The strong relaxation limit of the multidimensional Euler equations

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Abstract

This paper is devoted to the analysis of global smooth solutions to the multidimensional isentropic Euler equations with stiff relaxation. We show that the asymptotic behavior of the global smooth solution is governed by the porous media equation as the relaxation time tends to zero. The results are proved by combining some classical energy estimates with the so-called Shizuta-Kawashima condition.

Key words: Euler equations, Porous media equation, Relaxation, Initial value problem, Global existence.

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1 Introduction and main results

In this paper we are interested in the following multidimensional Euler equations with relaxation:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \mathbf{u}) &= \quad 0\,, \\ \partial_t (\rho\, \mathbf{u}) + \mathrm{Div}_x (\rho\, \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho) &= -\dfrac{1}{\varepsilon}\, \rho\, \mathbf{u}\,, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a relaxation time. The unknowns $\rho > 0$ and $\mathbf{u} \in \mathbb{R}^n$, which depend on the time variable $t \geq 0$, and the space variable $x \in \mathbb{R}^n$, are the density and velocity of the fluid respectively. The pressure p is related to the density by $p = p(\rho)$ under the classical assumption

$$p'(\rho) > 0\,,\ \forall\, \rho > 0\,. \tag{2}$$

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Note that when $p(\rho) = c^2\,\rho$ with $c > 0$, system (1) describes the isothermal compressible Euler equations, and for the isentropic compressible Euler equations with perfect gas law, the pressure p satisfies $p(\rho) = a\,\rho^\gamma$ with $a > 0$, $\gamma > 1$.

In one space dimension in Lagrangian coordinates, with fixed relaxation time ε, the solutions to isentropic compressible Euler equations follow Darcy's law asymptotically as time tends to infinity, as first proved by Hsiao and Liu in [4]. Then Nishihara [12] succeeded in improving the rates of convergence. After that, such problem was widely studied by many authors, see e.g. among many other works [20, 18, 9, 5, 13, 6] and references therein for further results.

A closely related - though not exactly equivalent - problem is to study the behavior of the solutions to the strong relaxation model as the relaxation time ε in (1) tends to 0. For the isothermal compressible Euler equations, namely $p(\rho) = c^2\,\rho$, by using a stream function, Junca and Rascle in [7] showed that the solutions to the damped isothermal Euler equations converge to those of the heat equation for large BV initial data. In several space dimensions ($n \geq 2$), Coulombel and Goudon in [2] studied the global existence of smooth solution and the convergence of the density to the solution of the heat equation as the relaxation time tends to 0. Stiff relaxation problems were also studied in connection with the approximation of a kinetic equation in [1]. Finally we mention some results for the isentropic Euler equations ($p(\rho) = a\,\rho^\gamma$, $\gamma > 1$). In one space dimension, the derivation of the porous media equation as the asymptotic behavior of the isentropic Euler equations was proved in [11]. In three space dimensions with a fixed relaxation time ε, the global existence of smooth solutions to the isentropic Euler equations (1) with the γ-law was proved in [15] by using an equivalent reformulation of the problem to obtain effective energy estimates. An analogous global existence result in a bounded domain was proved in [13]. However, it is not obvious that energy estimates in [15, 13] can be made uniform with respect to the relaxation time and our first main goal here is to obtain such a uniform existence result. Let us observe that the global existence of smooth solutions also follows from the abstract result of [19] but it is not more clear that this result can be made independent of the relaxation time ε.

As we will see in Remark 2.3, because of the nonlinearity of the pressure law, the analysis of [2] in the isothermal case can not be reproduced directly in the isentropic case. In this paper we extend the results in [2] for the isothermal compressible Euler equations to the general Euler equations. Our only assumption on the pressure law is (2), so our analysis applies in particular to the well-known γ-law for perfect gases but not only. The first aim of this paper is to construct global in time smooth solutions to (1) with small initial data, where the smallness condition is independent on $\varepsilon \in (0, 1]$. Our analysis shows that this result is not linked to any special symmetrization of the system based on algebraic properties of the pressure law. The second goal is to show that the density converges towards the solution of the porous media equation

$$\partial_\tau \rho - \triangle_x p(\rho) = 0\,,$$

on the accelerated time scale $\tau := \varepsilon\, t$. This justifies the approximation of (1) by Darcy's law:

$$\partial_\tau \rho + \nabla_x \cdot \mathbf{m} = 0\,, \quad \mathbf{m} = -\nabla_x p(\rho)\,.$$

Thus our main result is the following theorem concerning the global existence.

Theorem 1.1. *Let $\bar\rho > 0$ be a constant and let $s \in \mathbb{N}$ with $s > n/2 + 1$. There exist two positive constants δ and C such that for any fixed $\varepsilon \in (0, 1]$ and for any initial data $(\rho_0, \mathbf{u}_0)$ verifying $\|\rho_0 - \bar\rho\|_{H^s(\mathbb{R}^n)} \leq \delta$, $\|\mathbf{u}_0\|_{H^s(\mathbb{R}^n)} \leq \delta$, there exists a unique smooth solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ to system (1) with the initial data $(\rho_0, \mathbf{u}_0)$, and that satisfies $(\rho_\varepsilon - \bar\rho, \mathbf{u}_\varepsilon) \in \mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^n))$. Furthermore this* *global solution satisfies the following estimate*

$$\sup_{t\geq 0}\left(\|\rho_\varepsilon(t)-\bar\rho\|^2_{H^s(\mathbb{R}^n)}+\|\mathbf{u}_\varepsilon(t)\|^2_{H^s(\mathbb{R}^n)}\right)+\frac{1}{\varepsilon}\int_0^{+\infty}\|\mathbf{u}_\varepsilon(t)\|^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}t \leq C\left(\|\rho_0-\bar\rho\|^2_{H^s(\mathbb{R}^n)}+\|\mathbf{u}_0\|^2_{H^s(\mathbb{R}^n)}\right).$$

Next we study the convergence of the density ρ_ε in the strong relaxation limit. Here we choose the initial data $(\rho_0,\mathbf{u}_0)$ independent of ε and deal with the rescaled time variable $\tau := \varepsilon\, t$. We have the following theorem:

Theorem 1.2. *Let the assumptions of Theorem 1.1 be fulfilled and let $(\rho_\varepsilon,\mathbf{u}_\varepsilon)$ be the unique global solution obtained in Theorem 1.1 with the initial data $(\rho_0,\mathbf{u}_0)$ independent of ε. We denote by $R(\tau,x)\in\mathcal{C}(\mathbb{R}^+;H^s(\mathbb{R}^n)+\bar\rho)$ the unique solution to the following porous media equation:*

$$\begin{cases}\partial_\tau R-\triangle_x p(R)=0\,,\\ R|_{\tau=0}=\rho_0\,,\end{cases}$$

Define $\varrho_\varepsilon(\tau,x):=\rho_\varepsilon(\tau/\varepsilon,x)$. Then for any $0<T,X<+\infty$ and for any $0<s'<s$, ϱ_ε converges towards R in $\mathcal{C}([0,T];H^{s'}(B_X))$ where $B_X:=\{x\in\mathbb{R}^n/|x|<X\}$.

The proof of Theorem 1.2 does not follow from the exact same arguments than in [2] because the limit equation is nonlinear, while in [2] the limit equation is the heat equation which has a unique solution in the space of distributions (and this property was used in the conclusion of [2]). We thus need to pay some attention and adapt the argument to this nonlinear framework. Nevertheless, many arguments below are similar to those used in [2] and we have found it convenient to refer to this article several times rather than repeating the same arguments. This is mainly due to keep this article as short as possible.

2 Global existence

2.1 Preliminary transformations

It is convenient to introduce the momentum of the fluid $\mathbf{m}:=\rho\,\mathbf{u}$. System (1) can be written as

$$\begin{cases}\partial_t\rho+\nabla_x\cdot\mathbf{m}=0\,,\\ \partial_t\mathbf{m}+\mathrm{Div}\left(\dfrac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right)+\nabla_x p(\rho)=-\dfrac{\mathbf{m}}{\varepsilon},\end{cases}$$

which admits the following entropy

$$\eta(\rho,\mathbf{m}):=\frac{|\mathbf{m}|^2}{2\,\rho}+H(\rho)\,,\quad H''(\rho)=\frac{p'(\rho)}{\rho}\,.$$

By virtue of (2), H is a strictly convex function of ρ. The associated entropy flux is

$$q(\rho,\mathbf{m}):=\left[\frac{|\mathbf{m}|^2}{2\,\rho}+\rho\,H'(\rho)\right]\,\frac{\mathbf{m}}{\rho}\,.$$

Using the classical assumption (2) we have the following

Lemma 2.1. *1. The function $\eta(\rho,\mathbf{m})$ is strictly convex.*

2. For any smooth solution to (1) *away from vacuum, there holds the following equality:*

$$\partial_t \eta(\rho, \mathbf{m}) + \nabla_x \cdot q(\rho, \mathbf{m}) = -\frac{|\mathbf{m}|^2}{\rho\,\varepsilon}\,.$$

The proof of Lemma 2.1 can be obtained by a direct calculation and we omit the details here. Let $\bar\rho > 0$ be the constant given in Theorem 1.1. We introduce the following entropy variables

$$W = \begin{pmatrix} U \\ V \end{pmatrix} := \nabla \eta(\rho, \mathbf{m}) - \nabla \eta(\bar\rho, 0) = \begin{pmatrix} -\dfrac{|\mathbf{u}|^2}{2} + H'(\rho) - H'(\bar\rho) \\ \mathbf{u} \end{pmatrix}\,.$$

By virtue of Lemma (2.1), the mapping $(\rho, \mathbf{m}) \mapsto W$ is a diffeomorphism from $\mathbb{R}^+ \times \mathbb{R}^n$ onto its range (which contains the origin), and the norm $|W|$ is "equivalent" to $|(\rho, \mathbf{m}) - (\bar\rho, 0)|$ provided that $|(\rho, \mathbf{m}) - (\bar\rho, 0)|$ is sufficiently small. Then for smooth solutions $(\rho, \mathbf{m})$ away from vacuum, system (1) is equivalent to the following one in the entropy variable W:

$$A_0(W)\,\partial_t W + \sum_{j=1}^{n} A_j(W)\,\partial_{x_j} W = -\frac{\sigma(\rho)}{\varepsilon} \begin{pmatrix} 0 \\ V \end{pmatrix}\,, \tag{3}$$

with

$$\begin{aligned} A_0(W) \;&:=\; \begin{pmatrix} 1 & V^t \\ V & V \otimes V + \sigma(\rho)\,\mathbb{I}_n \end{pmatrix}, \\ A_j(W) \;&:=\; \begin{pmatrix} V_j & V^t\,V_j + \sigma(\rho)\,e_j^t \\ V\,V_j + \sigma(\rho)\,e_j & V_j\,(V \otimes V + \sigma(\rho)\,\mathbb{I}_n) + \sigma(\rho)\,(V \otimes e_j + e_j \otimes V) \end{pmatrix}, \end{aligned} \tag{4}$$

where $\sigma(\rho) := p'(\rho)$ can be viewed as function of W, $\mathbb{I}_n$ stands for the $n \times n$ unit matrix, and e_j denotes the j-th vector in the canonical basis of $\mathbb{R}^n$.

It is straightforward to check that $A_0(W)$ is symmetric and positive definite, and the matrices $A_j(W)$, $1 \le j \le n$, are symmetric. Furthermore, the matrices $A_0(W)$ and $A_j(W)$, $1 \le j \le n$, are independent of the relaxation time ε.

Remark 2.2. *For isothermal perfect gases, there holds* $H(\rho) = c^2\,\rho\ln\rho$, *and for isentropic perfect gases, there holds* $H(\rho) = \frac{a}{\gamma-1}\,\rho^\gamma$.

Remark 2.3. *Since ρ is a function of $U + |V|^2/2$, i.e of W and not V only, the matrices of coefficients truly depend on W, while for the isothermal compressible Euler equations considered in [2], namely for $p(\rho) = c^2\,\rho$, $\sigma(\rho)$ equals c^2 and the matrices $A_0, \ldots, A_d$ only depend on the relaxed variable V and not on U. This fact was used repeatedly in [2] and it simplifies the analysis of many terms arising in the energy estimates (because the relaxed quantity V has better decay properties than U). Therefore the analysis in the isothermal case cannot be used directly for the isentropic case. This is the crucial difference between the analysis of the two models.*

2.2 Global Well-posedness

This Section is devoted to the global existence result in Theorem 1.1. The local in time existence of smooth solutions may be proven by the standard iterative scheme and a fixed point argument, see e.g. [8, 10]. To show that the solutions of (3) are globally defined, we need further energy estimates.

We fix an integer $s > n/2+1$. Then for any positive time $T > 0$ and for any function $W = (U,V) \in \mathcal{C}([0,T]; H^s(\mathbb{R}^n))$, we define the energy functional

$$N_\varepsilon^2(T) := \sup_{0\le t\le T} \|W(t)\|^2_{H^s(\mathbb{R}^n)} + \frac{1}{\varepsilon}\int_0^T \|V(t)\|^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}t + \varepsilon \int_0^T \|\nabla_x W(t)\|^2_{H^{s-1}(\mathbb{R}^n)}\,\mathrm{d}t\,. \qquad (5)$$

The definition of the energy functional is the same as in [2], but it is different from the one used in [15], where the sound speed was chosen as a new dependent variable in order to symmetrize the system (while here we use the entropy variables).

Remark that since $s > n/2+1$, the Sobolev imbedding theorem yields

$$\|W(t)\|_{W^{1,\infty}(\mathbb{R}^n)} \le C\,N_\varepsilon(T), \quad \forall\, t\in[0,T]\,,$$

for some numerical constant C.

The following Theorem is essential for proving the global existence.

Theorem 2.4. *Assume that the hypothesis of Theorem 1.1 are fulfilled. Let $T > 0$ and let $W \in \mathcal{C}([0,T]; H^s(\mathbb{R}^n))$ be a solution to system (3). Then there exists a non-decreasing continuous function $C : \mathbb{R}^+ \to \mathbb{R}^+$ that is independent of $\varepsilon \in (0,1]$, such that the following nonlinear inequality holds:*

$$\forall\, t\in[0,T], \quad N_\varepsilon^2(t) \le C(N_\varepsilon(t))\left(N_\varepsilon^2(0) + N_\varepsilon^3(t)\right)\,. \qquad (6)$$

From Theorem 2.4 we can get the global existence result of Theorem 1.1 by repeating the (standard) arguments of [2]. Thus it remains to prove Theorem 2.4, which can be done in two steps. Firstly we show the $L^\infty(H^s)$ estimates of W. Secondly, the $L^2(H^{s-1})$ estimates of $\nabla_x W$ are obtained by using the so-called Kawashima-Shizuta stability condition.

2.2.1 $L^\infty(H^s)$ estimates

In this paragraph, we wish to prove the following proposition:

Proposition 2.5. *Under the assumptions stated in Theorem 2.4, there exists a non decreasing function $C : \mathbb{R}^+ \to \mathbb{R}^+$, such that for any $0 \le t \le T$, we have*

$$\|W(t)\|^2_{H^s(\mathbb{R}^n)} + \frac{1}{\varepsilon}\int_0^t \|V(\tau)\|^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}\tau \le C(N_\varepsilon(t))\left(N_\varepsilon(0)^2 + N_\varepsilon(t)^3\right)\,.$$

We begin with the L^2 estimate by using the entropy function η.

Lemma 2.6. *Let the assumptions of Theorem 2.4 be fulfilled. We have*

$$\|W(t)\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{\varepsilon}\int_0^t \|V(\tau)\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}\tau \le C(N_\varepsilon(t))\,N_\varepsilon(0)^2\,. \qquad (7)$$

Proof of Lemma 2.6. It is convenient to define from the entropy η the following relative entropy

$$\tilde\eta(\rho,\mathbf{m}) := \eta(\rho,\mathbf{m}) - \eta(\bar\rho,0) - \partial_\rho\eta(\bar\rho,0)(\rho-\bar\rho)\,.$$

It is a non-negative strictly convex quantity of $(\rho,\mathbf{m})$ which attains its minimum at $(\bar\rho,0)$. More precisely, we have

$$\tilde\eta(\rho,\mathbf{m}) \ge 0\,, \quad \tilde\eta(\bar\rho,0) = 0\,, \quad \nabla\tilde\eta(\bar\rho,0) = 0\,.$$

Since $\tilde{\eta}$ is a strictly convex function of $(\rho, \mathbf{m})$, $\tilde{\eta}(\rho, \mathbf{m})$ is equivalent to the quadratic function $|\rho-\bar{\rho}|^2+|\mathbf{m}|^2$, and then to $|W|^2$, provided $|\rho-\bar{\rho}|+|\mathbf{m}|$ remains in a bounded set. Hence we have

$$c\,|W|^2 \le \tilde{\eta}(\rho,\mathbf{m}) \le C\,|W|^2\,, \tag{8}$$

for some constants c, C depending on $\|(\rho,\mathbf{m})-(\bar{\rho},0)\|_{L^\infty([0,T]\times\mathbb{R}^n)}$ (and thus on $N_\varepsilon(T)$). Accordingly, the entropy flux $q_j(\rho,\mathbf{m})$ is modified as follows

$$\tilde{q}_j(\rho,\mathbf{m}) := q_j(\rho,\mathbf{m}) - \partial_\rho\eta(\bar{\rho},0)\,\mathbf{m}_j\,,$$

and we have

$$\partial_t\tilde{\eta}(\rho,\mathbf{m}) + \sum_{j=1}^n \partial_{x_j}\tilde{q}_j(\rho,\mathbf{m}) = -\frac{1}{\varepsilon}\,\rho\,|\mathbf{u}|^2\,.$$

Integrating this relation over $[0,t]\times\mathbb{R}^n$ yields

$$\int_{\mathbb{R}^n}\tilde{\eta}(\rho,\mathbf{m})\,\mathrm{d}x\Big|_0^t + \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}^n}\rho\,|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau = 0\,.$$

Using (8), we obtain (7) immediately. □

Next the estimates of the space derivatives of W are given as follows.

Lemma 2.7. *Let the assumptions of Theorem 2.4 be fulfilled and $\alpha\in\mathbb{N}^n$ verify $1\le|\alpha|\le s$. We have*

$$\|\partial_x^\alpha W(t)\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\varepsilon}\int_0^t \|\partial_x^\alpha V(\tau)\|^2_{L^2(\mathbb{R}^+)}\,\mathrm{d}\tau \le C(N_\varepsilon(t))\,\left(N_\varepsilon(0)^2 + N_\varepsilon(t)^3\right)\,. \tag{9}$$

Proof of Lemma 2.7. We apply ∂_x^α to the system (3), then take the scalar product with the vector $\partial_x^\alpha W$, and integrate the resulting equality over $[0,t]\times\mathbb{R}^n$. We obtain

$$\frac{1}{2}\int_{\mathbb{R}^n} A_0(W)\,\partial_x^\alpha W\cdot\partial_x^\alpha W\,\mathrm{d}x\Big|_0^t + \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}^n}\sigma(\rho)\,|\partial_x^\alpha V|^2\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t\!\!\int_{\mathbb{R}^n}\left(\frac{I_1+I_2}{2} - I_3 - I_4 - I_5\right)\mathrm{d}x\,\mathrm{d}\tau\,, \tag{10}$$

with

$$I_1 := \partial_t\{A_0(W)\}\,\partial_x^\alpha W\cdot\partial_x^\alpha W\,, \qquad I_2 := \sum_{j=1}^n \partial_{x_j}\{A_j(W)\}\,\partial_x^\alpha W\cdot\partial_x^\alpha W\,,$$

$$I_3 := [\partial_x^\alpha, A_0(W)]\,\partial_t W\cdot\partial_x^\alpha W\,, \qquad I_4 := \sum_{j=1}^n [\partial_x^\alpha, A_j(W)]\,\partial_{x_j} W\cdot\partial_x^\alpha W\,,$$

$$I_5 := \frac{1}{\varepsilon}\,[\partial_x^\alpha, \sigma(\rho)]\,V\cdot\partial_x^\alpha V\,.$$

(Note that for the isothermal model, $\sigma(\rho)$ is a constant and the term I_5 vanishes.)

We first estimate the integrals involving the terms I_1 and I_3. Then we shall estimate all other integrals. From the special form of $A_0(W)$, see (4), we can write the matrix $A_0(W)$ as the sum of two matrices, the first one depending only on V, and the second one having a single nonzero block matrix, namely

$$A_0(W) = \begin{pmatrix} 1 & V^t \\ V & V \otimes V \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma(\rho)\,\mathbb{I}_n \end{pmatrix} =: A_{01}(V) + A_{02}(\rho)\,. \tag{11}$$

From this decomposition of $A_0(W)$, the integral with I_1 can be written as the sum

$$\int_0^t\!\!\int_{\mathbb{R}^n} \partial_t\{A_{01}(V)\}\,\partial_x^\alpha W \cdot \partial_x^\alpha W \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_{\mathbb{R}^n} \partial_t\{A_{02}(W)\}\,\partial_x^\alpha W \cdot \partial_x^\alpha W \,\mathrm{d}x\,\mathrm{d}\tau =: \mathcal{H}_1 + \mathcal{H}_2\,.$$

The term $\mathcal{H}_1$ is exactly the same as the integral with I_1 for the isothermal model, see [2, page 642], since $\mathcal{H}_2$ vanishes in this case. Hence $\mathcal{H}_1$ can be estimated as

$$|\mathcal{H}_1| \le C(N_\varepsilon(t))\,N_\varepsilon^3(t)\,,$$

and we show now an estimate for $\mathcal{H}_2$. Since

$$\partial_t\{A_{02}(W)\} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma'(\rho)\,(D_W\rho(W)\,\partial_t W)\,\mathbb{I}_n \end{pmatrix}\,,$$

the term $\mathcal{H}_2$ can be estimated as

$$\begin{aligned} |\mathcal{H}_2| &\le C(N_\varepsilon(t)) \int_0^t\!\!\int_{\mathbb{R}^n} |\partial_t W|\,|\partial_x^\alpha V|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &\le C(N_\varepsilon(t)) \int_0^t \|\partial_x^\alpha V(\tau)\|^2_{L^2(\mathbb{R}^n)} \left(\|\nabla W(\tau)\|_{L^\infty(\mathbb{R}^n)} + \frac{1}{\varepsilon}\,\|V(\tau)\|_{L^\infty(\mathbb{R}^n)} \right) \mathrm{d}\tau\,, \end{aligned}$$

where we have used (3) to estimate the L^∞ norm of $\partial_t W$. Next we use the Sobolev imbedding Theorem and get

$$|\mathcal{H}_2| \le C(N_\varepsilon(t))\,N_\varepsilon^3(t)\,.$$

Using the estimates of $\mathcal{H}_i$, $i = 1, 2$, we finally get

$$\int_0^t\!\!\int_{\mathbb{R}^n} |I_1| \,\mathrm{d}x\,\mathrm{d}\tau \le C(N_\varepsilon(t))\,N_\varepsilon^3(t)\,. \tag{12}$$

To estimate the integral of I_3, we also use the decomposition (11) of $A_0(W)$. The integral of I_3 can be written as a sum

$$\int_0^t\!\!\int_{\mathbb{R}^n} [\partial_x^\alpha, A_{01}(V)]\,\partial_t W \cdot \partial_x^\alpha W \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_{\mathbb{R}^n} [\partial_x^\alpha, A_{02}(\rho)]\,\partial_t W \cdot \partial_x^\alpha W \,\mathrm{d}x\,\mathrm{d}\tau =: \mathcal{H}_3 + \mathcal{H}_4\,.$$

Once again, the estimate of $\mathcal{H}_3$ is just the same as the estimate of the corresponding integral in the isothermal model, see [2, page 643], and we obtain

$$|\mathcal{H}_3| \le C(N_\varepsilon(t))\,N_\varepsilon(t)^3\,.$$

Let us now focus on the term $\mathcal{H}_4$. By the special form of the matrix $A_{02}(W)$, we have

$$\begin{aligned} |\mathcal{H}_4| &= \left| \int_0^t\!\!\int_{\mathbb{R}^n} [\partial_x^\alpha, \sigma(\rho)]\,\partial_t V \cdot \partial_x^\alpha V \,\mathrm{d}x\,\mathrm{d}\tau \right| \\ &\le \int_0^t \|[\partial_x^\alpha, \sigma(\rho)]\,\partial_t V(\tau)\|_{L^2(\mathbb{R}^n)}\,\|\partial_x^\alpha V(\tau)\|_{L^2(\mathbb{R}^n)} \,\mathrm{d}\tau\,. \end{aligned}$$

Next, we use the classical tame estimate for the commutators (see for instance [10, Proposition 2.1]):

$$\|[\partial_x^\alpha, \sigma(\rho)]\, \partial_t V(\tau)\|_{L^2(\mathbb{R}^n)} \leq C\,\Big(\|\partial_t V(\tau)\|_{L^\infty(\mathbb{R}^n)}\,\|\nabla_x \sigma(\rho)\|_{H^{s-1}(\mathbb{R}^n)} + \|\partial_t V(\tau)\|_{H^{s-1}(\mathbb{R}^n)}\,\|\nabla_x \sigma(\rho)\|_{L^\infty(\mathbb{R}^n)}\Big)\,, \tag{13}$$

for any $0 \leq \tau \leq t$. By the expression of $\partial_t V$ in terms of $\nabla_x W$ and V, see (3), we have

$$\|\partial_t V(\tau)\|_{L^\infty(\mathbb{R}^n)} \leq C(N_\varepsilon(t))\,\left(\|\nabla_x W(\tau)\|_{L^\infty(\mathbb{R}^n)} + \frac{1}{\varepsilon}\,\|V(\tau)\|_{L^\infty(\mathbb{R}^n)}\right)\,,$$

and

$$\|\partial_t V(\tau)\|_{H^{s-1}(\mathbb{R}^n)} \leq C(N_\varepsilon(t))\,\left(\|\nabla_x W(\tau)\|_{H^{s-1}(\mathbb{R}^n)} + \frac{1}{\varepsilon}\,\|V(\tau)\|_{H^{s-1}(\mathbb{R}^n)}\right)\,,$$

where in the last estimate we have used the tame estimate of composed functions, see again [10]. Using the same classical inequalities, we also have

$$\|\nabla_x \sigma(\rho)\|_{L^\infty(\mathbb{R}^n)} \leq C(N_\varepsilon(t))\,\|\nabla_x W(\tau)\|_{L^\infty(\mathbb{R}^n)}\,,$$
$$\|\nabla_x \sigma(\rho)\|_{H^{s-1}(\mathbb{R}^n)} \leq C(N_\varepsilon(t))\,\|\nabla_x W(\tau)\|_{H^{s-1}(\mathbb{R}^n)}\,.$$

Inserting the above estimates in (13) and using Sobolev imbedding theorem, we get

$$\|[\partial_x^\alpha, \sigma(\rho)]\, \partial_t V(\tau)\|_{L^2(\mathbb{R}^n)} \leq C(N_\varepsilon(t))\,\|\nabla_x W(\tau)\|_{H^{s-1}(\mathbb{R}^n)}\,\left(\|\nabla_x W(\tau)\|_{H^{s-1}(\mathbb{R}^n)} + \frac{1}{\varepsilon}\,\|V(\tau)\|_{H^s(\mathbb{R}^n)}\right)\,,$$

and we can conlude:

$$|\mathcal{H}_4| \leq C(N_\varepsilon(t))\, N_\varepsilon^3(t)\,.$$

Using the estimates of both $\mathcal{H}_3$ and $\mathcal{H}_4$, we end up with the following estimates for the integral of I_3:

$$\int_0^t\!\!\int_{\mathbb{R}^n} |I_3|\,\mathrm{d}x\,\mathrm{d}\tau \leq C(N_\varepsilon(t))\, N_\varepsilon^3(t)\,.$$

To estimate the integrals with I_2 and I_4, we can first write the matrix $A_j(W)$ as the sum of two matrices; the first one only depends on V (as in the isothermal case studied in [2]), and the second one is a block matrix with a null block located at the first line and the first column. Then we can obtain similar estimates for I_2 and I_4 to those derived for I_1 and I_3. The derivation of the estimates is actually simpler because I_2 and I_4 only involve spatial derivatives. The details are omitted since they are very similar to [2, page 642-643] and to what we have already done. We end up with

$$\int_0^t\!\!\int_{\mathbb{R}^n} |I_2|\,\mathrm{d}x\,\mathrm{d}\tau \leq C(N_\varepsilon(t))\, N_\varepsilon^3(t)\,, \qquad \int_0^t\!\!\int_{\mathbb{R}^n} |I_4|\,\mathrm{d}x\,\mathrm{d}\tau \leq C(N_\varepsilon(t))\, N_\varepsilon^3(t)\,.$$

Eventually, the term I_5 has a $1/\varepsilon$ factor but it depends quadratically on the relaxed quantity V, so repeating more or less the same commutator estimates as above, see (13), we can derive an estimate for I_5 that has the same form as for the other terms $I_1 - I_4$. We feel free to omit the details in order to shorten the proof.

Under these preliminary results, we can go back to the equality (10). The left-hand side can be easily controlled from below and we have thus completed the proof of Lemma 2.7. □

The proof of Proposition 2.5 is just a matter of combining the estimates (9) with (7).

2.2.2 $L^2(H^{s-1})$ estimates of the function $\nabla_x W$

The next step consists in deriving the $L^2(H^{s-1})$ estimates of $\nabla_x W$. To this end, we follow the method introduced in [14] and further developed in [3, 19] (see also [1, 2] for other applications): we use the skew-symmetry of the compensating matrix in the Fourier space to show the estimate of $\nabla_x W$. We begin with the following existence of a compensating matrix which plays an essential role in this paragraph.

Lemma 2.8. *For any $\xi \in \mathbb{R}^n$, $\xi \neq 0$, let the matrix $K(\xi)$ (compensating matrix) be defined by*

$$K(\xi) := \begin{pmatrix} 0 & \frac{1}{\sigma(\bar\rho)}\,\xi^t/|\xi| \\ -\xi/|\xi| & 0 \end{pmatrix},$$

where $\bar\rho$ is the constant given in Theorem 1.1. Then $K(\xi)\,A_0(0)$ is skew-symmetric and there holds

$$K(\xi)\sum_{1\le j\le n}\xi_j\,A_j(0) = \begin{pmatrix} |\xi| & 0 \\ 0 & -\sigma(\bar\rho)\,\frac{\xi\,\xi^t}{|\xi|} \end{pmatrix}.$$

Proposition 2.9. *Let the assumptions of Theorem 2.4 be fulfilled. Then there exists a nondecreasing function $C : \mathbb{R}^+ \to \mathbb{R}^+$ that is independent of ε and such that, for any $0 \le t \le T$, we have*

$$\varepsilon \int_0^t \|\nabla_x W(\tau)\|^2_{H^{s-1}(\mathbb{R}^n)}\,\mathrm{d}\tau \le C(N_\varepsilon(t))\,\left(N_\varepsilon^2(0) + N_\varepsilon^3(t)\right). \tag{14}$$

Proof of Proposition 2.9. We linearize the system (3) around the constant state $W = 0$:

$$A_0(0)\,\partial_t W + \sum_{1\le j\le n} A_j(0)\,\partial_{x_j} W = -\frac{\sigma(\bar\rho)}{\varepsilon}\begin{pmatrix} 0 \\ V \end{pmatrix} + h,$$

with

$$h := -\sum_{1\le j\le n} A_0(0)\,[A_0^{-1}(W)\,A_j(W) - A_0^{-1}(0)\,A_j(0)]\,\partial_{x_j} W \\ - \frac{1}{\varepsilon}\,A_0(0)\,[\sigma(\rho)\,A_0^{-1}(W) - \sigma(\bar\rho)\,A_0^{-1}(0)]\begin{pmatrix} 0 \\ V \end{pmatrix}.$$

Since $s - 1 > n/2$, the Sobolev space $H^{s-1}(\mathbb{R}^n)$ is an algebra and we have an estimate of the form

$$\|h(t)\|_{H^{s-1}(\mathbb{R}^n)} \le C(N_\varepsilon(t))\left(\|W(t)\|_{H^{s-1}(\mathbb{R}^n)}\,\|\nabla_x W(t)\|_{H^{s-1}(\mathbb{R}^n)} + \frac{1}{\varepsilon}\,\|W(t)\|_{H^{s-1}}\,\|V(t)\|_{H^{s-1}(\mathbb{R}^n)}\right). \tag{15}$$

The definition of the source term h is not exactly the same as in [2] and the estimate of this source term is slightly weaker than the corresponding one in [2]. However, we still have

$$\varepsilon \int_0^t \|h(\tau)\|_{H^{s-1}(\mathbb{R}^n)}\,\mathrm{d}\tau \le C(N_\varepsilon(t))\,N_\varepsilon(t)^3,$$

which can be easily derived from (15) and the definition of the energy functional $N_\varepsilon(t)$.

The properties of our compensating matrix $K(\xi)$ are entirely similar to those of the corresponding one in [2]. So at this stage, the proof of Proposition 2.9 follows from the exact same computations than in [2] (see in particular the estimate (30) of this article). We thus feel free to skip the details. □

The combination of Propositions 2.5 and 2.9 give the result of Theorem 2.4 from which global existence of small smooth global solutions follows. The smallness condition is independent of the relaxation time ε because $N_\varepsilon(0)$ does not depend on ε.

3 Convergence to the porous media equation. Justification of Darcy's law

In §2 we have shown that, for any fixed $\varepsilon \in (0,1]$, if the initial data $(\rho_0, \mathbf{u}_0)$ are independent of ε and assumed to be in a small H^s-neighborhood of the constant state $(\bar{\rho}, 0)$, then the Cauchy problem (1) admits a unique global solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$. Here we use the lower index to denote the dependence of the solution on the relaxation time ε. In this section, we adapt the compactness argument developed in [2] and justify the convergence of the density towards the solution to the porous media equation as the relaxation time ε tends to zero. For this purpose, we introduce the rescaled time variable $\tau := \varepsilon\, t$ and define $(\varrho_\varepsilon, J_\varepsilon)(\tau, x) := (\rho_\varepsilon(t,x), \rho_\varepsilon(t,x)\,\mathbf{u}_\varepsilon(t,x)/\varepsilon)$. The system (1) can be written as

$$\partial_\tau \varrho_\varepsilon + \nabla_x \cdot J_\varepsilon \;=\; 0\,, \tag{16}$$

$$\varepsilon^2 \left[\partial_\tau J_\varepsilon + \mathrm{Div}_x \left(\frac{J_\varepsilon \otimes J_\varepsilon}{\rho_\varepsilon}\right)\right] + \nabla_x p(\varrho_\varepsilon) \;=\; -J_\varepsilon\,. \tag{17}$$

Furthermore the solutions $(\varrho_\varepsilon, J_\varepsilon)$ satisfy the following inequality:

$$\sup_\tau \left(\|\varrho_\varepsilon(\tau) - \bar{\rho}\|^2_{H^s(\mathbb{R}^n)} + \varepsilon^2\, \|J_\varepsilon(\tau)\|^2_{H^s(\mathbb{R}^n)}\right) + \int_0^\infty \|J_\varepsilon(\tau)\|^2_{H^s(\mathbb{R}^n)}\, \mathrm{d}\tau \le C\,, \tag{18}$$

for some constant C depending on the initial data, but that is independent of ε. The proof of Theorem 1.2 is performed in four steps.

Step 1: since ϱ_ε is uniformly bounded from above and from below by positive constants, the bound (18) shows that the sequence $(J_\varepsilon)_{\varepsilon\in(0,1]}$ is bounded in $L^2(\mathbb{R}^+; H^s(\mathbb{R}^n))$, and $(J_\varepsilon \otimes J_\varepsilon/\varrho_\varepsilon)_{\varepsilon\in(0,1]}$ is bounded in $L^1(\mathbb{R}^+; H^s(\mathbb{R}^n))$. Hence we can pass to the limit in (17) in the sense of distributions, that is

$$-J_\varepsilon - \nabla_x p(\varrho_\varepsilon) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^n)\,.$$

Together with (16) we thus get

$$\partial_\tau \varrho_\varepsilon - \triangle_x p(\varrho_\varepsilon) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^n)\,. \tag{19}$$

Step 2: let $T > 0$. Since $(\varrho_\varepsilon - \bar{\rho})$ is bounded in $\mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^n))$, this sequence is also bounded in $L^2(0,T; H^{s-1}(\mathbb{R}^n))$. Furthermore from (16) and (18), we know that $(\partial_\tau \varrho_\varepsilon)$ is bounded in $L^2(\mathbb{R}^+; H^{s-1}(\mathbb{R}^n))$. Thus $(\varrho_\varepsilon - \bar{\rho})$ is bounded in $H^1(0,T; H^{s-1}(\mathbb{R}^n))$ so there exists a subsequence ε_n that tends to 0, and there exists a function R such that $R \in H^1(0,T; \bar{\rho} + H^{s-1}(\mathbb{R}^n))$ and

$$\varrho_{\varepsilon_n} - \bar{\rho} \rightharpoonup R - \bar{\rho} \quad \text{weakly in } H^1(0,T; H^{s-1}(\mathbb{R}^n))\,.$$

Thus $R \in \mathcal{C}([0,T]; \bar{\rho} + H^{s-1}(\mathbb{R}^n))$, and $\varrho_{\varepsilon_n}|_{\tau=0} = \rho_0$ for all n, we obtain $R|_{\tau=0} = \rho_0$ by the same argument as in [2]. In particular we have

$$\varrho_{\varepsilon_n} \rightharpoonup R \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^n)\,. \tag{20}$$

Due to the bound of $(\varrho_\varepsilon - \bar{\rho})$ in $\mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^n))$, we can further assume $R \in L^\infty([0,T]; \bar{\rho} + H^s(\mathbb{R}^n))$.

Step 3: Let $T > 0$, and let $0 < s' < s$. We have seen in Step 2 that $(\partial_\tau \varrho_{\varepsilon_n})$ is bounded in $L^2(\mathbb{R}^+; H^{s-1}(\mathbb{R}^n))$, and moreover $(\varrho_{\varepsilon_n})$ is bounded in $\mathcal{C}(\mathbb{R}^+; H^s(B_X))$ for any $X > 0$. Here B_X denotes the ball $\{x \in \mathbb{R}^n / |x| < X\}$. Then by the compactness result of [16, Corollary 4], we deduce that $(\varrho_{\varepsilon_n})$ is relatively compact in $\mathcal{C}([0,T]; H^{s'}(B_X))$. Up to extracting a new subsequence and using a diagonal process, we can thus assume that $(\varrho_{\varepsilon_n})$ converges strongly in the $\mathcal{C}([0,T]; H^{s'}(B_X))$ topology for any $X > 0$. Thanks to the uniqueness of limits in the sense of distributions, we thus have

$$\forall X > 0\,, \quad \varrho_{\varepsilon_n} \to R \quad \text{in } \mathcal{C}([0,T]; H^{s'}(B_X))\,.$$

Step 4: by the strong convergence property proved in Step 3, we know that $(p(\varrho_{\varepsilon_n}))$ converges towards $p(R)$ in the $\mathcal{C}([0,T]; H^{s'}(B_X))$ topology for all $X > 0$. Using (19), we have therefore shown that R is a solution to the porous media equation

$$\begin{cases} \partial_\tau R - \triangle_x p(R) = 0\,, \\ R|_{\tau=0} = \rho_0\,. \end{cases}$$

Let us recall that R belongs to $\mathcal{C}([0,T]; H^{s-1}(\mathbb{R}^n))$. Classical results on parabolic differential equations, see e.g. [17, chapter 15], show that this equation has a unique solution in the space $\mathcal{C}([0,T]; H^{s-1}(\mathbb{R}^n))$ (here we use $s-1 > n/2$ and $s-1 \geq 1$). By standard arguments, we can therefore conclude that the whole sequence (ϱ_ε) converges weakly towards R in $H^1(0,T; H^{s-1}(\mathbb{R}^n))$, and strongly in the $\mathcal{C}([0,T]; H^{s'}(B_X))$ topology. The proof of Theorem 1.2 is complete.

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