Multivariate MA(∞) processes with heavy tails and random coefficients

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Abstract

Many interesting processes share the property of multivariate regular variation. This property is equivalent to existence of the tail process introduced by B. Basrak and J. Segers [1] to describe the asymptotic behavior for the extreme values of a regularly varying time series. We apply this theory to multivariate MA(∞) processes with random coefficients.

Key words : extremes, heavy tails, regular variation, tail process.

1 Introduction

We interested in multivariate infinite-order moving average process with random coefficient matrices, defined for $t \in \mathbb{Z}$ by

$$X_t = \sum_{i=0}^{\infty} C_i(t)\xi_{t-i}, \tag{1.1}$$

where $(\xi_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. regularly varying random vector in $\mathbb{R}^q$ and $\{C_i(t), i \geq 0, t \in \mathbb{Z}\}$ is an array of random $d \times q$ matrices. The tail behavior of such process is usually controlled by a moment condition on the matrix $C_i(t)$, this has been shown by Hult and Samorodnitsky [4]. We adapt the conditions as follows.

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Condition 1.1 *Suppose that* $\mathbf{P}(\bigcap_{i\geq 0}\{\|C_i(0)\| = 0\}) = 0$*, and there is some* $\varepsilon \in (0,\alpha)$ *such that*

$$\sum_i \mathbf{E}\|C_i(0)\|^{\alpha-\varepsilon} < \infty \text{ and } \sum_i \mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon} < \infty, \quad \text{if} \quad \alpha \in (0,1)\cup(1,2); \tag{1.2}$$

$$\mathbf{E}\left(\sum_i \|C_i(0)\|^{\alpha-\varepsilon}\right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} < \infty, \quad \text{if} \quad \alpha \in \{1,2\}; \tag{1.3}$$

$$\mathbf{E}\left(\sum_i \|C_i(0)\|^{2}\right)^{\frac{\alpha+\varepsilon}{2}} < \infty, \quad \text{if} \quad \alpha \in (2,\infty); \tag{1.4}$$

To apply the results in Hult and Samorodnitsky [4], a 'predictability' assumption is required.

Condition 1.2 *Suppose there is a filtration* $(\mathcal{F}_j, j \in \mathbb{Z})$ *such that for all* $t \in \mathbb{Z}$ *and* $i \in \mathbb{N}\cup\{0\}$

$$C_i(t) \in \mathcal{F}_{i-t}, \;\; \xi_{t-i} \in \mathcal{F}_{i-t+1}, \;\; (or \;\; \xi_j \in \mathcal{F}_{1-j}), \tag{1.5}$$

$$\mathcal{F}_j \;\; \text{is independent of} \;\; \sigma(\xi_{-j}, \xi_{-j-1}, \ldots). \tag{1.6}$$

If the sequence $(C_i(t))$ is independent of the sequence (ξ_i), we can set $\mathcal{F}_j = \sigma((C_k(t))_{k\geq 0, t\in\mathbb{Z}}, (\xi_k)_{k\geq 1-j})$.

Throughout the paper we will use the following notations, $\mathbb{E}^q = [-\infty,\infty]^q\backslash\{0\}$, $\mathbb{E}^q_u = \{x \in \mathbb{E}^q \mid \|x\| > u\}$ and $S^{q-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$. A function f is called *regularly varying at infinity with index* $\tau \in \mathbb{R}$, if $\lim_{x\to\infty} f(ux)/f(x) = u^\tau$ for all $u > 0$.

Definition 1.1. A random vector X in $\mathbb{R}^d$ is *regularly varying with index* $\alpha > 0$ if there exists a non-null Radon measure μ on $\mathbb{E}^d$ and a regularly varying function V of index $-\alpha$ such that, as $x \to \infty$,

$$\frac{\mathbf{P}\left\{x^{-1}X \in \cdot\right\}}{V(x)} \xrightarrow{v} \mu(\cdot), \tag{1.7}$$

where $\xrightarrow{v}$ denote the vague convergence of measures.

Relation (1.7) is equivalent to

$$\frac{\mathbf{P}\left\{\|X\| > ux, x^{-1}X \in \cdot\right\}}{\mathbf{P}\left\{\|X\| > x\right\}} \xrightarrow{v} u^{-\alpha}\mathbf{P}\left\{\Theta \in \cdot\right\},$$

as $x \to \infty$. The law of Θ is called *spectral measure.*

The random sequence (X_t) given by (1.1) is built from two components which satisfy the following *basic assumptions.*

A1 The q-dimensional random vector ξ_t is regularly varying of index $\alpha \in (0, \infty)$ and with spectral measure $\mathcal{L}(\Theta)$ on S^{q-1}, i.e. there exists a non-null Radon measure μ on $\mathbb{E}^q$ such that, as $x \to \infty$,

$$\frac{\mathbf{P}\left\{x^{-1}\xi_0 \in \cdot\right\}}{\mathbf{P}\left\{\|\xi_0\| > x\right\}} \xrightarrow{v} \mu(\cdot),$$

where the measure μ has the following representation, for all $u > 0$,

$$\mu\left\{x \;\middle|\; \|x\| > u, \frac{x}{\|x\|} \in \cdot\right\} = u^{-\alpha}\mathbf{P}\left\{\Theta \in \cdot\right\}.$$

A2 The array of random matrices $\{C_i(t), i \geq 0, t \geq \mathbb{Z}\}$ of dimension $d \times q$ is independent of the sequence (ξ_t) and stationary when index over $t \in \mathbb{Z}$.

2 Joint regular variation

The following theorem is adapted version of Theorem 3.1 in [4].

Theorem 2.1 *[4] Suppose that Condition 1.1 and 1.2 hold, then the series X_t given by (1.1) converges a.s. and*

$$\frac{\mathbf{P}\left\{x^{-1}X_0 \in \cdot\right\}}{\mathbf{P}\left\{\|\xi_0\| > x\right\}} \xrightarrow{v} \sum_{i=0}^{\infty} \mathbf{E}\left[\mu \circ C_i(0)^{-1}(\cdot)\right], \tag{2.8}$$

as $x \to \infty$ on $\mathbb{E}^q$.

Observe that by assumption **A1**, the function $u \mapsto \mathbf{P}\left\{\|\xi_0\| > u\right\}$ is regularly varying with index $-\alpha < 0$. Therefore if the limit on the right-hand side in (2.8) is a non-null Radon measure, then vector X_0 is regularly varying with index α. As we shall see slightly stronger condition is needed to ensure regular variation of vector X_0. Consider for $u > 0$

$$\begin{aligned}
\mathbf{E}\left[\mu \circ C_i(0)^{-1}(\mathbb{E}_u^d)\right] &= \mathbf{E}\left[\mu\left\{x \mid \|C_i(0)x\| > u\right\}\right] \\
&= \mathbf{E}\int\int \mathbb{I}_{\{\|C_i(0)r\theta\| > u\}} d(-r^{-\alpha})\mathbf{P}_\Theta(d\theta) \\
&= u^{-\alpha}\mathbf{E}\|C_i(0)\Theta\|^\alpha.
\end{aligned}$$

Thus if

$$0 < \sum_{i=0}^{\infty} \mathbf{E}\|C_i(0)\Theta\|^\alpha < \infty \tag{2.9}$$

the random vector X_0 is regularly varying.

Let us define a random vector of length $(t-s+1)d$ for $s,t \in \mathbb{Z}$ with $s \le t$

$$\tilde{X}(s,t) = (X_s, \ldots, X_t)'.$$

By the definition of the series X_t (1.1), we have

$$\tilde{X}(s,t) = \sum_{i=0}^{\infty} \tilde{C}_i(s,t)\xi_{t-i}, \tag{2.10}$$

where $\left\{\tilde{C}_i(s,t)\right\}$ is an array of random matrices of dimension $(t-s+1)d \times q$ defined by

$$\tilde{C}_i(s,t) = (C_{i-t+s}(s), C_{i-t+s+1}(s+1), \ldots, C_{i-1}(t-1), C_i(t))', \quad i \ge 0, \tag{2.11}$$

where $C_i(\cdot) = 0$, if $i < 0$. By condition (1.5) and the definition of the sequence $\left\{\tilde{C}_i(s,t)\right\}$ (2.11), we have

$$\tilde{C}_i(s,t) \in \mathcal{F}_{i-t}, \quad \text{for all } \ s,t \in \mathbb{Z}, i \in \mathbb{N} \cup \{0\}\,.$$

Hence Condition 1.2 holds for the sequence $\left\{\tilde{C}_i(s,t)\right\}$ and (ξ_t). In order to obtain the convergence of type (2.8) for the series $\tilde{X}(s,t)$ defined by (2.10), it is sufficient to show that the array $\left\{\tilde{C}_i(s,t)\right\}$ satisfy Condition 1.1. For this, the following lemma is needed.

Lemma 2.2 *Let* $A = (A_1, \ldots, A_n)'$ *be a block matrix of dimension* $nd \times q$ *with* n *blocks,* A_i *be a* $d \times q$ *matrix with entries in* $\mathbb{R}$, $i = 1, \ldots, n$. *Then*

$$\|A\| \le \sum_{i=1}^{n} \|A_i\|, \tag{2.12}$$

where $\|\cdot\|$ *denote the matrix norm.*

Proof. Without loss of generality, we consider the vector norm $\|(x_1, \ldots, x_d)'\|_d = \max\limits_{1 \le i \le d}\{\|x_i\|\}$ on $\mathbb{R}^d$. We may decompose the matrix A as

$$A = \sum_{i=1}^{n} B_i \tag{2.13}$$

with B_i is a $nd \times q$ matrix defined by

$$B_i = (0, \ldots, A_i, \ldots, 0)'$$

where A_i is in the ith position. We have, for all $x \in \mathbb{R}^d$

$$\|B_i x\|_{nd} = \|(0, \ldots, A_i x, \ldots, 0)'\|_{nd} = \|A_i x\|_d, \quad i = 1, \ldots, n.$$

Therefore

$$\|B_i\| = \sup_{\|x\|_q \le 1} \{\|B_i x\|_{nd}\} = \sup_{\|x\|_q \le 1} \{\|A_i x\|_d\} = \|A_i\|. \tag{2.14}$$

Inequality (2.12) follows from (2.13) and (2.14). ☐

By Lemma 2.2 and the definition of the sequence $\left\{\tilde{C}_i(s,t)\right\}$ (2.11), we obtain

$$\|\tilde{C}_i(s,t)\| \le \sum_{j=(i-t+s)\vee 0}^{i} \|C_j(t-i+j)\|. \tag{2.15}$$

Using (2.15) and the following inequality, for $a_i > 0$, $i = 1, \dots, n$,

$$\left(\sum_{i=1}^{n} a_i\right)^p \le c \sum_{i=1}^{n} a_i^p, \tag{2.16}$$

where $c = 1$ if $p \le 1$, $c = n^{p-1}$ if $p > 1$, we obtain, if $\alpha \in (0,1) \cup (1,2)$, for some $0 < \varepsilon < \alpha$,

$$\begin{aligned} \sum_{i=0}^{\infty} \mathbf{E}\|\tilde{C}_i(s,t)\|^{\alpha-\varepsilon} &\le \sum_{i=0}^{\infty} \mathbf{E}\left(\sum_{j=(i-t+s)\vee 0}^{i} \|C_j(t-i+j)\|\right)^{\alpha-\varepsilon} \\ &\le c_1 \sum_{i=0}^{\infty} \sum_{j=(i-t+s)\vee 0}^{i} \mathbf{E}\|C_j(t-i+j)\|^{\alpha-\varepsilon} \\ &= c_1 \sum_{j=0}^{t-s} \sum_{i=0}^{\infty} \mathbf{E}\|C_i(t-j)\|^{\alpha-\varepsilon}, \end{aligned} \tag{2.17}$$

where c_1 is a constant depending on $\alpha-\varepsilon$ and $t-s$. The latter inequality in combination with the condition (1.2) and the stationarity of the sequence $\{C_i(t)\}$ implies

$$\sum_{i=0}^{\infty} \mathbf{E}\|\tilde{C}_i(s,t)\|^{\alpha-\varepsilon} < \infty.$$

By the similar method for (2.17), we obtain, for some $0 < \varepsilon < \alpha$,

$$\sum_{i=0}^{\infty} \mathbf{E}\|\tilde{C}_i(s,t)\|^{\alpha+\varepsilon} \le c_2 \sum_{j=0}^{t-s} \sum_{i=0}^{\infty} \mathbf{E}\|C_i(t-j)\|^{\alpha+\varepsilon}, \text{ if } \alpha \in (0,1) \cup (1,2),$$

$$\mathbf{E}\left(\sum_{i=0}^{\infty} \|\tilde{C}_i(s,t)\|^{\alpha-\varepsilon}\right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} \le c_3 \sum_{j=0}^{t-s} \mathbf{E}\left(\sum_{i=0}^{\infty} \|C_i(t-j)\|^{\alpha-\varepsilon}\right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}, \text{ if } \alpha \in \{1,2\}$$

and

$$\mathbf{E}\left(\sum_{i=0}^{\infty}\|\tilde{C}_i(s,t)\|^2\right)^{\frac{\alpha+\varepsilon}{2}} \leq c_4 \sum_{j=0}^{t-s}\mathbf{E}\left(\sum_{i=0}^{\infty}\|C_i(t-j)\|^2\right)^{\frac{\alpha+\varepsilon}{2}}, \text{ if } \alpha \in (2,\infty),$$

where c_2, c_3 and c_4 are the constants depending on α, ε and $t-s$. As a consequence, under the conditions of Theorem 2.1, it is not only the marginal distribution of (X_t) that is regularly varying, the same holds for all finite dimensional distributions.

Corollary 2.3 *Under the conditions of Theorem 2.1, the stationary process (X_t) given by (1.1) satisfies, for $s, t \in \mathbb{Z}$ with $s \leq t$*

$$\frac{\mathbf{P}\left\{x^{-1}(X_s,\dots,X_t)' \in \cdot\right\}}{\mathbf{P}\left\{\|\xi_0\| > x\right\}} \xrightarrow{v} \sum_{i=0}^{\infty}\mathbf{E}\left[\mu \circ \tilde{C}_i(s,t)^{-1}(\cdot)\right] =: \nu_{s,t}(\cdot), \tag{2.18}$$

as $x \to \infty$ on $\mathbb{E}^{(t-s+1)d}$, where the sequence of random matrices $\left\{\tilde{C}_i(s,t)\right\}$ is defined by (2.11). In particular, if X_0 is regularly varying, then the process (X_t) is jointly regularly varying, i.e. for all $s, t \in \mathbb{Z}$ withs $\leq t$, random vector $(X_s,\dots,X_t)'$ are regularly varying.

3 Tail process

The joint regular variation of the sequence (X_t) in particular means that there exists a process $(Y_t)_{t\in\mathbb{Z}}$ in $\mathbb{R}^d$ with $\mathbf{P}\left\{\|Y_0\| > y\right\} = y^{-\alpha}$ for $y \geq 1$ such that for all $s, t \in \mathbb{Z}$ with $s \leq t$ and as $x \to \infty$

$$\mathcal{L}(x^{-1}(X_s,\dots,X_t) \mid \|X_0\| > x) \rightsquigarrow \mathcal{L}(Y_s,\dots,Y_t),$$

see Theorem 2.1 in [1]. The process (Y_t) is called the *tail process* of (X_t). Moreover the so-called *spectral process* $\left(\Theta_t = \frac{Y_t}{\|Y_0\|}\right)_{t\in\mathbb{Z}}$ is independent of $\|Y_0\|$, see Theorem 3.1 in [1].

To understand the tail process in this case, it is helpful for instance to consider a set B in $\mathbb{E}^{(t-s+1)d}$ of the form $B = \mathbb{E}^d_{b_s} \times \cdots \times \mathbb{E}^d_{b_t}$ for arbitrary real constant $b_i > 0$, $i = s,\dots,t$. The limit of the probabilities, as $x \to \infty$,

$$\frac{\mathbf{P}\left\{x^{-1}(X_s,\dots,X_t)' \in B\right\}}{\mathbf{P}\left\{\|\xi_0\| > x\right\}}$$

is the sum

$$\begin{aligned}
& \sum_{i=0}^{\infty} \mathbf{E}\left[\mu \circ \tilde{C}_i(s,t)^{-1}(B)\right] \\
= & \sum_{i=0}^{\infty} \mathbf{E}\left[\mu\left\{x \;\middle|\; \tilde{C}_i(s,t)x \in B\right\}\right] \\
= & \sum_{i=t-s}^{\infty} \mathbf{E}\left[\mu\left\{x \mid \|C_{i-t+s}(s)x\| > b_s, \dots, \|C_i(t)x\| > b_t\right\}\right] \\
= & \sum_{i=t-s}^{\infty} \mathbf{E}\left[\int \mathbf{P}_\Theta(d\theta) \int \mathbb{I}_{\{\|C_{i-t+s}(s)r\theta\| > b_s\}} \cdots \mathbb{I}_{\{\|C_i(t)r\theta\| > b_t\}} d(-r^{-\alpha})\right] \\
= & \sum_{i=t-s}^{\infty} \mathbf{E}\left[\int \mathbf{P}_\Theta(d\theta) \left(\max\left\{\frac{b_s}{\|C_{i-t+s}(s)\theta\|}, \dots, \frac{b_t}{\|C_i(t)\theta\|}\right\}\right)^{-\alpha}\right] \\
= & \sum_{i=t-s}^{\infty} \mathbf{E}\left[\min\left\{\frac{\|C_{i-t+s}(s)\Theta\|^\alpha}{b_s^\alpha}, \dots, \frac{\|C_i(t)\Theta\|^\alpha}{b_t^\alpha}\right\}\right].
\end{aligned}$$

In particular, if $s = t$ and $b_s = 1$, one can apply this to obtain the following limit, as $x \to \infty$,

$$\frac{\mathbf{P}\{\|X_0\| > x\}}{\mathbf{P}\{\|\xi_0\| > x\}} \to \sum_{i=0}^{\infty} \mathbf{E}\|C_i(0)\Theta\|^\alpha. \tag{3.19}$$

The following result is the multivariate version of Breiman's lemma [3] which appears as Proposition A. 1 in [2].

Lemma 3.1 *Let Z be a q-dimensional random vector and let A be a $d \times q$ random matrix, independent of Z. Assume that Z is multivariate regularly varying of index $\alpha \in (0, \infty)$, i.e. there exists a non-null Radon measure μ on $\mathbb{E}^q$ such that, as $x \to \infty$,*

$$\frac{\mathbf{P}\left\{x^{-1}Z \in \cdot\right\}}{\mathbf{P}\{\|Z\| > x\}} \xrightarrow{v} \mu(\cdot).$$

If $\mathbf{E}\|A\|^\beta < \infty$ for some $\beta > \alpha$, then in $\mathbb{E}^d$, as $x \to \infty$,

$$\frac{\mathbf{P}\left\{x^{-1}AZ \in \cdot\right\}}{\mathbf{P}\{\|Z\| > x\}} \xrightarrow{v} \mathbf{E}\left[\mu \circ A^{-1}(\cdot)\right].$$

Theorem 3.2 *Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process given by (1.1). Suppose that Condition 1.1 and 1.2 hold. If $\mathbf{P}\left(\cap_{i\geq 0}\{\|C_i(0)\Theta\| = 0\}\right) = 0$, then for $s, t \in \mathbb{Z}$ with $s \leq 0 \leq t$*

and bounded and continuous function $f : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R}$*,*

$$\begin{aligned}(3.20)\qquad & \mathbf{E}\left[f\left(\frac{X_s}{\|X_0\|},\dots,\frac{X_t}{\|X_0\|}\right)\,\middle|\,\|X_0\|>x\right] \\ & \to \frac{1}{\sum\limits_{i=0}^{\infty}\mathbf{E}\|C_i(0)\Theta\|^{\alpha}}\sum_{i=0}^{\infty}\mathbf{E}\left[f\left(\frac{C_{i+s}(s)\Theta}{\|C_i(0)\Theta\|},\dots,\frac{C_{i+t}(t)\Theta}{\|C_i(0)\Theta\|}\right)\|C_i(0)\Theta\|^{\alpha}\right],\end{aligned}$$

as $x \to \infty$*, where* $C_i(\cdot) = 0$ *if* $i < 0$*.*

Proof. Let $h : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R}$ be bounded and continuous. In view of convergence (3.19) and Corollary 2.3, as $x \to \infty$,

$$\begin{aligned}(3.21)\qquad & \mathbf{E}\left[f\left(\frac{X_s}{x},\dots,\frac{X_t}{x}\right)\,\middle|\,\|X_0\|>x\right] \\ & \to \frac{1}{\sum\limits_{i=0}^{\infty}\mathbf{E}\|C_i(0)\Theta\|^{\alpha}}\int h(y_s,\dots,y_t)\mathbb{1}_{\{\|y_0\|>1\}}\nu_{s,t}(d\mathbf{y}).\end{aligned}$$

Note

$$\tilde{X}(s,t) = (X_s,\dots,X_t)' = \sum_{i=0}^{\infty}\tilde{C}_i(s,t)\xi_{t-i},$$

where the sequence $\left\{\tilde{C}_i(s,t)\right\}$ is defined by (2.11). By (2.15), (2.16), Condition 1.1 and the stationarity of sequence $\{C_i(t)\}$, we have for some $\beta > \alpha$,

$$\mathbf{E}\|\tilde{C}_i(s,t)\|^{\beta} \le c\sum_{j=(i-t+s)\vee 0}^{i}\mathbf{E}\|C_j(t-i+j)\|^{\beta} = c\sum_{j=(i-t+s)\vee 0}^{i}\mathbf{E}\|C_j(0)\|^{\beta} < \infty,$$

where c is a constant depending on β and $t-s$. Using Lemma 3.1, we obtain, for each $i \ge 0$

$$(3.22)\qquad \frac{\mathbf{P}\left\{x^{-1}\tilde{C}_i(s,t)\xi_{t-i}\in\cdot\right\}}{\mathbf{P}\left\{\|\xi_0\|>x\right\}} \xrightarrow{v} \mathbf{E}\left[\mu\circ\tilde{C}_i(s,t)^{-1}(\cdot)\right] =: \nu_{s,t}^{(i)}(\cdot),$$

as $x \to \infty$, in $\mathbb{E}^{(t-s+1)d}$. By the definition of the measure $\nu_{s,t}$ in (2.18),

$$(3.23)\qquad \nu_{s,t}(\cdot) = \sum_{i=0}^{\infty}\nu_{s,t}^{(i)}(\cdot).$$

We denote

$$H(x) = \mathbf{E}\left[h\left(x^{-1}\tilde{C}_i(s,t)\xi_{t-i}\right)\mathbb{1}_{\{\|C_{i-t}(0)\xi_{t-i}\|>x\}}\right]/\mathbf{P}\left\{\|\xi_0\|>x\right\}.$$

On the one hand, it follows from (3.22), as $x \to \infty$,

$$\begin{aligned} H(x) &= \int h(y_s, \dots, y_t) \mathbb{1}_{\{\|y_0\|>1\}} \mathbf{P}_{x^{-1}\tilde{C}_i(s,t)\xi_{t-i}}(d\mathbf{y}) / \mathbf{P}\left\{\|\xi_0\| > x\right\} \\ &\to \int h(y_s, \dots, y_t) \mathbb{1}_{\{\|y_0\|>1\}} \nu_{s,t}^{(i)}(d\mathbf{y}). \tag{3.24} \end{aligned}$$

On the other hand, by independence between $\tilde{C}_i(s,t)$ and ξ_{t-i} and assumption **A1**, as $x \to \infty$,

$$\begin{aligned} H(x) &= \int \mathbf{E}\left[h\left(\tilde{C}_i(s,t)\mathbf{y}\right) \mathbb{1}_{\{\|C_{i-t}(0)\mathbf{y}\|>1\}}\right] \mathbf{P}_{x^{-1}\xi_{t-i}}(d\mathbf{y}) / \mathbf{P}\left\{\|\xi_0\| > x\right\} \\ &\to \int_0^\infty \mathbf{E}\left[h\left(\tilde{C}_i(s,t)\Theta r\right) \mathbb{1}_{\{\|C_{i-t}(0)\Theta\|r>1\}}\right] d(-r^{-\alpha}). \tag{3.25} \end{aligned}$$

Since for $i < t$, $\nu_{s,t}^{(i)}\left\{(y_s, \dots, y_t) \mid y_s = \dots = y_0 = 0\right\} = 1$,

$$\int h(y_s, \dots, y_t) \mathbb{1}_{\{\|y_0\|>1\}} \nu_{s,t}^{(i)}(d\mathbf{y}) = 0, \quad i = 0, \dots, t-1. \tag{3.26}$$

In combination with (3.23), (3.24), (3.25) and (3.26), it follows that

$$\begin{aligned} &\int h(y_s, \dots, y_t) \mathbb{1}_{\{\|y_0\|>1\}} \nu_{s,t}(d\mathbf{y}) \\ &= \sum_{i=t}^{\infty} \int h(y_s, \dots, y_t) \mathbb{1}_{\{\|y_0\|>1\}} \nu_{s,t}^{(i)}(d\mathbf{y}) \\ &= \sum_{i=t}^{\infty} \int_0^\infty \mathbf{E}\left[h\left(\tilde{C}_i(s,t)\Theta r\right) \mathbb{1}_{\{\|C_{i-t}(0)\Theta\|r>1\}}\right] d(-r^{-\alpha}). \tag{3.27} \end{aligned}$$

Considering (3.21) and (3.27), we have, as $x \to \infty$,

$$\mathbf{E}\left[f\left(\frac{X_s}{x}, \dots, \frac{X_t}{x}\right) \,\middle|\, \|X_0\| > x\right] \to I$$

where

$$I = \frac{1}{\sum\limits_{i=0}^{\infty} \mathbf{E}\|C_i(0)\Theta\|^\alpha} \sum_{i=t}^{\infty} \int_0^\infty \mathbf{E}\left[h\left(C_{i-t+s}(s)\Theta r, \dots, C_i(t)\Theta r\right) \mathbb{1}_{\{\|C_{i-t}(0)\Theta\|r>1\}}\right] d(-r^{-\alpha}).$$

Applying this relation to the function $h(x_s, \dots, x_t) = f\left(\frac{x_s}{\|x_0\|}, \dots, \frac{x_t}{\|x_0\|}\right)$ to see that, as $x \to \infty$, the left-hand side of (3.20) converges to

$$\frac{1}{\sum\limits_{i=0}^{\infty} \mathbf{E}\|C_i(0)\Theta\|^\alpha} \sum_{i=t}^{\infty} \int_0^\infty \mathbf{E}\left[f\left(\frac{C_{i-t+s}(s)\Theta}{\|C_{i-t}(0)\Theta\|}, \dots, \frac{C_i(t)\Theta}{\|C_{i-t}(0)\Theta\|}\right) \mathbb{1}_{\{\|C_{i-t}(0)\Theta\|r>1\}}\right] d(-r^{-\alpha}).$$

By Fubini's theorem, this is equal to the right-hand side of (3.20). □

4 Applied models

4.1 Heavy tailed multivariate ARMA(1,1) process

Assume that $((A_t, B_t, \xi_t))_t$ is an i.i.d. sequence of random vectors in $\mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \times \mathbb{R}^d$, for some $d \geq 1$. Throughout we assume that ξ_t's are regularly varying random vectors. Suppose that a stationary sequence $(X_t)_{t\in\mathbb{Z}}$ with value in $\mathbb{R}^d$ satisfies a multivariate random coefficient ARMA(1,1) equation of the following form

$$X_t = A_t X_{t-1} + B_t \xi_{t-1} + \xi_t. \tag{4.28}$$

Iterating this equation backwards we arrive at the following MA(∞) representation of this process

$$X_t = \xi_t + \sum_{i\geq 1} (A_{t-i+1} + B_{t-i+1}) \prod_{j=0}^{i-2} A_{t-j} \xi_{t-i}, \tag{4.29}$$

namely, the stationary solution can be represented as (1.1) with

$$C_i(t) = \begin{cases} Id, & i = 0, \\ (A_{t-i+1} + B_{t-i+1}) \prod_{j=0}^{i-2} A_{t-j}, & i \geq 1. \end{cases} \tag{4.30}$$

From Theorem 3.1 in Hult and Samorodnitsky [4] we obtain the following result.

Corollary 4.1 *Suppose that* $\xi \in RV(\alpha, \mu)$ *and there is some* $0 < \varepsilon < \alpha$ *such that*

$$\mathbf{E}\|A\|^{\alpha+\varepsilon} < 1,\ \mathbf{E}\|B\|^{\alpha+\varepsilon} < \infty. \tag{4.31}$$

Then the series (4.29) converges a.s. and

$$\frac{\mathbf{P}\{u^{-1}X_0 \in \cdot\}}{\mathbf{P}\{\|\xi_0\| > u\}} \xrightarrow{v} \mathbf{E}\left[\sum \mu \circ C_j(0)^{-1}(\cdot)\right] \tag{4.32}$$

as $u \to \infty$ *on* $\bar{\mathbb{R}}^d \backslash \{0\}$.

Proof. Since $C_0(0) = Id$, we have $\mathbf{P}(\bigcap_{j\geq 0}\{\|C_j(0)\| = 0\}) = 0$.
For $\alpha \in (0,1) \cup (1,2)$, we have

$$\begin{aligned} \sum \mathbf{E}\|C_i(0)\|^{\alpha-\varepsilon} &= 1 + \sum_{i\geq 1} \mathbf{E} \left\| (A_{-i+1} + B_{-i+1}) \prod_{j=0}^{i-2} A_{-j} \right\|^{\alpha-\varepsilon} \\ &\leq 1 + \sum_{i\geq 1} \mathbf{E}\|A+B\|^{\alpha-\varepsilon} \mathbf{E} \prod_{j=0}^{i-2} \|A_{-j}\|^{\alpha-\varepsilon} \\ &= 1 + \mathbf{E}\|A+B\|^{\alpha-\varepsilon} \sum_{i\geq 1} (\mathbf{E}\|A\|^{\alpha-\varepsilon})^{i-1} \\ &= 1 + \mathbf{E}\|A+B\|^{\alpha-\varepsilon} \sum_{i\geq 0} (\mathbf{E}\|A\|^{\alpha-\varepsilon})^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon} i \frac{\alpha-\varepsilon}{\alpha+\varepsilon}}. \qquad (4.33) \end{aligned}$$

By Jensen's inequality, the last term is bounded by

$$1+\mathbf{E}\|A+B\|^{\alpha-\varepsilon}\sum_{i\geq 0}(\mathbf{E}\|A\|^{\alpha+\varepsilon})^{i\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}. \tag{4.34}$$

In the case of $\alpha\in(0,1)$,

$$\mathbf{E}\|A+B\|^{\alpha-\varepsilon}<\mathbf{E}\|A\|^{\alpha-\varepsilon}+\mathbf{E}\|B\|^{\alpha-\varepsilon}. \tag{4.35}$$

In the case of $\alpha\in(1,2)$

$$\mathbf{E}\|A+B\|^{\alpha-\varepsilon}\leq(\mathbf{E}\|A+B\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}\leq\left[\left(\mathbf{E}\|A\|^{\alpha+\varepsilon}\right)^{\frac{1}{\alpha+\varepsilon}}+\left(\mathbf{E}\|B\|^{\alpha+\varepsilon}\right)^{\frac{1}{\alpha+\varepsilon}}\right]^{\alpha-\varepsilon}. \tag{4.36}$$

Combining (4.33), (4.34), (4.35) and (4.36) proves

$$\sum\mathbf{E}\|C_i(0)\|^{\alpha-\varepsilon}<\infty.$$

Similarly as in (4.33), we have

$$\sum\mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon}\leq 1+\mathbf{E}\|A+B\|^{\alpha+\varepsilon}\sum_{i\geq 0}(\mathbf{E}\|A\|^{\alpha+\varepsilon})^{i}. \tag{4.37}$$

If $\alpha+\varepsilon<1$, then

$$\mathbf{E}\|A+B\|^{\alpha+\varepsilon}<\mathbf{E}\|A\|^{\alpha+\varepsilon}+\mathbf{E}\|B\|^{\alpha+\varepsilon}, \tag{4.38}$$

else

$$\mathbf{E}\|A+B\|^{\alpha+\varepsilon}\leq\left[(\mathbf{E}\|A\|^{\alpha+\varepsilon})^{\frac{1}{\alpha+\varepsilon}}+(\mathbf{E}\|B\|^{\alpha+\varepsilon})^{\frac{1}{\alpha+\varepsilon}}\right]^{\alpha+\varepsilon}. \tag{4.39}$$

Combining (4.37), (4.38) and (4.39) proves

$$\sum\mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon}<\infty.$$

For $\alpha\in\{1,2\}$, using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

$$\begin{aligned}
&\mathbf{E}\left(\sum\|C_i(0)\|^{\alpha-\varepsilon}\right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}\\
\leq\ &\left[\sum(\mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}\right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}\\
=\ &\left[1+\sum_{i\geq 1}\left\{\mathbf{E}\left\|(A_{-i+1}+B_{-i+1})\prod_{j=0}^{i-2}A_{-j}\right\|^{\alpha+\varepsilon}\right\}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}\right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}\\
\leq\ &\left[1+\sum_{i\geq 1}\left\{\mathbf{E}\|A+B\|^{\alpha+\varepsilon}\mathbf{E}\prod_{j=0}^{i-2}\|A_{-j}\|^{\alpha+\varepsilon}\right\}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}\right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}\\
=\ &\left[1+(\mathbf{E}\|A+B\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}\sum_{i\geq 0}(\mathbf{E}\|A\|^{\alpha+\varepsilon})^{i\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}\right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}. \qquad (4.40)
\end{aligned}$$

Considering (4.39) and (4.40) we get

$$\mathbf{E}\left(\sum \|C_i(0)\|^{\alpha-\varepsilon}\right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} < \infty.$$

For $\alpha \in (2,\infty)$, using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

$$\begin{aligned}\mathbf{E}\left(\sum \|C_i(0)\|^2\right)^{\frac{\alpha+\varepsilon}{2}} &\leq \left[\sum (\mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon})^{\frac{2}{\alpha+\varepsilon}}\right]^{\frac{\alpha+\varepsilon}{2}} \\ &\leq \left[1 + (\mathbf{E}\|A+B\|^{\alpha+\varepsilon})^{\frac{2}{\alpha+\varepsilon}} \sum_{i\geq 0} (\mathbf{E}\|A\|^{\alpha+\varepsilon})^{i\frac{2}{\alpha+\varepsilon}}\right]^{\frac{\alpha+\varepsilon}{2}}. \qquad (4.41)\end{aligned}$$

Considering (4.39) and (4.41) we get

$$\mathbf{E}\left(\sum \|C_i(0)\|^2\right)^{\frac{\alpha+\varepsilon}{2}} < \infty.$$

☐

Here is another proof of Corollary 4.1.

Proof. We denote

$$\tilde{X}_t = (X_t, \xi_t, \xi_{t+1})', \qquad (4.42)$$

$$\tilde{A}_t = \begin{pmatrix} A_t & B_t & Id \\ 0 & 0 & Id \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Z_t = (0, 0, \xi_{t+1})'.$$

Then the relation (4.28) can be wrote as

$$\tilde{X}_t = \tilde{A}_t \tilde{X}_{t-1} + Z_t. \qquad (4.43)$$

Iterating this equation we get

$$\tilde{X}_t = \sum_{i=0}^{\infty} C_i(t) Z_{t-i}$$

where $C_0(t) = Id$, $C_1(t) = \tilde{A}_t$ and

$$C_i(t) = \begin{pmatrix} \prod_{j=0}^{i-1} A_{t-j} & B_{t-i+1}\prod_{j=0}^{i-2} A_{t-j} & (A_{t-i+2} + B_{t-i+2})\prod_{j=0}^{i-3} A_{t-j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i \geq 2.$$

It is equivalent to prove the convergence (4.32) of vector $\tilde{X}_t$ defined by (4.42).

By Lemma 2.2 and the properties of matrix norm, for $i \geq 1$

$$\|C_i(t)\| \leq \prod_{j=0}^{i-1} \|A_{t-j}\| + \|B_{t-i+1}\| \prod_{j=0}^{i-2} \|A_{t-j}\| + \|A_{t-i+2} + B_{t-i+2}\| \prod_{j=0}^{i-3} \|A_{t-j}\| \tag{4.44}$$

where $A_i = B_i = Id$ if $i > t$. It follows from (2.16), if $\alpha \in (0,1) \cup (2,1)$

$$\begin{aligned} & \sum_{i\geq 0} \mathbf{E}\|C_i(0)\|^{\alpha-\varepsilon} \\ \leq \quad & 1 + c\sum_{i\geq 1} \left[\mathbf{E}\prod_{j=0}^{i-1} \|A_{t-j}\|^{\alpha-\varepsilon} + \mathbf{E}\left(\|B_{t-i+1}\| \prod_{j=0}^{i-2} \|A_{t-j}\| \right)^{\alpha-\varepsilon} \right. \\ & \left. + \mathbf{E}\left(\|A_{t-i+2} + B_{t-i+2}\| \prod_{j=0}^{i-3} \|A_{t-j}\| \right)^{\alpha-\varepsilon} \right] \\ = \quad & 1 + c\left(\sum_{i\geq 1} \left(\mathbf{E}\|A\|^{\alpha-\varepsilon}\right)^i + \mathbf{E}\|B\|^{\alpha-\varepsilon} \sum_{i\geq 0} \left(\mathbf{E}\|A\|^{\alpha-\varepsilon}\right)^i \right. \\ & \left. + \mathbf{E}\|A+B\|^{\alpha-\varepsilon} \sum_{i\geq 0} \left(\mathbf{E}\|A\|^{\alpha-\varepsilon}\right)^i + 2^{\alpha-\varepsilon} \right), \end{aligned} \tag{4.45}$$

where c is a constant depending on $\alpha - \varepsilon$. By Jensen's inequality,

$$\mathbf{E}\|A\|^{\alpha-\varepsilon} \leq \left(\mathbf{E}\|A\|^{\alpha+\varepsilon}\right)^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}. \tag{4.46}$$

Combining (4.45), (4.46), (4.38) and (4.39) proves

$$\sum_{i\geq 0} \mathbf{E}\|C_i(0)\|^{\alpha-\varepsilon} < \infty.$$

Similarly we can prove

$$\sum_{i\geq 0} \mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon} < \infty.$$

For $\alpha \in \{1,2\}$, using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

$$\mathbf{E}\left(\sum \|C_i(0)\|^{\alpha-\varepsilon}\right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} \leq \left[\sum (\mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}\right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}. \tag{4.47}$$

By (2.16) and (4.44), $\sum (\mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}$ is bounded by the right-hand side of (4.45) which is bounded. Hence

$$\mathbf{E}\left(\sum \|C_i(0)\|^{\alpha-\varepsilon}\right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} < \infty. \tag{4.48}$$

By similar method of (4.48), we get

$$\mathbf{E}\left(\sum \|C_i(0)\|^2\right)^{\frac{\alpha+\varepsilon}{2}} < \infty.$$

□

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