

On the counting function of the embedded eigenvalues for some manifold with cusps, and magnetic Laplacian

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Abstract

We consider a non compact, complete manifold \mathbf{M} of finite area with cuspidal ends. The generic cusp is isomorphic to $\mathbf{X} \times]1, +\infty[$ with metric $ds^2 = (h + dy^2)/y^{2\delta}$. \mathbf{X} is a compact manifold with nonzero first Betti number equipped with the metric h . For a one-form A on \mathbf{M} such that in each cusp A is a non exact one-form on the boundary at infinity, we prove that the magnetic Laplacian $-\Delta_A = (id + A)^*(id + A)$ satisfies the Weyl asymptotic formula with sharp remainder. We deduce an upper bound for the counting function of the embedded eigenvalues of the Laplace-Beltrami operator $-\Delta = -\Delta_0$.¹

1 Introduction

We consider a smooth, connected n -dimensional Riemannian manifold (\mathbf{M}, \mathbf{g}) , ($n \geq 2$), such that

$$\mathbf{M} = \bigcup_{j=0}^J \mathbf{M}_j \quad (J \geq 1), \quad (1.1)$$

where the \mathbf{M}_j are open sets of \mathbf{M} . We assume that the closure of \mathbf{M}_0 is compact and that the other \mathbf{M}_j are cuspidal ends of \mathbf{M} .

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This means that $\mathbf{M}_j \cap \mathbf{M}_k = \emptyset$, if $1 \leq j < k$, and that there exists, for any j , $1 \leq j \leq J$, a closed compact $(n-1)$ -dimensional Riemannian manifold $(\mathbf{X}_j, \mathbf{h}_j)$ such that \mathbf{M}_j is isometric to $\mathbf{X}_j \times]a_j^2, +\infty[$, ($a_j > 0$) equipped with the metric

$$ds_j^2 = y^{-2\delta_j}(\mathbf{h}_j + dy^2); \quad (1/n < \delta_j \leq 1). \quad (1.2)$$

So there exists a smooth real one-form $A_j \in T^*(\mathbf{X}_j)$, non exact, such that

$$\begin{cases} i) dA_j \neq 0 \\ \text{or} \\ ii) dA_j = 0 \text{ and } [A_j] \text{ is not integer.} \end{cases} \quad (1.3)$$

In *ii*) we mean that there exists a smooth closed curve γ in \mathbf{X}_j such that

$$\int_{\gamma} A_j \notin 2\pi\mathbb{Z}.$$

Then one can always find a smooth real one-form $A \in T^*(\mathbf{M})$ such that

$$\forall j, 1 \leq j \leq J, \quad A = A_j \quad \text{on } \mathbf{M}_j. \quad (1.4)$$

We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A = (i d + A)^*(i d + A), \quad (1.5)$$

($i = \sqrt{-1}$, $(i d + A)u = i du + uA$, $\forall u \in C_0^\infty(\mathbf{M}; \mathbb{C})$, the upper star, $*$, stands for the adjoint in $L^2(\mathbf{M})$).

As \mathbf{M} is a complete metric space, by Hopf-Rinow theorem \mathbf{M} is geodesically complete, so it is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M}; \mathbb{C})$, the space of smooth and compactly supported functions. The spectrum of $-\Delta_A$ is gauge invariant : for any $f \in C^1(\mathbf{M}; \mathbb{R})$, $-\Delta_A$ and $-\Delta_{A+df}$ are unitary equivalent, hence they have the same spectrum.

For a self-adjoint operator P on a Hilbert space H , $\text{sp}(P)$, $\text{sp}_{\text{ess}}(P)$, $\text{sp}_p(P)$ and $\text{sp}_d(P)$ will denote respectively the spectrum, the essential spectrum, the point spectrum and the discrete spectrum of P . We recall that $\text{sp}(P) = \text{sp}_{\text{ess}}(P) \cup \text{sp}_d(P)$, $\text{sp}_d(P) \subset \text{sp}_p(P)$ and $\text{sp}_{\text{ess}}(P) \cap \text{sp}_d(P) = \emptyset$.

Theorem 1.1 *Under the above assumptions on \mathbf{M} , the essential spectrum of the Laplace-Beltrami operator on \mathbf{M} , $-\Delta = -\Delta_0$ is given by*

$$\begin{cases} \text{sp}_{\text{ess}}(-\Delta) = [0, +\infty[, & \text{if } 1/n < \delta < 1 \\ \text{sp}_{\text{ess}}(-\Delta) = [\frac{(n-1)^2}{4}, +\infty[, & \text{if } \delta = 1 \end{cases}. \quad (1.6)$$

When (1.3) and (1.4) are satisfied, the magnetic Laplacian $-\Delta_A$ has a compact resolvent. The spectrum $\text{sp}(-\Delta_A) = \text{sp}_d(-\Delta_A)$ is a sequence of non-decreasing eigenvalues $(\lambda_j)_{j \in \mathbb{N}^*}$, $\lambda_j \leq \lambda_{j+1}$, $\lim_{j \rightarrow +\infty} \lambda_j = +\infty$, such that the sequence of normalized eigenfunctions $(\varphi_j)_{j \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(\mathbf{M})$. Moreover $\lambda_0 > 0$.

For any self-adjoint operator P with compact resolvent, and for any real λ , $N(\lambda, P)$ will denote the number of eigenvalues, (repeated according to their multiplicity), of P less than λ ,

$$N(\lambda, P) = \text{trace}(\chi_{]-\infty, \lambda[}(P)), \quad (1.7)$$

(for any $I \subset \mathbb{R}$, $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ if $x \in \mathbb{R} \setminus I$).

The asymptotic behavior of $N(\lambda, -\Delta_A)$ satisfies the Weyl formula with the following sharp remainder.

Theorem 1.2 *Under the above assumptions on \mathbf{M} and on A , we have the Weyl formula with remainder as $\lambda \rightarrow +\infty$,*

$$N(\lambda, -\Delta_A) = |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + \mathbf{O}(r(\lambda)), \quad (1.8)$$

with

$$r(\lambda) = \begin{cases} \lambda^{(n-1)/2} \ln(\lambda), & \text{if } 1/(n-1) \leq \delta \\ \lambda^{1/(2\delta)}, & \text{if } 1/n < \delta < 1/(n-1) \end{cases}, \quad (1.9)$$

$\delta = \min_{1 \leq j \leq J} \delta_j$, $|\mathbf{M}|$ is the Riemannian measure of \mathbf{M} and ω_d is the euclidian

volume of the unit ball of \mathbb{R}^d , $\omega_d = \frac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})}$.

The asymptotic formula (1.8) without remainder is given in [Go-Mo], and with remainder but only for $n = 2$ (and $\delta_j = 1$ for any $1 \leq j \leq J$) in [Mo-Tr].

The Laplace-Beltrami operator $-\Delta = -\Delta_0$ may have embedded eigenvalues in its essential spectrum $\text{sp}_{\text{ess}}(-\Delta)$. Let $N_{\text{ess}}(\lambda, -\Delta)$ denote the number of eigenvalues of $-\Delta$, (counted according to their multiplicity), less than λ .

Theorem 1.3 *There exists a constant $C_{\mathbf{M}}$ such that, for any $\lambda \gg 1$,*

$$N_{\text{ess}}(\lambda, -\Delta) \leq |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + C_{\mathbf{M}} r_0(\lambda), \quad (1.10)$$

with $r_0(\lambda)$ defined by

$$r_0(\lambda) = \begin{cases} \lambda^{\frac{n-1}{2}} \ln(\lambda), & \text{if } 2/n \leq \delta \leq 1 \\ \lambda^{\frac{n-(n\delta-1)}{2}}, & \text{if } 1/n < \delta < 2/n \end{cases} ; \quad (1.11)$$

δ is the one defined in Theorem 1.2 .

The above upper bound proves that any eigenvalue of $-\Delta$ has finite multiplicity.

The estimate (1.10) is sharp when $n = 2$. There exist hyperbolic surfaces \mathbf{M} of finite area so that

$$N_{\text{ess}}(\lambda, -\Delta) = |\mathbf{M}| \frac{\omega_2}{(2\pi)^2} \lambda + \Gamma_{\mathbf{M}} \lambda^{1/2} \ln(\lambda) + \mathbf{O}(\lambda^{1/2}),$$

for some constant $\Gamma_{\mathbf{M}}$. See [Mul] for such examples.

Still in the case of surfaces, a compact perturbation of the metric of non compact hyperbolic surface \mathbf{M} of finite area can destroy all embedded eigenvalues, see [Col1].

2 Proof

2.1 Proof of Theorem 1.1

Since the essential spectrum of an elliptic operator on a manifold is invariant by compact perturbation of the manifold, (see for example [Do-Li], Proposition 2.1), we can write

$$\text{sp}_{\text{ess}}(-\Delta_A) = \bigcup_{j=1}^J \text{sp}_{\text{ess}}(-\Delta_A^{\mathbf{M}_j, D}), \quad (2.1)$$

where $-\Delta_A^{\mathbf{M}_j, D}$ denotes the self-adjoint operator on $L^2(\mathbf{M}_j)$ associated to $-\Delta_A$ with Dirichlet boundary conditions on the boundary $\partial\mathbf{M}_j$ of \mathbf{M}_j .

Let us consider a cusp $\mathbf{M}_j = \mathbf{X}_j \times]a_j^2, +\infty[$ equipped with the metric (1.2). Then for any $u \in C^2(\mathbf{M}_j)$,

$$-\Delta_A u = -y^{2\delta_j} \Delta_{A_j}^{\mathbf{X}_j} u - y^{n\delta_j} \partial_y (y^{(2-n)\delta_j} \partial_y u), \quad (2.2)$$

where $\Delta_{A_j}^{\mathbf{X}_j}$ is the magnetic Laplacian on \mathbf{X}_j : if for local coordinates $\mathbf{h}_j =$

$\sum_{k,\ell} G_{k\ell} dx_k dx_\ell$ and $A_j = \sum_{k=1}^{n-1} a_{j,k} dx_k$, then

$$-\Delta_{A_j}^{\mathbf{X}_j} = \frac{1}{\sqrt{\det(G)}} \sum_{k,\ell} (i\partial_{x_k} + a_{j,k}) \left(\sqrt{\det(G)} G^{k\ell} (i\partial_{x_\ell} + a_{j,\ell}) \right).$$

We perform the change of variables $y = e^t$, and define the unitary operator $U : L^2(\mathbf{X}_j \times]2 \ln(a_j), +\infty[) \rightarrow L^2(\mathbf{M}_j)$, where $]2 \ln(a_j), +\infty[$ is equipped with the standard euclidian metric dt^2 , by $U(f) = y^{(n\delta_j-1)/2} f$. Thus $L^2(\mathbf{M}_j)$ is unitary equivalent to $L^2(\mathbf{X}_j \times]2 \ln(a_j), +\infty[)$, and

$$-U^* \Delta_A U f = -e^{2\delta_j t} \Delta_{A_j}^{\mathbf{X}_j} f + \frac{(n\delta_j - 1)[3 + \delta_j(n - 4)]}{4} e^{2t(\delta_j-1)} f - \partial_t (e^{2t(\delta_j-1)} \partial_t f). \quad (2.3)$$

Let us denote by $(\mu_\ell(j))_{\ell \in \mathbb{N}}$ the increasing sequence of eigenvalues of $-\Delta_{A_j}^{\mathbf{X}_j}$, each eigenvalue repeated according to its multiplicity. Then $-\Delta_A^{\mathbf{M}_j, D}$

is unitary equivalent to $\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D$,

$$\text{sp}(-\Delta_A^{\mathbf{M}_j, D}) = \text{sp}\left(\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D\right), \quad (2.4)$$

where $L_{j,\ell}^D$ is the Dirichlet operator on $L^2(]2 \ln(a_j), +\infty[)$ associated to

$$L_{j,\ell} = e^{2\delta_j t} \mu_\ell(j) + \frac{(n\delta_j - 1)}{4} [3 + \delta_j(n - 4)] e^{2t(\delta_j-1)} - \partial_t (e^{2t(\delta_j-1)} \partial_t). \quad (2.5)$$

If $\mu_\ell(j) > 0$ then $\text{sp}(L_{j,\ell}^D) = \text{sp}_d(L_{j,\ell}^D) = \{\mu_{\ell,k}(j); k \in \mathbb{N}\}$, where $(\mu_{\ell,k}(j))_{k \in \mathbb{N}}$ is the increasing sequence of eigenvalues of $L_{j,\ell}^D$, $\lim_{k \rightarrow +\infty} \mu_{\ell,k}(j) = +\infty$.

If $\mu_\ell(j) = 0$ then $\text{sp}(L_{j,\ell}^D) = \text{sp}_{\text{ess}}(L_{j,\ell}^D) = [\alpha_n, +\infty[$, with $\alpha_n = 0$ if $\delta_j < 1$, and $\alpha_n = (n - 1)^2/4$ if $\delta_j = 1$.

Since we have $\mu_0(j) = 0$ when $A = 0$, we get that $\text{sp}_{\text{ess}}(-\Delta_0) = [\alpha_n, +\infty[$.

If A satisfies assumptions (1.3) and (1.4), then $0 < \mu_0(j) \leq \mu_\ell(j)$ for all j and ℓ , (see for example [Hel]), so $\text{sp}(-\Delta_A^{\mathbf{M}_j, D}) = \{\mu_{\ell,k}(j); (\ell, k) \in \mathbb{N}^2\}$. As $\lim_{\ell \rightarrow +\infty} \mu_{\ell,k}(j) = +\infty$, each $\mu_{\ell,k}(j)$ is an eigenvalue of $-\Delta_A^{\mathbf{M}_j, D}$ of finite multiplicity, so $\text{sp}(-\Delta_A^{\mathbf{M}_j, D}) = \text{sp}_d(-\Delta_A^{\mathbf{M}_j, D})$. Therefore, we get that $\text{sp}_{\text{ess}}(-\Delta_A) = \emptyset$ \square

2.2 Proof of Theorem 1.2

We proceed as in [Mo-Tr].

We begin by establishing formula (1.8) for \mathbf{M}_j , with $-\Delta_A^{\mathbf{M}_j, D}$ defined in (2.1), instead of $-\Delta_A$. When $\delta_j = 1$ we make the same change of variables and functions as in the proof of Theorem 1.1, but when $1/n < \delta_j < 1$, we set $y = [(1 - \delta_j)t]^{1/(1-\delta_j)}$, and define the unitary operator

$$U : L^2(\mathbf{X}_j \times]\frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, +\infty[) \rightarrow L^2(\mathbf{M}_j), \text{ by } U(f) = y^{(n-1)\delta_j/2} f.$$

Then when $1/n < \delta_j < 1$,

$$-U^* \Delta_A U f = -[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} \Delta_{A_j}^{\mathbf{X}_j} f + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2 t^2} f - \partial_t^2 f. \quad (2.6)$$

As a matter of fact,

$$-U^* y^{n\delta_j} \partial_y [y^{(2-n)\delta_j} \partial_y U(f)] = -y^{(n+1)\delta_j/2} \partial_y [y^{(3-n)\delta_j/2} \partial_y f] - \frac{(n-1)\delta_j}{2} y^{2\delta_j-1} \partial_y f + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4} y^{-2(1-\delta_j)} f,$$

then using that $y^{\delta_j} \partial_y = \partial_t$ and that $t^\rho \partial_t = \partial_t(t^\rho \cdot) - \rho t^{\rho-1}$, we get easily (2.6).

Equality (2.4) still holds when $L_{j,\ell}^D$ is the Dirichlet operator on $L^2(] \frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, +\infty[)$ associated to

$$L_{j,\ell} = \mu_\ell(j) [(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2 t^2} - \partial_t^2. \quad (2.7)$$

From now on, any constant depending only on δ_j and on $\min_j \mu_0(j)$ will be invariably denoted by C .

As in [Mo-Tr], we will follow Titchmarsh's method. Using Theorem 7.4 in [Tit] page 146, we prove the following Lemma.

Lemma 2.1 *There exists $C > 1$ so that for any $\lambda \gg 1$ and any $\ell \in K_\lambda = \{l \in \mathbb{N}; \mu_\ell(j) \in [0, \lambda / \min_j a_j^2]\}$,*

$$|N(\lambda, L_{j,\ell}^D) - \frac{1}{\pi} w_{j,\ell}(\lambda)| \leq C \ln(\lambda), \quad (2.8)$$

$$\text{with } w_{j,\ell}(\mu) = \int_{\alpha_j}^{+\infty} [\mu - V_{j,\ell}(t)]_+^{1/2} dt = \int_{\alpha_j}^{T_j(\mu)} [\mu - V_{j,\ell}(t)]_+^{1/2} dt.$$

The potential $V_{j,\ell}$ is defined as following:

$$\begin{cases} \text{if } \delta_j = 1 & V_{j,\ell}(t) = \mu_\ell(j)e^{2t} + \frac{(n-1)^2}{4} \\ \text{if } 1/n < \delta_j < 1 & V_{j,\ell}(t) = \mu_\ell(j)[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2}t^{-2} \end{cases}, \quad (2.9)$$

and

$$\begin{cases} \text{if } \delta_j = 1 & \alpha_j = 2 \ln(a_j), \quad T_j(\mu) = \frac{1}{2} \ln(\mu/\mu_0(j)) \\ \text{if } 1/n < \delta_j < 1 & \alpha_j = \frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, \quad T_j(\mu) = \frac{1}{1-\delta_j} \left(\frac{\mu}{\mu_0(j)} \right)^{\frac{1-\delta_j}{2\delta_j}} \end{cases}. \quad (2.10)$$

Proof of Lemma 2.1

When $1/n < \delta_j < 1$, by enlarging \mathbf{M}_0 and reducing \mathbf{M}_j , we can take α_j large enough so that $V_{j,\ell}(t)$ is an increasing function on $[\alpha_j, +\infty[$ and $\lambda/\mu_\ell(j) \gg 1$ when $\ell \in K_\lambda$. Then, if $\alpha_j \leq Y < X(\lambda) = V_{j,\ell}^{-1}(\lambda)$, following the proof of Theorem 7.4 in [Tit] pages 146-147, we get that

$$|N(\lambda, L_{j,\ell}^D) - \frac{1}{\pi} w_{j,\ell}(\lambda)| \leq \quad (2.11)$$

$$C[\ln(\lambda - V_{j,\ell}(\alpha_j)) - \ln(\lambda - V_{j,\ell}(Y)) + (X(\lambda) - Y)(\lambda - V_{j,\ell}(Y)) + 1].$$

When $\delta_j = 1$, we choose $Y = X(\lambda) - \frac{\sqrt{\ln \lambda}}{\sqrt{\lambda}}$.

When $1/n < \delta_j < 1$, we choose $Y = X(\lambda) - \frac{\sqrt{\ln \lambda}}{\sqrt{\lambda}} \left(\frac{\lambda}{\mu_\ell(j)} \right)^{\frac{1-\delta_j}{4\delta_j}}$;

$$(X(\lambda) \sim \frac{1}{1-\delta_j} \left(\frac{\lambda}{\mu_\ell(j)} \right)^{\frac{1-\delta_j}{2\delta_j}}) \square$$

Let us recall the sharp asymptotic Weyl formula of L. Hörmander [Hor1] (see also [Hor2]).

Theorem 2.2 *There exists $C > 0$ so that for any $\mu \gg 1$*

$$|N(\mu, -\Delta_{A_j}^{\mathbf{X}_j}) - \frac{\omega_{n-1}}{(2\pi)^{n-1}} |\mathbf{X}_j| \mu^{(n-1)/2}| \leq C \mu^{(n-2)/2}. \quad (2.12)$$

Lemma 2.3 *There exists $C > 0$ such that for any $\lambda \gg 1$*

$$|N(\lambda, -\Delta_A^{\mathbf{M}_j, D}) - \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_j| \lambda^{n/2}| \leq \quad (2.13)$$

$$C \begin{cases} \lambda^{(n-1)/2} \ln(\lambda), & \text{if } 1/(n-1) \leq \delta_j \leq 1 \\ \lambda^{1/(2\delta_j)}, & \text{if } 1/n < \delta_j < 1/(n-1) \end{cases}.$$

Proof of Lemma 2.3 By the formula (2.4),

$$N(\lambda, -\Delta_A^{\mathbf{M}_j, D}) = \sum_{\ell=0}^{+\infty} N(\lambda, L_{j,\ell}^D). \quad (2.14)$$

As $N(\lambda, L_{j,\ell}^D) = 0$ when $\ell \notin K_\lambda$, (K_λ is defined in Lemma 2.1), the estimates (2.8), (2.12) and formula (2.14) prove that

$$|N(\lambda, -\Delta_A^{\mathbf{M}_j, D}) - \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda)| \leq C\lambda^{(n-1)/2} \ln(\lambda). \quad (2.15)$$

Let us denote

$$\Theta_j(\lambda) = \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda) \quad \text{and} \quad R_j(\mu) = \sum_{\ell=0}^{+\infty} [\mu - \mu_\ell(j)]_+^{1/2}. \quad (2.16)$$

As $R_j(\mu) = \frac{1}{2} \int_0^{+\infty} [\mu - s]_+^{-1/2} N(s, -\Delta_{A_j}^{\mathbf{X}_j}) ds$,

the Hörmander estimate (2.12) entails the following one.

There exists a constant $C > 0$ such that, for any $\mu \gg 1$,

$$|R_j(\mu) - \frac{\omega_{n-1}}{2(2\pi)^{n-1}} |\mathbf{X}_j| \int_0^{+\infty} [\mu - s]_+^{-1/2} s^{(n-1)/2} ds| \leq C\mu^{(n-1)/2}. \quad (2.17)$$

Writing in (2.9)

$$V_{j,\ell}(t) = \mu_\ell(j) V_j(t) + W_j(t), \quad (2.18)$$

we get that $\Theta_j(\lambda) = \frac{1}{\pi} \int_{\alpha_j}^{T_j(\lambda)} V_j^{1/2}(t) R_j\left(\frac{\lambda - W_j(t)}{V_j(t)}\right) dt$.

So according to (2.17)

$$|\Theta_j(\lambda) - \frac{\omega_{n-1} \Gamma(\frac{1}{2}) \Gamma(\frac{n+1}{2})}{(2\pi)^n \Gamma(1 + \frac{n}{2})} |\mathbf{X}_j| \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{n/2}}{V_j^{(n-1)/2}(t)} dt| \leq \quad (2.19)$$

$$C \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{(n-1)/2}}{V_j^{(n-2)/2}(t)} dt.$$

From the definitions (2.9) and (2.18) we get that

$$\left| \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{n/2}}{V_j^{(n-1)/2}(t)} dt - \lambda^{n/2} \frac{1}{(\delta_j n - 1) a_j^{2(\delta_j n - 1)}} \right| \leq C \lambda^{(n-1)/2}, \quad (2.20)$$

and

$$\int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{(n-1)/2}}{V_j^{(n-2)/2}(t)} dt \leq \quad (2.21)$$

$$C \begin{cases} \lambda^{(n-1)/2} & \text{if } 1/(n-1) < \delta_j \leq 1 \\ \lambda^{(n-1)/2} \ln \lambda & \text{if } 1/(n-1) = \delta_j \\ \lambda^{1/(2\delta_j)} & \text{if } 1/n < \delta \leq 1/(n-1) \end{cases}.$$

As $|\mathbf{M}_j| = \frac{|\mathbf{X}_j|}{(\delta_j n - 1) a_j^{2(\delta_j n - 1)}}$, we get (2.13) from (2.15), (2.16) and (2.19)—
(2.21) \square

To achieve the proof of Theorem 1.2, we proceed as in [Mo-Tr].

We denote $\mathbf{M}_0^0 = \mathbf{M} \setminus \left(\bigcup_{j=1}^J \overline{\mathbf{M}_j} \right)$, then

$$\mathbf{M} = \overline{\mathbf{M}_0^0} \cup \left(\bigcup_{j=1}^J \overline{\mathbf{M}_j} \right). \quad (2.22)$$

Let us denote respectively by $-\Delta_A^{\Omega, D}$ and by $-\Delta_A^{\Omega, N}$ the Dirichlet operator and the Neumann-like operator on an open set Ω of \mathbf{M} associated to $-\Delta_A$. $-\Delta_A^{\Omega, N}$ is the Friedrichs extension defined by the associated quadratic form $q_A^\Omega(u) = \int_{\Omega} |idu + Au|^2 d\mathbf{m}$, $u \in C^\infty(\overline{\Omega}; \mathbb{C})$, u with compact support in $\overline{\Omega}$. ($d\mathbf{m}$ is the n -form volume of \mathbf{M}).

The minimax principle and (2.22) imply that

$$\begin{aligned} N(\lambda, -\Delta_A^{\mathbf{M}_0^0, D}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_A^{\mathbf{M}_j, D}) &\leq N(\lambda, -\Delta_A) \quad (2.23) \\ &\leq N(\lambda, -\Delta_A^{\mathbf{M}_0^0, N}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_A^{\mathbf{M}_j, N}) \end{aligned}$$

The Weyl formula with remainder, (see [Hor2] for Dirichlet boundary condition and [Sa-Va] p. 9 for Neumann-like boundary condition), gives that

$$N(\lambda, -\Delta_A^{\mathbf{M}_0^0, Z}) = \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_0^0| \lambda^{n/2} + \mathbf{O}(\lambda^{(n-1)/2}); \quad (\text{for } Z = D \text{ and for } Z = N). \quad (2.24)$$

For $1 \leq j \leq J$, the asymptotic formula for $N(\lambda, -\Delta_A^{\mathbf{M}_j, N})$,

$$N(\lambda, -\Delta_A^{\mathbf{M}_j, N}) = \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_j| \lambda^{n/2} + \mathbf{O}(r(\lambda)), \quad (2.25)$$

is obtained as for the Dirichlet case (2.13) by noticing that $N(\lambda, L_{j,\ell}^D) \leq N(\lambda, L_{j,\ell}^N) \leq N(\lambda, L_{j,\ell}^D) + 1$, where $L_{j,\ell}^D$ and $L_{j,\ell}^N$ are Dirichlet and Neumann-like operators on a half-line $I =]\alpha_j, +\infty[$, associated to the same differential Schrödinger operator $L_{j,\ell}$ defined by (2.5) when $\delta_j = 1$, and by (2.7) otherwise. (The Neumann-like boundary condition is of the form $\partial_t u(\alpha_j) + \beta_j u(\alpha_j) = 0$ because of the change of functions performed by U^*).

We get (1.8) from (2.13) and (2.23)—(2.25) \square

2.3 Proof of Theorem 1.3

Lemma 2.4 *For any $j \in \{1, \dots, J\}$, there exists a one-form A_j satisfying (1.3) and the following property.*

There exists $\tau_0 = \tau_0(A_j) > 0$ and $C = C(A_j) > 0$ such that for any $\lambda \gg 1$, if $e(\tau, j) = \inf_{u \in C^\infty(\mathbf{X}_j), \|u\|_{L^2(\mathbf{X}_j)}=1} \|idu + \tau u A_j\|_{L^2(\mathbf{X}_j)}^2$ denotes the first eigenvalue of $-\Delta_{\tau A_j}^{\mathbf{X}_j}$, then

$$e(\tau, j) \geq C\tau^2; \quad \forall \tau \in]0, \tau_0]. \quad (2.26)$$

Proof of Lemma 2.4. When $n = 2$, we can take $A_j = \omega_j d\mathbf{x}_j$, ($d\mathbf{x}_j$ is the $(n-1)$ -form volume of \mathbf{X}_j), for some constant $\omega_j \in \mathbb{R} \setminus \frac{2\pi}{|\mathbf{X}_j|} \mathbb{Z}$, then $e(\tau, j) = \tau^2 \omega_j^2$ for small $|\tau|$.

When $n \geq 3$, we have $e(0, j) = 0$, $\partial_\tau e(0, j) = 0$ and

$$\partial_\tau^2 e(0, j) = \int_{\mathbf{X}_j} \left[|A_j|^2 - (-\Delta_0^{\mathbf{X}_j})^{-1} (d^* A_j) \cdot (d^* A_j) \right] d\mathbf{x}_j.$$

(d^* is the adjoint of d defined on functions, and $(-\Delta_0^{\mathbf{X}_j})^{-1}$ is the inverse of the Laplace-Beltrami operator on functions, which is well-defined on the space $\{f \in L^2(\mathbf{X}_j); \int_{\mathbf{X}_j} f d\mathbf{x}_j = 0\}$).

To the non-negative quadratic form $A_j \rightarrow \partial_\tau^2 e(0, j)$, we associate a self-adjoint operator P on $T^*(\mathbf{X}_j)$, which is a pseudodifferential operator of order 0 with principal symbol, the square matrix $p_0(x, \xi) = (p_0^{ik}(x, \xi))_{1 \leq i, k \leq n-1}$ defined as follows. In local coordinates, if $\mathbf{h}_j = G_{ik}(x) dx_i dx_k$, then

$$p_0^{ik}(x, \xi) = G^{ik}(x) - \sum_{\ell, m} G^{im}(x) G^{\ell k}(x) \frac{\xi_m}{|\xi|} \frac{\xi_\ell}{|\xi|}; \quad (|\xi|^2 = \sum_{\ell, m} G^{m\ell}(x) \xi_m \xi_\ell).$$

As the non-negative symmetric matrix $p_0(x, \xi)$ is not the zero matrix, there exists A_j such that $P(A_j) \neq 0$ and by the positivity of P , $\partial_\tau^2 e(0, j) = \int_{\mathbf{X}_j} \langle P(A_j) | A_j \rangle d\mathbf{x}_j > 0 \square$

Lemma 2.5 *For a one-form A satisfying (1.4), there exists a constant $C_A > 0$ such that, if u is a function in $L^2(\mathbf{M})$ such that $du \in L^2(\mathbf{M})$ and*

$$\forall j = 1, \dots, J, \quad \int_{\mathbf{X}_j} u(x_j, y) d\mathbf{x}_j = 0, \quad \forall y \in]a_j^2, +\infty[, \quad (2.27)$$

then $\forall \tau \in]0, 1]$,

$$\|idu + \tau u A\|_{L^2(\mathbf{M})}^2 \leq (1 + \tau C_A) \|idu\|_{L^2(\mathbf{M})}^2 + C_A \|u\|_{L^2(\mathbf{M})}^2. \quad (2.28)$$

Proof of Lemma 2.5. First we remark that the inequality

$$|idu + \tau u A|^2 \leq (1 + \rho) |du|^2 + (1 + \rho^{-1}) |\tau u A|^2 \quad (2.29)$$

is satisfied for any $\rho > 0$.

For $\rho = \tau$ we get that there exists a constant $C_A^0 > 0$, depending only on A/\mathbf{M}_0 , such that

$$\|idu + \tau u A\|_{L^2(\mathbf{M}_0)}^2 \leq (1 + \tau) \|idu\|_{L^2(\mathbf{M}_0)}^2 + \tau C_A^0 \|u\|_{L^2(\mathbf{M}_0)}^2. \quad (2.30)$$

We get also for $\rho = \tau$ that for any $j \in \{1, \dots, J\}$,

$$\int_{a_j^2}^{+\infty} \|idu + \tau u A\|_{L^2(\mathbf{X}_j)}^2 y^{(2-n)\delta_j} dy \leq \quad (2.31)$$

$$\int_{a_j^2}^{+\infty} \left((1 + \tau) \|idu\|_{L^2(\mathbf{x}_j)}^2 + \tau C_A^j \|u\|_{L^2(\mathbf{x}_j)}^2 \right) y^{(2-n)\delta_j} dy ,$$

for some constant C_A^j depending only on A/X_j .

But (2.27) implies that

$$\|u\|_{L^2(\mathbf{x}_j)}^2 \leq \frac{1}{\mu_1(j)} \|idu\|_{L^2(\mathbf{x}_j)}^2 , \quad (2.32)$$

with $(\mu_\ell(j))_{\ell \in \mathbb{N}}$ the sequence of eigenvalues of Laplace-Beltrami operator on \mathbf{X}_j , $\mu_0(j) = 0 < \mu_1(j) \leq \mu_2(j) \leq \dots$. So if (2.27) is satisfied then (2.31) and (2.32) imply that

$$\|idu + \tau u A\|_{L^2(\mathbf{M}_j)}^2 \leq (1 + \tau c_A^j) \|idu\|_{L^2(\mathbf{M}_j)}^2 , \quad (2.33)$$

for some constant c_A^j depending only on A/X_j .

The existence of a constant $C_A > 0$ satisfying the inequality (2.28) follows from (2.30) and (2.33) for $j = 1, \dots, J$ \square

Lemma 2.6 *When A satisfies (1.3), (1.4) and Lemma 2.4 , then as $\lambda \rightarrow +\infty$, the following Weyl formula is satisfied.*

$$N(\lambda, -\Delta_{(\lambda^{-\rho} A)}) = |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + \mathbf{O}(r_0(\lambda)) , \quad (2.34)$$

with

$$\rho = \begin{cases} 1/2, & \text{if } 2/n \leq \delta \leq 1 \\ (n\delta - 1)/2, & \text{if } 1/n < \delta < 2/n \end{cases} , \quad (2.35)$$

δ and ω_d are as in Theorem 1.2, and the function $r_0(\lambda)$ is the one defined by (1.11) .

Proof of Lemma 2.6. Since A satisfies Lemma 2.4, we have

$$C/\lambda^{2\rho} \leq \mu_0(j) \quad \text{and} \quad C \leq \mu_1(j) ,$$

where $(\mu_\ell(j))_{\ell \in \mathbb{N}}$ denotes now the increasing sequence of eigenvalues of $-\Delta_{\lambda^{-\rho} A_j}$. Hence we can mimick the proof of Theorem 1.2. More precisely Lemma 2.1 holds for any $\ell \in K_\lambda$, $\ell \neq 0$, and to get the result it only remains to prove that we have, for $L_{j,0}$ defined by (2.5) if $\delta_j = 1$, and by (2.7) otherwise,

$$N(\lambda, L_{j,0}^D) = \mathbf{O}(r_0(\lambda)) .$$

This can easily be done as follows.

When $\delta_j = 1$, ($\rho = 1/2$), it is easy to see that

$$N(\lambda, L_{j,0}^D) \leq N(\lambda + C, L^{D,\lambda}) \leq C\lambda^{1/2} \ln(\lambda),$$

where $L^{D,\lambda}$ is the Dirichlet operator on $]0, +\infty[$ associated to $\frac{C}{\lambda}e^{2t} - \partial_t^2$.

When $0 < \delta_j < 1$, by scaling we have that

$$N(\lambda, L_{j,0}^D) \leq N((\lambda + C)^{1+2\rho(1-\delta_j)}, L^D) \leq C\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)},$$

where L^D is the Dirichlet operator on $]0, +\infty[$ associated to $\frac{1}{C^2}t^{\frac{2\delta_j}{1-\delta_j}} - \partial_t^2$.

When $2/n \leq \delta < 1$, as $2/n \leq \delta \leq \delta_j$, then

$$\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)} = \lambda^{(2-\delta_j)/(2\delta_j)} \leq \lambda^{(2-\delta)/(2\delta)} \leq \lambda^{(n-1)/2} = \mathbf{O}(r_0(\lambda)).$$

When $1/n < \delta < 2/n$, as $\delta \leq \delta_j$, then

$$\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)} \leq \lambda^{(1+2\rho(1-\delta))/(2\delta)} = \lambda^{(n-(n\delta-1))/2} = \mathbf{O}(r_0(\lambda)) \quad \square$$

To achieve the proof of Theorem 1.3, we take a one-form A satisfying the assumptions of Lemma 2.6.

We remark that any eigenfunction u of the Laplace-Beltrami operator $-\Delta$ on \mathbf{M} associated to an eigenvalue in $] \inf \text{sp}_{\text{ess}}(-\Delta), +\infty[$, satisfies (2.27). So if H_λ is the subspace of $L^2(\mathbf{M})$ spanned by eigenfunctions of $-\Delta$ associated to eigenvalues in $]0, +\infty[$, then, by (2.28) of Lemma 2.5 with $\tau = 1/\lambda^\rho$, with ρ defined by (2.35), we have

$$\forall u \in H_\lambda, \quad \|idu + \frac{1}{\lambda^\rho}uA\|_{L^2(\mathbf{M})}^2 \leq (1 + \frac{C_A}{\lambda^\rho})\|du\|_{L^2(\mathbf{M})}^2 + C_A\|u\|_{L^2(\mathbf{M})}^2 \quad (2.36)$$

So

$$\dim(H_\lambda) \leq N((1 + \frac{C_A}{\lambda^\rho})\lambda + C_A, -\Delta_{(\lambda^{-\rho}A)}). \quad (2.37)$$

The estimates (2.34) and (2.37) prove (1.10), by noticing that $\lambda^{n/2}/\lambda^\rho = \mathbf{O}(r_0(\lambda)) \quad \square$

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