Kernel density estimation for stationary random fields

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Abstract

This paper establishes the asymptotic normality of the Parzen-Rosenblatt density estimator for stationary random fields under natural and easily verifiable conditions. We deal with random fields of the form $X_k = g\left(\varepsilon_{k-s}, s \in \mathbb{Z}^d\right)$, $k \in \mathbb{Z}^d$, where $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ are i.i.d random variables and g is a measurable function. Such kind of spatial processes provides a general framework for stationary ergodic random fields. In particular, in the one-dimensional case, this class of processes includes linear as well as many widely used nonlinear time series models as special cases.

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1 Introduction and main results

The kernel density estimator introduced by Rosenblatt [14] and Parzen [13] has received considerable attention in nonparametric estimation of probability densities for time series. If $(X_i)_{i\in\mathbb{Z}}$ is a stationary sequence of real random variables with a marginal density f then the kernel density estimator of f is defined for any positive integer n and any x in $\mathbb{R}$ by

$$f_n(x) = \frac{1}{nb_n}\sum_{i=1}^{n} \mathrm{K}\left(\frac{x - X_i}{b_n}\right)$$

where K is a probability kernel and the bandwidth b_n is a parameter which converges slowly to zero such that nb_n goes to infinity. The literature dealing with the asymptotic properties of f_n when the observations $(X_i)_{i\in\mathbb{Z}}$ are independent is very extensive (see Silverman [15]). In particular, Parzen [13] proved that when $(X_i)_{i\in\mathbb{Z}}$ is i.i.d. and the bandwidth b_n goes to zero such that nb_n goes to infinity then $(nb_n)^{1/2}(f_n(x_0) -$

$\mathbb{E}f_n(x_0))$ converges in distribution to the normal law with zero mean and variance $f(x_0)\int_{\mathbb{R}}\mathrm{K}^2(t)dt$. Under the same conditions on the bandwidth, this result was recently extended by Wu an Mielniczuk [19] for causal linear processes with i.i.d. innovations and by Dedecker and Merlevède [6] for strongly mixing sequences. In this paper, we are interested by the kernel density estimation problem in the setting of dependent random fields indexed by $\mathbb{Z}^d$ where d is a positive integer. The question is not trivial since $\mathbb{Z}^d$ does not have a natural ordering for $d\geq 2$. In recent years, there is a growing interest in asymptotic properties of kernel density estimators for spatial processes. One can refer for example to Carbon et al. ([2], [3]), Cheng et al. [4], El Machkouri [8], Hallin et al. [10] and Tran [16]. In [16], the asymptotic normality of the kernel density estimator for strongly mixing random fields was obtained using the Bernstein's blocking technique and coupling arguments. Using the same method, the case of linear random fields with i.i.d. innovations was handled in [10]. In [8], the central limit theorem for the Parzen-Rosenblatt estimator given in [16] was improved using the Lindeberg's method (see [12]). In particular, a simple criterion on the strong mixing coefficients is provided and the only condition imposed on the bandwith is $n^d b_n\to\infty$ which is identical to the usual condition imposed in the independent case (see Condition **(A2)** below). It is interesting to note that in [4], the asymptotic normality of the kernel density estimator for linear random fields with i.i.d. innovations is obtained using a martingale approximation method (initiated by Cheng and Ho [5]). To our knowledge, [5] and [4] are the first two papers where a central limit theorem is obtained by martingale techniques in the spatial context. Since the mixing property is often unverifiable and might be too restrictive, it is important to provide limit theorems for nonmixing and possibly nonlinear spatial processes. If d is a positive integer, we consider in this work a field $(X_i)_{i\in\mathbb{Z}^d}$ of identically distributed real random variables with a marginal density f such that

$$X_i=g\left(\varepsilon_{i-s};\,s\in\mathbb{Z}^d\right),\quad i\in\mathbb{Z}^d, \tag{1}$$

where $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ are i.i.d. random variables and g is a measurable function. In the one-dimensional case ($d=1$), the class (1) includes linear as well as many widely used nonlinear time series models as special cases. More importantly, it provides a very general framework for asymptotic theory for statistics of stationary time series (see [17] and the review paper [18]). Let $(\varepsilon_j^{'})_{j\in\mathbb{Z}^d}$ be an i.i.d. copy of $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ and consider for any positive integer n the coupled version X_i^* of X_i defined by $X_i^*=g\left(\varepsilon_{i-s}^*;\,s\in\mathbb{Z}^d\right)$ where $\varepsilon_j^*=\varepsilon_j\,\mathbb{1}_{\{j\neq 0\}}+\varepsilon_0^{'}\,\mathbb{1}_{\{j=0\}}$ for any j in $\mathbb{Z}^d$. Following Wu [17], we introduce appropriate dependence measures: let i in $\mathbb{Z}^d$ and $p > 0$ be fixed. If X_i belongs to $\mathbb{L}_p$, we define the physical dependence measure $\delta_{i,p} = \|X_i - X_i^*\|_p$. In the sequel, we denote δ_i for $\delta_{i,2}$ and we assume that

$$\sum_{i \in \mathbb{Z}^d} |i|^{\frac{5d}{2}} \delta_i < \infty \tag{2}$$

where $|i| = \max_{1 \leq k \leq d} |i_k|$ for any $i = (i_1, ..., i_d) \in \mathbb{Z}^d$. We consider the density estimator of f defined for any positive integer n and any x in $\mathbb{R}$ by

$$f_n(x) = \frac{1}{n^d b_n} \sum_{i \in \Lambda_n} \mathrm{K}\left(\frac{x - X_i}{b_n}\right)$$

where b_n is the bandwidth parameter, Λ_n denotes the set $\{1, ..., n\}^d$ and K is a probability kernel. Our aim is to prove that condition (2) is sufficient for the $\mathbb{L}_1$ distance between f_n and f to converge to zero in probability (Theorem 1) and for $(n^d b_n)^{1/2}(f_n(x_i) - \mathbb{E}f_n(x_i))_{1 \leq i \leq k}$, $(x_i)_{1 \leq i \leq k} \in \mathbb{R}^k$, $k \in \mathbb{N}^*$, to converge in law to a multivariate normal distribution (Theorem 2) under minimal conditions on the bandwidth parameter. We consider the following assumptions:

(A1) K is Lipschitzian, with compact support, $\int \mathrm{K}(u)\, du = 1$ and $\int \mathrm{K}^2(u)\, du < \infty$.

(A2) The bandwidth b_n converges to zero and $n^d b_n$ goes to infinity.

Theorem 1 *Assume that* **(A1)** *and* **(A2)** *hold. If* (2) *is satisfied then*

$$\lim_{n \to +\infty} \mathbb{E} \int_{\mathbb{R}} |f_n(x) - f(x)|\, dx = 0.$$

Remark 1. The above convergence result was obtained also by Hallin et al. ([11], Theorem 2.1) under a more restrictive condition on the bandwith parameter related to the rate of convergence to zero of the stability coefficients $(v(m))_{m \geq 1}$ defined by $v(m) = \|X_0 - \overline{X}_0\|_2^2$ where $\overline{X}_0 = \mathbb{E}(X_0 | \mathcal{H}_m)$ and $\mathcal{H}_m = \sigma(\varepsilon_s\,,\, |s| \leq m)$ with $|s| = \max_{1 \leq k \leq d} |s_k|$ for any $s = (s_1, ..., s_d)$ in $\mathbb{Z}^d$. Arguing as in the proof of Lemma 4 below, one can notice that $v(m) \leq \sum_{|i| > m} \delta_i^2$.

In order to establish the asymptotic normality of f_n, we need additional assumptions:

(A3) The marginal probability distribution of each X_k is absolutely continuous with continuous positive density function f.

(A4) The joint probability distribution of each (X_0, X_k) is absolutely continuous with continuous joint density $f_{0,k}$.

Theorem 2 *Assume that* **(A1)**, **(A2)**, **(A3)** *and* **(A4)** *hold. If* (2) *is satisfied then for any positive integer k and any distinct points $x_1, ..., x_k$ in $\mathbb{R}$,*

$$(n^d b_n)^{1/2} \begin{pmatrix} f_n(x_1) - \mathbb{E} f_n(x_1) \\ \vdots \\ f_n(x_k) - \mathbb{E} f_n(x_k) \end{pmatrix} \xrightarrow[n \to +\infty]{\mathcal{L}} \mathcal{N}(0, \Gamma) \tag{3}$$

where Γ is a diagonal matrix with diagonal elements $\gamma_{ii} = f(x_i) \int_{\mathbb{R}} K^2(u) du$.

Remark 2. A replacement of $\mathbb{E} f_n(x_i)$ by $f(x_i)$ for any $1 \leq i \leq k$ in (3) is a classical problem in density estimation theory. For example, if f is assumed to be Lipschitz and if $\int_{\mathbb{R}} |u||\mathrm{K}(u)|du < \infty$ then $|\mathbb{E} f_n(x_i) - f(x_i)| = O(b_n)$ and thus the centering $\mathbb{E} f_n(x_i)$ may be changed to $f(x_i)$ without affecting the above result provided that $n^d b_n^3$ converges to zero.

Remark 3. If $(X_i)_{i \in \mathbb{Z}^d}$ is a linear random field of the form $X_i = \sum_{j \in \mathbb{Z}^d} a_j \varepsilon_{i-j}$ where $(a_j)_{j \in \mathbb{Z}^d}$ are real numbers such that $\sum_{j \in \mathbb{Z}^d} a_j^2 < \infty$ and $(\varepsilon_j)_{j \in \mathbb{Z}^d}$ are i.i.d. real random variables with zero mean and finite variance then $\delta_i = |a_i| \|\varepsilon_0 - \varepsilon_0^{'}\|_2$ and Theorem 2 holds provided that $\sum_{i \in \mathbb{Z}^d} |i|^{\frac{5d}{2}} |a_i| < \infty$. Cheng et al. [4] obtained a similar result for linear random fields under the condition $\sum_{j \in \mathbb{Z}^d} \sqrt{|a_j|} < \infty$ and a condition on the joint probability distribution of each (X_0, X_k), $k \in \mathbb{Z}^d$ which seems to be very restrictive (see Condition (C2)(ii) in [4]). Finally, Hallin et al. [10] obtained also the same result when $|a_j| = O\left(|j|^{-\gamma}\right)$ with $\gamma > \max\{d + 3, 2d + 0.5\}$ and $n^d b_n^{(2\gamma - 1 + 6d)/(2\gamma - 1 - 4d)}$ goes to infinity. So, in the particular case of linear random fields, our condition (2) is more restrictive than the condition obtained by Hallin et al. [10] but our result is valid for a larger class of random fields (namely, the class of spatial processes of the form (1)) and under only minimal conditions on the bandwidth (i.e. **(A2)** holds).

Remark 4. Let $(\Gamma_n)_{n \geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^d$ such that $|\Gamma_n|$ goes to infinity and $|\partial \Gamma_n|/|\Gamma_n|$ goes to zero as n goes to infinity. A carefull reading of the proof of Theorem 2 shows that the result still hold if the density estimator f_n is defined for any x in $\mathbb{R}$ by

$$f_n(x) = \frac{1}{|\Gamma_n| b_n} \sum_{i \in \Gamma_n} \mathrm{K}\left(\frac{x - X_i}{b_n}\right)$$

and provided that b_n goes to zero and $|\Gamma_n| b_n$ goes to infinity as n goes to infinity.

2 Proofs

In the sequel, for any $i=(i_1,...,i_d)\in\mathbb{Z}^d$, we denote $|i|=\max_{1\leq k\leq d}|i_k|$ and we consider the sequence $(m_n)_{n\geq 1}$ defined by

$$m_n=\max\left\{v_n,\left[\left(\frac{1}{b_n^3}\sum_{|i|>v_n}|i|^{\frac{5d}{2}}\,\delta_i\right)^{\frac{1}{3d}}\right]+1\right\}\tag{4}$$

where $v_n=\left[b_n^{\frac{-1}{2d}}\right]$ and $[\,.\,]$ denotes the integer part function. The following technical lemma is a spatial version of a result by Bosq, Merlevède and Peligrad ([1], pages 88-89). Its proof is postponned to the appendix.

Lemma 1 *If* (2) *holds then*

$$m_n\to\infty,\quad m_n^d b_n\to 0\quad\textit{and}\quad\frac{1}{(m_n^d b_n)^{3/2}}\sum_{|i|>m_n}|i|^{\frac{5d}{2}}\,\delta_i\to 0.$$

For any z in $\mathbb{R}$, we denote

$$\mathrm{K}_i(z)=\mathrm{K}\left(\frac{z-X_i}{b_n}\right),\ \overline{\mathrm{K}}_i(z)=\mathbb{E}\left(\mathrm{K}_i(z)|\mathcal{F}_{n,i}\right)\ \text{and}\ \mathcal{F}_{n,i}=\sigma\left(\varepsilon_{i-s}\,;\,|s|\leq m_n\right)\tag{5}$$

So, denoting $M_n=2m_n+1$, $(\overline{\mathrm{K}}_i(x))_{i\in\mathbb{Z}^d}$ is an M_n-dependent random field (i.e. $\overline{\mathrm{K}}_i(x)$ and $\overline{\mathrm{K}}_j(x)$ are independent as soon as $|i-j|\geq M_n$).

Lemma 2 *For any $p\geq 2$, any positive integer n and any $(a_i)_{i\in\mathbb{Z}^d}$ in $\mathbb{R}^{\mathbb{Z}^d}$,*

$$\left\|\sum_{i\in\Lambda_n}a_i\left(K_i(x)-\overline{K}_i(x)\right)\right\|_p\leq\frac{8m_n^d}{b_n}\left(p\sum_{i\in\Lambda_n}a_i^2\right)^{1/2}\sum_{|i|>m_n}\delta_{i,p}.$$

Proof of Lemma 2. We are going to follow the proof of Proposition 1 in [9]. For any i in $\mathbb{Z}^d$ and any x in $\mathbb{R}$, we denote $R_i=\mathrm{K}_i(x)-\overline{\mathrm{K}}_i(x)$. Since there exists a measurable function h such that $R_i=h(\varepsilon_{i-s}\,;\,s\in\mathbb{Z}^d)$, we denote by $(\delta_{i,p}^{(n)})_{i\in\mathbb{Z}^d}$ the physical dependence measure coefficients associated to the random field $(R_i)_{i\in\mathbb{Z}^d}$. Let $\tau:\mathbb{Z}\to\mathbb{Z}^d$ be a bijection. For any $l\in\mathbb{Z}$, for any $i\in\mathbb{Z}^d$,

$$P_lR_i:=\mathbb{E}(R_i|\mathcal{F}_l)-\mathbb{E}(R_i|\mathcal{F}_{l-1})\tag{6}$$

where $\mathcal{F}_l=\sigma\left(\varepsilon_{\tau(s)};s\leq l\right)$.

Lemma 3 *For any l in $\mathbb{Z}$ and any i in $\mathbb{Z}^d$, we have $\|P_lR_i\|_p\leq\delta_{i-\tau(l),p}^{(n)}$.*

Proof of Lemma 3. Let l in $\mathbb{Z}$ and i in $\mathbb{Z}^d$ be fixed.

$$\begin{aligned}
\|P_lR_i\|_p &= \|\mathbb{E}(R_i|\mathcal{F}_l) - \mathbb{E}(R_i|\mathcal{F}_{l-1})\|_p \\
&= \left\|\mathbb{E}(R_0|T^i\mathcal{F}_l) - \mathbb{E}(R_0|T^i\mathcal{F}_{l-1})\right\|_p \quad \text{where } T^i\mathcal{F}_l = \sigma\left(\varepsilon_{\tau(s)-i}; s \le l\right) \\
&= \left\|\mathbb{E}\left(h\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d}\right)|T^i\mathcal{F}_l\right) - \mathbb{E}\left(h\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\backslash\{i-\tau(l)\}};\varepsilon'_{\tau(l)-i}\right)|T^i\mathcal{F}_l\right)\right\|_p \\
&\le \left\|h\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d}\right) - h\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\backslash\{i-\tau(l)\}};\varepsilon'_{\tau(l)-i}\right)\right\|_p \\
&= \left\|h\left((\varepsilon_{i-\tau(l)-s})_{s\in\mathbb{Z}^d}\right) - h\left((\varepsilon_{i-\tau(l)-s})_{s\in\mathbb{Z}^d\backslash\{i-\tau(l)\}};\varepsilon'_0\right)\right\|_p \\
&= \left\|R_{i-\tau(l)} - R^*_{i-\tau(l)}\right\|_p \\
&= \delta^{(n)}_{i-\tau(l),p}.
\end{aligned}$$

The proof of Lemma 3 is complete. For all i in $\mathbb{Z}^d$, $R_i = \sum_{l\in\mathbb{Z}} P_lR_i$. Consequently,

$$\left\|\sum_{i\in\Lambda_n} a_iR_i\right\|_p = \left\|\sum_{i\in\Lambda_n} a_i\sum_{l\in\mathbb{Z}} P_lR_i\right\|_p = \left\|\sum_{l\in\mathbb{Z}}\sum_{i\in\Lambda_n} a_iP_lR_i\right\|_p.$$

Since $\left(\sum_{i\in\Lambda_n} a_iP_lR_i\right)_{l\in\mathbb{Z}}$ is a martingale-difference sequence, by Burkholder inequality, we have

$$\left\|\sum_{i\in\Lambda_n} a_iR_i\right\|_p \le \left(2p\sum_{l\in\mathbb{Z}}\left\|\sum_{i\in\Lambda_n} a_iP_lR_i\right\|_p^2\right)^{\frac{1}{2}} \le \left(2p\sum_{l\in\mathbb{Z}}\left(\sum_{i\in\Lambda_n}|a_i|\,\|P_lR_i\|_p\right)^2\right)^{\frac{1}{2}} \qquad (7)$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i\in\Lambda_n}|a_i|\,\|P_lR_i\|_p\right)^2 \le \left(\sum_{i\in\Lambda_n} a_i^2\,\|P_lR_i\|_p\right)\times\left(\sum_{i\in\Lambda_n}\|P_lR_i\|_p\right)$$

and by Lemma 3,

$$\sum_{i\in\mathbb{Z}^d}\|P_lR_i\|_p \le \sum_{j\in\mathbb{Z}^d}\delta^{(n)}_{j,p}.$$

So, we obtain

$$\left\|\sum_{i\in\Lambda_n} a_iR_i\right\|_p \le \left(2p\sum_{j\in\mathbb{Z}^d}\delta^{(n)}_{j,p}\sum_{i\in\Lambda_n} a_i^2\sum_{l\in\mathbb{Z}}\|P_lR_i\|_p\right)^{\frac{1}{2}}.$$

Applying again Lemma 3, for any i in $\mathbb{Z}^d$, we have

$$\sum_{l\in\mathbb{Z}}\|P_lR_i\|_p \le \sum_{j\in\mathbb{Z}^d}\delta^{(n)}_{j,p},$$

Finally, we derive

$$\left\|\sum_{i\in\Lambda_n} a_iR_i\right\|_p \leq \left(2p\sum_{i\in\Lambda_n} a_i^2\right)^{\frac{1}{2}} \sum_{i\in\mathbb{Z}^d} \delta_{i,p}^{(n)}$$

where

$$\delta_{i,p}^{(n)} = \|\mathrm{K}_i(x) - \overline{\mathrm{K}}_i(x) - \left(\mathrm{K}_i(x) - \overline{\mathrm{K}}_i(x)\right)^* \|_p.$$

Since $\overline{K}_i^* = \mathbb{E}\left(\mathrm{K}_i^*(x)\middle|\mathcal{F}_{n,i}^*\right)$ where $\mathcal{F}_{n,i}^* = \sigma\left(\varepsilon_{i-s}^*\,;\,|s|\leq m_n\right)$ and $\left(\mathrm{K}_i(x) - \overline{\mathrm{K}}_i(x)\right)^* = \mathrm{K}_i^*(x) - \overline{\mathrm{K}}_i^*(x)$, we derive $\delta_{i,p}^{(n)} \leq 2\|\mathrm{K}_i(x) - \mathrm{K}_i^*(x)\|_p$. Since K is Lipschitzian, we obtain

$$\delta_{i,p}^{(n)} \leq \frac{2\delta_{i,p}}{b_n} \tag{8}$$

where $\delta_{i,p} = \|X_i - X_i^*\|_p$. Morever, we have also $\delta_{i,p}^{(n)} \leq 2\|\mathrm{K}_0(x) - \overline{\mathrm{K}}_0(x)\|_p$. The proof of the following lemma is postponned to the appendix.

Lemma 4 *For any $p\geq 2$, any positive integer n and any x in $\mathbb{R}$,*

$$\|\mathrm{K}_0(x) - \overline{\mathrm{K}}_0(x)\|_p \leq \frac{\sqrt{8p}}{b_n}\sum_{|j|>m_n}\delta_{j,p}.$$

Applying Lemma 4, we derive

$$\delta_{i,p}^{(n)} \leq \frac{2\sqrt{8p}}{b_n}\sum_{|j|>m_n}\delta_{j,p}. \tag{9}$$

Combining (8) and (9), we obtain

$$\sum_{i\in\mathbb{Z}^d}\delta_{i,p}^{(n)} \leq 2\left(\frac{m_n^d\sqrt{8p}}{b_n}\sum_{|j|>m_n}\delta_{j,p} + \frac{1}{b_n}\sum_{|j|>m_n}\delta_{j,p}\right).$$

The proof of Lemma 2 is complete.

2.1 Proof of Theorem 1

The proof follows the same lines of the proof of Theorem 2.1 in [2]. For any positive integer n, denote

$$J_n = \int_{\mathbb{R}} |f_n(x) - f(x)|\,dx.$$

For any positive real A, we have $J_n = J_{n,1}(A) + J_{n,2}(A)$ where

$$J_{n,1}(A) = \int_{|x|>A} |f_n(x) - f(x)|\,dx \quad\text{and}\quad J_{n,2}(A) = \int_{|x|\leq A} |f_n(x) - f(x)|\,dx.$$

Lemma 5 (Hallin et al. (2004)) *For any* $\varepsilon > 0$*, there exists* $N(\varepsilon) > 0$ *such that* $\lim_{n\to+\infty} \mathbb{E}J_{n,1}(A) < \varepsilon$ *for all* A *larger than* $N(\varepsilon)$*.*

Proof of Lemma 5. See Hallin et al. ([11], Lemma 4.1).

Now, $J_{n,2}(A) \leq J_{n,2}^{(1)}(A) + J_{n,2}^{(2)}(A)$ where

$$J_{n,2}^{(1)}(A) = \int_{|x|\leq A} |f_n(x) - \mathbb{E}f_n(x)|\,dx \quad \text{and} \quad J_{n,2}^{(2)} = \int_{\mathbb{R}} |\mathbb{E}f_n(x) - f(x)|\,dx$$

Lemma 6 (Carbon et al. (1996)) $J_{n,2}^{(2)}$ *goes to zero as* n *goes to infinity.*

Proof of Lemma 6. See Lemma 2.1 of Carbon et al. ([2], p. 159) or Lemma 1 of Devroye ([7], p. 897)

So, it suffices to show that $\mathbb{E}J_{n,2}^{(1)}(A)$ goes to zero as n goes to infinity. We consider

$$J_{n,2}^{(1)}(A) = \mathrm{I}_{n,1}(A) + \mathrm{I}_{n,2}(A) + \mathrm{I}_{n,3}(A)$$

where

$$\mathrm{I}_{n,1}(A) = \int_{|x|\leq A} |f_n(x) - \overline{f}_n(x)|\,dx$$
$$\mathrm{I}_{n,2}(A) = \int_{|x|\leq A} |\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x)|\,dx$$
$$\mathrm{I}_{n,3}(A) = \int_{|x|\leq A} |\mathbb{E}\overline{f}_n(x) - \mathbb{E}f_n(x)|\,dx$$

and

$$\overline{f}_n(x) = \frac{1}{n^d b_n} \sum_{i\in\Lambda_n} \overline{\mathrm{K}}_i(x).$$

Lemma 7 *Let* $A > 0$ *be fixed. We have*

$$\mathbb{E}I_{n,i}(A) = O\left(\frac{A}{\sqrt{n^d b_n}}\right) \quad \forall i \in \{1,2,3\}.$$

Proof of Lemma 7. Let n in $\mathbb{N}^*$ and x in $\mathbb{R}$ be fixed. Applying Lemmas 1 and 2, we have

$$\left\|f_n(x) - \overline{f}_n(x)\right\|_2 \leq \frac{8\sqrt{2}}{\sqrt{n^d b_n}(m_n^d b_n)^{3/2}} \sum_{|i|>m_n} |i|^{\frac{5d}{2}} \delta_i = o\left(\frac{1}{\sqrt{n^d b_n}}\right).$$

So, we obtain the result for $\mathbb{E}\mathrm{I}_{n,1}(A)$ and $\mathbb{E}\mathrm{I}_{n,3}(A)$. Now, we are going to control $\mathbb{E}\mathrm{I}_{n,2}(A)$, denoting

$$\overline{Z}_i(x) = \frac{1}{\sqrt{b_n}}\left(\overline{\mathrm{K}}_i(x) - \mathbb{E}\overline{\mathrm{K}}_i(x)\right),$$

then $\left\|\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x)\right\|_2^2$ equals to

$$\frac{1}{n^{2d}b_n}\left(n^d\mathbb{E}(\overline{Z}_0^2(x)) + \sum_{\substack{j\in\mathbb{Z}^d\backslash\{0\}\\ |j|<M_n}} |\Lambda_n \cap (\Lambda_n - j)|\mathbb{E}(\overline{Z}_0(x)\overline{Z}_j(x))\right)$$

where we recall that $M_n = 2m_n + 1$. The proof of the following lemma is done in the appendix.

Lemma 8 *Let x, s and t be fixed in $\mathbb{R}$. Then $\mathbb{E}(\overline{Z}_0^2(x))$ converges to $f(x)\int_{\mathbb{R}} K^2(u)du$ and $\sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| = o(M_n^{-d})$.*

Consequently,

$$\left\|\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x)\right\|_2^2 = O\left(\frac{1}{n^d b_n} + \frac{o(1)}{n^d b_n}\right).$$

The proof of Lemma 7 is complete. Combining Lemmas 5, 6 and 7, we obtain Theorem 1.

2.2 Proof of Theorem 2

Without loss of generality, we consider only the case $k = 2$ and we refer to x_1 and x_2 as x and y ($x \neq y$). Let λ_1 and λ_2 be two constants such that $\lambda_1^2 + \lambda_2^2 = 1$ and note that

$$\lambda_1(n^d b_n)^{1/2}(f_n(x) - \mathbb{E}f_n(x)) + \lambda_2(n^d b_n)^{1/2}(f_n(y) - \mathbb{E}f_n(y)) = \sum_{i\in\Lambda_n}\frac{\Delta_i}{n^{d/2}}$$

$$\lambda_1(n^d b_n)^{1/2}(\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x)) + \lambda_2(n^d b_n)^{1/2}(\overline{f}_n(y) - \mathbb{E}\overline{f}_n(y)) = \sum_{i\in\Lambda_n}\frac{\overline{\Delta}_i}{n^{d/2}}$$

where

$$\Delta_i = \lambda_1 Z_i(x) + \lambda_2 Z_i(y) \quad \text{and} \quad \overline{\Delta}_i = \lambda_1\overline{Z}_i(x) + \lambda_2\overline{Z}_i(y)$$

and for any z in $\mathbb{R}$,

$$Z_i(z) = \frac{1}{\sqrt{b_n}}\left(\mathrm{K}_i(z) - \mathbb{E}\mathrm{K}_i(z)\right) \quad \text{and} \quad \overline{Z}_i(z) = \frac{1}{\sqrt{b_n}}\left(\overline{\mathrm{K}}_i(z) - \mathbb{E}\overline{\mathrm{K}}_i(z)\right)$$

where $\mathrm{K}_i(z)$ and $\overline{\mathrm{K}}_i(z)$ are defined by (5). Applying Lemma 1 and Lemma 2, we know that

$$\frac{1}{n^{d/2}}\left\|\sum_{i\in\Lambda_n}\left(\Delta_i-\overline{\Delta}_i\right)\right\|_2\leq\frac{8\sqrt{2}(\lambda_1+\lambda_2)}{(m_n^d b_n)^{3/2}}\sum_{|i|>m_n}|i|^{\frac{5d}{2}}\delta_i=o(1). \tag{10}$$

So, it suffices to prove the asymptotic normality of the sequence $\left(n^{-d/2}\sum_{i\in\Lambda_n}\overline{\Delta}_i\right)_{n\geq 1}$. We consider the notations

$$\eta=(\lambda_1^2 f(x)+\lambda_2^2 f(y))\sigma^2\quad\text{and}\quad\sigma^2=\int_{\mathbb{R}}\mathrm{K}^2(u)du. \tag{11}$$

The proof of the following technical result is also postponed to the appendix.

Lemma 9 $\mathbb{E}(\overline{\Delta}_0^2)$ *converges to* η *and* $\sup_{i\in\mathbb{Z}^d\backslash\{0\}}\mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i|=o(M_n^{-d})$.

Let p be fixed in $\mathbb{N}^*$. Let φ be a one to one map from $[1,p]\cap\mathbb{N}^*$ to a finite subset of $\mathbb{Z}^d$ and $(\xi_i)_{i\in\mathbb{Z}^d}$ a real random field. For all integers k in $[1,p]$, we denote

$$S_{\varphi(k)}(\xi)=\sum_{i=1}^{k}\xi_{\varphi(i)}\quad\text{and}\quad S^c_{\varphi(k)}(\xi)=\sum_{i=k}^{p}\xi_{\varphi(i)}$$

with the convention $S_{\varphi(0)}(\xi)=S^c_{\varphi(p+1)}(\xi)=0$. To describe the set $\Lambda_n=\{1,...,n\}^d$, we define the one to one map φ from $[1,n^d]\cap\mathbb{N}^*$ to Λ_n by: φ is the unique function such that $\varphi(k)<_{\text{lex}}\varphi(l)$ for $1\leq k<l\leq n^d$ where $<_{\text{lex}}$ denotes the lexicographic order on $\mathbb{Z}^d$. From now on, we consider a field $(\xi_i)_{i\in\mathbb{Z}^d}$ of i.i.d. random variables independent of $(X_i)_{i\in\mathbb{Z}^d}$ such that ξ_0 has the standard normal law $\mathcal{N}(0,1)$. We introduce the fields Y and γ defined for any i in $\mathbb{Z}^d$ by

$$Y_i=\frac{\overline{\Delta}_i}{n^{d/2}}\quad\text{and}\quad\gamma_i=\frac{\xi_i\sqrt{\eta}}{n^{d/2}}$$

where η is defined by (11). Note that Y is an M_n-dependent random field where $M_n=2m_n+1$ and m_n is defined by (4).
Let h be any function from $\mathbb{R}$ to $\mathbb{R}$. For $0\leq k\leq l\leq n^d+1$, we introduce $h_{k,l}(Y)=h(S_{\varphi(k)}(Y)+S^c_{\varphi(l)}(\gamma))$. With the above convention we have that $h_{k,n^d+1}(Y)=h(S_{\varphi(k)}(Y))$ and also $h_{0,l}(Y)=h(S^c_{\varphi(l)}(\gamma))$. In the sequel, we will often write $h_{k,l}$ instead of $h_{k,l}(Y)$. We denote by $B_1^4(\mathbb{R})$ the unit ball of $C_b^4(\mathbb{R})$: h belongs to $B_1^4(\mathbb{R})$ if and only if it belongs to $C^4(\mathbb{R})$ and satisfies $\max_{0\leq i\leq 4}\|h^{(i)}\|_\infty\leq 1$.
It suffices to prove that for all h in $B_1^4(\mathbb{R})$,

$$\mathbb{E}\left(h\left(S_{\varphi(n^d)}(Y)\right)\right)\xrightarrow[n\to+\infty]{}\mathbb{E}\left(h\left(\xi_0\sqrt{\eta}\right)\right).$$

We use Lindeberg's decomposition:

$$\mathbb{E}\left(h\left(S_{\varphi(n^d)}(Y)\right) - h\left(\xi_0\sqrt{\eta}\right)\right) = \sum_{k=1}^{n^d} \mathbb{E}\left(h_{k,k+1} - h_{k-1,k}\right).$$

Now,

$$h_{k,k+1} - h_{k-1,k} = h_{k,k+1} - h_{k-1,k+1} + h_{k-1,k+1} - h_{k-1,k}.$$

Applying Taylor's formula we get that:

$$h_{k,k+1} - h_{k-1,k+1} = Y_{\varphi(k)} h^{'}_{k-1,k+1} + \frac{1}{2} Y^2_{\varphi(k)} h^{''}_{k-1,k+1} + R_k$$

and

$$h_{k-1,k+1} - h_{k-1,k} = -\gamma_{\varphi(k)} h^{'}_{k-1,k+1} - \frac{1}{2} \gamma^2_{\varphi(k)} h^{''}_{k-1,k+1} + r_k$$

where $|R_k| \leq Y^2_{\varphi(k)}(1 \wedge |Y_{\varphi(k)}|)$ and $|r_k| \leq \gamma^2_{\varphi(k)}(1 \wedge |\gamma_{\varphi(k)}|)$. Since $(Y, \xi_i)_{i \neq \varphi(k)}$ is independent of $\xi_{\varphi(k)}$, it follows that

$$\mathbb{E}\left(\gamma_{\varphi(k)} h^{'}_{k-1,k+1}\right) = 0 \quad \text{and} \quad \mathbb{E}\left(\gamma^2_{\varphi(k)} h^{''}_{k-1,k+1}\right) = \mathbb{E}\left(\frac{\eta}{n^d} h^{''}_{k-1,k+1}\right)$$

Hence, we obtain

$$\begin{aligned}\mathbb{E}\left(h(S_{\varphi(n^d)}(Y)) - h\left(\xi_0\sqrt{\eta}\right)\right) &= \sum_{k=1}^{n^d} \mathbb{E}(Y_{\varphi(k)} h^{'}_{k-1,k+1}) \\ &+ \sum_{k=1}^{n^d} \mathbb{E}\left(\left(Y^2_{\varphi(k)} - \frac{\eta}{n^d}\right) \frac{h^{''}_{k-1,k+1}}{2}\right) \\ &+ \sum_{k=1}^{n^d} \mathbb{E}\left(R_k + r_k\right).\end{aligned}$$

Let $1 \leq k \leq n^d$ be fixed. Noting that $\overline{\Delta}_0$ is bounded by $4\|\mathrm{K}\|_\infty/\sqrt{b_n}$ and applying Lemma 9, we derive

$$\mathbb{E}|R_k| \leq \frac{\mathbb{E}|\overline{\Delta}_0|^3}{n^{3d/2}} = O\left(\frac{1}{(n^{3d}\, b_n)^{1/2}}\right)$$

and

$$\mathbb{E}|r_k| \leq \frac{\mathbb{E}|\gamma_0|^3}{n^{3d/2}} \leq \frac{\eta^{3/2}\mathbb{E}|\xi_0|^3}{n^{3d/2}} = O\left(\frac{1}{n^{3d/2}}\right).$$

Consequently, we obtain

$$\sum_{k=1}^{n^d} \mathbb{E}\left(|R_k| + |r_k|\right) = O\left(\frac{1}{(n^d b_n)^{1/2}} + \frac{1}{n^{d/2}}\right) = o(1).$$

Now, it is sufficient to show

$$\lim_{n\to+\infty}\sum_{k=1}^{n^d}\left(\mathbb{E}(Y_{\varphi(k)}h^{'}_{k-1,k+1})+\mathbb{E}\left(\left(Y^2_{\varphi(k)}-\frac{\eta}{n^d}\right)\frac{h^{''}_{k-1,k+1}}{2}\right)\right)=0. \tag{12}$$

On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i=(i_1,...,i_d)$ and $j=(j_1,...,j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i<_{\text{lex}} j$ means that either $i_1<j_1$ or for some p in $\{2,3,...,d\}$, $i_p<j_p$ and $i_q=j_q$ for $1\le q<p$. Let the sets $\{V_i^k\,;\, i\in\mathbb{Z}^d\,,\, k\in\mathbb{N}^*\}$ be defined as follows:

$$V_i^1=\{j\in\mathbb{Z}^d\,;\, j<_{lex} i\},$$

and for $k\ge 2$

$$V_i^k=V_i^1\cap\{j\in\mathbb{Z}^d\,;\,|i-j|\ge k\}\quad\text{where}\quad |i-j|=\max_{1\le k\le d}|i_k-j_k|.$$

First, we focus on $\sum_{k=1}^{n^d}\mathbb{E}\left(Y_{\varphi(k)}h^{'}_{k-1,k+1}\right)$. For all n in $\mathbb{N}^*$ and all integer k in $[1,n^d]$, we define

$$\mathrm{E}_k^{M_n}=\varphi([1,k]\cap\mathbb{N}^*)\cap V^{M_n}_{\varphi(k)}\quad\text{and}\quad S^{M_n}_{\varphi(k)}(Y)=\sum_{i\in\mathrm{E}_k^{M_n}}Y_i.$$

For any function Ψ from $\mathbb{R}$ to $\mathbb{R}$, we define $\Psi^{M_n}_{k-1,l}=\Psi(S^{M_n}_{\varphi(k)}(Y)+S^c_{\varphi(l)}(\gamma))$ (we are going to apply this notation to the successive derivatives of the function h). Our aim is to show that

$$\lim_{n\to+\infty}\sum_{k=1}^{n^d}\mathbb{E}\left(Y_{\varphi(k)}h^{'}_{k-1,k+1}-Y_{\varphi(k)}\left(S_{\varphi(k-1)}(Y)-S^{M_n}_{\varphi(k)}(Y)\right)h^{''}_{k-1,k+1}\right)=0. \tag{13}$$

First, we use the decomposition

$$Y_{\varphi(k)}h^{'}_{k-1,k+1}=Y_{\varphi(k)}h^{'M_n}_{k-1,k+1}+Y_{\varphi(k)}\left(h^{'}_{k-1,k+1}-h^{'M_n}_{k-1,k+1}\right).$$

Applying again Taylor's formula,

$$Y_{\varphi(k)}(h^{'}_{k-1,k+1}-h^{'M_n}_{k-1,k+1})=Y_{\varphi(k)}(S_{\varphi(k-1)}(Y)-S^{M_n}_{\varphi(k)}(Y))h^{''}_{k-1,k+1}+R^{'}_k,$$

where

$$|R^{'}_k|\le 2|Y_{\varphi(k)}(S_{\varphi(k-1)}(Y)-S^{M_n}_{\varphi(k)}(Y))(1\wedge|S_{\varphi(k-1)}(Y)-S^{M_n}_{\varphi(k)}(Y)|)|.$$

Since $(Y_i)_{i\in\mathbb{Z}^d}$ is M_n-dependent, we have

$$\mathbb{E}\left(Y_{\varphi(k)}h^{'M_n}_{k-1,k+1}\right)=0$$

and consequently (13) holds if and only if $\lim_{n\to+\infty}\sum_{k=1}^{n^d}\mathbb{E}|R_k'|=0$. In fact, denoting $W_n=\{-M_n+1,...,M_n-1\}^d$ and $W_n^*=W_n\backslash\{0\}$, it follows that

$$\begin{aligned}\sum_{k=1}^{n^d}\mathbb{E}|R_k'| &\leq 2\mathbb{E}\left(|\overline{\Delta}_0|\left(\sum_{i\in W_n}|\overline{\Delta}_i|\right)\left(1\wedge\frac{1}{n^{d/2}}\sum_{i\in W_n}|\overline{\Delta}_i|\right)\right)\\
&= 2\mathbb{E}\left(\left(\overline{\Delta}_0^2+\sum_{i\in W_n^*}|\overline{\Delta}_0\overline{\Delta}_i|\right)\left(1\wedge\frac{1}{n^{d/2}}\sum_{i\in W_n}|\overline{\Delta}_i|\right)\right)\\
&\leq \frac{2}{n^{d/2}}\sum_{i\in W_n}\mathbb{E}(\overline{\Delta}_0^2|\overline{\Delta}_i|)+2\sum_{i\in W_n^*}\mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i|\\
&\leq \frac{8\|\mathrm{K}\|_\infty}{(n^db_n)^{1/2}}\sum_{i\in W_n}\mathbb{E}(|\overline{\Delta}_0\overline{\Delta}_i|)+2\sum_{i\in W_n^*}\mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i|\quad\text{since } \overline{\Delta}_0\leq\frac{4\|K\|_\infty}{\sqrt{b_n}}\text{ a.s.}\\
&= \frac{8\mathbb{E}(\overline{\Delta}_0^2)\|\mathrm{K}\|_\infty}{(n^db_n)^{1/2}}+2^{d+1}\left(1+\frac{4\|\mathrm{K}\|_\infty}{(n^db_n)^{1/2}}\right)M_n^d\sup_{i\in\mathbb{Z}^d\backslash\{0\}}\mathbb{E}(|\overline{\Delta}_0\overline{\Delta}_i|)\\
&= o(1)\quad\text{(by Lemma 9).}\end{aligned}$$

In order to obtain (12) it remains to control

$$F_0=\mathbb{E}\left(\sum_{k=1}^{n^d}h_{k-1,k+1}''\left(\frac{Y_{\varphi(k)}^2}{2}+Y_{\varphi(k)}\left(S_{\varphi(k-1)}(Y)-S_{\varphi(k)}^{M_n}(Y)\right)-\frac{\eta}{2n^d}\right)\right).$$

Let μ be the law of the stationary i.i.d. real random field $(\varepsilon_k)_{k\in\mathbb{Z}^d}$ and consider the projection π_0 from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$ defined by $\pi_0(\omega)=\omega_0$ and the family of translation operators $(T^k)_{k\in\mathbb{Z}^d}$ from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}^{\mathbb{Z}^d}$ defined by $(T^k(\omega))_i=\omega_{i+k}$ for any $k\in\mathbb{Z}^d$ and any ω in $\mathbb{R}^{\mathbb{Z}^d}$. Denote by $\mathcal{B}$ the Borel σ-algebra of $\mathbb{R}$. The random field $(\pi_0\circ T^k)_{k\in\mathbb{Z}^d}$ defined on the probability space $(\mathbb{R}^{\mathbb{Z}^d},\mathcal{B}^{\mathbb{Z}^d},\mu)$ is stationary with the same law as $(\varepsilon_k)_{k\in\mathbb{Z}^d}$, hence, without loss of generality, one can suppose that $(\Omega,\mathcal{F},\mathbb{P})=(\mathbb{R}^{\mathbb{Z}^d},\mathcal{B}^{\mathbb{Z}^d},\mu)$ and $\varepsilon_k=\pi_0\circ T^k$. Let ρ be the metric defined for any finite subsets Γ_1 and Γ_2 of $\mathbb{Z}^d$ by $\rho(\Gamma_1,\Gamma_2)=\min\{|i-j|\,;\,i\in\Gamma_1,\,j\in\Gamma_2\}$. We consider the following sets:

$$\Lambda_n^{M_n}=\{i\in\Lambda_n\,;\,\rho(\{i\},\partial\Lambda_n)\geq M_n\}\quad\text{and}\quad \mathrm{I}_n^{M_n}=\{1\leq i\leq n^d\,;\,\varphi(i)\in\Lambda_n^{M_n}\},$$

and the function Ψ from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$ such that

$$\Psi(\overline{\Delta})=\overline{\Delta}_0^2+\sum_{i\in V_0^1\cap W_n}2\overline{\Delta}_0\overline{\Delta}_i\quad\text{where } W_n=\{-M_n+1,...,M_n-1\}^d.$$

For $1\leq k\leq n^d$, we set $D_k^{(n)}=\eta-\Psi\circ T^{\varphi(k)}(\overline{\Delta})$. By definition of Ψ and of the set $\mathrm{I}_n^{M_n}$, we have for any k in $\mathrm{I}_n^{M_n}$

$$\Psi\circ T^{\varphi(k)}(\overline{\Delta})=\overline{\Delta}_{\varphi(k)}^2+2\overline{\Delta}_{\varphi(k)}(S_{\varphi(k-1)}(\overline{\Delta})-S_{\varphi(k)}^{M_n}(\overline{\Delta})).$$

Therefore for k in $\mathrm{I}_n^{M_n}$

$$\frac{D_k^{(n)}}{n^d} = \frac{\eta}{n^d} - Y_{\varphi(k)}^2 - 2Y_{\varphi(k)}(S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y)).$$

For any finite subset Γ of $\mathbb{Z}^d$, we denote by $|\Gamma|$ the number of elements in Γ. Since $\lim_{n\to+\infty} n^{-d}|\mathrm{I}_n^{M_n}| = 1$, it remains to consider

$$F_1 = \left|\mathbb{E}\left(\frac{1}{n^d}\sum_{k=1}^{n^d} h''_{k-1,k+1} D_k^{(n)}\right)\right|.$$

Applying Lemma 9, we have

$$\begin{aligned} F_1 &\leq \left|\mathbb{E}\left(\frac{1}{n^d}\sum_{k=1}^{n^d} h''_{k-1,k+1}(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2))\right)\right| + |\eta - \mathbb{E}(\overline{\Delta}_0^2)| + 2\sum_{j\in V_0^1\cap W_n} \mathbb{E}|\overline{\Delta}_0\overline{\Delta}_j| \\ &\leq \left|\mathbb{E}\left(\frac{1}{n^d}\sum_{k=1}^{n^d} h''_{k-1,k+1}(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2))\right)\right| + o(1), \end{aligned}$$

it suffices to prove that

$$F_2 = \left|\mathbb{E}\left(\frac{1}{n^d}\sum_{k=1}^{n^d} h''_{k-1,k+1}(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2))\right)\right|$$

goes to zero as n goes to infinity. In fact,

$$F_2 \leq \frac{1}{n^d}\sum_{k=1}^{n^d} \left(J_k^1(n) + J_k^2(n)\right)$$

where

$$J_k^1(n) = \left|\mathbb{E}\left(h''^{M_n}_{k-1,k+1}\left(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}\left(\overline{\Delta}_0^2\right)\right)\right)\right| = 0$$

since $h''^{M_n}_{k-1,k+1}$ is $\sigma\left(Y_i\,;\,i\in V^{M_n}_{\varphi(k)}\right)$-measurable and

$$\begin{aligned} J_k^2(n) &= \left|\mathbb{E}\left(\left(h''_{k-1,k+1}-h''^{M_n}_{k-1,k+1}\right)\left(\overline{\Delta}^2_{\varphi(k)}-\mathbb{E}\left(\overline{\Delta}^2_0\right)\right)\right)\right| \\ &\leq \mathbb{E}\left(\left(2\wedge\sum_{|i|<M_n}\frac{|\overline{\Delta}_i|}{n^{d/2}}\right)\overline{\Delta}^2_0\right) \\ &\leq \frac{4\|\mathrm{K}\|_\infty\,\mathbb{E}(\overline{\Delta}^2_0)}{(n^d b_n)^{1/2}}+\frac{4\|\mathrm{K}\|_\infty}{(n^d b_n)^{1/2}}\sum_{\substack{|i|<M_n\\ i\neq 0}}\mathbb{E}|\overline{\Delta}_i\overline{\Delta}_0| \quad \text{since } \overline{\Delta}_0\leq\frac{4\|K\|_\infty}{\sqrt{b_n}} \text{ a.s.} \\ &= O\left(\frac{1}{(n^d b_n)^{1/2}}+\frac{M_n^d\sup_{i\in\mathbb{Z}^d\backslash\{0\}}\mathbb{E}(|\overline{\Delta}_0\overline{\Delta}_i|)}{(n^d b_n)^{1/2}}\right) \\ &= o(1) \quad \text{(by Lemma 9).} \end{aligned}$$

The proof of Theorem 2 is complete.

3 Appendix

Proof of Lemma 1. We follow the proof by Bosq, Merlevède and Peligrad ([1], pages 88-89). First, m_n goes to infinity since $v_n=\left[b_n^{-\frac{1}{2d}}\right]$ goes to infinity and $m_n\geq v_n$. For any positive integer m, we consider

$$\psi(m)=\sum_{|i|>m}|i|^{\frac{5d}{2}}\,\delta_i.$$

Since the condition (2) holds, we know that $\psi(m)$ converges to zero as m goes to infinity. Moreover,

$$m_n^d b_n\leq\max\left\{\sqrt{b_n},\psi(v_n)^{1/3}+b_n\right\}\xrightarrow[n\to+\infty]{}0.$$

We have also

$$m_n^d\geq\frac{1}{b_n}\left(\psi\left(v_n\right)\right)^{1/3}\geq\frac{1}{b_n}\left(\psi\left(m_n\right)\right)^{1/3}\quad\text{since } v_n\leq m_n.$$

Finally, we obtain

$$\frac{1}{\left(m_n^d b_n\right)^{3/2}}\sum_{|i|>m_n}|i|^{\frac{5d}{2}}\,\delta_i\leq\sqrt{\psi(m_n)}\xrightarrow[n\to+\infty]{}0.$$

The proof of Lemma 1 is complete.

Proof of Lemma 4 . We consider the sequence $(\Gamma_n)_{n\geq 0}$ of finite subsets of $\mathbb{Z}^d$ defined by $\Gamma_0 = \{(0,...,0)\}$ and for any n in $\mathbb{N}^*$, $\Gamma_n = \{i = (i_1,...,i_d) \in \mathbb{Z}^d\,;\,|i| = n\}$. The cardinality of the set Γ_n is $|\Gamma_n| = 2d(2n+1)^{d-1}$ for $n \geq 1$. Let $\tau : \mathbb{N}^* \to \mathbb{Z}^d$ be the bijection defined by

- $\tau(1) = (0,...,0)$,
- for any n in $\mathbb{N}^*$, if $l \in\,]a_{n-1}, a_n]$ then $\tau(l) \in \Gamma_n$,
- for any n in $\mathbb{N}^*$, if $(p,q) \in\,]a_{n-1}, a_n]^2$ and $p < q$ then $\tau(p) <_{\text{lex}} \tau(q)$

where $a_n = \sum_{j=0}^n |\Gamma_j|$ goes to infinity as n goes to infinity. Let $(m_n)_{n\geq 1}$ be the sequence of positive integers defined by (4). For any n in $\mathbb{N}^*$, we recall that $\mathcal{F}_{n,0} = \sigma\left(\varepsilon_{-s}\,;\,|s| \leq m_n\right)$ (see (5)) and we consider also the σ-algebra $\mathcal{G}_n := \sigma\left(\varepsilon_{\tau(p)}\,;\,1 \leq p \leq n\right)$. By the definition of the bijection τ, for any n in $\mathbb{N}$, $1 \leq p \leq a_n$ if and only if $|\tau(p)| \leq n$. So, we have $\mathcal{G}_{a_{m_n}} = \mathcal{F}_{n,0}$. Consequently,

$$\mathrm{K}_0 - \overline{\mathrm{K}}_0 = \sum_{l > a_{m_n}} \underbrace{\mathbb{E}\left(\mathrm{K}_0|\mathcal{G}_l\right) - \mathbb{E}\left(\mathrm{K}_0|\mathcal{G}_{l-1}\right)}_{D_l}.$$

Since $(D_l)_{l\in\mathbb{Z}}$ is a martingale-difference sequence, we have

$$\|\mathrm{K}_0 - \overline{\mathrm{K}}_0\|_p \leq \left(2p \sum_{l > a_{m_n}} \|D_l\|_p^2\right)^{1/2}.$$

Denoting

$$\mathrm{K}_0' = \mathrm{K}\left(\frac{x - g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\setminus\{-\tau(l)\}}; \varepsilon'_{\tau(l)}\right)}{b_n}\right),$$

we have

$$\begin{aligned}
\|D_l\|_p &= \|\mathbb{E}\left(\mathrm{K}_0|\mathcal{G}_l\right) - \mathbb{E}\left(\mathrm{K}_0'|\mathcal{G}_l\right)\|_p \leq \|\mathrm{K}_0 - \mathrm{K}_0'\|_p \\
&\leq \frac{1}{b_n}\left\|g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d}\right) - g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\setminus\{-\tau(l)\}}; \varepsilon'_{\tau(l)}\right)\right\|_p \\
&= \frac{1}{b_n}\left\|g\left((\varepsilon_{-\tau(l)-s})_{s\in\mathbb{Z}^d}\right) - g\left((\varepsilon_{-\tau(l)-s})_{s\in\mathbb{Z}^d\setminus\{-\tau(l)\}}; \varepsilon'_0\right)\right\|_p \\
&= \frac{1}{b_n}\left\|X_{-\tau(l)} - X^*_{-\tau(l)}\right\|_p = \frac{\delta_{-\tau(l),p}}{b_n}.
\end{aligned}$$

Consequently, we obtain

$$\|\mathrm{K}_0-\overline{\mathrm{K}}_0\|_p \leq \frac{1}{b_n}\left(2p\sum_{l>a_{m_n}}\delta^2_{-\tau(l),p}\right)^{1/2} \leq \frac{\sqrt{2p}}{b_n}\sum_{|j|>m_n}\delta_{j,p}.$$

The proof of Lemma 4 is complete.

Proof of Lemma 8. For any z in $\mathbb{R}$, we have

$$\mathbb{E}\mathrm{K}_0^2(z) = b_n\int_{\mathbb{R}}\mathrm{K}^2(v)f(z-vb_n)dv = O(b_n). \tag{14}$$

Let s and t be fixed in $\mathbb{R}$. We have

$$\left|\mathbb{E}\left(\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_0(t)\right)-\mathbb{E}\left(\mathrm{K}_0(s)\mathrm{K}_0(t)\right)\right| \leq \|\mathrm{K}_0(s)\|_2\|\mathrm{K}_0(t)-\overline{\mathrm{K}}_0(t)\|_2+\|\overline{\mathrm{K}}_0(t)\|_2\|\mathrm{K}_0(s)-\overline{\mathrm{K}}_0(s)\|_2.$$

In the sequel, the letter C will denote constants whose values are not important. Using (14) and Lemma 4, we have

$$\left|\mathbb{E}\left(\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_0(t)\right)-\mathbb{E}\left(\mathrm{K}_0(s)\mathrm{K}_0(t)\right)\right| \leq \frac{C}{\sqrt{b_n}}\sum_{|j|>m_n}\delta_j.$$

Since $b_n|\mathbb{E}(Z_0(s)Z_0(t))-\mathbb{E}(\overline{Z}_0(s)\overline{Z}_0(t)| = \left|\mathbb{E}\left(\mathrm{K}_0(s)\mathrm{K}_0(t)\right)-\mathbb{E}\left(\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_0(t)\right)\right|$, we have

$$M_n^d|\mathbb{E}(Z_0(s)Z_0(t))-\mathbb{E}(\overline{Z}_0(s)\overline{Z}_0(t)| \leq \frac{C}{(m_n^d b_n)^{3/2}}\sum_{|j|>m_n}|j|^{\frac{5d}{2}}\delta_j. \tag{15}$$

Moreover, keeping in mind Assumptions **(A2)** and **(A3)**, we have

$$\lim_n\frac{1}{b_n}\mathbb{E}\left(\mathrm{K}_0(s)\mathrm{K}_0(t)\right) = \lim_n\int_{\mathbb{R}}\mathrm{K}\left(v\right)\mathrm{K}\left(v+\frac{t-s}{b_n}\right)f(s-vb_n)dv = u(s,t)\,f(s)\int_{\mathbb{R}}\mathrm{K}^2(u)du \tag{16}$$

where $u(s,t)=1$ if $s=t$ and $u(s,t)=0$ if $s\neq t$. We have also

$$\lim_n\frac{1}{b_n}\mathbb{E}\mathrm{K}_0(s)\mathbb{E}\mathrm{K}_0(t) = \lim_n b_n\int_{\mathbb{R}}\mathrm{K}(v)f(s-vb_n)dv\int_{\mathbb{R}}\mathrm{K}(w)f(t-wb_n)dw = 0. \tag{17}$$

In the other part, let $i\neq 0$ be fixed in $\mathbb{Z}^d$. We have

$$\mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| \leq \frac{1}{b_n}\mathbb{E}\left|\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_i(t)\right|+\frac{3}{b_n}\mathbb{E}\left|\mathrm{K}_0(s)\right|\mathbb{E}\left|\mathrm{K}_0(t)\right|. \tag{18}$$

Keeping in mind that $||\alpha|-|\beta||\leq|\alpha-\beta|$ for any (α,β) in $\mathbb{R}^2$ and applying the Cauchy-Schwarz inequality, we obtain

$$\left|\mathbb{E}|\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_i(t)|-\mathbb{E}|\mathrm{K}_0(s)\mathrm{K}_i(t)|\right| \leq \|\overline{\mathrm{K}}_0(s)\|_2\|\overline{\mathrm{K}}_0(t)-\mathrm{K}_0(t)\|_2+\|\mathrm{K}_0(t)\|_2\|\overline{\mathrm{K}}_0(s)-\mathrm{K}_0(s)\|_2$$

Using (14) and Lemma 4, we obtain

$$\frac{M_n^d}{b_n}\big|\mathbb{E}|\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_i(t)| - \mathbb{E}|\mathrm{K}_0(s)\mathrm{K}_i(t)|\big| \leq \frac{C}{(m_n^d b_n)^{3/2}}\sum_{|j|>m_n}|j|^{\frac{5d}{2}}\delta_j. \tag{19}$$

Since Assumptions **(A3)** and **(A4)** hold and $M_n^d b_n = o(1)$, we have

$$\frac{M_n^d}{b_n}\mathbb{E}\big|\mathrm{K}_0(s)\mathrm{K}_i(t)\big| = M_n^d b_n \iint_{\mathbb{R}^2}\big|\mathrm{K}\left(w_1\right)\mathrm{K}\left(w_2\right)\big|f_{0,i}(s-w_1b_n, t-w_2b_n)dw_1dw_2 = o(1) \tag{20}$$

and

$$\frac{M_n^d}{b_n}\mathbb{E}\big|\mathrm{K}_0(s)\big|\,\mathbb{E}\big|\mathrm{K}_0(t)\big| = M_n^d b_n\int_{\mathbb{R}}|\mathrm{K}(u)|f(s-ub_n)du\int_{\mathbb{R}}|\mathrm{K}(v)|f(t-vb_n)dv = o(1). \tag{21}$$

Combining (18), (19), (20), (21) and Lemma 1, we obtain

$$M_n^d \sup_{i\in\mathbb{Z}^d\backslash\{0\}}\mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| = o(1). \tag{22}$$

Let x be fixed in $\mathbb{R}$. Choosing $s=t=x$ and combining (15), (16), (17) and Lemma 1, we obtain $\mathbb{E}(\overline{Z}_0^2(x))$ goes to $f(x)\int_{\mathbb{R}}\mathrm{K}^2(u)du$ as n goes to infinity. The proof of Lemma 8 is complete.

Proof of Lemma 9. Let x and y be two distinct real numbers. Noting that

$$\mathbb{E}(\Delta_0^2) = \lambda_1^2\mathbb{E}(Z_0^2(x)) + \lambda_2^2\mathbb{E}(Z_0^2(y)) + 2\lambda_1\lambda_2\mathbb{E}(Z_0(x)Z_0(y))$$
$$\mathbb{E}(\overline{\Delta}_0^2) = \lambda_1^2\mathbb{E}(\overline{Z}_0^2(x)) + \lambda_2^2\mathbb{E}(\overline{Z}_0^2(y)) + 2\lambda_1\lambda_2\mathbb{E}(\overline{Z}_0(x)\overline{Z}_0(y))$$

and using (15) and Lemma 1, we obtain

$$\lim_{n\to+\infty} M_n^d|\mathbb{E}(\Delta_0^2) - \mathbb{E}(\overline{\Delta}_0^2)| = 0. \tag{23}$$

Combining (16) and (23), we derive that $\mathbb{E}(\overline{\Delta}_0^2)$ converges to $\eta = (\lambda_1^2 f(x) + \lambda_2^2 f(y))\int_{\mathbb{R}}\mathrm{K}^2(u)du$. Let $i\neq 0$ be fixed in $\mathbb{Z}^d$. Combining (22) and

$$\mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i| \leq \lambda_1^2\mathbb{E}|\overline{Z}_0(x)\overline{Z}_i(x)| + \lambda_2^2\mathbb{E}|\overline{Z}_0(y)\overline{Z}_i(y)| + \lambda_1\lambda_2\mathbb{E}|\overline{Z}_0(x)\overline{Z}_i(y)| + \lambda_1\lambda_2\mathbb{E}|\overline{Z}_0(y)\overline{Z}_i(x)|, \tag{24}$$

we obtain $M_n^d\sup_{i\in\mathbb{Z}^d\backslash\{0\}}\mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i| = o(1)$. The proof of Lemma 9 is complete.

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