SPLITTING METHODS FOR THE NONLOCAL FOWLER EQUATION

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ABSTRACT. We consider a nonlocal scalar conservation law proposed by Andrew C. Fowler to describe the dynamics of dunes, and we develop a numerical procedure based on splitting methods to approximate its solutions. We begin by proving the convergence of the well-known Lie formula, which is an approximation of the exact solution of order one in time. We next use the split-step Fourier method to approximate the continuous problem using the fast Fourier transform and the finite difference method. Our numerical experiments confirm the theoretical results.

1. INTRODUCTION

We consider the Fowler equation [10, 11]:

$$\begin{cases} \partial_t u(t,x) + \partial_x \left(\frac{u^2}{2}\right)(t,x) + \mathcal{I}[u(t,\cdot)](x) - \partial_x^2 u(t,x) = 0, & x \in \mathbf{R}, t>0, \\ u(0,x) = u_0(x), & x\in \mathbf{R}, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ represents the dune height and $\mathcal{I}$ is a nonlocal operator defined as follows: for any Schwartz function $\varphi \in \mathcal{S}(\mathbf{R})$ and any $x \in \mathbf{R}$,

$$\mathcal{I}[\varphi](x) := \int_0^{+\infty} |\xi|^{-\frac{1}{3}} \varphi''(x-\xi)\, d\xi. \tag{1.2}$$

We refer to [1, 2, 6] for theoretical results on this equation.

Remark 1.1. The nonlocal term $\mathcal{I}$ is anti-diffusive. Indeed, it has been proved in [1] that

$$\mathcal{F}\left(\mathcal{I}[\varphi]\right)(\xi) = -4\pi^2 \Gamma\left(\frac{2}{3}\right)\left(\frac{1}{2} - i\,\mathrm{sgn}(\xi)\frac{\sqrt{3}}{2}\right)|\xi|^{4/3}, \tag{1.3}$$

where $\mathcal{F}$ denotes the Fourier transform normalized in (1.12). Thus, $\mathcal{I}$ can been seen as a fractional power of order $2/3$ of the Laplacian, with the "bad" sign. It will be clear from the analysis below that our results can easily be extended to the case where $\mathcal{I}$ is replaced with a Fourier multiplier homogeneous of degree $\lambda \in]0,2[$, as in [4], and not only $\lambda = 4/3$.

We assume that the initial data u_0 belongs to $H^3(\mathbf{R})$, and thus (1.1) has a unique solution belonging to $C([0,t], H^3(\mathbf{R}))$ for all $t>0$, from [1]. We will denote $u(t,\cdot)$ by $S^t u_0$; S^t maps $H^3(\mathbf{R})$ to itself. Duhamel's formula for the continuous problem (1.1) reads

$$u(t,\cdot) := S^t u_0 = K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * \left(S^s u_0\right)^2 \, ds, \tag{1.4}$$

Key words and phrases. Nonlocal operator, numerical time integration, operator splitting, split-step Fourier method, stability, error analysis.

2010 *Mathematics Subject Classification.* Primary 65M15; Secondary 35K59, 86A05.

This work was supported by the French ANR project MATHOCEAN, ANR-08-BLAN-0301-02.

where $K(t,\cdot)=\mathcal{F}^{-1}\left(e^{-t\psi_{\mathcal{I}}}\right)$ is the kernel of the operator $\mathcal{I}-\partial_x^2$, and $\psi_{\mathcal{I}}$ is defined by

$$\psi_{\mathcal{I}}(\xi)=4\pi^2\xi^2-a_{\mathcal{I}}|\xi|^{4/3}+ib_{\mathcal{I}}\xi|\xi|^{1/3}, \tag{1.5}$$

where $a_{\mathcal{I}}, b_{\mathcal{I}}$ are positive constants.

Recently, to solve the Fowler equation some numerical experiments have been performed using mainly finite difference approximation schemes [3, 4]. However, these schemes are not effective because if we opt for an explicit scheme, numerical stability requires that the time step Δt is limited by $O(\Delta x^2)$. And, if we choose an implicit scheme, we have to solve a large system which is a computationally expensive operation. Thus, the splitting method becomes an interesting alternative to solve the Fowler model. To our knowledge, there is no convergence result in the literature for the splitting method associated to the Fowler equation. This method is more commonly used to split different physical terms, such as reaction and diffusion terms, see for instance [16]. Splitting methods have also been employed for solving a wide range of nonlinear wave equations. The basic idea of this method is to decompose the original problem into sub-problems and then to approximate the solution of the original problem by solving successively the sub-problems. Various versions of this method have been developed for the nonlinear Schrödinger, Korteweg-de-Vries and modified Korteweg-de-Vries equations, see for instance [15, 17, 19].
For the Fowler model (1.1), we consider, separately, the linear Cauchy problem

$$\frac{\partial v}{\partial t}+\mathcal{I}[v(t,\cdot)]-\eta\,\partial_x^2 v=0;\quad v(0,x)=v_0(x), \tag{1.6}$$

and the nonlinear Cauchy problem

$$\frac{\partial w}{\partial t}+\partial_x\left(\frac{w^2}{2}\right)-\varepsilon\,\partial_x^2 w=0;\quad w(0,x)=w_0(x), \tag{1.7}$$

where ε,η are fixed positive parameters such that $\varepsilon+\eta=1$. Equation (1.7) is simply the viscous Burgers' equation. We denote by X^t and Y^t, respectively, the evolution operator associated with (1.6) and (1.7):

$$v(t,\cdot):=X^t v_0=D(t,\cdot)*v_0,$$

where $D(t,\cdot)=\mathcal{F}^{-1}\left(e^{-t\,\phi_{\mathcal{I}}}\right)$ with $\phi_{\mathcal{I}}(\xi)=4\pi^2\eta\xi^2-a_{\mathcal{I}}|\xi|^{4/3}+b_{\mathcal{I}}\xi|\xi|^{1/3}$, and

$$w(t,\cdot):=Y^t w_0=G(t,\cdot)*w_0-\frac{1}{2}\int_0^t\partial_x G(t-s,\cdot)*\left(Y^s w_0\right)^2\,ds, \tag{1.8}$$

where G is the heat kernel defined by

$$G(t,\cdot)=\mathcal{F}^{-1}(e^{-t(4\pi^2\varepsilon|.|^2)})=\frac{1}{\sqrt{4\pi\varepsilon t}}e^{-\frac{|.|^2}{4\varepsilon t}}.$$

Furthermore, the following L^2-estimate holds

$$\|Y^t w\|_{L^2(\mathbf{R})}\leqslant\|w\|_{L^2(\mathbf{R})}. \tag{1.9}$$

Let us explain the choice of this decomposition. First, we can remark that if we do not consider the nonlinear term in (1.1), the analytical solutions are available using the Fourier transform. Thus, the linear part may be computed efficiently using a fast Fourier transform (hereafter FFT) algorithms. Note also that the Laplacian and the fractional term $\mathcal{I}$ cannot be treated separately. Indeed, the equation $u_t+\mathcal{I}[u]=0$ is ill-posed. We next decide to handle the nonlinear term by adding a bit of viscosity in order to avoid shock problems in the standard Burgers' equation. Therefore, the splitting approach presented in this article differs from e.g. the one analyzed in [9], which corresponds to assuming $\varepsilon=0$ in the above definitions. The splitting operators associated to this approach when $\mathcal{I}=0$ (which amount to considering alternatively the heat equation and the Burgers equation) have been studied in [13] (as well as other equations involving the Burgers nonlinearity, such as the KdV equation, see also [12]). Note however that the regularity required on the initial data in [13] is higher than in the present paper, where smoothing effects associated to the viscous Burgers equation are used. It should be clear that using the methods from [12, 13], the analysis of splitting operators in the limiting case $\varepsilon = 0$ (Burgers equation instead of the viscous Burgers equation) could be achieved. We also motivate this choice by the presence of artificial diffusion in classical numerical schemes used to solve the convection equations. An alternative to reduce this effect is to consider numerical schemes of high order which are usually computationally expensive and do not seem to be very useful for the Fowler model because of the diffusion term.

We consider the Lie formula defined by

$$Z_L^t = X^t Y^t. \tag{1.10}$$

The alternative definition $Z_L^t = Y^t X^t$ could be studied as well, leading to a similar result. Also, the following evolution operators

$$Z_S^t = X^{t/2} Y^t X^{t/2} \quad \text{or} \quad Z_S^t = Y^{t/2} X^t Y^{t/2},$$

corresponding to the Strang method [18] could be considered. Following the computations detailed in the present paper for the case (1.10), it would be possible to show that the other Lie formula generates a scheme of order one, and to prove that the Strang method is of order two (for smooth initial data), in the same fashion as in, e.g., [5, 15]. This fact is simply illustrated numerically in Section 6, to avoid a lengthy presentation. With Z_L^t given by (1.10), our main result is:

Theorem 1.2. *For all $u_0 \in H^3(\mathbf{R})$ and for all $T > 0$, there exist two positive constants C_T and Δt_0 such that for all $\Delta t \in]0, \Delta t_0]$ and for all $n \in \mathbf{N}$ such that $0 \leqslant n\Delta t \leqslant T$,*

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2(\mathbf{R})} \leqslant C_T(m)\, \Delta t, \tag{1.11}$$

where $m = \max_{t\in[0,T]} \|S^t u_0\|_{H^3(\mathbf{R})}$, and $C_T(m)$ depends only on T and m.

Remark 1.3. It will follow from Lemma 3.12 that

$$m = \max_{t\in[0,T]} \|S^t u_0\|_{H^3(\mathbf{R})} \leqslant C_T(\|u_0\|_{H^2(\mathbf{R})}) \|u_0\|_{H^3(\mathbf{R})},$$

for some nonlinear (increasing) function C_T depending on T.

In this paper, we begin by estimating the L^2-stability for error propagation. We next prove that the local error of the Lie formula is an approximation of order two in time. Finally we prove that this evolution operator represents a good approximation, of order one in time, of the evolution operator S^t in the following sense:

$$S^t \approx \left[Z_L^{t/N} \right]^N,$$

for N large and t fixed. For that, we use the standard argument of Lady Windermere's fan.

Furthermore, we apply Lie and Strang approximations in order to make some numerical simulations using the split-step Fourier experiments.

This paper is organised as follows. In the next section we give some properties related to the kernels G and K. In Section 3, we prove two fractional Gronwall Lemmas, and some estimates on X^t, Y^t, Z_L^t and S^t. In Section 4, we give an estimate of the local error and an L^2 stability property for the Lie formula. Theorem 1.2 is proved in Section 5. We finally perform some numerical experiments which show that the Lie and Strang methods have a convergence rate in $\mathcal{O}(\Delta t)$ and $\mathcal{O}\left(\Delta t^2\right)$, respectively.

Notations.
- We denote by $\mathcal{F}$ the Fourier transform of f which is defined by: for all $\xi \in \mathbf{R}$,

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbf{R}} e^{-2i\pi x\xi} f(x)dx. \tag{1.12}$$

We denote by $\mathcal{F}^{-1}$ its inverse.
- We denote by $C_T(c_1, c_2, \cdots)$ a generic constant, strictly positive, which depends on parameters $c_1, c_2, \cdots$, and T. C is assumed to be a monotone increasing function of its arguments.

2. Preliminaries

We begin by recalling the properties of kernels K of $\mathcal{I} - \partial_x^2$ and G (the heat kernel).

Proposition 2.1 (Main properties of K, [1])**.** *The kernel K satisfies:*

(1) $\forall t > 0$, $K(t,\cdot) \in L^1(\mathbf{R})$ *and* $K \in C^\infty(]0,\infty[\times\mathbf{R})$.
(2) $\forall s,t > 0$, $K(s,\cdot) * K(t,\cdot) = K(s+t,\cdot)$.
(3) $\forall T > 0, \exists C_T > 0$ *such that for all* $t \in]0,T]$, $\|\partial_x K(t,\cdot)\|_{L^2(\mathbf{R})} \leqslant C_T\, t^{-3/4}$.
(4) $\forall T > 0, \exists C_T > 0$ *such that for all* $t \in]0,T]$, $\|\partial_x K(t,\cdot)\|_{L^1(\mathbf{R})} \leqslant C_T\, t^{-1/2}$.
(5) *For any* $u_0 \in L^2(\mathbf{R})$ *and* $t > 0$,
$$\|K(t,\cdot) * u_0\|_{L^2(\mathbf{R})} \leqslant e^{\alpha_0 t}\|u_0\|_{L^2(\mathbf{R})},$$
where $\alpha_0 = -\min \mathrm{Re}(\psi_{\mathcal{I}}) > 0$.

Proposition 2.2 (Main properties of G, [8])**.** *The kernel G satisfies:*

(1) $G \in C^\infty(]0,\infty[\times\mathbf{R})$.
(2) $\forall s,t > 0$, $G(s,\cdot) * G(t,\cdot) = G(s+t,\cdot)$.
(3) $\forall t > 0$, $\|G(t,\cdot)\|_{L^1(\mathbf{R})} = 1$.
(4) $\exists C_0 > 0$ *such that for all* $t > 0$, $\|\partial_x G(t,\cdot)\|_{L^2(\mathbf{R})} \leqslant C_0 t^{-3/4}$.
(5) $\exists C_1 > 0$ *such that for all* $t > 0$, $\|\partial_x G(t,\cdot)\|_{L^1(\mathbf{R})} \leqslant C_1 t^{-1/2}$.

Remark 2.3. The kernel D of $\mathcal{I} - \eta\,\partial_x^2$ has similar properties to the kernel K. Moreover, for all $t > 0$, we have

$$D(t,\cdot) * G(t,\cdot) = K(t,\cdot). \tag{2.1}$$

3. Some useful estimates

3.1. **Gronwall type lemmas.**

Lemma 3.1 (Fractional Gronwall Lemma)**.** *Let $\phi : [0,T] \to \mathbf{R}_+$ be a bounded measurable function, and suppose that there are positive constants A, L and $\theta \in]0,1[$ such that for all $t \in [0,T]$,*

$$\phi(t) \leqslant A + L\frac{d^{-\theta}}{dt^{-\theta}}\phi(t), \tag{3.1}$$

where $\frac{d^{-\theta}}{dt^{-\theta}}$ is the Riemann–Liouville operator defined by

$$\frac{d^{-\theta}}{dt^{-\theta}}\phi(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\phi(s)\,ds.$$

Then there exists $C_T(\theta)$ *such that*

$$\phi(t) \leqslant C_T(\theta)A, \quad \forall t \in [0,T].$$

Proof. Iterating inequality (3.1) once, we have

$$\begin{aligned}\phi(t) &\leqslant A + \frac{L}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\phi(s)\,ds \\ &\leqslant A + \frac{L}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\left(A + \frac{L}{\Gamma(\theta)}\int_0^s (s-r)^{\theta-1}\phi(r)\,dr\right)ds \\ &= A\left(1 + \frac{L}{\theta\Gamma(\theta)}T^\theta\right) + \frac{L^2}{\Gamma(\theta)^2}\int_0^t (t-s)^{\theta-1}\int_0^s (s-r)^{\theta-1}\phi(r)\,dr\,ds.\end{aligned}$$

From Fubini's Theorem, we get

$$\begin{aligned}\int_0^t (t-s)^{\theta-1}\int_0^s (s-r)^{\theta-1}\phi(r)\,dr\,ds &= \int_0^t \phi(r)\int_r^t (t-s)^{\theta-1}(s-r)^{\theta-1}ds\,dr \\ &= \int_0^t \phi(r)(t-r)^{2\theta-1}\left(\int_0^1 (1-\tau)^{\theta-1}\tau^{\theta-1}d\tau\right)dr \\ &= \beta(\theta,\theta)\int_0^t \phi(r)(t-r)^{2\theta-1}dr,\end{aligned}$$

where β is the beta function. Therefore, we have

$$\phi(t) \leqslant C_T(\theta)A + \frac{L^2}{\Gamma(\theta)^2}\beta(\theta,\theta)\int_0^t \phi(s)(t-s)^{2\theta-1}\,ds. \tag{3.2}$$

Iterating the estimate (3.2) n times, with $n\theta \geqslant 1$, we get the following estimate:

$$\phi(t) \leqslant \tilde{C}_T(\theta)A + \tilde{L}_T(\theta)\int_0^t \phi(s)(t-s)^\alpha ds,$$

with $\alpha \geqslant 0$, and where $\tilde{L}_T(\theta)$ is a positive constant which depends on T and θ. The lemma then follows from the classical Gronwall Lemma. □

Lemma 3.2 (Modified fractional Gronwall Lemma)**.** *Let* $\phi : [0,T] \to \mathbf{R}_+$ *be a bounded measurable function and* P *be a polynomial with positive coefficients and no constant term. We assume there exists two positive constants* C *and* $\theta \in]0,1[$ *such that for all* $t \in [0,T]$*,*

$$0 \leqslant \phi(t) \leqslant \phi(0) + P(t) + C\frac{d^{-\theta}}{dt^{-\theta}}\phi(t). \tag{3.3}$$

Then there exists $C_T(\theta)$ *such that for all* $t \in [0,T]$*,*

$$\phi(t) \leqslant C_T(\theta)\,\phi(0) + C_T(\theta)\,P(t).$$

Proof. Arguing as in the proof of Lemma 3.1, we iterate the previous inequality. After one iteration, we get

$$\begin{aligned}\phi(t) \leqslant \tilde{C}\phi(0) + P(t) &+ \frac{C}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}P(s)ds \\ &+ \frac{C^2}{\Gamma(\theta)^2}\int_0^t (t-s)^{\theta-1}\left(\int_0^s (s-r)^{\theta-1}\phi(r)dr\right)ds \\ \leqslant \tilde{C}\phi(0) + \tilde{C}P(t) &+ \frac{C^2}{\Gamma(\theta)^2}\beta(\theta,\theta)\int_0^t \phi(s)(t-s)^{2\theta-1}ds,\end{aligned}$$

where we have used the assumptions on P, and Fubini's Theorem again for the last term. Iterating sufficiently many times, we infer like in the proof of Lemma 3.1:

$$\phi(t) \leqslant c_0\phi(0) + c_0 P(t) + \underline{C}\int_0^t \phi(s)(t-s)^\alpha ds, \tag{3.4}$$

with $\alpha > 0$. Set

$$\psi(t) = \left(c_0\phi(0) + c_0 P(t) + \underline{C}\int_0^t \phi(s)(t-s)^\alpha ds\right) e^{-C_1 t}.$$

Then

$$\begin{aligned}\psi'(t) = \Big(c_0 P'(t) &+ \underline{C}\alpha\int_0^t \phi(s)(t-s)^{\alpha-1}ds \\ &- C_1\Big(c_0\phi(0) + c_0 P(t) + \underline{C}\int_0^t \phi(s)(t-s)^\alpha ds\Big)\Big)e^{-C_1 t}.\end{aligned}$$

Using (3.4) to control the second term, and choosing C_1 sufficiently large, we infer:

$$\psi'(t) \leqslant c_0 P'(t) e^{-C_1 t}.$$

Since $P(0) = 0$, $\psi(0) = c_0\phi(0)$, for all $t \in [0,T]$,

$$\begin{aligned}\phi(t) \leqslant \psi(t)e^{C_1 t} &\leqslant c_0\phi(0)e^{C_1 t} + c_0\int_0^t P'(s)e^{C_1(t-s)}ds \\ &\leqslant c_0\phi(0)e^{C_1 T} + c_0 e^{C_1 T}\int_0^t P'(s)\, ds \leqslant c_0 e^{C_1 T}\left(\phi(0) + P(t)\right).\end{aligned}$$

This completes the proof. □

3.2. **Estimates on linear flows.** In this paragraph, we collect several estimates concerning the convolutions with D, K and G, which will be useful in the estimates of the local error of the scheme.

Proposition 3.3. *Let $s \in \mathbf{R}$ and $\varphi \in H^s(\mathbf{R})$. Then $\mathcal{I}[\varphi] \in H^{s-4/3}(\mathbf{R})$ and we have*

$$\|\mathcal{I}[\varphi]\|_{H^{s-4/3}(\mathbf{R})} \leqslant 4\pi^2\Gamma\left(\frac{2}{3}\right)\|\varphi\|_{H^s(\mathbf{R})}. \tag{3.5}$$

Proof. For all $s \in \mathbf{R}$ and all $\varphi \in H^s(\mathbf{R})$, we have, using (1.3)

$$\begin{aligned}\|\mathcal{I}[\varphi]\|_{H^{s-4/3}(\mathbf{R})} &= \left(\int_{\mathbf{R}}(1+|\xi|^2)^{s-4/3}|\mathcal{F}(\mathcal{I}[\varphi])(\xi)|^2\, d\xi\right)^{1/2} \\ &= 4\pi^2\Gamma\left(\frac{2}{3}\right)\left(\int_{\mathbf{R}}(1+|\xi|^2)^{s-4/3}\left|\frac{1}{2} - i\,\mathrm{sgn}(\xi)\frac{\sqrt{3}}{2}\right|^2 |\xi|^{8/3}|\mathcal{F}(\varphi)(\xi)|^2\, d\xi\right)^{1/2} \\ &= 4\pi^2\Gamma\left(\frac{2}{3}\right)\left(\int_{\mathbf{R}}\left(\frac{|\xi|^2}{1+|\xi|^2}\right)^{4/3}(1+|\xi|^2)^s|\mathcal{F}(\varphi)(\xi)|^2\, d\xi\right)^{1/2} \\ &\leqslant 4\pi^2\Gamma\left(\frac{2}{3}\right)\left[\int_{\mathbf{R}}(1+|\xi|^2)^s|\mathcal{F}(\varphi)(\xi)|^2\, d\xi\right]^{1/2} = 4\pi^2\Gamma\left(\frac{2}{3}\right)\|\varphi\|_{H^s(\mathbf{R})},\end{aligned}$$

hence the result. □

Lemma 3.4. (1) *Let* $n \in \mathbf{N}$. *Then, for all* $v \in H^n(\mathbf{R})$ *and all* $t > 0$,

$$\|X^t v\|_{H^n(\mathbf{R})} \leqslant e^{\beta_0 t} \|v\|_{H^n(\mathbf{R})},$$

where $\beta_0 = -\min \mathrm{Re}(\phi_{\mathcal{I}}) > 0$.
(2) *There exists* C *such that for all* $v \in H^2(\mathbf{R})$ *and all* $t > 0$,

$$\|X^t v - v\|_{L^2(\mathbf{R})} \leqslant C\, t\, e^{\beta_0 t} \|v\|_{H^2(\mathbf{R})}. \tag{3.6}$$

Proof. Using Plancherel formula, we have

$$\begin{aligned}
\|X^t v\|^2_{L^2(\mathbf{R})} &= \|D(t,\cdot) * v\|^2_{L^2(\mathbf{R})} \\
&= \|\mathcal{F}\left(D(t,\cdot)\right) \mathcal{F}v\|^2_{L^2(\mathbf{R})} = \int_{\mathbf{R}} |\mathcal{F}\left(D(t,\cdot)\right)(\xi)|^2 |\mathcal{F}v(\xi)|^2 \, d\xi \\
&= \int_{\mathbf{R}} e^{-2t\phi_{\mathcal{I}}(\xi)} |\mathcal{F}v(\xi)|^2 \, d\xi \leqslant e^{2\beta_0 t} \|v\|^2_{L^2(\mathbf{R})}.
\end{aligned}$$

Moreover, since

$$\partial_x^n X^t v = D(t,\cdot) * \partial_x^n v$$

then, from again Plancherel formula, we have

$$\|\partial_x^n X^t v\|_{L^2(\mathbf{R})} \leqslant e^{\beta_0 t} \|\partial_x^n v\|_{L^2(\mathbf{R})},$$

hence the first point of the lemma.

Let $v \in H^2(\mathbf{R})$. We have

$$\|X^t v - v\|_{L^2(\mathbf{R})} = \left\| \int_0^t \dot{X^s}\; v \, ds \right\|_{L^2(\mathbf{R})}.$$

But from the definition of X^t, $\dot{X^s}$ is given by

$$\dot{X^s}\; v = \eta \partial_x^2 X^s v - \mathcal{I}[X^s v] = \eta X^s \partial_x^2 v - \mathcal{I}[X^s v],$$

since $X^s \partial_x^2 v = D(s,\cdot) * \partial_x^2 v = \partial_x^2 \left(D(s,\cdot) * v\right)$. Thus, using Proposition 3.3 and the first point of this lemma, we get

$$\begin{aligned}
\|X^t v - v\|_{L^2(\mathbf{R})} &\leqslant \eta \int_0^t \|X^s \partial_x^2 v\|_{L^2(\mathbf{R})} \, ds + \int_0^t \|\mathcal{I}[X^s v]\|_{L^2(\mathbf{R})} \, ds \\
&\leqslant \eta\, t\, e^{\beta_0 t} \|v\|_{H^2(\mathbf{R})} + \int_0^t \|\mathcal{I}[X^s v]\|_{L^2(\mathbf{R})} ds \\
&\leqslant \eta\, t\, e^{\beta_0 t} \|v\|_{H^2(\mathbf{R})} + 4\pi^2 \Gamma\left(\frac{2}{3}\right) \int_0^t \|X^s v\|_{H^{4/3}(\mathbf{R})} ds \\
&\leqslant \eta\, t\, e^{\beta_0 t} \|v\|_{H^2(\mathbf{R})} + 4\pi^2 \Gamma\left(\frac{2}{3}\right) \int_0^t \|X^s v\|_{H^2(\mathbf{R})} ds \\
&\leqslant \left(\eta + 4\pi^2 \Gamma\left(\frac{2}{3}\right)\right) t\, e^{\beta_0 t} \|v\|_{H^2(\mathbf{R})},
\end{aligned}$$

hence the result. □

Recalling that K corresponds to D in the case $\eta = 1$, we readily infer:

Corollary 3.5. *For all* $w \in H^2(\mathbf{R})$ *and all* $t > 0$,

$$\|K(t,\cdot) * w - w\|_{L^2(\mathbf{R})} \leqslant C\, t\, e^{\alpha_0 t} \|w\|_{H^2(\mathbf{R})}, \tag{3.7}$$

where C *is a positive constant independent of* t *and* w.

We conclude this paragraph with an analogous result on the heat kernel G:

Lemma 3.6. *For all $w \in H^2(\mathbf{R})$ and all $t > 0$,*

$$\|G(t,\cdot) * w - w\|_{L^2(\mathbf{R})} \leqslant \varepsilon\, t\, \|w\|_{H^2(\mathbf{R})}.$$

Proof. Proceeding as above, we have:

$$G(t,\cdot) * w - w = \int_0^t \frac{\partial}{\partial t} \left(G(s,\cdot) * w\right)\, ds = \varepsilon \int_0^t \partial_x^2 \left(G(s,\cdot) * w\right)\, ds$$
$$= \varepsilon \int_0^t G(s,\cdot) * \partial_x^2 w\, ds.$$

Taking the norm L^2 and using Proposition 2.2, Young's inequality yields

$$\|G(t,\cdot) * w - w\|_{L^2(\mathbf{R})} \leqslant \varepsilon \int_0^t \|G(s,\cdot)\|_{L^1(\mathbf{R})} \|\partial_x^2 w\|_{L^2(\mathbf{R})}\, ds \leqslant \varepsilon\, t\, \|w\|_{H^2(\mathbf{R})},$$

hence the result. □

3.3. **Estimates on Y^t.** We now turn to the viscous Burgers' equation (1.7):

$$\partial_t w - \varepsilon \partial_x^2 w + w \partial_x w = 0; \quad w_{|t=0} = w_0. \tag{3.8}$$

Remark 3.7 (Hopf–Cole transform). The change of unknown function

$$w = -2\varepsilon \frac{1}{\phi} \partial_x \phi = -2\varepsilon \partial_x \left(\ln \phi\right),$$

turns the viscous Burgers' equation into the heat equation ([7, 14]):

$$\partial_t \phi - \varepsilon \partial_x^2 \phi = 0.$$

We infer the explicit formula:

$$w(t,x) = -2\varepsilon \partial_x \ln \left(\frac{1}{\sqrt{4\pi \varepsilon t}} \int_{-\infty}^{+\infty} \exp \left(-\frac{(x-y)^2}{4\varepsilon t} - \frac{1}{2\varepsilon} \int_0^y w_0(z) dz \right) dy \right).$$

However, this formula does not seem very helpful in order to establish Proposition 3.8.

Proposition 3.8. *Let $w_0 \in H^1(\mathbf{R})$. Then* (3.8) *has a unique solution $w \in C(\mathbf{R}_+; H^1(\mathbf{R}))$. In addition, there exists $C = C(\varepsilon, \|w_0\|_{H^1(\mathbf{R})})$ such that for all $t \geqslant 0$,*

$$\|w(t)\|_{L^2(\mathbf{R})} \leqslant \|w_0\|_{L^2(\mathbf{R})}, \quad \|\partial_x w(t)\|_{L^2(\mathbf{R})} \leqslant \|w_0'\|_{L^2(\mathbf{R})} e^{C(t^{5/8}+t)}.$$

If in addition $w_0 \in H^2(\mathbf{R})$, then $w \in C(\mathbf{R}_+; H^2(\mathbf{R}))$ and for all $T > 0$, there exists $M = M(\varepsilon, T, \|w_0\|_{H^2(\mathbf{R})})$ such that for all $t \in [0,T]$,

$$\|\partial_x w(t)\|_{L^2(\mathbf{R})} \leqslant \|w_0'\|_{L^2(\mathbf{R})} e^{Mt}, \quad \|\partial_x^2 w(t)\|_{L^2(\mathbf{R})} \leqslant \|w_0''\|_{L^2(\mathbf{R})} e^{Mt}.$$

Proof. The existence and uniqueness part being standard, we focus on the estimates. The L^2 estimate yields (formally, multiply (3.8) by w and integrate)

$$\frac{1}{2}\frac{d}{dt} \|w(t)\|_{L^2}^2 + \varepsilon \|\partial_x w(t)\|_{L^2}^2 = 0, \tag{3.9}$$

and the H^1 estimate (differentiate (3.8) with respect to x, multiply by $\partial_x w$ and integrate),

$$\frac{1}{2}\frac{d}{dt} \|\partial_x w(t)\|_{L^2}^2 + \varepsilon \|\partial_x^2 w(t)\|_{L^2}^2 = -\frac{1}{4} \int_{\mathbf{R}} \left(\partial_x w(t,x)\right)^3 dx. \tag{3.10}$$

The L^2 estimate (3.9) shows that the map $t \mapsto \|w(t)\|_{L^2}^2$ is non-increasing:

$$\|w(t)\|_{L^2} \leqslant \|w_0\|_{L^2}, \quad \forall t \geqslant 0.$$

An integration by parts and Cauchy–Schwarz inequality then yield

$$\|\partial_x w(t)\|_{L^2}^2 \leqslant \|w(t)\|_{L^2}\|\partial_x^2 w(t)\|_{L^2} \leqslant \|w_0\|_{L^2}\|\partial_x^2 w(t)\|_{L^2}. \tag{3.11}$$

In order to take advantage of the smoothing effect provided by the viscous part, integrate (3.10) in time and write

$$\varepsilon\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds \leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + \frac{1}{4}\int_0^t \|\partial_x w(s)\|_{L^3}^3 ds.$$

Gagliardo–Nirenberg inequality yields

$$\|\partial_x w\|_{L^3} \leqslant C\|\partial_x w\|_{L^2}^{5/6}\|\partial_x^2 w\|_{L^2}^{1/6},$$

so using (3.11), we infer:

$$\begin{aligned}\varepsilon\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds &\leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + C\int_0^t \|\partial_x w(s)\|_{L^2}^{5/2}\|\partial_x^2 w(s)\|_{L^2}^{1/2} ds\\ &\leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + C\|w_0\|_{L^2}^{5/4}\int_0^t \|\partial_x^2 w(s)\|_{L^2}^{7/4} ds.\end{aligned}$$

In view of Hölder inequality in the last integral in time,

$$\begin{aligned}\varepsilon\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds &\leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + C\|w_0\|_{L^2}^{5/4}\left(\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds\right)^{7/8} t^{1/8}\\ &\leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + \frac{\varepsilon}{2}\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds + C\left(\|w_0\|_{L^2}\right) t,\end{aligned}$$

where we have used Young inequality $ab \lesssim a^{8/7} + b^8$. We infer

$$\varepsilon\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds \leqslant \|w_0'\|_{L^2}^2 + C\left(\|w_0\|_{L^2}\right) t. \tag{3.12}$$

Gagliardo–Nirenberg inequality $\|f\|_{L^\infty} \leqslant \sqrt{2}\|f\|_{L^2}^{1/2}\|f'\|_{L^2}^{1/2}$ now yields

$$\begin{aligned}\int_0^t \|\partial_x w(s)\|_{L^\infty} ds &\leqslant \sqrt{2}\int_0^t \|\partial_x w(s)\|_{L^2}^{1/2}\|\partial_x^2 w(s)\|_{L^2}^{1/2} ds\\ &\leqslant C\left(\varepsilon, \|w_0\|_{L^2}\right)\int_0^t \|\partial_x^2 w(s)\|_{L^2}^{3/4} ds\\ &\leqslant C\left(\varepsilon, \|w_0\|_{L^2}\right)\left(\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds\right)^{3/8} t^{5/8}\\ &\leqslant C\left(\varepsilon, \|w_0\|_{H^1}\right)(1+t)^{3/8} t^{5/8} \leqslant C\left(\varepsilon, \|w_0\|_{H^1}\right)\left(t^{5/8}+t\right),\end{aligned}$$

where we have used (3.11), Hölder inequality and (3.12), successively.

Integrate the H^1 estimate (3.10) with respect to time, and now discard the viscous part whose contribution is non-negative:

$$\begin{aligned}\|\partial_x w(t)\|_{L^2}^2 &\leqslant \|w_0'\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\partial_x w(s)\|_{L^3}^3 ds\\ &\leqslant \|w_0'\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\partial_x w(s)\|_{L^\infty}\|\partial_x w(s)\|_{L^2}^2 ds.\end{aligned} \tag{3.13}$$

The first part of the proposition then follows from the Gronwall lemma.

To complete the proof of the proposition, we use the H^2 estimate which yields, in a similar fashion as for the above estimate:

$$\|\partial_x^2 w(t)\|_{L^2}^2 \leqslant \|w_0''\|_{L^2}^2 + C\int_0^t \|\partial_x w(s)\|_{L^\infty}\|\partial_x^2 w(s)\|_{L^2}^2 ds. \tag{3.14}$$

Gronwall lemma implies $\|\partial_x^2 w(t)\|_{L^2} \leqslant \|w_0''\|_{L^2} e^{C(t^{5/8}+t)}$, where $C = C(\varepsilon, \|w_0\|_{H^1})$. We bootstrap, thanks to Gagliardo–Nirenberg inequality again:

$$\|\partial_x w(t)\|_{L^\infty} \leqslant \sqrt{2}\|\partial_x w(t)\|_{L^2}^{1/2}\|\partial_x^2 w(t)\|_{L^2}^{1/2} \leqslant \sqrt{2}\|w_0\|_{H^2} e^{C(t^{5/8}+t)}.$$

Therefore, for $t \in [0,T]$,

$$\int_0^t \|\partial_x w(s)\|_{L^\infty} ds \leqslant \sqrt{2}\|w_0\|_{H^2} \times t \times e^{C(T^{5/8}+T)}.$$

The last estimates of the proposition then follow from the above inequality, along with Gronwall lemma applied to (3.13) and (3.14). ☐

Lemma 3.9. *Let $n \in \mathbf{N}^*$ and $T > 0$. For all $w \in H^n(\mathbf{R})$, there exists $C_T(\|w\|_{H^{n-1}(\mathbf{R})})$ such that for all $t \in [0,T]$,*

$$\|Y^t w\|_{H^n(\mathbf{R})} \leqslant C_T(\|w\|_{H^{n-1}(\mathbf{R})})\|w\|_{H^n(\mathbf{R})}. \tag{3.15}$$

Proof. Differentiating the Duhamel formula (1.8) in space, we have

$$\partial_x Y^t w = G(t,\cdot) * \partial_x w - \int_0^t \partial_x G(t-s,\cdot) * (Y^s w)\partial_x(Y^s w)\, ds.$$

Using Young inequality and inequality (1.9), we infer, for any integer $n \geqslant 1$:

$$\|\partial_x Y^t w\|_{H^{n-1}(\mathbf{R})} \leqslant \|\partial_x w\|_{H^{n-1}(\mathbf{R})} + \int_0^t \|\partial_x G(t-s,\cdot)\|_{L^2(\mathbf{R})}\|(Y^s w)\partial_x(Y^s w)\|_{W^{n-1,1}(\mathbf{R})}\, ds.$$

In view of Proposition 2.2, this implies:

$$\|\partial_x Y^t w\|_{H^{n-1}(\mathbf{R})} \leqslant \|\partial_x w\|_{H^{n-1}(\mathbf{R})} + C_0\int_0^t (t-s)^{-3/4}\|(Y^s w)\partial_x(Y^s w)\|_{W^{n-1,1}(\mathbf{R})}\, ds.$$

For $n = 1$, we use

$$\|(Y^s w)\partial_x(Y^s w)\|_{L^1} \leqslant \|Y^s w\|_{L^2}\|\partial_x(Y^s w)\|_{L^2} \leqslant \|w\|_{L^2}\|\partial_x(Y^s w)\|_{L^2},$$

and the fractional Gronwall Lemma 3.1 with $\theta = 1/4$, to obtain

$$\|\partial_x Y^t w\|_{L^2(\mathbf{R})} \leqslant C_T(\|w\|_{L^2(\mathbf{R})})\|\partial_x w\|_{L^2(\mathbf{R})}.$$

From (1.9), this implies the lemma in the case $n = 1$.

For $n \geqslant 1$, Leibniz rule and Cauchy–Schwarz inequality yield

$$\|(Y^s w)\partial_x(Y^s w)\|_{W^{n-1,1}(\mathbf{R})} \leqslant C(n)\|Y^s w\|_{H^{n-1}(\mathbf{R})}\|\partial_x(Y^s w)\|_{H^{n-1}(\mathbf{R})}.$$

The lemma then easily follows by induction on n. ☐

3.4. **Estimates on the splitting operator Z_L^t.** Combining the estimates on X^t and Y^t established in the previous two sections, we infer:

Corollary 3.10. (1) *For all $u \in L^2(\mathbf{R})$ and all $t > 0$,*

$$\|Z_L^t\, u\|_{L^2(\mathbf{R})} \leqslant e^{\beta_0 t}\, \|u\|_{L^2(\mathbf{R})},$$

where $\beta_0 = -\min \mathrm{Re}(\phi_{\mathcal{I}}) > 0$.

(2) *Let $T > 0, n \in \mathbf{N}^*$ and $u \in H^n(\mathbf{R})$. There exists $C_T(\|u\|_{H^{n-1}(\mathbf{R})})$ such that for all $t \in [0, T]$,*

$$\|Z_L^t\, u\|_{H^n(\mathbf{R})} \leqslant C_T(\|u\|_{H^{n-1}(\mathbf{R})})\|u\|_{H^n(\mathbf{R})}.$$

Proof. The first point is a direct consequence of the relation (1.9) and Lemma 3.4.
The second point is readily established with Lemmas 3.4 and 3.9. □

3.5. **Estimates on the exact flow S^t.**

Lemma 3.11 (L^2-a priori estimate)**.** *Let $u_0 \in L^2(\mathbf{R})$ and $T > 0$. Then, the unique mild solution $u \in C([0, T]; L^2(\mathbf{R})) \cap C(]0, T]; H^2(\mathbf{R}))$ of* (1.1) *satisfies, for all $t \in [0, T]$*

$$\|u(t, \cdot)\|_{L^2(\mathbf{R})} \leqslant e^{\alpha_0 t}\|u_0\|_{L^2(\mathbf{R})},$$

where $\alpha_0 = -\min \mathrm{Re}(\psi_{\mathcal{I}}) > 0$.

Proof. Multiplying (1.1) by u and integrating with respect to the space variable, we get:

$$\int_{\mathbf{R}} u_t u\, dx + \int_{\mathbf{R}} (\mathcal{I}[u] - u_{xx})\, u\, dx = 0$$

because the nonlinear term is zero. Using (1.3) and the fact that u and $\int_{\mathbf{R}} (\mathcal{I}[u] - \partial_{xx}^2 u) u\, dx$ are real, we get

$$\int_{\mathbf{R}} (\mathcal{I}[u] - \partial_{xx}^2 u) u\, dx = \int_{\mathbf{R}} \mathcal{F}^{-1}(\psi_{\mathcal{I}} \mathcal{F} u) u\, dx = \int_{\mathbf{R}} \psi_{\mathcal{I}} |\mathcal{F} u|^2\, d\xi = \int_{\mathbf{R}} \mathrm{Re}(\psi_{\mathcal{I}}) |\mathcal{F} u|^2\, d\xi.$$

We infer

$$\frac{1}{2}\frac{d}{dt}\|u(t, \cdot)\|_{L^2}^2 \quad \leqslant \quad \alpha_0 \|u(t)\|_{L^2}^2$$

where $\alpha_0 = -\min \mathrm{Re}(\psi_{\mathcal{I}}) > 0$. The result then follows from the Gronwall lemma. □

Lemma 3.12. *Let $n \in \mathbf{N}^*$, $u_0 \in H^n(\mathbf{R})$ and $T > 0$. There exists $C_T(\|u_0\|_{H^{n-1}(\mathbf{R})})$ such that the unique mild solution $u \in C([0, T]; H^n(\mathbf{R}))$ satisfies*

$$\|u(t, \cdot)\|_{H^n(\mathbf{R})} \leqslant C_T(\|u_0\|_{H^{n-1}(\mathbf{R})})\|u_0\|_{H^n(\mathbf{R})}. \tag{3.16}$$

Proof. The proof is similar to the one given in Lemma 3.9, replacing G with K, and is obtained using Proposition 2.1, and Lemmas 3.1 and 3.11. □

4. Stability property on Z_L^t and local error

Proposition 4.1 (L^2-stability)**.** *Let $T > 0$, $u_0, v_0 \in H^2(\mathbf{R})$ and M_0 a constant with*

$$\|u_0\|_{H^2(\mathbf{R})} \leqslant M_0,\ \|v_0\|_{H^2(\mathbf{R})} \leqslant M_0.$$

Then there exists c_0 depending of M_0 and T such that for all $t \in [0, T]$,

$$\|Z_L^t u_0 - Z_L^t v_0\|_{L^2(\mathbf{R})} \leqslant e^{c_0\, t}\|u_0 - v_0\|_{L^2(\mathbf{R})}.$$

Proof. We only need to compare $Y^t u_0$ and $Y^t v_0$, solutions at time t of the viscous Burgers' equation:

$$\partial_t u - \varepsilon \partial_x^2 u + u\partial_x u = 0; \quad u_{|t=0} = u_0, \tag{4.1}$$

$$\partial_t v - \varepsilon \partial_x^2 v + v\partial_x v = 0; \quad v_{|t=0} = v_0, \tag{4.2}$$

where $u(t,\cdot) := Y^t u_0$ and $v(t,\cdot) := Y^t v_0$. Setting $w = u - v$, this error solves

$$\partial_t w - \varepsilon \partial_x^2 w + (u\partial_x u - v\partial_x v) = 0. \tag{4.3}$$

Arguing as in Proposition 3.8 and Lemma 3.11, the L^2 energy estimate yields:

$$\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbf{R})} \leqslant \int_{\mathbf{R}} w(u\partial_x u - v\partial_x v)\,dx.$$

Writing $u = v + w$ and using integrations by parts, we have

$$\int_{\mathbf{R}} w(u\partial_x u - v\partial_x v)\,dx = \int_{\mathbf{R}} w(v\partial_x w + w\partial_x v + w\partial_x w)\,dx = \frac{1}{2}\int_{\mathbf{R}} w^2 \partial_x v\,dx,$$

where we have used $w^2\partial_x w = \frac{1}{3}\partial_x(w^3)$, so by Hölder inequality,

$$\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbf{R})} \leqslant \frac{1}{2}\|\partial_x v(t,\cdot)\|_{L^\infty(\mathbf{R})}\|w(t,\cdot)\|^2_{L^2(\mathbf{R})}.$$

We finally get

$$\|w(t,\cdot)\|_{L^2(\mathbf{R})} \leqslant \|w_0\|_{L^2(\mathbf{R})} \exp\left(\frac{1}{2}\int_0^t \|\partial_x v(s,\cdot)\|_{L^\infty(\mathbf{R})} ds\right).$$

From Gagliardo–Nirenberg inequality and Proposition 3.8 we have

$$\begin{aligned}\int_0^t \|\partial_x v(s,\cdot)\|_{L^\infty(\mathbf{R})}\,ds &\leqslant \sqrt{2}\int_0^t \|\partial_x v(s,\cdot)\|^{1/2}_{L^2(\mathbf{R})}\,\|\partial_x^2 v(s,\cdot)\|^{1/2}_{L^2(\mathbf{R})}\,ds\\ &\leqslant \sqrt{2}e^{MT}\|v_0\|_{H^2(\mathbf{R})}t,\end{aligned}$$

where M is a positive constant which depends on T and M_0. Hence for all $t\in[0,T]$,

$$\|Y^t u_0 - Y^t v_0\|_{L^2(\mathbf{R})} \leqslant e^{c_0 t}\|u_0 - v_0\|_{L^2(\mathbf{R})},$$

where c_0 is a positive constant which depends on M_0 and T. Therefore,

$$\|Z^t_L u_0 - Z^t_L v_0\|_{L^2(\mathbf{R})} = \|X^t(Y^t u_0 - Y^t v_0)\|_{L^2(\mathbf{R})} \leqslant e^{\beta_0 t}e^{c_0 t}\|u_0 - v_0\|_{L^2(\mathbf{R})},$$

which completes the proof of this proposition. □

Proposition 4.2 (Local error). *Let $u_0 \in H^3(\mathbf{R})$. There exists $C\left(\|u_0\|_{H^2(\mathbf{R})}\right)$ such that for all $t\in[0,1]$,*

$$\|Z^t_L u_0 - S^t u_0\|_{L^2(\mathbf{R})} \leqslant C\left(\|u_0\|_{H^2(\mathbf{R})}\right) t^2\,\|u_0\|^2_{H^3(\mathbf{R})}. \tag{4.4}$$

Proof. From the definition of Z^t_L and Remark 2.3, we have

$$\begin{aligned} Z^t_L u_0 = X^t Y^t u_0 &= X^t\left(G(t)*u_0 - \frac{1}{2}\int_0^t G(t-s)*\partial_x(Y^s u_0)^2\,ds\right)\\ &= D(t)*G(t)*u_0 - \frac{1}{2}\int_0^t D(t)*G(t-s)*\partial_x(Y^s u_0)^2\,ds\\ &= K(t)*u_0 - \frac{1}{2}\int_0^t D(t)*G(t-s)*\partial_x(Y^s u_0)^2\,ds. \end{aligned} \tag{4.5}$$

Thus, from Duhamel formula for the Fowler equation (1.4) and the Lie formula (4.5), we have:

$$Z_L^t u_0 - S^t u_0 = \frac{1}{2}\int_0^t \partial_x K(t-s) * (S^s u_0)^2\, ds - \frac{1}{2}\int_0^t D(t) * \partial_x G(t-s) * (Y^s u_0)^2\, ds$$

$$= \frac{1}{2}\int_0^t \partial_x K(t-s) * \left((S^s u_0)^2 - (Z_L^s u_0)^2\right)\, ds + R(t), \tag{4.6}$$

where the remainder $R(t)$ is written as

$$R(t) = \frac{1}{2}\int_0^t R_1(s) ds, \quad \text{with } R_1(s) = \partial_x K(t-s) * (Z_L^s u_0)^2 - D(t) * \partial_x G(t-s,\cdot) * (Y^s u_0)^2.$$

Then, from Proposition 2.1, Corollary 3.10 and Lemma 3.11, we have, for $t \in [0,1]$:

$$\begin{aligned}
\|Z_L^t u_0 - S^t u_0\|_{L^2(\mathbf{R})} &\leqslant \frac{1}{2}\int_0^t \|\partial_x K(t-s,\cdot)\|_{L^2(\mathbf{R})} \|(S^s u_0)^2 - (Z_L^s u_0)^2\|_{L^1(\mathbf{R})}\, ds \\
&\quad + \|R(t)\|_{L^2(\mathbf{R})} \\
&\leqslant C\int_0^t (t-s)^{-3/4} \|S^s u_0 - Z_L^s u_0\|_{L^2(\mathbf{R})} \|S^s u_0 + Z_L^s u_0\|_{L^2(\mathbf{R})}\, ds + \|R(t)\|_{L^2(\mathbf{R})} \\
&\leqslant C(e^{\alpha_0 t} + e^{\beta_0 t})\|u_0\|_{L^2(\mathbf{R})} \int_0^t (t-s)^{-3/4} \|S^s u_0 - Z_L^s u_0\|_{L^2(\mathbf{R})}\, ds + \|R(t)\|_{L^2(\mathbf{R})},
\end{aligned}$$

where C is a positive constant. To estimate the remainder, we decompose it as follows

$$R_1(s) \quad = \quad T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned}
T_1 &= K(t-s,\cdot) * \partial_x (Z_L^s u_0)^2 - \partial_x (Z_L^s u_0)^2, \\
T_2 &= \partial_x (Y^s u_0)^2 - G(t-s,\cdot) * \partial_x (Y^s u_0)^2, \\
T_3 &= G(t-s,\cdot) * \partial_x (Y^s u_0)^2 - D(t,\cdot) * G(t-s,\cdot) * \partial_x (Y^s u_0)^2, \\
T_4 &= \partial_x \left(Z_L^s u_0\right)^2 - \partial_x \left(Y^s u_0\right)^2.
\end{aligned}$$

Let us first study the term T_1. From Corollaries 3.5 and 3.10, we have

$$\begin{aligned}
\|T_1\|_{L^2(\mathbf{R})} &= \|K(t-s,\cdot) * \partial_x (Z_L^s u_0)^2 - \partial_x (Z_L^s u_0)^2\|_{L^2(\mathbf{R})} \\
&\leqslant C e^{\alpha_0 (t-s)} (t-s) \|\partial_x (Z_L^s u_0)^2\|_{H^2(\mathbf{R})} \\
&\leqslant C e^{\alpha_0 (t-s)} (t-s) \|Z_L^s u_0\|^2_{H^3(\mathbf{R})} \\
&\leqslant C\left(\|u_0\|_{H^2(\mathbf{R})}\right) e^{\alpha_0 (t-s)} (t-s) \|u_0\|^2_{H^3(\mathbf{R})}.
\end{aligned}$$

In the same way, from Lemmas 3.6 and 3.9, we control the term T_2 as

$$\begin{aligned}
\|T_2\|_{L^2(\mathbf{R})} &= \|\partial_x (Y^s u_0)^2 - G(t-s,\cdot) * \partial_x (Y^s u_0)^2\|_{L^2(\mathbf{R})} \\
&\leqslant \varepsilon\,(t-s) \|\partial_x (Y^s u_0)^2\|_{H^2(\mathbf{R})} \leqslant C\left(\|u_0\|_{H^2(\mathbf{R})}\right)(t-s)\|u_0\|^2_{H^3(\mathbf{R})}.
\end{aligned}$$

From Lemmas 3.4 and 3.9,

$$\begin{aligned}
\|T_3\|_{L^2(\mathbf{R})} &= \|G(t-s,\cdot) * \partial_x (Y^s u_0)^2 - D(t,\cdot) * G(t-s,\cdot) * \partial_x (Y^s u_0)^2\|_{L^2(\mathbf{R})} \\
&\leqslant C\, e^{\beta_0 t}\, t\, \|G(t-s,\cdot) * \partial_x (Y^s u_0)^2\|_{H^2(\mathbf{R})} \\
&\leqslant C\, e^{\beta_0 t}\, t\, \|\partial_x (Y^s u_0)^2\|_{H^2(\mathbf{R})} \\
&\leqslant C\, \left(\|u_0\|_{H^2(\mathbf{R})}\right)\, e^{\beta_0 t}\, t\, \|u_0\|^2_{H^3(\mathbf{R})}.
\end{aligned}$$

For the term T_4, write

$$\begin{aligned}\|T_4\|_{L^2(\mathbf{R})} &= \|\partial_x\left(Z_L^s u_0\right)^2 - \partial_x\left(Y^s u_0\right)^2\|_{L^2(\mathbf{R})}\\ &= 2\|(Z_L^s u_0)\partial_x\left(Z_L^s u_0\right) - (Y^s u_0)\partial_x\left(Y^s u_0\right)\|_{L^2(\mathbf{R})}.\end{aligned}$$

By linearity of the evolution operator X^t, we have

$$\partial_x\left(Z_L^s u_0\right) = X^s \partial_x\left(Y^s u_0\right),$$

hence

$$\begin{aligned}\|T_4\|_{L^2(\mathbf{R})} &= 2\|(Z_L^s u_0)X^s\partial_x\left(Y^s u_0\right) - (Y^s u_0)\partial_x\left(Y^s u_0\right)\|_{L^2(\mathbf{R})}\\ &\leqslant 2\left\|X^s\partial_x\left(Y^s u_0\right)\left(X^s Y^s u_0 - Y^s u_0\right)\right\|_{L^2(\mathbf{R})}\\ &\quad + 2\left\|(Y^s u_0)\left(X^s\partial_x\left(Y^s u_0\right) - \partial_x\left(Y^s u_0\right)\right)\right\|_{L^2(\mathbf{R})}.\end{aligned}$$

Now from Sobolev embedding, Lemmas 3.4 and 3.9, we get:

$$\begin{aligned}\|T_4\|_{L^2(\mathbf{R})} &\leqslant 2\|X^s\partial_x\left(Y^s u_0\right)\|_{L^\infty(\mathbf{R})}\|X^s Y^s u_0 - Y^s u_0\|_{L^2(\mathbf{R})}\\ &\quad + 2\|Y^s u_0\|_{L^\infty(\mathbf{R})}\|X^s\partial_x\left(Y^s u_0\right) - \partial_x\left(Y^s u_0\right)\|_{L^2(\mathbf{R})}\\ &\leqslant C\|X^s\partial_x\left(Y^s u_0\right)\|_{H^1(\mathbf{R})}\, e^{\beta_0 s}\, s\,\|Y^s u_0\|_{H^2(\mathbf{R})}\\ &\quad + C\|Y^s u_0\|_{H^1(\mathbf{R})}\, e^{\beta_0 s}\, s\,\|\partial_x\left(Y^s u_0\right)\|_{H^2(\mathbf{R})}\\ &\leqslant C\, e^{2\beta_0 s}\, s\,\|Y^s u_0\|^2_{H^2(\mathbf{R})} + C\, e^{\beta_0 s}\, s\,\|Y^s u_0\|^2_{H^3(\mathbf{R})}\\ &\leqslant C(\|u_0\|_{H^2(\mathbf{R})})\, e^{2\beta_0 s}\, s\,\|u_0\|^2_{H^3(\mathbf{R})}.\end{aligned}$$

Finally, since $R_1(s) = T_1 + T_2 + T_3 + T_4$ then for $0 \leqslant s \leqslant t \leqslant 1$,

$$\|R_1(s)\|_{L^2(\mathbf{R})} \leqslant C\left(\|u_0\|_{H^2(\mathbf{R})}\right)\, t\,\|u_0\|^2_{H^3(\mathbf{R})},$$

and by integration for $s \in [0, t]$,

$$\|R(t)\|_{L^2(\mathbf{R})} \leqslant C\left(\|u_0\|_{H^2(\mathbf{R})}\right)\, t^2\,\|u_0\|^2_{H^3(\mathbf{R})}.$$

We conclude by applying the modified fractional Lemma 3.2.
For $\phi(t) = \|Z_L^t u_0 - S^t u_0\|_{L^2(\mathbf{R})}$ and $P(t) = C\left(\|u_0\|_{H^2(\mathbf{R})}\right)\, t^2\|u_0\|^2_{H^3(\mathbf{R})}$, we obtain the following estimate

$$\|Z_L^t u_0 - S^t u_0\|_{L^2(\mathbf{R})} \leqslant C\left(\|u_0\|_{H^2(\mathbf{R})}\right) t^2\|u_0\|^2_{H^3(\mathbf{R})},$$

since $\phi(0) = 0$. This completes the proof of the proposition. □

5. Proof of Theorem 1.2

To prove this result we use the argument of Lady Windermere's fan. The approach is now rather standard, and is recalled for the sake of completeness. We use the formula

$$\begin{aligned}(Z_L^{\Delta t})^n u_0 - S^{n\Delta t}u_0 \\ &= \sum_{j=1}^{n}\left((Z_L^{\Delta t})^{n-j-1} Z_L^{\Delta t} S^{(j-1)\Delta t}u_0 - (Z_L^{\Delta t})^{n-j-1} S^{\Delta t} S^{(j-1)\Delta t}u_0\right),\end{aligned}$$

together with the L^2-stability property (see Proposition 4.1) and the estimate of the local error (see Proposition 4.2) established in Section 4. Indeed, the triangle inequality yields

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2} \leqslant \sum_{j=1}^{n} \left\| (Z_L^{\Delta t})^{n-j-1} Z_L^{\Delta t} S^{(j-1)\Delta t} u_0 - (Z_L^{\Delta t})^{n-j-1} S^{\Delta t} S^{(j-1)\Delta t} u_0 \right\|_{L^2}.$$

We next iterate the L^2-stability property, which requires the boundedness in H^2 norm of numerical solutions. This boundedness is ensured by Corollary 3.10, so

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2} \leqslant \sum_{j=1}^{n} e^{c_0(n-j-1)\Delta t} \left\| Z_L^{\Delta t} S^{(j-1)\Delta t} u_0 - S^{\Delta t} S^{(j-1)\Delta t} u_0 \right\|_{L^2},$$

for some uniform c_0. The local error estimate (Proposition 4.2) yields

$$\begin{aligned}
\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2} &\leqslant \sum_{j=1}^{n} e^{c_0(n-j-1)\Delta t} C(m)\,(\Delta t)^2 \left\| S^{(j-1)\Delta t} u_0 \right\|_{L^2} \\
&\leqslant \sum_{j=1}^{n} e^{c_0(n-j-1)\Delta t} \widetilde{C}(m)\,(\Delta t)^2 \\
&\leqslant \widetilde{C}(m)\,(\Delta t)^2 \frac{e^{c_0(n-1)\Delta t}}{1-e^{-c_0\Delta t}} \leqslant C_T(m)\Delta t \frac{\Delta t}{1-e^{-c_0\Delta t}}.
\end{aligned}$$

The result then follows form the estimate $1 - e^{-y} \geqslant cy$ for $y \geqslant 0$.

6. Numerical Experiments

The aim of this section is to numerically verify the Lie method convergence rate in $\mathcal{O}(\Delta t)$ for the Fowler equation (1.1).

To solve the linear sub-equation (1.6), discrete Fourier transform is used and for the non-linear sub-equation (1.7), different numerical approximations can be used. Here, we use the finite difference method.

Since the discrete Fourier transform plays a key role in these schemes, we briefly review its definition, which can be found in most books. In some situation, when the mesh nodes number N is chosen to be $N = 2^p$ for some integer p, a fast Fourier transform (FFT) algorithm is used to further decrease the computation time. In this work we will use a subroutine implemented in Matlab. In this program, the interval $[0, 1]$ is discretized by N equidistant points, with spacing $\Delta x = 1/N$. The spatial grid points are then given by $x_j = j/N, j = 0, ..., N$. If $u_j(t)$ denotes the approximate solution to $u(t, x_j)$, the discrete Fourier transform of the sequence $\{u_j\}_{j=0}^{N-1}$ is defined by

$$\hat{u}(k) = \mathcal{F}_k^d(u_j) = \sum_{j=0}^{N-1} u_j e^{-2i\pi jk/N},$$

for $k = 0, \cdots, N-1$, and the inverse discrete Fourier transform is given by

$$u_j = \mathcal{F}_j^{-d}(\hat{u}_k) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{u}_k e^{2i\pi k x_j},$$

for $j = 0, \cdots, N-1$. Here $\mathcal{F}^d$ denotes the discrete Fourier transform and $\mathcal{F}^{-d}$ its inverse.

In what follows, the linear equation (1.6) is solved using the discrete Fourier transform and time marching is performed exactly according to

$$u_j^{n+1} = \mathcal{F}_j^{-d}\left(e^{-\phi_{\mathcal{I}}(k)\Delta t}\mathcal{F}_k^d(u_j^n)\right). \tag{6.1}$$

To approximate the viscous Burgers' equation (1.7), we use the following explicit centered scheme:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left[\left(\frac{u^2}{2}\right)_{j+1}^n - \left(\frac{u^2}{2}\right)_{j-1}^n\right] + \varepsilon\,\Delta t\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \tag{6.2}$$

which is stable under the CFL-Peclet condition

$$\Delta t = \min\left(\frac{\Delta x}{|v|}, \frac{\Delta x^2}{2\varepsilon}\right), \tag{6.3}$$

where v is an average value of u in the neighbourhood of (t^n, x_j).

Remark 6.1. In the case where the linear sub-equation (1.6) is solved using a finite difference scheme instead of a FFT computation, an additional stability condition is required, see [3].

We have proved that the Lie formulation is of order one in time for initial data in H^3. To perform numerical simulations, we use initial data displayed in Figure 1. To determine

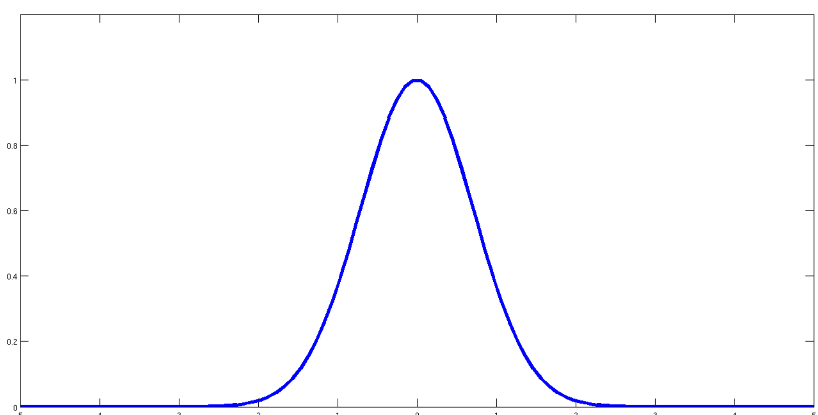


FIGURE 1. Initial data used for numerical experiments.

the numerical order, we consider the following number

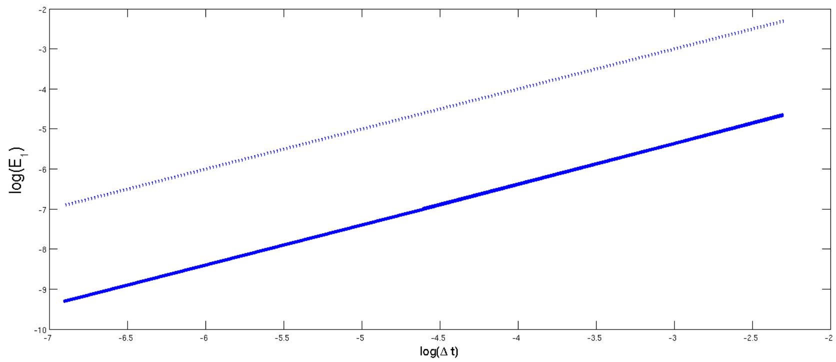
$$E_1 = \frac{1}{N}\sum_{n=0}^{N}|u_j^1(T) - u_j^2(T)|, \tag{6.4}$$

where u^1 and u^2 are, respectively, computed for time steps $\Delta t/2$ and $\Delta t/4$, until the final time T. Hence, the numerical order corresponds to the slope of the $\log(E_1)$ plotted in function of Δt, see Figure 2. Table 1 displays the results for different CFL. We emphasize the fact that both formulas defining a Lie operator, as well as both formulas defining a Strang operator, lead to the same results.

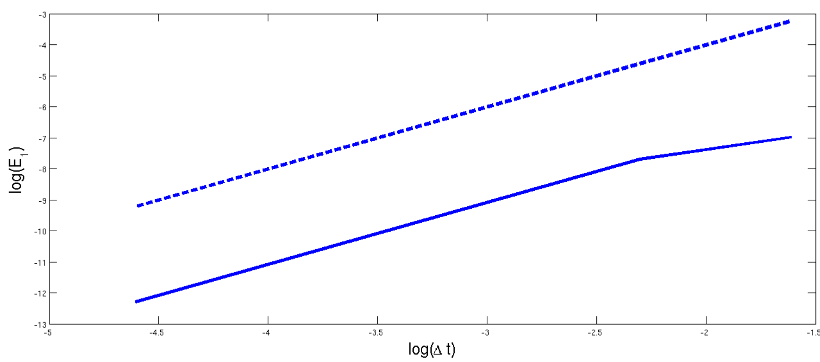
Acknowledgements. The first author is grateful to Pascal Azerad and Bijan Mohammadi for helpful comments.

	Lie	Strang
CFL = 0.5	1.001	2.0081
CFL = 1.0	1.0004	2.0002

TABLE 1. Numerical order of accuracy.



(a) Lie method. Dotted line has slope one.



(b) Strang method. Dotted line has slope two.

FIGURE 2. Numerical convergence rates in Δt and Δt^2.

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