+x^2y}{1-xy-x^2y}.
\end{aligned}$$

$\square$

Corollary 5.17. *For all* n, k *with* $n > k \geq 1$,

$$\ell_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k-1} + \varepsilon_{n,k} \tag{5.4}$$

where

$$\varepsilon_{n,k} = \begin{cases} -1 & \text{if } n = 2k, \\ 1 & \text{if } n = 2k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $\ell_{0,0} = \ell_{1,0} = 1$, $\ell_{n,0} = 0$ *for* $n > 1$ *and*

$$\ell_{n,n} = \begin{cases} 2 & \text{if } n \text{ is even } (n \geq 2), \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By the previous theorem,

$$\ell(x,y) = \frac{1}{1-xy-x^2y} + \frac{x^2y}{1-xy-x^2y} + \frac{1}{1+xy} - \frac{1-x}{1-x^2y}.$$

We will analyse each term of this sum separately.

$$\begin{aligned} \frac{1}{1-xy-x^2y} &= \sum_{b\geq 0}(xy(1+x))^b = \sum_{b\geq 0} x^b y^b \sum_{a=0}^{b} x^a \binom{b}{a} \\ &= \sum_{b\geq 0}\sum_{a=0}^{b} x^{a+b} y^b \binom{b}{a} = \sum_{n\geq 0}\sum_{k=0}^{n} \binom{k}{n-k} x^n y^k \end{aligned} \tag{5.5}$$

$$\begin{aligned} \frac{x^2y}{1-xy-x^2y} &= x^2y\sum_{b\geq 0}(xy(1+x))^b = x^2y\sum_{b\geq 0}\sum_{a=0}^{b} x^{a+b}y^b\binom{b}{a} \\ &= \sum_{b\geq 0}\sum_{a=0}^{b} x^{a+b+2}y^{b+1}\binom{b}{a} = \sum_{n\geq 2}\sum_{k=1}^{n-1}\binom{k-1}{n-k-1}x^n y^k. \end{aligned} \tag{5.6}$$

The third term of the sum is

$$\frac{1}{1+xy} = \sum_{b\geq 0}(-xy)^b = \sum_{n\geq 0}(-1)^n x^n y^n. \tag{5.7}$$

Finally, the last term will be decomposed as follows:

$$-\frac{1-x}{1-x^2y} = \frac{x}{1-x^2y} - \frac{1}{1-x^2y},$$

$$\frac{x}{1-x^2y} = x\sum_{a\geq 0}(x^2y)^a = \sum_{k\geq 0}(x^{2k+1})y^k, \tag{5.8}$$

and the second sub-term

$$\frac{-1}{1-x^2y} = -\sum_{a\geq 0}(x^2y)^a = -\sum_{k\geq 0}(x^{2k})y^k. \tag{5.9}$$

Equations (5.5), (5.6), (5.8) and (5.9) give us the desired result when $k \neq 0$, $k \neq n$.

When $k = 0$, equation (5.5) contributes with 1 when $n = 0$; equation (5.7) contributes with 1 when $n = 0$; equation (5.8) contributes with 1 when $n = 1$ and equation (5.9) contributes with -1 for $n = 0$.

When $k = n \geq 1$, equation (5.5) contributes with 1 and equation (5.7) contributes with $(-1)^n$. □

Notice that for $n \geq 2$,

$$\sum_{k=0}^{n} \ell_{n,k} = \sum_{k=1}^{n-1}\left[\binom{k}{n-k} + \binom{k-1}{n-k-1}\right] + \varepsilon_{n,\lfloor \frac{n}{2}\rfloor} + \ell_{n,0} + \ell_{n,n},$$

where

$$\varepsilon_{n,\lfloor \frac{n}{2}\rfloor} = (-1)^{n+1}, \quad \ell_{n,0} = 0 \quad \text{and} \quad \ell_{n,n} = 1 + (-1)^n.$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n} \ell_{n,k} &= \sum_{k=0}^{n}\binom{k}{n-k} + \sum_{k=0}^{n-2}\binom{k}{n-2-k} \\ &= F_{n+1} + F_{n-1} = L_n = |V(\Lambda_n)|.\end{aligned}$$

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