Characterization and Detection of Loops in n-Dimensional Discrete Toric Spaces

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Abstract. Toric spaces being non-simply connected, it is possible to find in such spaces some loops which are not homotopic to a point: we call them *toric loops*. Some applications, such as the study of the relationship between the geometrical characteristics of a material and its physical properties, rely on three dimensional discrete toric spaces and require detecting objects having a toric loop.

In this work, we study objects embedded in discrete toric spaces, and propose a new definition of loops and equivalence of loops. Moreover, we introduce a characteristic of loops that we call *wrapping vector*: relying on this notion, we propose a linear time algorithm which detects whether an object has a toric loop or not.

1 Introduction

Topology is used in various domains of image processing in order to perform geometric analysis of objects. In porous material analysis, different topological transformations, such as skeletonisation, are used to study the relationships between the geometrical characteristics of a material and its physical properties.

When simulating a fluid flow through a porous material, the whole material can be approximated by the tessellation of the space made up by copies of one of its samples, under the condition that the volume of the sample is superior to the so-called Representative Elementary Volume (REV) of the material [1]. When the whole Euclidean space is tiled this way, one can remark that the result of the fluid flow simulation is itself the tessellation of the local flow obtained inside any copy of the sample (see Fig. 1-**a**). When considering the flow obtained inside the sample, it appears that the flow leaving the sample by one side comes back by the opposite side (see Fig. 1-**b**). Thus, it is possible to perform the fluid flow simulation only on the sample, under the condition that its opposite sides are joined: with this construction, the sample is embedded inside a toric space [2] [3]. In order to perform geometric analysis of fluid flow through porous materials, we therefore use topological tools adapted to toric spaces.

Considering the sample inside a toric space leads to new difficulties. In a real fluid flow, grains of a material (pieces of the material which are not connected with the borders of the sample) do not have any effect on the final results, as

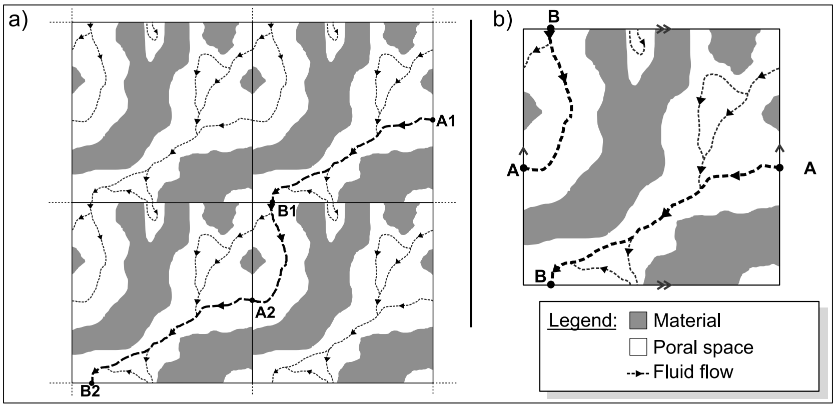

Fig. 1. Simulating a fluid flow - When simulating a fluid flow, a porous material (in gray) can be approximated by the tessellation of one of its samples (see **a**). When the results of the simulation are obtained (the dotted lines), one can see that the fluid flow through the mosaic is the tessellation of the fluid flow simulation results obtained in one sample. For example, one can look at the bold dotted line in **a)**: the flow going from $A1$ to $B1$ is the same than the flow going from $A2$ to $B2$. It is therefore possible to perform the fluid flow simulation through only one sample and, in order to obtain the same results than in **a)**, connect the opposite sides of the sample (see **b**): the sample is embedded inside a toric space.

these grains eventually either evacuate the object with the flow or get blocked and connect with the rest of the material. Thus, before performing a fluid flow simulation on a sample, it is necessary to remove its grains (typically, in a finite subset S of $\mathbb{Z}^n$, a grain is a connected component which does not 'touch' the borders of S). However, characterizing a grain inside a toric space, which does not have any border, is more difficult than in $\mathbb{Z}^n$. On the contrary of the discrete space $\mathbb{Z}^n$, n-dimensional discrete toric spaces are not simply connected spaces [3]: some loops, called *toric loops*, are not homotopic to a point (this can be easily seen when considering a 2D torus). In a toric space, a connected component may be considered as a grain if it contains no toric loop. Indeed, when considering a sample embedded inside a toric space, and a tessellation of the Euclidean space made up by copies of this sample, one can remark that the connected components of the sample which do not contain toric loops produce grains in the tessellation, while the connected components containing toric loops cannot be considered as grains in the tiling (see Fig. 2).

In this work, we give a new definition of loops and homotopy class, adapted to n-dimensional discrete toric spaces. Relying on these notions, we also introduce *wrapping vectors*, a new characteristic of loops in toric spaces which is the same for all homotopic loops. Thanks to wrapping vectors, we give a linear time algorithm which allows to decide whether an n-dimensional object contains a toric loop or not.

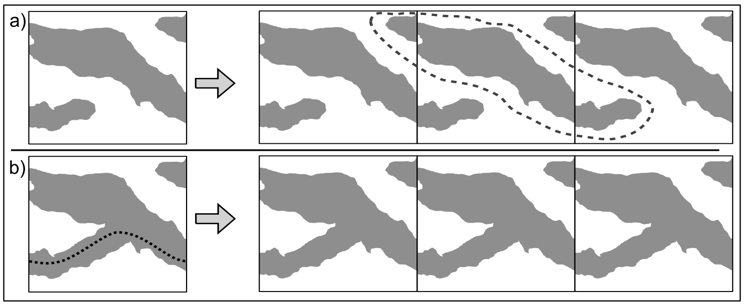

Fig. 2. Grains in toric spaces - The image in **a)** contains no grain based on the 'border criterion'; when the Euclidean space is tessellated with copies of the image, grains appear (the circled connected component is an example of grain). In **b)**, the connected component has toric loops (e.g. the dotted line) and when the Euclidean space is tessellated with copies of the image, no grain appear.

This paper is an extension of a paper submitted for a conference [4]. In addition, it contains an algorithm which not only allows to detect when an object contains a toric loop (as the algorithm proposed in [4]) but also allows to build a basis characterizing all toric loops contained in an object. Furthermore, it contains a comparison between loop homotopy defined in this article and loop equivalence defined in [5].

2 Basic Notions

2.1 Discrete Toric Spaces

A n-dimensional torus is classically defined as the direct product of n circles (see [2]). In the following, we give a discrete definition of toric space, based on modular arithmetic (see [6]).

Given d a positive integer, we set $\mathbb{Z}_d = \{0, ..., d-1\}$. We denote by $\oplus_d$ the operation such that for all $a, b \in \mathbb{Z}$, $(a \oplus_d b)$ is the element of $\mathbb{Z}_d$ congruent to $(a+b)$ modulo d. We point out that $(\mathbb{Z}_d, \oplus_d)$ forms a cyclic group of order d.

Let n be a positive integer, $\boldsymbol{d} = (d_1, ..., d_n) \in \mathbb{N}^n$, and $\mathbb{T}^n = \mathbb{Z}_{d_1} \times ... \times \mathbb{Z}_{d_n}$, we denote by $\oplus_{\boldsymbol{d}}$ the operation such that for all $\boldsymbol{a} = (a_1, ..., a_n) \in \mathbb{Z}^n$ and $\boldsymbol{b} = (b_1, ..., b_n) \in \mathbb{Z}^n$, $\boldsymbol{a} \oplus_{\boldsymbol{d}} \boldsymbol{b} = (a_1 \oplus_{d_1} b_1, ..., a_n \oplus_{d_n} b_n)$. The group $(\mathbb{T}^n, \oplus_d)$ is the direct product of the n groups $(\mathbb{Z}_{d_i}, \oplus_{d_i})_{(1 \leq i \leq n)}$, and is an *n-dimensional discrete toric space* [2].

The scalar d_i is *the size of the i-th dimension* of $\mathbb{T}^n$, and $\boldsymbol{d}$ is the *size (vector)* of $\mathbb{T}^n$. For simplicity, the operation $\oplus_{\boldsymbol{d}}$ will be also denoted by $\oplus$.

2.2 Neighbourhoods in Toric Spaces

As in $\mathbb{Z}^n$, various adjacency relations may be defined in a toric space.

Definition 1. *A vector* $\boldsymbol{s} = (s_1, ..., s_n)$ *of* $\mathbb{Z}^n$ *is an* m-step $(0 < m \leq n)$ *if, for all* $i \in [1;n], s_i \in \{-1, 0, 1\}$ *and* $\sum_{i=1}^{n} |s_i| \leq m$.

Two points $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{T}^n$ *are* m-adjacent *if there exists an m-step* $\boldsymbol{s}$ *such that* $\boldsymbol{a} \oplus \boldsymbol{s} = \boldsymbol{b}$.

In 2D, the 1- and 2-adjacency relations respectively correspond to the 4- and 8-neighbourhood [7] adapted to bidimensional toric spaces. In 3D, the 1-, 2- and 3-adjacency relations can be respectively seen as the 6-, 18- and 26-neighbourhood [7] adapted to three-dimensional toric spaces.

Based on the *m-adjacency relation* previously defined, we can introduce the notion of *m-connectedness.*

Definition 2. *A set of points* X *of* $\mathbb{T}^n$ *is* m-connected *if, for all* $\boldsymbol{a}, \boldsymbol{b} \in X$, *there exists a sequence* $(\boldsymbol{x_1}, ..., \boldsymbol{x_k})$ *of elements of* X *such that* $\boldsymbol{x_1} = \boldsymbol{a}$, $\boldsymbol{x_k} = \boldsymbol{b}$ *and for all* $i \in [1; k-1]$, $\boldsymbol{x_i}$ *and* $\boldsymbol{x_{i+1}}$ *are m-adjacent.*

2.3 Loops in Toric Spaces

Classically, in $\mathbb{Z}^n$, an m-loop is defined as a sequence of m-adjacent points such that the first point and the last point of the sequence are equal [5]. However, this definition does not suit discrete toric spaces: in small discrete toric spaces, two different loops can be written as the same sequence of points, as shown in the following example.

Example 3. Let us consider the bidimensional toric space $\mathbb{T}^2 = \mathbb{Z}_3 \times \mathbb{Z}_2$, and the 2-adjacency relation on $\mathbb{T}^2$. Let us also consider $\boldsymbol{x_1} = (1, 0)$ and $\boldsymbol{x_2} = (1, 1)$ in $\mathbb{T}^2$.

There are two ways of interpreting the sequence of points $\mathcal{L} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_1})$ as a loop of $\mathbb{T}^2$: either $\mathcal{L}$ is the loop passing by $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ and doing a 'u-turn' to come back to $\boldsymbol{x_1}$, or $\mathcal{L}$ is the loop passing by $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, and 'wrapping around' the toric space in order to reach $\boldsymbol{x_1}$ without making any 'u-turn', as shown on Fig. 3.

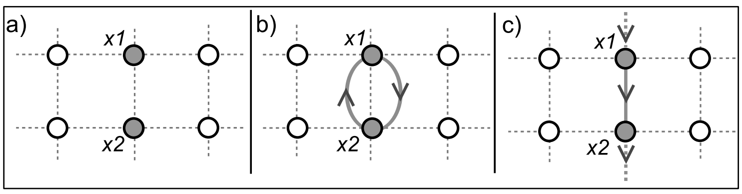

Fig. 3. Loops in toric spaces - In the toric space $\mathbb{Z}_3 \times \mathbb{Z}_2$ (see **a**), the sequence of points $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_1})$ can be interpreted in two different ways: **b)** and **c)**.

Thus, when considering discrete toric spaces, loops cannot be considered as sequences of points as it can lead to such ambiguities. This is why we propose the following definition.

Definition 4. *Given* $\boldsymbol{p} \in \mathbb{T}^n$*, an* m-loop (of base point $\boldsymbol{p}$) *is a pair* $\mathcal{L} = (\boldsymbol{p}, V)$*, where* $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$ *is a sequence of m-steps such that* $(\boldsymbol{p} \oplus \boldsymbol{v_1} \oplus ... \oplus \boldsymbol{v_k}) = \boldsymbol{p}$.
The number k *is the* length of $\mathcal{L}$.
We call i-th point of $\mathcal{L}$*, with* $1 \leq i \leq k+1$*, the point* $(\boldsymbol{p} \oplus \boldsymbol{v_1} \oplus ... \oplus \boldsymbol{v_{i-1}})$.
The loop $(\boldsymbol{p}, ())$ *is called the* trivial loop of base point $\boldsymbol{p}$.

Remark 5. In the previous definition, the $(k+1)$-th point of $\mathcal{L}$ is $\boldsymbol{p}$, and has been defined in order to make some propositions and proofs more simple.

The ambiguity pinpointed in Ex. 3 is removed with this definition of loops: let $\boldsymbol{v}$ be the vector $(0, 1)$, the loop passing by $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ and making a u-turn is $(\boldsymbol{x_1}, (\boldsymbol{v}, -\boldsymbol{v}))$ (see Fig. 3-**b**), while the loop wrapping around the toric space is $(\boldsymbol{x_1}, (\boldsymbol{v}, \boldsymbol{v}))$ (see Fig. 3-**c**).

3 Loop Homotopy in Toric Spaces

3.1 Homotopic Loops

In this section, we define an equivalence relation between loops, corresponding to an homotopy, inside a discrete toric space. An equivalence relation between loops inside $\mathbb{Z}^2$ and $\mathbb{Z}^3$ has been defined in [5], however, it cannot be adapted to discrete toric spaces (see Sec. 7). Observe that the following definition does not constrain the loops to lie in a subset of the space, on the contrary of the definition given in [5].

Definition 6. *Let* $\mathcal{K} = (\boldsymbol{p}, U)$ *and* $\mathcal{L} = (\boldsymbol{p}, V)$ *be two m-loops of base point* $\boldsymbol{p} \in \mathbb{T}^n$*, with* $U = (\boldsymbol{u_1}, ..., \boldsymbol{u_k})$ *and* $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_l})$*. The two m-loops* $\mathcal{K}$ and $\mathcal{L}$ are directly homotopic *if one of the three following conditions is satisfied:*

1. *There exists* $j \in [1; l]$ *such that* $\boldsymbol{v_j} = 0$ *and* $U = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_l})$.
2. *There exists* $j \in [1; k]$ *such that* $\boldsymbol{u_j} = 0$ *and* $V = (\boldsymbol{u_1}, ..., \boldsymbol{u_{j-1}}, \boldsymbol{u_{j+1}}, ..., \boldsymbol{u_k})$.
3. *There exists* $j \in [1; k-1]$ *such that*
   - $V = (\boldsymbol{u_1}, ..., \boldsymbol{u_{j-1}}, \boldsymbol{v_j}, \boldsymbol{v_{j+1}}, \boldsymbol{u_{j+2}}, ..., \boldsymbol{u_k})$*, and*
   - $\boldsymbol{u_j} + \boldsymbol{u_{j+1}} = \boldsymbol{v_j} + \boldsymbol{v_{j+1}}$*, and*
   - $(\boldsymbol{u_j} - \boldsymbol{v_j})$ *is an n-step.*

Remark 7. In the case 1 (resp. 2 and 3), we have $k = l - 1$ (resp. $(l = k - 1)$ and $(l = k)$).

Remark 8. It may be observed that in the above definition, the parameter m allows to define an m-loop, but is not taken into account in order to decide if two m-loops are directly homotopic.

Definition 9. *Two m-loops* $\mathcal{K}$ *and* $\mathcal{L}$ *of base point* $\boldsymbol{p} \in \mathbb{T}^n$ are homotopic *if there exists a sequence of m-loops* $(\mathcal{C}_1, ..., \mathcal{C}_k)$ *such that* $\mathcal{C}_1 = \mathcal{K}$, $\mathcal{C}_k = \mathcal{L}$ *and for all* $j \in [1; k-1], \mathcal{C}_j$ *and* $\mathcal{C}_{j+1}$ *are directly homotopic.*

Example 10. In the toric space $\mathbb{Z}_4 \times \mathbb{Z}_2$, let us consider the point $\boldsymbol{p} = (0, 0)$, the 1-steps $\boldsymbol{v_1} = (1, 0)$ and $\boldsymbol{v_2} = (0, 1)$, and the 1-loops $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ and $\mathcal{L}_d$ (see Fig. 4). The loops $\mathcal{L}_a$ and $\mathcal{L}_b$ are homotopic, the loops $\mathcal{L}_b$ and $\mathcal{L}_c$ are directly homotopic, and the loops $\mathcal{L}_c$ and $\mathcal{L}_d$ are also directly homotopic.

On the other hand, it may be seen that the 1-loops depicted on Fig. 3-**b** and on Fig. 3-**c** are not homotopic.

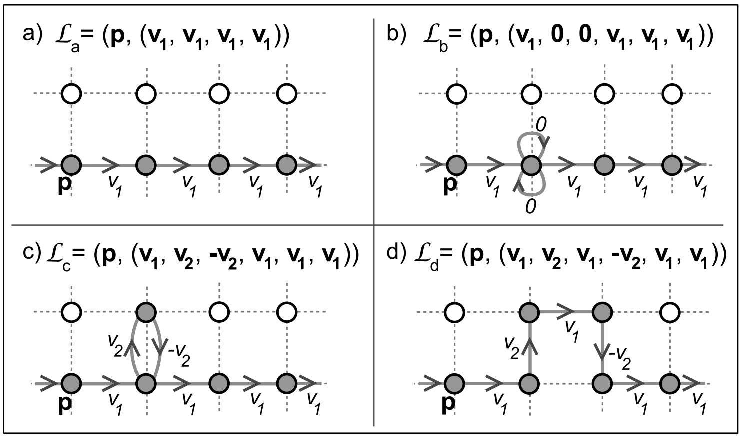

Fig. 4. Homotopic Loops - The 1-loops $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$ and $\mathcal{L}_d$ are homotopic.

We propose an adaptation of the definition of loop homotopy to $\mathbb{Z}^2$ and $\mathbb{Z}^3$ in the Annex, and we show that the resulting definition is equivalent to the loop equivalence defined in [5].

3.2 Fundamental Group

Initially defined in the continuous space by Henri Poincaré in 1895 [8], the fundamental group is an essential concept of topology, based on the homotopy relation, which has been transposed in different discrete frameworks (see e.g. [5], [9], [10]).

Given two m-loops $\mathcal{K} = (\boldsymbol{p}, (\boldsymbol{u_1}, ..., \boldsymbol{u_k}))$ and $\mathcal{L} = (\boldsymbol{p}, (\boldsymbol{v_1}, ..., \boldsymbol{v_l}))$ of same base point $\boldsymbol{p} \in \mathbb{T}^n$, the *product of* $\mathcal{K}$ *and* $\mathcal{L}$ is the m-loop $\mathcal{K}.\mathcal{L} = (\boldsymbol{p}, (\boldsymbol{u_1}, ..., \boldsymbol{u_k}, \boldsymbol{v_1}, ..., \boldsymbol{v_l}))$. The identity element of this product operation is the trivial loop $(\boldsymbol{p}, ())$, and for each m-loop $\mathcal{K} = (\boldsymbol{p}, (\boldsymbol{u_1}, ..., \boldsymbol{u_k}))$, we define the inverse of $\mathcal{K}$ as the m-loop $\mathcal{K}^{-1} = (p, (-\boldsymbol{u_k}, ..., -\boldsymbol{u_1}))$.

Remark 11. The symbol $\prod$ will be used for the iteration of the product operation on loops.

Given a positive integer w, and an m-loop $\mathcal{K}$ of base point $\boldsymbol{p}$, we set $\mathcal{K}^w = \prod_{i=1}^{w} \mathcal{K}$ and $\mathcal{K}^{-w} = \prod_{i=1}^{w} \mathcal{K}^{-1}$. We also define $\mathcal{K}^0 = (\boldsymbol{p}, ())$.

The homotopy of m-loops is a reflexive, symmetric and transitive relation: it is therefore an equivalence relation and the equivalence class, called *homotopy class*, of an m-loop $\mathcal{L}$ is denoted by $[\mathcal{L}]$. The product operation can be extended to the homotopy classes of m-loops of same base point: the product of $[\mathcal{K}]$ and $[\mathcal{L}]$ is $[\mathcal{K}].[\mathcal{L}] = [\mathcal{K}.\mathcal{L}]$. It may be easily seen that this binary operation is well defined since, if $\mathcal{K}' \in [\mathcal{K}]$ and $\mathcal{L}' \in [\mathcal{L}]$, then $(\mathcal{K}'.\mathcal{L}') \in [\mathcal{K}.\mathcal{L}]$.

We now define the fundamental group of $\mathbb{T}^n$.

Definition 12. *Given an m-adjacency relation on $\mathbb{T}^n$ and a point $\boldsymbol{p} \in \mathbb{T}^n$, the* m-fundamental group of $\mathbb{T}^n$ with base point $\boldsymbol{p}$ *is the group formed by the homotopy classes of all m-loops of base point $\boldsymbol{p} \in \mathbb{T}^n$ under the product operation.*

The identity element of this group is the homotopy class of the trivial loop, and for each m-loop $\mathcal{K}$ of base point $\boldsymbol{p}$, the inverse of $[\mathcal{K}]$ is $[\mathcal{K}^{-1}]$, since $[\mathcal{K}.\mathcal{K}^{-1}] = [(\boldsymbol{p}, ())]$.

The choice of the base point leads to different fundamental groups which are all isomorphic to each other. Thus, in the following, we sometimes talk about the m-fundamental group of $\mathbb{T}^n$, without specifying the base point.

4 Wrapping Vector and Homotopy Classes in $\mathbb{T}^n$

Deciding if two loops $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to the same homotopy class can be difficult, as it involves building a sequence of directly homotopic loops in order to 'link' $\mathcal{L}_1$ and $\mathcal{L}_2$. However, this problem may be solved using the *wrapping vector*, a characteristic which can be easily computed on each loop.

4.1 Wrapping Vector of a Loop

The *wrapping vector* of a loop is the sum of all the elements of the m-step sequence associated to the loop.

Definition 13. *Let $\mathcal{L} = (\boldsymbol{p}, V)$ be an m-loop, with $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$. The* wrapping vector of $\mathcal{L}$ *is* $\sum_{i=1}^{k} \boldsymbol{v_i}$.

Remark 14. In Def. 13, the symbol $\sum$ stands for the iteration of the classical addition operation on $\mathbb{Z}^n$, not of the operation $\oplus$ defined in Sec. 2.1.

Example 15. In $\mathbb{T}^2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, depicted on Fig. 5, the loop $\mathcal{K} = (\boldsymbol{p}, (\boldsymbol{v_3}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_1}, \boldsymbol{v_3}))$ (see Fig. 5-**a**) has a wrapping vector equal to $(4, 4)$, while the loop $\mathcal{L} = (\boldsymbol{p}, (\boldsymbol{v_3}, \boldsymbol{v_1}, \boldsymbol{v_1}, -\boldsymbol{v_2}, -\boldsymbol{v_1}, -\boldsymbol{v_3}, -\boldsymbol{v_3}, -\boldsymbol{v_1}, -\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_1}, \boldsymbol{v_1}, \boldsymbol{v_2}))$ has a wrapping vector equal to $(0, 0)$ (see Fig. 5-**b**).

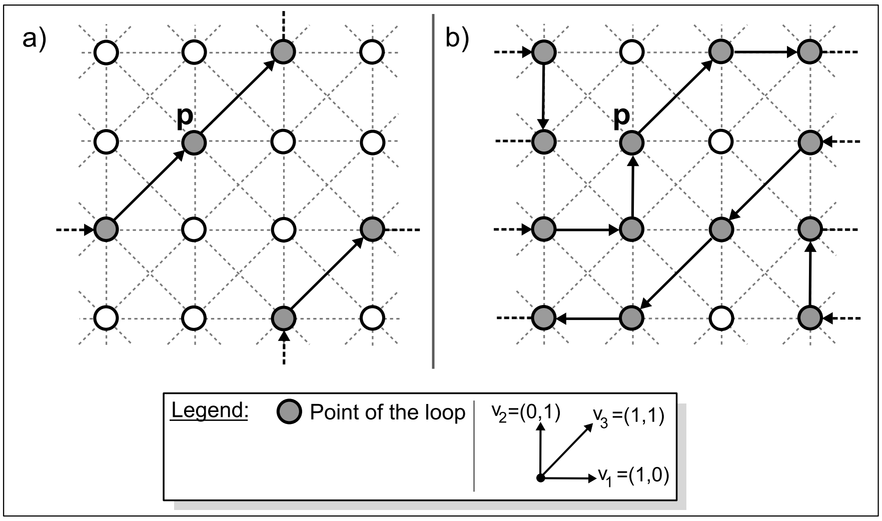

Fig. 5. Wrapping vector - In $\mathbb{T}^2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, the 2-loop in **a)** has a wrapping vector equal to $(4, 4)$, and the 2-loop in **b)** has a wrapping vector equal to $(0, 0)$.

We now define the notion of 'basic loops', which will be used for the proof of Prop. 18 and for building, in Def. 22, a canonical loop of a given wrapping vector.

Definition 16. *Let $\mathbb{T}^n$ be an n-dimensional toric space of size vector $\boldsymbol{d} = (d_1, ..., d_n)$. We denote, for each $i \in [1; n]$, by $\boldsymbol{b_i}$ the 1-step whose i-th coordinate is equal to* 1*, and by B_i the 1-step sequence composed of exactly d_i 1-steps $\boldsymbol{b_i}$.*

Given $\boldsymbol{p} \in \mathbb{T}^n$, for all $i \in [1; n]$, we define the i-th basic loop of base point $\boldsymbol{p}$ *as the 1-loop $(\boldsymbol{p}, B_i)$.*

Remark 17. For all $i \in [1; n]$, the wrapping vector of the i-th basic loop of base point $\boldsymbol{p}$ is equal to $(d_i.\boldsymbol{b_i})$.

The next property establishes that the wrapping vector of any m-loop can only take specific values in $\mathbb{Z}^n$.

Proposition 18. *Let $\mathbb{T}^n$ be an n-dimensional toric space of size vector $\boldsymbol{d} = (d_1, ..., d_n)$. A vector $\boldsymbol{w} = (w_1, ..., w_n)$ of $\mathbb{Z}^n$ is the wrapping vector of an m-loop of $\mathbb{T}^n$ if and only if, for all $i \in [1; n]$, w_i is a multiple of d_i.*

Proof. First, let $\mathcal{L} = (\boldsymbol{p}, V)$ be an m-loop of wrapping vector $\boldsymbol{w} = (w_1, ..., w_n)$, with $\boldsymbol{p} = (p_1, ..., p_n)$. As $\mathcal{L}$ is a loop, for all $i \in [1; n]$, $p_i \oplus_{d_i} w_i = p_i$. Hence, for all $i \in [1; n]$, $w_i \equiv 0(mod\ d_i)$, proving that w_i is a multiple of d_i for all $i \in [1; n]$.

Let $\boldsymbol{w} = (w_1, ..., w_n)$ be a vector of $\mathbb{Z}^n$ such that for all $i \in [1; n]$, w_i is a multiple of d_i. For all $i \in [1; n]$, we set $w_i^* = (w_i/d_i)$: we have $w_i^* \in \mathbb{Z}$. Therefore, given $\boldsymbol{p} \in \mathbb{T}^n$, if we denote by $(\boldsymbol{p}, B_i)$ the i-th basic loop of base point $\boldsymbol{p}$, we see that $(\prod_{i=1}^{n}(\boldsymbol{p}, B_i)^{w_i^*})$ is an m-loop whose wrapping vector is equal to $\boldsymbol{w}$. □

Thanks to Prop. 18, we can now define the *normalized wrapping vector* of an m-loop.

Definition 19. *Given $\mathbb{T}^n$ of size vector $\boldsymbol{d} = (d_1, ..., d_n)$, let $\mathcal{L}$ be an m-loop of wrapping vector $\boldsymbol{w} = (w_1, ..., w_n)$. The* normalized wrapping vector *of $\mathcal{L}$ is $\boldsymbol{w^*} = (\frac{w_1}{d_1}, ..., \frac{w_n}{d_n})$.*

Example 20. The wrapping vector and the normalized wrapping vector give information on how a loop 'wraps around' each dimension of a toric space before 'coming back to its starting point'. For example, let $\mathbb{T}^3 = \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_7$ (hence, the size vector of $\mathbb{T}^3$ is $(2, 5, 7)$). A loop with wrapping vector (4,5,0) has a normalized wrapping vector equal to (2,1,0): it wraps two times in the first dimension, one time in the second, and does not wrap in the third dimension.

On Fig. 5, the reduced wrapping vector of loop $\mathcal{K}$ (see Ex. 15), depicted on Fig. 5-**a**, is equal to $(1, 1)$, while the reduced wrapping vector of $\mathcal{L}$ (see Ex. 15), depicted on Fig. 5-**b**, is equal to $(0, 0)$.

It may easily be seen that, in $\mathbb{T}^n$, for each $i \in [1; n]$, the normalized wrapping vector of the i-th basic loop of any base point is equal to $\boldsymbol{b_i}$ (see Def. 16).

4.2 Equivalence Between Homotopy Classes and Wrapping Vector

It can be seen that two directly homotopic m-loops have the same wrapping vector, as their associated m-step sequences have the same sum. Therefore, we have the following property.

Proposition 21. *Two homotopic m-loops of $\mathbb{T}^n$ have the same wrapping vector.*

The following definition and the two next lemmas are necessary in order to understand Prop. 26 and its demonstration, leading to the main theorem of this article.

Definition 22. *Let $\boldsymbol{p}$ be an element of $\mathbb{T}^n$, and $\boldsymbol{w^*} = (w_1^*, ..., w_n^*) \in \mathbb{Z}^n$.*

The canonical loop of base point $\boldsymbol{p}$ and normalized wrapping vector $\boldsymbol{w^*}$ *is the 1-loop $\prod_{i=1}^{n}(\boldsymbol{p}, B_i)^{w_i^*}$, where $(\boldsymbol{p}, B_i)$ is the i-th basic loop of base point $\boldsymbol{p}$.*

Example 23. Consider $\mathbb{T}^4 = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_1 \times \mathbb{Z}_2$, $\boldsymbol{w^*} = (1, 0, 1, -2)$ and $\boldsymbol{p} = (0, 0, 0, 0)$. The canonical loop of base point $\boldsymbol{p}$ and normalized wrapping vector $\boldsymbol{w^*}$ is the 1-loop $(\boldsymbol{p}, V)$ with:

$$V = \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix} \right)$$

Lemma 24. *Any m-loop $\mathcal{L} = (\boldsymbol{p}, V)$ is homotopic to a 1-loop.*

Proof. Let us write $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$ and let $j \in [1; n]$ be such that $\boldsymbol{v_j}$ is not a 1-step. The m-loop $\mathcal{L}$ is directly homotopic to $\mathcal{L}_1 = (\boldsymbol{p}, V_1)$, with $V_1 = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_j}, \boldsymbol{0}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$. As $\boldsymbol{v_j}$ is not a 1-step, there exists an (m-1)-step $\boldsymbol{v'_j}$ and a 1-step $\boldsymbol{v_{j1}}$ such that $\boldsymbol{v_j} = (\boldsymbol{v_{j1}} + \boldsymbol{v'_j})$. The m-loop $\mathcal{L}_1$ is directly homotopic to $\mathcal{L}_2 = (\boldsymbol{p}, V_2)$, with $V_2 = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_{j1}}, \boldsymbol{v'_j}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$. By iteration, it can be shown that $\mathcal{L}$ is homotopic to a 1-loop. □

Lemma 25. *Let $\mathcal{L}_A = (\boldsymbol{p}, V_A)$ and $\mathcal{L}_B = (\boldsymbol{p}, V_B)$ be two m-loops such that $V_A = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_{j1}}, \boldsymbol{v_{j2}}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$ and $V_B = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_{j2}}, \boldsymbol{v_{j1}}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$, where $\boldsymbol{v_{j1}}$ and $\boldsymbol{v_{j2}}$ are 1-steps. Then, $\mathcal{L}_A$ and $\mathcal{L}_B$ are homotopic.*

Proof. As $\boldsymbol{v_{j1}}$ and $\boldsymbol{v_{j2}}$ are 1-steps, they have at most one non-null coordinate. If $(\boldsymbol{v_{j1}} - \boldsymbol{v_{j2}})$ is an n-step, the two loops are directly homotopic. If $(\boldsymbol{v_{j1}} - \boldsymbol{v_{j2}})$ is not an n-step, then necessarily $\boldsymbol{v_{j1}} = (-\boldsymbol{v_{j2}})$. Therefore, $\mathcal{L}_A$ is directly homotopic to $\mathcal{L}_C = (\boldsymbol{p}, V_C)$, with $V_C = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$. Furthermore, $\mathcal{L}_C$ is also directly homotopic to $\mathcal{L}_B$. □

Proposition 26. *Any m-loop of base point $\boldsymbol{p} \in \mathbb{T}^n$ and of normalized wrapping vector $\boldsymbol{w^*} \in \mathbb{Z}^n$ is homotopic to the canonical loop of base point $\boldsymbol{p}$ and of normalized wrapping vector $\boldsymbol{w^*}$.*

Proof. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-null 1-steps, and let i and j be the respective indexes of the non-null coordinate of $\boldsymbol{a}$ and $\boldsymbol{b}$. We say that $\boldsymbol{a}$ *is index-smaller than* $\boldsymbol{b}$ if $i < j$.

Let $\mathcal{L} = (\boldsymbol{p}, V)$ be an m-loop of normalized wrapping vector $\boldsymbol{w^*} \in \mathbb{Z}^n$.

- **1 -** The m-loop $\mathcal{L}$ is homotopic to a 1-loop $\mathcal{L}_1 = (\boldsymbol{p}, V_1)$ (see Lem. 24).
- **2 -** By Def. 6 and 9, the 1-loop $\mathcal{L}_1$ is homotopic to a 1-loop $\mathcal{L}_2 = (\boldsymbol{p}, V_2)$, where V_2 contains no null vector.
- **3 -** Let $\mathcal{L}_3 = (\boldsymbol{p}, V_3)$ be such that V_3 is obtained by iteratively permuting all pairs of consecutive 1-steps $(\boldsymbol{v_j}, \boldsymbol{v_{j+1}})$ in V_2 such that $\boldsymbol{v_{j+1}}$ is index-smaller than $\boldsymbol{v_j}$. Thanks to Lem. 25, $\mathcal{L}_3$ is homotopic to $\mathcal{L}_2$.
- **4 -** Consider $\mathcal{L}_4 = (\boldsymbol{p}, V_4)$, where V_4 is obtained by iteratively replacing all pairs of consecutive 1-steps $(\boldsymbol{v_j}, \boldsymbol{v_{j+1}})$ in V_3 such that $\boldsymbol{v_{j+1}} = (-\boldsymbol{v_j})$ by two null vectors, and then removing these two null vectors. The loop $\mathcal{L}_4$ is homotopic to $\mathcal{L}_3$.

The 1-loop $\mathcal{L}_4$ is homotopic to $\mathcal{L}$, it has therefore the same normalized wrapping vector $\boldsymbol{w^*} = (w_1^*, ..., w_n^*)$ (see Prop. 21). By construction, each pair of consecutive 1-steps $(\boldsymbol{v_j}, \boldsymbol{v_{j+1}})$ of V_4 is such that $\boldsymbol{v_j}$ and $\boldsymbol{v_{j+1}}$ are non-null and either $\boldsymbol{v_j} = \boldsymbol{v_{j+1}}$ or $\boldsymbol{v_j}$ is index-smaller than $\boldsymbol{v_{j+1}}$.

Let $\boldsymbol{d} = (d_1, ..., d_n)$ be the size vector of $\mathbb{T}^n$. As the normalized wrapping vector of $\mathcal{L}_4$ is equal to $\boldsymbol{w^*}$, we deduce that the $(d_1.|w_1^*|)$ first elements of V_4 are equal to $(\frac{w_1^*}{|w_1^*|}.\boldsymbol{b_1})$ (see Def. 16). Moreover, the $(d_2.|w_2^*|)$ next elements are equal to $(\frac{w_2^*}{|w_2^*|}.\boldsymbol{b_2})$, etc. Therefore, we have $\mathcal{L}_4 = (\prod_{i=1}^{n} (\boldsymbol{p}, B_i)^{w_i^*})$. □

The previous lemma shows that the canonical loop of base point $\boldsymbol{p}$ and of normalized wrapping vector $\boldsymbol{w}^*$ can be seen as a canonical form for all loops of base point $\boldsymbol{p}$ and normalized wrapping vector $\boldsymbol{w}^*$.

From this, we deduce that two m-loops of same base point $\boldsymbol{p}$ and same normalized wrapping vector $\boldsymbol{w}^*$ are homotopic, as they both belong to the homotopy class of the canonical loop of base point $\boldsymbol{p}$ and of normalized wrapping vector $\boldsymbol{w}^*$.

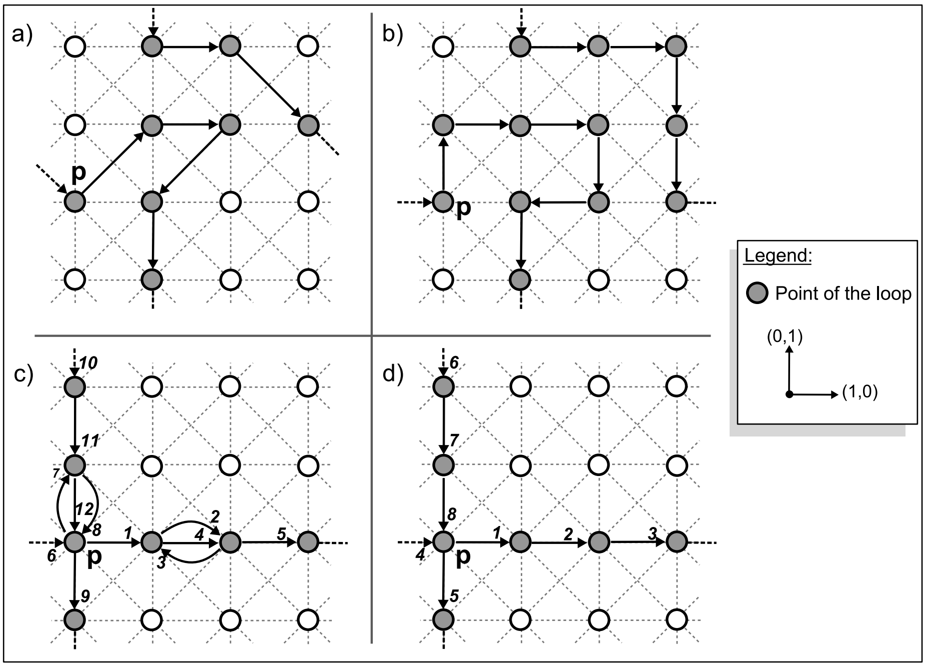

Fig. 6. In $\mathbb{T}^2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, the 2-loop in **a)** has a normalized wrapping vector equal to $(1, -1)$. The 2-loop in **a)** and the 1-loops in **b)**, **c)** and **d)** are homotopic. The 1-loop in **d)** is the canonical loop of base point $\boldsymbol{p}$ and normalized wrapping vector $(1, -1)$. On **c)** and **d)**, the numbers represent the positions of the 1-steps in the 1-step sequence associated to the loops.

Example 27. The following example illustrates the proof of Prop. 26 and uses the same notations. In $\mathbb{T}^2 = \mathbb{Z}^4 \times \mathbb{Z}^4$, let $\mathcal{L}$ be the 2-loop of base point $\boldsymbol{p}$ and of normalized wrapping vector $(1, -1)$, represented on Fig. 6-**a**: the 1-loop $\mathcal{L}_1$ is represented on Fig. 6-**b**, the 1-loop $\mathcal{L}_3$ is represented on Fig. 6-**c**, and the 1-loop $\mathcal{L}_4$, which is the canonical loop of base point $\boldsymbol{p}$ and of normalized wrapping vector $(1, -1)$, is represented on Fig. 6-**d**.

The loops $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_3$ and $\mathcal{L}_4$ are homotopic.

We can now state the main theorem of this article, which is a direct consequence of Prop. 21 and Prop. 26:

Theorem 28. *Two m-loops of $\mathbb{T}^n$ of same base point are homotopic if and only if their wrapping vectors are equal.*

Remark 29. According to Th. 28, the homotopy class of the trivial loop $(\boldsymbol{p}, ())$ is the set of all m-loops of base point $\boldsymbol{p}$ which have a null wrapping vector.

The loop depicted on Fig. 5-**b** belongs to the homotopy class of the trivial loop.

4.3 Wrapping Vector and Fundamental Group

Given a point $\boldsymbol{p} \in \mathbb{T}^n$, we introduce $\Omega = \{\boldsymbol{w^*} \in \mathbb{Z}^n /$ *there exists an m-loop in* $\mathbb{T}^n$ *of base point* $\boldsymbol{p}$ *and of normalized wrapping vector* $\boldsymbol{w^*}\}$. From Prop. 18, it is plain that $\Omega = \mathbb{Z}^n$. Therefore, $(\Omega, +)$ is precisely $(\mathbb{Z}^n, +)$

Theorem 28 states that there exists a bijection between the set of the homotopy classes of all m-loops of base point $\boldsymbol{p}$ and Ω. The product (see Sec. 3.2) of two m-loops $\mathcal{K}$ and $\mathcal{L}$ of same base point $\boldsymbol{p}$ and of respective wrapping vectors $\boldsymbol{w_k}$ and $\boldsymbol{w_l}$ is the loop $(\mathcal{K}.\mathcal{L})$ of base point $\boldsymbol{p}$. The wrapping vector of $(\mathcal{K}.\mathcal{L})$ is $(\boldsymbol{w_k} + \boldsymbol{w_l})$, therefore we can state that there exists an isomorphism between the fundamental group of $\mathbb{T}^n$ and $(\Omega, +)$.

Consequently, we retrieve in our discrete framework a well-known property of the fundamental group of toric spaces [2].

Proposition 30. *The fundamental group of $\mathbb{T}^n$ is isomorphic to $(\mathbb{Z}^n, +)$.*

5 Toric Loops in Subsets of $\mathbb{T}^n$

The toric loops, introduced in Sec. 1, can now be formally defined using previous notions.

Definition 31. *In $\mathbb{T}^n$, we say that an m-loop is a* toric m-loop *if it does not belong to the homotopy class of a trivial loop.*

A connected subset of $\mathbb{T}^n$ is wrapped in $\mathbb{T}^n$ *if it contains a toric m-loop.*

Remark 32. The notion of *grain* introduced informally in Sec. 1 may now be defined: a connected component of $\mathbb{T}^n$ is as *a grain* if it is not wrapped in $\mathbb{T}^n$.

5.1 Algorithm for Detecting Wrapped Subsets of $\mathbb{T}^n$

In order to know whether a given subset of $\mathbb{T}^n$ is wrapped or not, it is not necessary to build all the m-loops which can be found in the subset: the Wrapped Subset Descriptor (WSD) algorithm (see Alg. 1) allows to answer this question in linear time, as stated by the following proposition.

Algorithm 1: WSD(n,m,$\mathbb{T}^n$,$\boldsymbol{d}$,X)

Data: An n-dimensional toric space $\mathbb{T}^n$ of dimension vector $\boldsymbol{d}$ and a non-empty m-connected subset X of $\mathbb{T}^n$.

Result: A set B of elements of Z^n

```
1  Let p ∈ X; Coord(p) = 0^n; S = {p }; B = ∅;
2  forall x ∈ X do HasCoord(x) = false;
3  HasCoord(p) = true;
4  while there exists x ∈ S do
5      S = S \ {x};
6      forall n-dimensional m-steps v do
7          y=x ⊕_d v;
8          if y ∈ X and HasCoord(y) = true then
9              if Coord(y) ≠ Coord(x) + v then
10                 B = B ∪ ((Coord(x) + v - Coord(y))/ d);
11         else if y ∈ X and HasCoord(y) = false then
12             Coord(y) = Coord(x) + v;
13             S = S ∪ {y};
14             HasCoord(y) = true;
15 return B
```

Proposition 33. *Let $\mathbb{T}^n$ be an n-dimensional toric space of size vector $\boldsymbol{d}$. A non-empty m-connected subset X of $\mathbb{T}^n$ is wrapped in $\mathbb{T}^n$ if and only if WSD(n,m, $\mathbb{T}^n$,$\boldsymbol{d}$,X) is non-empty.*

Remark 34. In Alg. 1, the division operation performed on line 10 is a 'coordinate by coordinate' division between elements of $\mathbb{Z}^n$.

Before proving Prop. 33, let us study an example of the execution of Alg. 1 on an object.

Example 35. Let us consider a subset X of points of $\mathbb{Z}_4 \times \mathbb{Z}_4$ (see Fig. 7-**a**) and the 2-adjacency relation. In Fig. 7-**a**, one element of X is chosen as $\boldsymbol{p}$ and is given the coordinates of the origin (see l. 1 of Alg. 1); then we set $\boldsymbol{x} = \boldsymbol{p}$. In Fig. 7-**b**, every neighbour $\boldsymbol{y}$ of $\boldsymbol{x}$ (l. 6,7) which is in X (l. 11) is given coordinates depending on its position relative to $\boldsymbol{x}$ (l. 12) and is added to the set S (l. 13).

Then, in Fig. 7-**c**, one element of S is chosen as $\boldsymbol{x}$ (l. 4). Every neighbour $\boldsymbol{y}$ of $\boldsymbol{x}$ is scanned (l. 6,7). If $\boldsymbol{y}$ is in X and has already been given some coordinates (l. 8), it is compared with $\boldsymbol{x}$: as the coordinates of $\boldsymbol{x}$ and $\boldsymbol{y}$ are compatible in $\mathbb{Z}^2$ (the test achieved l. 9 returns false), the set B remains empty. Else, if $\boldsymbol{y}$ is in X and has not previously been given coordinates (l. 11) (see Fig. 7-**d**), then it is given coordinates depending on its position relative to $\boldsymbol{x}$ (l. 12) and added to the set S.

Finally, in Fig. 7-**e**, another element of S is chosen as $\boldsymbol{x}$. The algorithm tests one of the neighbours $\boldsymbol{y}$ of $\boldsymbol{x}$ (the left neighbour) which is in X and has already

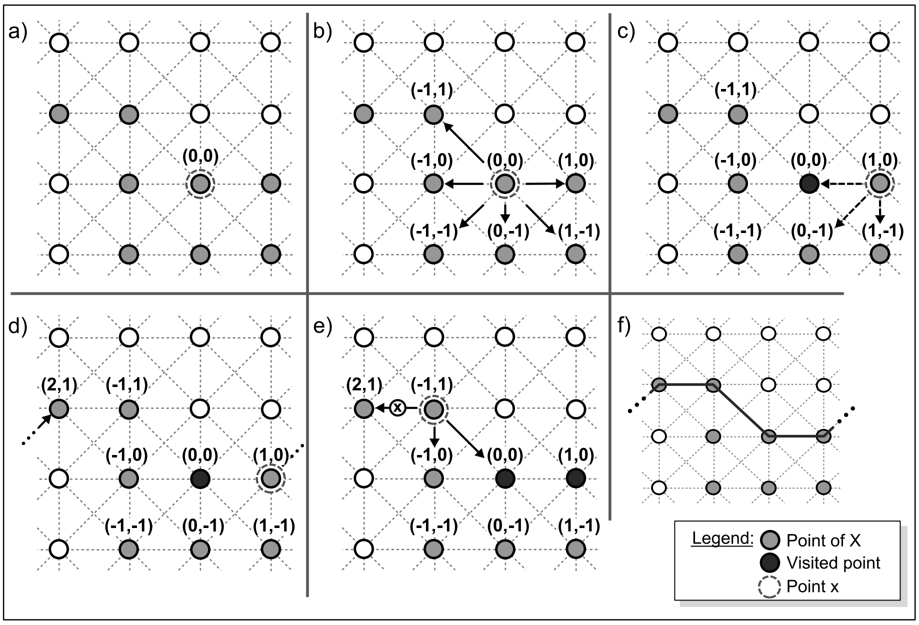

Fig. 7. Example of execution of WSD - see Ex. 35 for a complete description.

some coordinates (l. 8). As the coordinates of $\boldsymbol{y}$ and $\boldsymbol{x}$ are incompatible in $\mathbb{Z}^2$ (the points $(-1, 1)$ and $(2, 1)$ are not neighbours in $\mathbb{Z}^2$), the algorithm adds $(\boldsymbol{x}, \boldsymbol{y})$ to B (l. 10): according to Prop. 33, the subset X is wrapped in $\mathbb{T}^n$.

To summarize, Alg. 1 'tries to embed' the subset X of $\mathbb{T}^n$ in $\mathbb{Z}^n$: if some incompatible coordinates are detected by the test achieved on l. 9 of Alg. 1, then the object has a feature (a toric loop) which is incompatible with $\mathbb{Z}^n$. A toric 2-loop lying in X is depicted in Fig. 7-**f**.

Proof. (of Prop. 33) For all $\boldsymbol{y} \in X$ such that $\boldsymbol{y} \neq \boldsymbol{p}$, there exists a point $\boldsymbol{x}$ such that the test performed on l. 11 of Alg. 1 is true: we call $\boldsymbol{x}$ *the label predecessor of* $\boldsymbol{y}$.

– At the end of the execution of Alg. 1, if the set B is empty, then the test performed l. 9 was never true. Let $\mathcal{L} = (\boldsymbol{p}, V)$ be an m-loop contained in X, with $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$, and let us denote by $\boldsymbol{x_i}$ the i-th point of $\mathcal{L}$. As the test performed l. 9 was always false, we have the following:

$$\begin{cases} \text{for all } i \in [1; k-1], \boldsymbol{v_i} = Coord(\boldsymbol{x_{i+1}}) - Coord(\boldsymbol{x_i}) \\ \boldsymbol{v_k} = Coord(\boldsymbol{x_1}) - Coord(\boldsymbol{x_k}) \end{cases}$$

The wrapping vector of $\mathcal{L}$ is

$$\boldsymbol{w} = \sum_{i=1}^{k-1} (Coord(\boldsymbol{x_{i+1}}) - Coord(\boldsymbol{x_i})) + Coord(\boldsymbol{x_1}) - Coord(\boldsymbol{x_k}) = \boldsymbol{0}$$

Thus, if the algorithm returns false, each m-loop of X has a null wrapping vector and, according to Th. 28, belongs to the homotopy class of a trivial loop: there is no toric m-loop in X which is therefore not wrapped in $\mathbb{T}^n$.

– If B is not empty, then, there exists $(\boldsymbol{x}, \boldsymbol{y}) \in B$ and an m-step $\boldsymbol{a}$, such that $\boldsymbol{x} \oplus \boldsymbol{a} = \boldsymbol{y}$ and $Coord(\boldsymbol{y}) - Coord(\boldsymbol{x}) \neq \boldsymbol{a}$.
It is therefore possible to find two sequences γ_x and γ_y of m-adjacent points in X, with $\gamma_x = (\boldsymbol{p} = \boldsymbol{x_1}, \boldsymbol{x_2}, ..., \boldsymbol{x_q} = \boldsymbol{x})$ and $\gamma_y = (\boldsymbol{y} = \boldsymbol{y_t}, ..., \boldsymbol{y_2}, \boldsymbol{y_1} = \boldsymbol{p})$, such that, for all $i \in [1; q-1], \boldsymbol{x_i}$ is the label predecessor of $\boldsymbol{x_{i+1}}$, and for all $i \in [1; t-1], \boldsymbol{y_i}$ is the label predecessor of $\boldsymbol{y_{i+1}}$. Therefore, we can set

$$\begin{cases} \text{. for all } i \in [1; q-1], \boldsymbol{u_i} = Coord(\boldsymbol{x_{i+1}}) - Coord(\boldsymbol{x_i}) \\ \text{ is an m-step such that } \boldsymbol{x_i} \oplus \boldsymbol{u_i} = \boldsymbol{x_{i+1}} \\ \text{. for all } i \in [1; t-1], \boldsymbol{v_i} = Coord(\boldsymbol{y_i}) - Coord(\boldsymbol{y_{i+1}}) \\ \text{ is an m-step such that } \boldsymbol{y_{i+1}} \oplus \boldsymbol{v_i} = \boldsymbol{y_i} \end{cases}$$

Let $\mathcal{N}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}} = (\boldsymbol{p}, V)$ be the m-loop such that $V = (\boldsymbol{u_1}, ..., \boldsymbol{u_{q-1}}, \boldsymbol{a}, \boldsymbol{v_{t-1}}, ..., \boldsymbol{v_1})$. The m-loop $\mathcal{N}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}}$ is lying in X and its wrapping vector $\boldsymbol{w}$ is equal to:

$$\boldsymbol{w} = \sum_{i=1}^{q-1} \boldsymbol{u_i} + \boldsymbol{a} + \sum_{i=1}^{t-1} \boldsymbol{v_i} = \boldsymbol{a} - (Coord(\boldsymbol{y}) - Coord(\boldsymbol{x})) \neq \boldsymbol{0}$$

Thus, when the algorithm returns true, it is possible to find, inside X, an m-loop with a non-null wrapping vector: by Th. 28, there is a toric m-loop in X which is therefore wrapped in $\mathbb{T}^n$. □

The algorithm proposed in [4] returns a boolean telling whether the subset X is a wrapped subset of $\mathbb{T}^n$ or not. To obtain this algorithm from the code given in Alg. 1, it is sufficient to replace l. 10 by 'return true' and to replace l. 15 by 'return false'. We chose to give, in this article, a version of the algorithm returning a set, as it allows to get more information on the toric loops lying inside a wrapped subset X, as shown in Sec. 5.2.

5.2 Computing a Basis For Toric Loops in a Subset of $\mathbb{T}^n$

In this section, we show that Alg. 1 allows to build a basis for all normalized wrapping vector of all toric m-loops contained in a subset of $\mathbb{T}^n$.

Given $\mathbb{T}^n$ of size vector $\boldsymbol{d}$ and an m-connected subset X of $\mathbb{T}^n$, we consider having run WSD$(n, m, \mathbb{T}^n, \boldsymbol{d}, X)$, and we will use *Coord*, the function built on l. 12 of Alg. 1.

Given an m-step $\boldsymbol{v}$ and two points $\boldsymbol{x}, \boldsymbol{y} \in X$ such that $\boldsymbol{x} \oplus \boldsymbol{v} = \boldsymbol{y}$, *the points* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are conflictive through* $\boldsymbol{v}$ if $Coord(\boldsymbol{x}) + \boldsymbol{v} \neq Coord(\boldsymbol{y})$. Observe that, for all conflictive pairs of points x, y through v contained in the subset X of $\mathbb{T}^n$, the vector $(Coord(\boldsymbol{x}) + \boldsymbol{v} - Coord(\boldsymbol{y}))$ is added to the set B built on l. 10 of Alg. 1.

The next lemma establishes that, in order to calculate the wrapping vector of an m-loop (and therefore, its homotopy class, as stated by Th. 28), only the conflictive pairs of points in the loop need to be considered:

Lemma 36. *Given $\boldsymbol{p} \in X$ and an m-loop $\mathcal{K} = (\boldsymbol{p}, V)$ in X, with $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$, we denote, for all $i \in [1; k+1]$, by $\boldsymbol{x_i}$ the i-th point of $\mathcal{K}$, and we set $C = \{i \in [1; k] \mid \boldsymbol{x_i}$ and $\boldsymbol{x_{i+1}}$ are conflictive through $\boldsymbol{v_i}\}$. Let $\boldsymbol{w}$ be the wrapping vector of $\mathcal{K}$. We have:*

$$\boldsymbol{w} = \sum_{j \in C} (Coord(\boldsymbol{x_j}) + \boldsymbol{v_j} - Coord(\boldsymbol{x_{j+1}}))$$

Proof. The wrapping vector w of $\mathcal{K}$ is by definition:

$$\boldsymbol{w} = \sum_{j=1}^{k} \boldsymbol{v_j} = \sum_{j \notin C} \boldsymbol{v_j} + \sum_{j \in C} \boldsymbol{v_j} = \sum_{j \notin C} (Coord(\boldsymbol{x_{j+1}}) - Coord(\boldsymbol{x_j})) + \sum_{j \in C} \boldsymbol{v_j}$$

$$= \sum_{j=1}^{k} (Coord(\boldsymbol{x_{j+1}}) - Coord(\boldsymbol{x_j})) - \sum_{j \in C} (Coord(\boldsymbol{x_{j+1}}) - Coord(\boldsymbol{x_j})) + \sum_{j \in C} \boldsymbol{v_j}$$

As $\sum_{j=1}^{k} (Coord(\boldsymbol{x_{j+1}}) - Coord(\boldsymbol{x_j}))) = Coord(\boldsymbol{x_{k+1}}) - Coord(\boldsymbol{x_1}) = 0$, we get the lemma proved. □

We now focus on the set B, result of WSD$(n, m, \mathbb{T}^n, \boldsymbol{d}, X)$. For all $\boldsymbol{x}, \boldsymbol{y} \in X$ which are conflictive through an m-step $\boldsymbol{v}$, the vector $(\boldsymbol{v} + Coord(\boldsymbol{x}) - Coord(\boldsymbol{y}))$ is in B. Therefore, by Lem. 36, we obtain the following proposition.

Proposition 37. *Let the set $B = (\boldsymbol{w_1}, ..., \boldsymbol{w_k})$ be the result of WSD(n,m, $\mathbb{T}^n$,$\boldsymbol{d}$, X). A vector $\boldsymbol{w^*} \in \mathbb{Z}^n$ is the normalized wrapping vector of an m-loop of X if and only if there exists k non-negative integers $\alpha_1, ..., \alpha_k$ such that*

$$\boldsymbol{w^*} = \sum_{i=1}^{k} \alpha_i . \boldsymbol{w_i} \tag{1}$$

Remark 38. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are conflictive through $\boldsymbol{v}$, then $\boldsymbol{y}$ and $\boldsymbol{x}$ are conflictive through $(-\boldsymbol{v})$: therefore, if $\boldsymbol{u}$ belongs to B, then $-\boldsymbol{u}$ also belongs to B. This is why it is possible, in Prop. 37, to restrain the choice of the coefficients $\alpha_1, ..., \alpha_k$ to the set of non-negative integers.

Proof. If $\mathcal{L}$ is an m-loop in X of normalized wrapping vector w^*, then, by Lem. 36 and by construction of B, we deduce that w^* satisfies Equ. 1.

Now, let w^* be a vector which satisfies Equ. 1. For each $\boldsymbol{b} \in B$, there exists $\boldsymbol{x}$ and $\boldsymbol{y}$ in X and an m-step $\boldsymbol{a}$ such that $\boldsymbol{x}$ and $\boldsymbol{y}$ are conflictive through $\boldsymbol{a}$ and such that $\boldsymbol{b} = \frac{Coord(\boldsymbol{x}) + \boldsymbol{a} - Coord(\boldsymbol{y})}{\boldsymbol{d}}$. Consider the m-loop $\mathcal{N}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}}$ (see the second part of proof of Prop. 33), lying inside X, and whose wrapping vector is equal to $(Coord(\boldsymbol{x}) + \boldsymbol{a} - Coord(\boldsymbol{y}))$: the normalized wrapping vector of $\mathcal{N}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}}$ is $\boldsymbol{b}$.

Therefore, for each $\boldsymbol{b} \in B$, there exists an m-loop $\mathcal{L}_{\boldsymbol{b}}$ inside X, whose normalized wrapping vector is equal to $\boldsymbol{b}$. Let $\mathcal{L}^* = \prod_{i=1}^{k} (\mathcal{L}_{\boldsymbol{w_i}})^{\alpha_i}$. By construction, $\mathcal{L}^*$ is contained in X, and its wrapping vector is equal to $\boldsymbol{w^*}$. □

Thus, algorithm 1 builds a (non-minimal) basis allowing to compute the normalized wrapping vector of any m-loop of X: the normalized wrapping vector of any m-loop lying inside X is the linear combination of elements of B with non-negative coefficients. The set B, result of Alg. 1, allows to get information on how X wraps inside the toric space.

6 Conclusion

In this article, we give a formal definition of loops and homotopy inside discrete toric spaces in order to define various notions such as the fundamental group and the wrapping vector. Moreover, we show that wrapping vectors completely characterize toric loops (see Th. 28) and lead to build a linear time algorithm for the detection of such loops in a subset X of $\mathbb{T}^n$. In addition, this algorithm allows to build, for each subset X of $\mathbb{T}^n$, a basis of vectors which characterizes all toric loops contained in X and describes how X wraps around $\mathbb{T}^n$.

In Sec. 1, we have seen that detecting toric loops is important in order to filter grains from a material's sample and perform a fluid flow simulation on the sample. The WSD algorithm proposed in this article, detects which subsets of a sample, embedded inside a toric space, will create grains and should be removed. Future works will include analysis of the relationship between other topological characteristics of materials and their physical properties: for example, studying the skeleton of the pore space of a material could help to find new methods for performing fluid flow analysis.

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7 Annex: More About Loop Homotopy

An homotopy relation between loops in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, called *loop equivalence*, was defined in [5]. This article [5] has become a reference in digital topology, and the reader may wonder why a new definition of homotopy relation between loops in toric spaces is given in this work.

In this section, we first recall (using the notations previously introduced) the definition of *loop equivalence* given in [5], and we show that, when adapted to toric spaces, this definition gives unwanted results, therefore explaining why a new definition of homotopy was necessary for toric spaces. Then, we show that when adapted to $\mathbb{Z}^2$ and $\mathbb{Z}^3$, our definition of *loop homotopy* is equivalent to the *loop equivalence* defined in [5].

For more fluent reading, we will call *loop homotopy* the equivalence relation between loops defined in Def. 9, and *loop equivalence* the equivalence relation between loops defined in [5].

7.1 Loop Equivalence [5]

In [5], the loops must lie in a subset X of $\mathbb{Z}^2$ or $\mathbb{Z}^3$; X is *the set of black points of the space*. A *black m-loop of base point* $\boldsymbol{p} \in X$ is an m-loop $\mathcal{L} = (\boldsymbol{p}, U)$ (see Def. 4), with $U = (\boldsymbol{u_1}, ..., \boldsymbol{u_k})$, such that, for all $j \in [1;k]$, the j-th point of $\mathcal{L}$ is in X.

Definition 39. *Let $\mathcal{K} = (p, U)$ and $\mathcal{L} = (p, V)$ be two black m-loops, with $(n, m) \in \{(2,1); (2,2); (3,1); (3,3)\}$. Let k and l be respectively the length of $\mathcal{K}$ and $\mathcal{L}$. We say that $\mathcal{K}$ and $\mathcal{L}$* are directly equivalent *if one of these two conditions is matched:*

- *Considering $\tilde{U}$ and $\tilde{V}$, the sequences obtained from U and V respectively by removing all null steps, we have $\tilde{U} = \tilde{V}$.*

- *We have $k = l$ and, if we denote, for all $j \in [2;k]$, $\boldsymbol{x_j}$ as the j-th point of $\mathcal{K}$ and $\boldsymbol{y_j}$ as the j-th point of $\mathcal{L}$, and if we define*

$$D_K = \{\boldsymbol{x} \in \mathbb{T}^n | \text{ there exists } j \in [2;k] \text{ such that } \boldsymbol{x} = \boldsymbol{x_j} \text{ and } \boldsymbol{x_j} \neq \boldsymbol{y_j}\}$$
$$\text{and } D_L = \{\boldsymbol{y} \in \mathbb{T}^n | \text{ there exists } j \in [2;k] \text{ such that } \boldsymbol{y} = \boldsymbol{y_j} \text{ and } \boldsymbol{x_j} \neq \boldsymbol{y_j}\},$$

  then, $(D_K \cup D_L)$ is included in a unit lattice square or a unit lattice cube of the space which, if $m = 1$ and $n = 3$, does not contain two diametrically opposite white points.

Definition 40. *Two black loops $\mathcal{K}$ and $\mathcal{L}$* are equivalent *if there exists a sequence $(\mathcal{K} = \mathcal{C}_1, ..., \mathcal{C}_i = \mathcal{L})$ of black loops such that, for all $j \in [1; i-1], \mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are directly equivalent.*

7.2 Loop Equivalence in Toric Spaces Gives Unwanted Results

In this article, the loops we consider are contained inside a toric space whose points are all black. Therefore, in order to adapt Def. 39 to our discrete toric framework, it is necessary to replace all occurrences of '$\mathbb{Z}^n$' by '$\mathbb{T}^n$' and 'unit lattice' by 'toric unit lattice'. Moreover, all conditions depending on the colours of the points of the space can be removed (this means that, in the end of Def. 39, the condition stating that the toric unit lattice cube, for $m = 1$, must not contain two diametrically opposite white points, can be ignored).

The following example pinpoints that Def. 39, adapted to our discrete toric framework, can produce unwanted results.

Example 41. Given a bidimensional toric space $(\mathbb{T}^2, \oplus)$, with $\mathbb{T}^2 = \mathbb{Z}_3 \times \mathbb{Z}_3$, let us consider the element $\boldsymbol{p} = (0;1)$, the 2-steps $\boldsymbol{v_1} = (1;0)$, $\boldsymbol{v_2} = (0;1)$, $\boldsymbol{v_3} = (-1;-1)$ and the 2-loops $\mathcal{K} = (\boldsymbol{p}, (\boldsymbol{v_1}, \boldsymbol{v_1}, \boldsymbol{v_1}))$ and $\mathcal{L} = (\boldsymbol{p}, (\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}))$. It can be seen on Fig. 8 that $\mathcal{K}$ and $\mathcal{L}$ do not belong to the same homotopy class in $\mathbb{T}^2$ ($\mathcal{K}$ wraps around the toric space, but $\mathcal{L}$ does not), however, based on Def. 39 adapted to toric spaces in an obvious way, they are equivalent.

This example proves that Def. 39 gives unwanted results: in some toric spaces, like the one pinpointed in Ex. 41, the fundamental group obtained from Def. 39 is trivial, resulting in the fact that the space is simply connected. In order to avoid such results, we introduced a new definition of loop homotopy for toric spaces in this article (see Def. 6,9).

7.3 Comparing Black Loop Homotopy and Black Loop Equivalence in $\mathbb{Z}^2$ and $\mathbb{Z}^3$

We will now work in the 'classical' discrete frameworks $\mathbb{Z}^3$ or $\mathbb{Z}^2$, and we will study the black loops homotopy. From this point, the points of the space are either black or white, and the loops are constrained to lie in the subset of the space which contains the black points [5].

It is possible to adapt all definitions given previously in this article to the classical space $\mathbb{Z}^n$, by replacing the operation '$\oplus$' by the usual operation '+'. This way, we can define black m-loops direct homotopy in $\mathbb{Z}^n$: two black m-loops of same base point $p \in \mathbb{Z}^n$ *are directly homotopic* if they are directly homotopic in the sense of definition 6 adapted to $\mathbb{Z}^n$.

We can now define black m-loops homotopy in $\mathbb{Z}^n$: two black m-loops $\mathcal{K}$ and $\mathcal{L}$ of same base point $p \in \mathbb{Z}^n$ *are homotopic* if there exists a sequence $(\mathcal{K} = \mathcal{C}_1, ..., \mathcal{C}_i = \mathcal{L})$ of black m-loops such that, for all $j \in [1; i-1], \mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are directly homotopic.

The next proposition establishes that, in $\mathbb{Z}^n$, black m-loop homotopy and black m-loop equivalence defined in [5] (see Def. 39), with $(n, m) \in \{(2,1); (2,2); (3,1); (3,3)\}$, are equivalent.

Proposition 42. *Two black m-loops* $\mathcal{K} = (\boldsymbol{p}, U)$ *and* $\mathcal{L} = (\boldsymbol{p}, V)$ *in* $\mathbb{Z}^n$ *(*$(n, m) \in \{(2,1); (2,2); (3,1); (3,3)\}$*) are equivalent if and only if they are homotopic.*

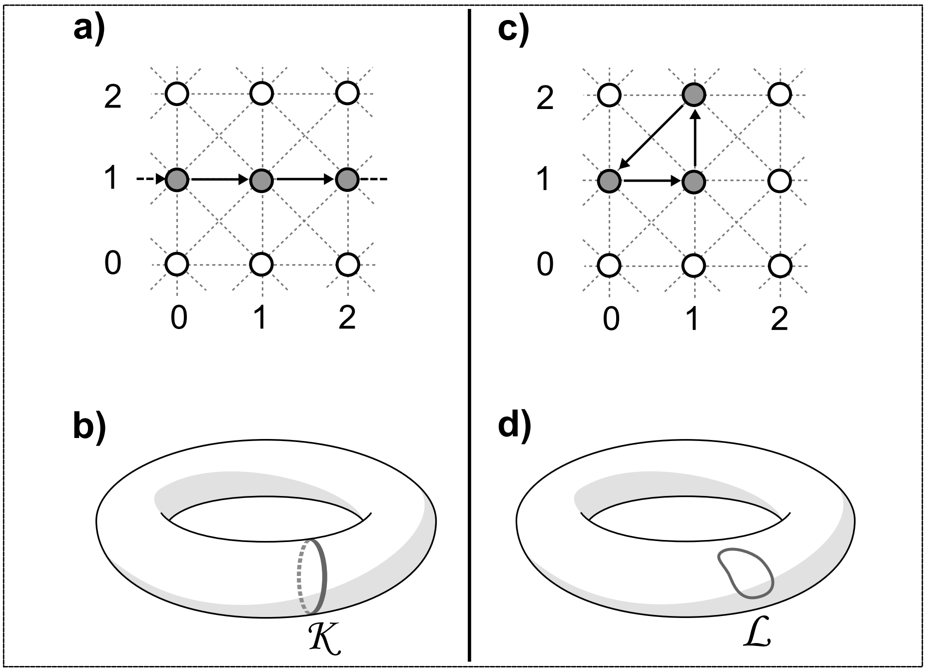

Fig. 8. Equivalent loops - In **a** and **c**: in $\mathbb{T}^2 = \mathbb{Z}_3 \times \mathbb{Z}_3$, the loops in **a)** and **c)** are equivalent (see Def. 39) but not homotopic (see Def. 6). In **b** and **d**: the two loops do not belong to the same homotopy class in $\mathbb{T}^2$, as one wraps around the toric space, and not the other.

Proof. In the following proof, we set $U = (\boldsymbol{u_1}, ..., \boldsymbol{u_k})$, $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_l})$, and we use the same notations than in Def. 39.

If $k \neq l$ and $\mathcal{K}$ and $\mathcal{L}$ are directly equivalent ($\tilde{U} = \tilde{V}$), then it can be easily seen that $\mathcal{K}$ and $\mathcal{L}$ are homotopic. Reciprocally, if $k \neq l$ and $\mathcal{K}$ and $\mathcal{L}$ are directly homotopic, then they are directly equivalent.

Therefore, let us consider the case where $k = l$. In the following, for all $j \in [1; k+1]$, we denote by x_j (resp. y_j) the j-th point of $\mathcal{K}$ (resp. $\mathcal{L}$). It may be easily seen that, as we are working in $\mathbb{Z}^n$, for all $j \in [1; k]$, $u_j = x_{j+1} - x_j$ and $v_j = y_{j+1} - y_j$.

- In the case where $\mathcal{K}$ and $\mathcal{L}$ are directly homotopic (see Def. 6, case 3), then there exists $j \in [1; k-1]$ such that $V = (\boldsymbol{u_1}, ..., \boldsymbol{u_{j-1}}, \boldsymbol{v_j}, \boldsymbol{v_{j+1}}, \boldsymbol{u_{j+2}}, ..., \boldsymbol{u_k})$, with $(u_j - v_j)$ being an m-step and $(u_j + u_{j+1} = v_j + v_{j+1})$. Therefore, we have $D_K \cup D_L = \{\boldsymbol{x_{j+1}}, \boldsymbol{y_{j+1}}\}$.
  - If $(n, m) \in \{(2,1), (2,2), (3,3)\}$, then, as $(\boldsymbol{x_{j+1}} - \boldsymbol{y_{j+1}}) = (\boldsymbol{u_j} - \boldsymbol{v_j})$, the points $\boldsymbol{x_{j+1}}$ and $\boldsymbol{y_{j+1}}$ lie in a same unit lattice square or cube, proving that $\mathcal{K}$ and $\mathcal{L}$ are directly equivalent.
  - If $(n, m) \in \{(3, 1)\}$, then $\boldsymbol{u_j}$ and $\boldsymbol{v_j}$ are both 1-steps. Therefore, $(\boldsymbol{u_j} - \boldsymbol{v_j})$ is a 2-step, proving that $\boldsymbol{x_{j+1}}$ and $\boldsymbol{y_{j+1}}$ lie in a same unit lattice square.

Therefore, $\mathcal{K}$ and $\mathcal{L}$ are directly equivalent.

– Reciprocally, suppose that $\mathcal{K}$ and $\mathcal{L}$ are directly equivalent.
  - In the case where $(n, m) \in \{(2,2),(3,3)\}$, we set, for all $h \in [1;k]$, $R_h = (\boldsymbol{v_1}, ..., \boldsymbol{v_{h-1}}, \boldsymbol{x_{h+1}} - \boldsymbol{y_h}, \boldsymbol{u_{h+1}}, ..., \boldsymbol{u_k})$ and $\mathcal{C}_h = (p, R_h)$.
    First, we prove that for all $h \in [1;k]$, $\mathcal{C}_h$ is an m-loop of base point p, by proving that $(\boldsymbol{x_{h+1}} - \boldsymbol{y_h})$ is an m-step. As $\mathcal{K}$ and $\mathcal{L}$ are directly equivalent, we either have $\boldsymbol{x_h} = \boldsymbol{y_h}$ or $\boldsymbol{x_{h+1}} = \boldsymbol{y_{h+1}}$ (the result is then directly obtained), or we have $\boldsymbol{x_h}, \boldsymbol{y_h}, \boldsymbol{x_{h+1}}$ and $\boldsymbol{y_{h+1}}$ lying in a same unit lattice cube or square: $(\boldsymbol{x_{h+1}} - \boldsymbol{y_h})$ is therefore an n-step, and also an m-step since $n = m$.
    We are going to prove that for all $h \in [1; k-1], \mathcal{C}_h$ and $\mathcal{C}_{h+1}$ are directly homotopic by proving that they match the case 3 of Def. 6:
    * $\boldsymbol{x_{h+1}} - \boldsymbol{y_h} + \boldsymbol{u_{h+1}} = \boldsymbol{x_{h+2}} - \boldsymbol{y_h} = \boldsymbol{v_h} + \boldsymbol{x_{h+2}} - \boldsymbol{y_{h+1}}$,
    * $\boldsymbol{x_{h+1}} - \boldsymbol{y_h} - \boldsymbol{v_h} = \boldsymbol{x_{h+1}} - \boldsymbol{y_{h+1}}$ is an n-step, as either $\boldsymbol{x_{h+1}} = \boldsymbol{y_{h+1}}$ or $\boldsymbol{x_{h+1}}$ and $\boldsymbol{y_{h+1}}$ belong to a same unit lattice cube or square, and also an m-step since $n = m$.

    Finally, by pointing out that $\mathcal{C}_1$ is equal to $\mathcal{K}$ and that $\mathcal{C}_k$ is equal to $\mathcal{L}$, we conclude that $\mathcal{K}$ and $\mathcal{L}$ are homotopic.
  - In the case where $m = 1$ and $n = 3$, let us assume that the set D_K (resp. D_L) contains only consecutive points of the loop $\mathcal{K}$ (resp. $\mathcal{L}$): if it was not the case, the following reasoning could still be performed on each consecutive elements of D_K and D_L in order to obtain the same result.
    There exists $i \in [2;k]$ and $j \in [i;k]$ such that $(D_K \cup D_L) = \{\boldsymbol{x_i}, ..., \boldsymbol{x_j}, \boldsymbol{y_i}, ..., \boldsymbol{y_j}\}$ is included in a unit lattice square or a unit lattice cube which does not contain two diametrically opposite white points. Therefore, we have $V = (\boldsymbol{u_1}, ..., \boldsymbol{u_{i-2}}, \boldsymbol{v_{i-1}}, ..., \boldsymbol{v_j}, \boldsymbol{u_{j+1}}, ..., \boldsymbol{u_k})$. It is possible to simplify the problem in two ways:
    * As $m = 1$, $\boldsymbol{y_i} - \boldsymbol{x_{i-1}}$ and $\boldsymbol{x_i} - \boldsymbol{x_{i-1}}$ are 1-steps. Therefore, $\boldsymbol{x_{i-1}}, \boldsymbol{x_i}$ and $\boldsymbol{y_i}$ are in a same unit lattice square and, as $\boldsymbol{x_i} \neq \boldsymbol{y_i}$, we find that $\boldsymbol{x_{i-1}}$ lie in the same unit lattice cube or square than the elements of $(D_K \cup D_L)$. The same way, we prove that $\boldsymbol{x_{j+1}}$ lie in the same unit lattice cube or square than the elements of $(D_K \cup D_L)$.
      It may easily be seen that $\mathcal{K}$ is homotopic to the black 1-loop $\mathcal{K}' = (\boldsymbol{p}, (\boldsymbol{u_1}, ..., \boldsymbol{u_j}, -\boldsymbol{v_j}, ..., -\boldsymbol{v_{i-1}}, \boldsymbol{v_{i-1}}, ..., \boldsymbol{v_j}, \boldsymbol{u_{j+1}}, ..., \boldsymbol{u_k}))$.
      Hence, proving that $\mathcal{K}'$ and $\mathcal{L}$ are homotopic can be achieved by proving that the black 1-loop $(\boldsymbol{x_{i-1}}, (\boldsymbol{u_{i-1}}, ..., \boldsymbol{u_j}, -\boldsymbol{v_j}, ..., -\boldsymbol{v_{i-1}}))$, whose points are contained inside the same unit lattice cube or square than $(D_K \cup D_L)$, is homotopic to the trivial loop $(\boldsymbol{x_{i-1}}, ())$.
    * Let $\mathcal{C} = (\boldsymbol{p}, (\boldsymbol{w_1}, ..., \boldsymbol{w_i}, ..., \boldsymbol{w_j}, ..., \boldsymbol{w_k}))$ be a self-intersecting black 1-loop such that $\boldsymbol{p} + \boldsymbol{w_1} + ... + \boldsymbol{w_i} = \boldsymbol{p} + \boldsymbol{w_1} + ... + \boldsymbol{w_j}$. The problem of showing that $\mathcal{C}$ is homotopic to $(\boldsymbol{p}, ())$ can be decomposed into two smaller problems: proving that $\mathcal{C}' = (\boldsymbol{p} + \boldsymbol{w_1} + ... + \boldsymbol{w_i}, (\boldsymbol{w_{i+1}}, ..., \boldsymbol{w_j}))$ is homotopic to $(\boldsymbol{p} + \boldsymbol{w_1} + ... + \boldsymbol{w_i}, ())$, and then proving that $\mathcal{C}'' = (\boldsymbol{p}, (\boldsymbol{w_1}, ..., \boldsymbol{w_i}, \boldsymbol{w_{j+1}}, ..., \boldsymbol{w_k}))$ is homotopic to $(\boldsymbol{p}, ())$. Therefore, in

order to prove that a black 1-loop is homotopic to a trivial loop, we can consider only, without loss of generality, non self-intersecting black 1-loops.

Therefore, in order to prove that the two black 1-loops $\mathcal{K}$ and $\mathcal{L}$ are homotopic, it is sufficient to prove that any non self-intersecting black 1-loop, contained in a unit lattice cube which does not contain two diametrically opposite white points, is homotopic to a trivial loop.
A program building all possible configurations of black points inside a unit lattice cube which does not contain two diametrically opposite white points (52 configuration according to our program), and building for each of these configurations all the non self-intersecting black 1-loops, was written. A greedy algorithm is used to build, for each loop, a sequence of directly homotopic black 1-loop inside the unit lattice cube, in order to prove that every non self-intersecting black 1-loop in the cube is homotopic to a trivial loop: at each step, the newly built black 1-loop contains less points than the previous black 1-loop in the sequence, until a single point is reached.
As the programs successfully proves that each such non self-intersecting black 1-loop is equivalent to a trivial loop, and as the case $(n, m) = (2, 1)$ is included in the case $(n, m) = (3, 1)$, it can be concluded that $\mathcal{K}$ and $\mathcal{L}$ are homotopic. $\square$