

# Eigenvalue bounds for radial magnetic bottles on the disk

Françoise Truc\*

September 9, 2011

## Abstract

We consider a Schrödinger operator  $H_A^D$  with a non-vanishing radial magnetic field  $B = dA$  and Dirichlet boundary conditions on the unit disk. We assume growth conditions on  $B$  near the boundary which guarantee in particular the compactness of the resolvent of this operator. Under some assumptions on an additional radial potential  $V$  the operator  $H_A^D - V$  has a discrete negative spectrum and we obtain an upper bound of the number of negative eigenvalues. As a consequence we get an upper bound of the number of eigenvalues of  $H_A^D$  smaller than any positive value  $\lambda$ , which involves the minimum of  $B$  and the square of the  $L^2$ -norm of  $A(r)/r$ , where  $A(r)$  is the specific magnetic potential defined as the flux of the magnetic field through the disk of radius  $r$  centered in the origin.

## 1 Introduction

Let us consider a particle in a domain  $\Omega$  in  $\mathbb{R}^2$  in the presence of a magnetic field  $B$ . We define the 2-dimensional magnetic Laplacian associated to this particle as follows: Let  $A$  be a magnetic potential associated to  $B$ ; it means that  $A$  is a smooth real one-form on  $\Omega \subset \mathbb{R}^2$ , given by  $A = \sum_{j=1}^2 a_j dx_j$ , and that the magnetic field  $B$  is the two-form  $B = dA$ . We have  $B(x) = \mathbf{b}(x) dx_1 \wedge dx_2$  with  $\mathbf{b}(x) = \partial_1 a_2(x) - \partial_2 a_1(x)$ . The magnetic connection  $\nabla = (\nabla_j)$  is the differential operator defined by

$$\nabla_j = \frac{\partial}{\partial x_j} - ia_j .$$

The 2-dimensional magnetic Schrödinger operator  $H_A$  is defined by

$$H_A = - \sum_{j=1}^2 \nabla_j^2 .$$

---

\*Institut Fourier, francoise.truc@ujf-grenoble.fr Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France)

The magnetic Dirichlet integral  $h_A = \langle H_A \cdot | \cdot \rangle$  is given, for  $u \in C_0^\infty(\Omega)$ , by

$$h_A(u) = \int_{\Omega} \sum_{j=1}^2 |\nabla_j u|^2 |dx|. \quad (1.1)$$

From the previous definitions and the fact that the formal adjoint of  $\nabla_j$  is  $-\nabla_j$ , it is clear that the operator  $H_A$  is symmetric on  $C_0^\infty(\Omega)$ .

In [5] we discuss the essential self-adjointness of this operator. The result in dimension 2 is the following

**Theorem 1.1** *Assume that  $\partial\Omega$  is compact and that  $B(x)$  satisfies near  $\partial\Omega$*

$$\mathbf{b}(x) \geq (D(x))^{-2}, \quad (1.2)$$

*then the Schrödinger operator  $H_A$  is essentially self-adjoint. ( $D(x)$  denotes the distance to the boundary). This still holds true for any gauge  $A'$  such that  $dA' = dA = B$ .*

We have, using Cauchy-Schwarz inequality,

$$|\langle \mathbf{b}(x)u, u \rangle| = |\langle [\nabla_1, \nabla_2]u, u \rangle| \leq \|\nabla_1 u\|^2 + \|\nabla_2 u\|^2 \quad u \in C_0^\infty(\Omega).$$

This gives the well-known lower bound

$$\forall u \in C_0^\infty(\Omega), \quad h_A(u) \geq \left| \int_{\Omega} \mathbf{b}(x)|u|^2 |dx| \right|. \quad (1.3)$$

In this paper, we do not use the conditions (1.2) but we assume nevertheless that  $\mathbf{b}(\mathbf{x})$  grows to infinity as  $x$  approaches the boundary. The operator  $H_A^D$  defined by Friedrichs extension of the quadratic form  $h_A$  has a compact resolvent. By analogy with magnetic bottles on the whole space (see [1, 4, 19]), such an operator is called a magnetic bottle on the disk.

We will deal with spectral estimates for the operator  $H_A^D$ , using a perturbative method: introducing an additional non-negative bounded and radial potential  $V$ , we obtain an upper bound of the number  $N(A, V)$  of negative eigenvalues of the operator  $H_A^D - V$  (Theorem 2.1) and deduce, for any  $\lambda > 0$ , an upper bound of the number  $N(H_A^D, \lambda)$  of eigenvalues of the operator  $H_A^D$  smaller than  $\lambda$  (Theorem 2.2). Theorem 2.1 can be seen as a magnetic version of the Cwikel-Lieb-Rosenblum inequality (see [6, 16, 18]). The CLR inequality provides a bound on the number of negative eigenvalues of Schrödinger operators in  $\mathbb{R}^d$  for  $d \geq 3$  (without magnetic field) and is a particular case of Lieb-Thirring inequalities (see [15, 17]).

Eigenvalue bounds were recently studied for magnetic Hamiltonians on  $\mathbb{R}^2$ , for constant magnetic fields (see [10]), for Aharonov-Bohm magnetic fields (see [3, 14]) and for a large class of magnetic fields (see [12]). However, in [12], the total magnetic flux  $\phi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathbf{b}(x)dx$  has to be finite and the dependence on the magnetic field is not explicit even in the radial case. In our result, the total flux is not necessarily finite

(see example 2.4) and the upper bound involves explicitly the square of the magnetic potential.

Magnetic Lieb-Thirring inequalities were also obtained for Pauli operators (see [7, 8]), and links between magnetic and non-magnetic Lieb-Thirring inequalities were discussed in [9].

## Aknowledgements

The author would like to thank A. Laptev for fruitful discussions, Y. Colin de Verdière for useful comments, J.P. Truc for the communication about Proposition 5.2 and the referee for careful reading and helpful suggestions.

## 2 Main results

We consider a smooth magnetic field  $B = \mathbf{b}(x)dx_1 \wedge dx_2$  and a scalar potential  $V$  on the unit disk  $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = r^2 < 1\}$  so that

- $(H_1)$   $K = \inf_{x \in \Omega} \mathbf{b}(x) > 0$  and  $\mathbf{b}(x) \rightarrow +\infty$  as  $D(x) \rightarrow 0$  (i.e as  $x$  approaches the boundary.)
- $(H_2)$   $B$  is radially symmetric ( consequently we write  $\mathbf{b}(r)$  instead of  $\mathbf{b}(x)$ )
- $(H_3)$   $V \in L^1(\Omega)$ ,  $V$  radial and non-negative,  $V$  bounded from above .

From assumption  $(H_1)$  and from inequality (1.3) we deduce that for any gauge  $A$  associated to  $B$ , the operator  $H_A^D$  has a compact resolvent, and assumption  $(H_3)$  entails that the negative spectrum of  $H_A^D - V$  is discrete, where  $H_A^D - V$  denotes the operator defined by Friedrichs extension of the quadratic form  $h_A - V$ .

Using assumption  $(H_2)$  we introduce polar coordinates  $(r, \theta)$ , ( $r \in \mathbb{R}^+$ ,  $\theta \in [0, 2\pi[$ ) and consider the following magnetic potential :

$$A = -a(r) \sin \theta dx_1 + a(r) \cos \theta dx_2, \quad a(r) = \frac{1}{r} \int_0^r \mathbf{b}(t) t dt . \quad (2.1)$$

We have  $dA = B$  and

$$A = A(r) d\theta \quad \text{with} \quad A(r) = r a(r) = \int_0^r \mathbf{b}(t) t dt . \quad (2.2)$$

$A(r)$  is the flux of the magnetic field through the disk of radius  $r$  centered in the origin. The function  $a(r) = A(r)/r$  is well-defined (and smooth) at the origin and it is the amplitude of the magnetic potential  $A$  in cartesian coordinates .

The first theorem provides an upper bound of the number  $N(A, V)$  of negative eigenvalues of the operator  $H_A^D - V$  where  $A$  is the magnetic potential defined by (2.2).

From now on,  $A$  denotes this specific potential.

Noticing that we have  $N(A', V) = N(A, V)$  for any gauge  $A'$  so that  $dA' = dA = B$ , we will prove the following

**Theorem 2.1** *If assumptions  $(H_1)(H_2)(H_3)$  are verified and if moreover*

$$\mathbf{b}(x) \leq M(D(x))^{-\beta}, \quad 0 < \beta < \frac{3}{2} \quad (2.3)$$

for some  $M > 0$ , then

$$N(A, V) \leq \frac{1}{\sqrt{1-\alpha}} \int_0^1 \left[ \left( \frac{1}{\alpha} - 1 \right) \frac{A^2(r)}{r^2} + V(r) \right] r dr + 2 \int_0^1 \left[ 1 + |\log[r\sqrt{K}]| \right] V(r) r dr$$

for any  $\alpha \in ]0, 1[$ .

This inequality still holds when we replace in the left-hand side  $N(A, V)$  by  $N(A', V)$ , where  $A'$  is any gauge verifying  $dA' = dA = B$ .

The second theorem is a consequence of the first one and provides an explicit upper bound of the number  $N(H_A^D, \lambda)$  of the eigenvalues of  $H_A^D$  smaller than any positive value  $\lambda$  :

**Theorem 2.2** *If assumptions  $(H_1)$  and  $(H_2)$  are verified and if moreover*

$$\mathbf{b}(x) \leq M(D(x))^{-\beta}, \quad 0 < \beta < \frac{3}{2}$$

for some  $M > 0$ , then the number of eigenvalues of the operator  $H_A^D$  smaller than  $\lambda$  satisfies, for any  $\alpha \in ]0, 1[$ , the following inequality

$$N(H_A^D, \lambda) \leq c_K \lambda + \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} \int_0^1 \left( \frac{A(r)}{r} \right)^2 r dr \quad (2.4)$$

with

- $c_K = \frac{3 - \log K}{2}$  if  $0 < K \leq 1$
- $c_K = \left[ \frac{1 + \log K}{2} + \frac{1}{K} \right]$  if  $K > 1$ ,

Inequality (2.4) still holds when we replace in the left-hand side  $N(H_A^D, \lambda)$  by  $N(H_{A'}^D, \lambda)$ , where  $A'$  is any gauge verifying  $dA' = dA = B$ .

**Remark 2.3** *The minimum of the right-hand side is obtained by taking*

$$\alpha_\lambda = \frac{-3I + \sqrt{I^2 + 4I\lambda}}{\lambda - 2I}$$

with  $I := \int_0^1 \left( \frac{A(r)}{r} \right)^2 r dr$ .

**Example 2.4** Consider a magnetic field  $B$  as in the definition (3.2) below, and assume  $b(r) \equiv 1$  and  $\beta = 1$ . Then  $c_K = \frac{3}{2}$ , the chosen gauge is  $A(r) = \int_0^r \mathbf{b}(t)tdt = -\ln(1-r) - r$  and the corresponding value of  $I$  is

$$I = \int_0^1 \frac{[\ln(1-r) + r]^2}{r} dr = 2\zeta(3) - \frac{3}{2}. \quad (2.5)$$

### 3 Proofs

#### 3.1 Proof of Theorem 2.1

Let us introduce the polar coordinates  $x = (r, \theta)$ ,  $r \in \mathbb{R}^+$ ,  $\theta \in [0, 2\pi[$ . We have denoted by  $A$  the following vector potential :

$$A = A(r)d\theta \quad \text{with} \quad A(r) = ra(r) = \int_0^r \mathbf{b}(t)tdt. \quad (3.1)$$

Due to assumption (2.3) the magnetic field we consider is of the type

$$\mathbf{b}(r) = \frac{b(r)}{(1-r)^\beta}, \quad \text{with} \quad \max_{[0,1[} b(r) \leq M \quad \text{and} \quad \beta < \frac{3}{2}. \quad (3.2)$$

We first prove the following

**Lemma 3.1** *If  $B$  satisfies (3.2), then we can find some constant  $C$  so that  $A$  writes  $A = A(r)d\theta = ra(r)d\theta$  where*

- if  $\beta < 1$   $\max_{[0,1[} a(r) \leq C$ .
- if  $\beta = 1$   $a(r) = \tilde{a}(r) \ln(1-r)$ , with  $\max_{[0,1[} \tilde{a}(r) \leq C$ .
- if  $\beta > 1$   $a(r) = \frac{\tilde{a}(r)}{(1-r)^{\beta-1}}$ , with  $\max_{[0,1[} \tilde{a}(r) \leq C$ .

In particular  $\int_0^1 \left(\frac{A(r)}{r}\right)^2 r dr < \infty$ .

*Proof.*–

Let us explain the case  $\beta > 1$ . The method for the case  $\beta = 1$  is the same.

From (3.2) we get

$$0 \leq \frac{1}{r} \int_0^r \mathbf{b}(t)tdt \leq \frac{1}{r} \int_0^r b(t)t(1-t)^{-\beta} dt \leq M \int_0^r (1-t)^{-\beta} dt \leq M \frac{(1-r)^{-\beta+1}}{\beta-1} \quad \text{and the result follows.}$$

The case  $\beta < 1$  is straightforward.

□

We come now to the proof of Theorem 2.1, following the method of [13]. The quadratic form associated to  $H_A^D - V$  can be rewritten as

$$h_{A,V}(u) = \int_0^1 \int_0^{2\pi} \left[ \left| \frac{\partial u}{\partial r} \right|^2 - V(r)|u|^2 + r^{-2} \left[ \left[ \frac{\partial}{\partial \theta} - iA(r) \right] u \right]^2 \right] r dr d\theta \quad (3.3)$$

for any  $u \in C_0^\infty([0, 1[ \times [0, 2\pi[)$ . Changing variables  $r = e^t$  and denoting  $w(t, \theta) = u(e^t, \theta)$  for  $t \in ]-\infty, 0[$  and  $\theta \in [0, 2\pi[$  we transfer the form  $h_{A,V}(u)$  to

$$\tilde{h}_{A,V}(w) = \int_{-\infty}^0 \int_0^{2\pi} \left[ \left| \frac{\partial w}{\partial t} \right|^2 - \tilde{V}(t)|w|^2 + \left[ \left[ \frac{\partial}{\partial \theta} - if(t) \right] w \right]^2 \right] dt d\theta \quad (3.4)$$

with

$$\tilde{V}(t) = e^{2t}V(e^t), \quad f(t) = A(e^t).$$

By expanding a given function  $w \in C_0^\infty([-\infty, 0[ \times [0, 2\pi[)$  into a Fourier series we obtain that  $\tilde{h}_{A,V}(w) = \bigoplus_{\ell \in \mathbb{Z}} h_{A,V}^\ell(w_\ell)$  with

$$h_{A,V}^\ell(v) = \int_{-\infty}^0 \left| \frac{\partial v}{\partial t} \right|^2 + \left[ (\ell - f(t))^2 - \tilde{V}(t) \right] |v|^2 dt,$$

and  $w_\ell = \Pi_\ell(w)$  where  $\Pi_\ell$  is the projector acting as

$$\Pi_\ell(w)(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\ell(\theta - \theta')} w(r, \theta') d\theta'.$$

We write, for any  $\alpha \in ]0, 1[$  and any  $\ell \in \mathbb{Z}^*$

$$h_{A,V}^\ell(v) \geq \int_{-\infty}^0 \left| \frac{\partial v}{\partial t} \right|^2 + \left[ \left(1 - \frac{1}{\alpha}\right) f^2(t) - \tilde{V}(t) + (1 - \alpha)\ell^2 \right] |v|^2 dt.$$

Let us denote by  $L_\alpha$  the operator associated via Friedrichs extension to the quadratic form

$$q_\alpha(v) = \int_{-\infty}^0 \left| \frac{\partial v}{\partial t} \right|^2 + \left[ \left(1 - \frac{1}{\alpha}\right) f^2(t) - \tilde{V}(t) \right] |v|^2 dt.$$

$L_\alpha$  and  $q_\alpha$  depend on  $V$  and  $A$  but we skip the reference to  $V$  and  $A$  in notations for the sake of simplicity. Since

$$h_{A,V}^\ell \geq q_\alpha + (1 - \alpha)\ell^2,$$

the number  $N(h_{A,V}^\ell)$  of negative eigenvalues of  $h_{A,V}^\ell$  is smaller than the number of negative eigenvalues of  $L_\alpha + (1 - \alpha)\ell^2$ . Hence denoting by  $\{-\mu_k^\alpha\}$  the negative eigenvalues of  $L_\alpha$  and by  $I_\ell$  the set  $\{k \in \mathbb{N}; -\mu_k^\alpha + (1 - \alpha)\ell^2 < 0\}$  for any  $\ell \in \mathbb{Z}^*$ , we get

$$N(A, V) \leq \sum_{\ell \in \mathbb{Z}^*} \sum_{k \in I_\ell} 1 + N(h_{A,V}^0).$$

Noticing that the sum in the right-hand side is taken over the  $(\ell, k)$  so that  $0 < |\ell| \leq \frac{1}{\sqrt{1-\alpha}} \sqrt{\mu_k^\alpha}$  we write

$$N(A, V) \leq \frac{2}{\sqrt{1-\alpha}} \sum_{k \in \mathbb{N}} \sqrt{\mu_k^\alpha} + N(h_{A,V}^0). \quad (3.5)$$

Let us extend the functions  $f$  and  $\tilde{V}$  to  $\mathbb{R}$  by zero and denote respectively by  $f_1$  and  $\tilde{V}_1$  these extensions.

Since  $C_0^\infty([-\infty, 0]) \subset C_0^\infty(\mathbb{R})$ , the negative eigenvalues  $\{-\nu_k^\alpha\}$  of the operator  $L_1^\alpha$  associated via Friedrichs extension to the quadratic form

$$q_1^\alpha(v) = \int_{-\infty}^{+\infty} \left| \frac{\partial v}{\partial t} \right|^2 + \left[ \left(1 - \frac{1}{\alpha}\right) f_1^2(t) - \tilde{V}_1(t) \right] |v|^2 dt$$

verify

$$\sum_{k \in \mathbb{N}} \sqrt{\nu_k^\alpha} \leq \sum_{k \in \mathbb{N}} \sqrt{\nu_k^\alpha}. \quad (3.6)$$

Applying the sharp inequality of Hundertmarkt-Lieb-Thomas [11] (see Appendix ) to the operator  $L_1^\alpha$  we get

$$\begin{aligned} \sum_{k \in \mathbb{N}} \sqrt{\nu_k^\alpha} &\leq \frac{1}{2} \int_{-\infty}^{+\infty} \left[ \left(\frac{1}{\alpha} - 1\right) f_1^2(t) + \tilde{V}_1(t) \right] dt \\ &\leq \frac{1}{2} \int_{-\infty}^0 \left[ \left(\frac{1}{\alpha} - 1\right) f^2(t) + \tilde{V}(t) \right] dt \\ &\leq \frac{1}{2} \int_0^1 \left[ \left(\frac{1}{\alpha} - 1\right) \frac{A^2(r)}{r^2} + V(r) \right] r dr. \end{aligned} \quad (3.7)$$

To conclude we need the following

**Lemma 3.2** *Assume that  $K = \inf_{x \in \Omega} \mathbf{b}(x) > 0$ . Then for any  $\varepsilon \in ]0, 1[$*

$$N(h_{A,V}^0) = N(h_{A,0}^0 - V) \leq \frac{1}{\varepsilon} \int_0^1 \left[ 1 + \left| \log \left( \sqrt{\frac{(1-\varepsilon)K}{\varepsilon}} r \right) \right| \right] V(r) r dr. \quad (3.8)$$

*In particular*

$$N(h_{A,V}^0) \leq 2 \int_0^1 \left[ 1 + \left| \log(\sqrt{K}r) \right| \right] V(r) r dr. \quad (3.9)$$

*Proof.*–

- Step 1 : From (1.3) we get that  $h_A(u) \geq K \int_{\Omega} |u|^2 |dx| \quad \forall u \in C_0^\infty(\Omega)$ , which implies for  $h_{A,0}^0$  (returning to the variable  $r$  and considering  $V \equiv 0$ ),

$$\begin{aligned} h_{A,0}^0(w) &= \int_0^1 \left[ \left| \frac{\partial w}{\partial r} \right|^2 + r^{-2} A^2(r) |w^2| \right] r dr \\ &\geq K \int_0^1 |w|^2 r dr \quad \forall w \in C_0^\infty([0, 1]) . \end{aligned}$$

We write for any  $\varepsilon \in ]0, 1[$

$$N(h_{A,0}^0 - V) \leq N(\varepsilon h_{A,0}^0 + (1-\varepsilon)K - V) \leq N\left(h_{A,0}^0 + \frac{(1-\varepsilon)K}{\varepsilon} - \frac{V}{\varepsilon}\right), \quad (3.10)$$

where we have used the fact that multiplying an operator by a positive constant does not change the number of its negative eigenvalues.

- Step 2 : We establish the following upper bound :

$$N(h_{A,0}^0 + 1 - V) = N(h_{A,V}^0 + 1) \leq \int_0^1 [1 + |\log r|] V(r) r dr . \quad (3.11)$$

We have

$$\begin{aligned} h_{A,V}^0(w) &= \int_0^1 \left[ \left| \frac{\partial w}{\partial r} \right|^2 + [r^{-2} A^2(r) - V(r)] |w^2| \right] r dr \\ &\geq \int_0^1 \left[ \left| \frac{\partial w}{\partial r} \right|^2 - V(r) |w^2| \right] r dr \quad \forall w \in C_0^\infty([0, 1]) . \end{aligned}$$

By the variational principle,

$$N(h_{A,V}^0 + 1) \leq N(P_0 + 1 - V), \quad (3.12)$$

where  $P_0$  is the operator generated by the closure, in  $L^2([0, 1], r dr)$  of the quadratic form

$$\int_0^1 \left| \frac{\partial w}{\partial r} \right|^2 r dr, \quad w \in C_0^\infty([0, 1]) .$$

Considering the mapping  $U : L^2([0, 1], r dr) \rightarrow L^2([0, 1], dr)$  defined by  $(Uf)(r) = r^{1/2} f(r)$  we get that

$$N(P_0 + 1 - V) \leq N(T_0 + 1 - V) \quad (3.13)$$

where the operator  $T_0 = UP_0U^{-1}$  is the Sturm-Liouville operator on  $L^2([0, 1], dr)$  acting on its domain by

$$(T_0 u)(r) = -u''(r) - \frac{u(r)}{4r^2}, \quad u(0) = u(1) = 0 . \quad (3.14)$$

The upper bound (3.11) will follow from the properties of  $G(r, r, 1)$ , the diagonal element of the integral kernel of  $(T_0 + 1)^{-1}$ . Precisely we have

$$G(r, r, 1) \leq r(1 + |\log r|), \quad r \in [0, 1[. \quad (3.15)$$

The proof of (3.15) is given in Appendix . The Birman-Schwinger principle then yields

$$N(T_0 + 1 - V) \leq \int_0^1 G(r, r, 1)V(r)dr \leq \int_0^1 [1 + |\log r|] V(r)rdr. \quad (3.16)$$

This ends the proof of (3.11), together with the inequalities (3.12) and (3.13).

- Step 3 : We mimick the previous method to get, for any strictly positive number  $k$

$$N(h_{A,0}^0 + k^2 - V) \leq \int_0^1 [1 + |\log(kr)|] V(r)rdr. \quad (3.17)$$

Due to the Birman-Schwinger principle it suffices to prove that, for any strictly positive number  $k$

$$G(r, r, k^2) \leq r(1 + |\log(kr)|), \quad r \in [0, 1[. \quad (3.18)$$

This is done in Appendix .

- Step 4 : Returning to (3.10) and applying (3.17) with  $k^2 = \frac{(1-\varepsilon)K}{\varepsilon}$  and  $\frac{V}{\varepsilon}$  instead of  $V$  we get, for any  $\varepsilon \in ]0, 1[$

$$N(h_{A,0}^0 - V) \leq N\left(h_{A,0}^0 + \frac{(1-\varepsilon)K}{\varepsilon} - \frac{V}{\varepsilon}\right) \quad (3.19)$$

$$\leq \frac{1}{\varepsilon} \int_0^1 \left[1 + \left|\log\left(\sqrt{\frac{(1-\varepsilon)K}{\varepsilon}}r\right)\right|\right] V(r)rdr, \quad (3.20)$$

and taking  $\varepsilon = \frac{1}{2}$  we obtain Lemma 3.2. □

Theorem 2.1 follows from Lemma 3.2 together with inequalities (3.5), (3.6), and (3.7).

## 3.2 Proof of Theorem 2.2

Noticing that for any  $\lambda > 0$  the constant potential  $V(x) \equiv \lambda$  is in  $L^1(\Omega)$ , and that  $N(A, \lambda)$  denotes the number of eigenvalues of the operator  $H_A^D$  less than  $\lambda$ , we apply

Theorem 2.1 to  $V(x) \equiv \lambda$ . To get the result it suffices to compute  $\int_0^1 [1 + |\log(kr)|] r dr$ . We get after computation that

$$\int_0^1 [1 + |\log(kr)|] r dr = \gamma_k, \quad (3.21)$$

with

- $\gamma_k = \frac{3 - 2 \log k}{4}$  if  $k \leq 1$
- $\gamma_k = \frac{1 + 2 \log k}{4} + \frac{1}{2k^2}$  if  $k > 1$ .

### 3.3 Proof of Remark 2.3

To get the minimum over the values of  $\alpha$  we study the sign of the expression, for any  $\alpha \in ]0, 1[$ , of

$$g_\lambda(\alpha) := \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} I.$$

A direct computation shows that the value  $\alpha_\lambda$  which realizes the minimum of  $g_\lambda(\alpha)$  is the positive solution of

$$\alpha^2(\lambda - 2I) + 6\alpha I - 4I = 0. \quad (3.22)$$

## 4 An asymptotic eigenvalue upper bound

From Theorem 2.2 we get easily an asymptotic estimate for the right-hand side of (2.4) when  $\lambda$  tends to  $\infty$  :

**Corollary 4.1** *If assumptions  $(H_1)$  and  $(H_2)$  are satisfied and if moreover*

$$\mathbf{b}(x) \leq M(D(x))^{-\beta}, \quad 0 < \beta < \frac{3}{2}$$

*for some  $M > 0$ , then the number of eigenvalues of the operator  $H_A^D$  smaller than  $\lambda$  satisfies, as  $\lambda \rightarrow \infty$*

$$N(H_A^D, \lambda) \leq \left(\frac{1}{2} + c_K\right)\lambda + \sqrt{\lambda}\sqrt{I} + O(1), \quad (4.1)$$

where

$$I = \int_0^1 \left(\frac{A(r)}{r}\right)^2 r dr,$$

and

- $c_K = \frac{3 - \log K}{2}$  if  $0 < K \leq 1$

- $c_K = \left[ \frac{1 + \log K}{2} + \frac{1}{K} \right]$  if  $K > 1$ .

*Inequality (4.1) still holds when we replace in the left-hand side  $N(H_A^D, \lambda)$  by  $N(H_{A'}^D, \lambda)$ , where  $A'$  is any gauge verifying  $dA' = dA = B$ .*

*Proof.*–

We define as previously, for any  $\alpha \in ]0, 1[$ ,

$$g_\lambda(\alpha) := \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} I$$

and we want to determine the asymptotic behavior as  $\lambda$  tends to  $\infty$  of  $g_\lambda(\alpha_\lambda)$ , where  $\alpha_\lambda$  is the minimum of  $g_\lambda(\alpha)$ .

From (3.22) we compute the following asymptotics

$$\alpha_\lambda = \frac{2\sqrt{I}}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)$$

$$\sqrt{1-\alpha_\lambda} = 1 - \frac{\sqrt{I}}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right),$$

and this gives the result. □

**Remark 4.2** *The leading term in the estimate (4.1) is of the same order than the leading term in the Weyl formula for the Dirichlet Laplacian (corresponding to the case  $A \equiv 0$ ) in the unit disk.*

## 5 Appendix

### 5.1 The inequality of Hundertmarkt-Lieb-Thomas

We recall the sharp inequality of Hundertmarkt-Lieb-Thomas [11]

**Theorem 5.1** *Let*

$$Lv(t) = -v''(t) - W(t)v(t), \quad W \geq 0 \quad W \in L^1(\mathbb{R})$$

*be defined in the sense of quadratic forms on  $\mathbb{R}$ , and assume that the negative spectrum of  $L$  is discrete. Denote by  $\{-\nu_k, k \in \mathbb{N}\}$  the negative eigenvalues of  $L$ . Then*

$$\sum_{k \in \mathbb{N}} \sqrt{\nu_k} \leq \frac{1}{2} \int_{-\infty}^{+\infty} W(t) dt .$$

## 5.2 The Green function $G(r, r', 1)$ of the operator $T_0$ .

Let us compute the diagonal element for the Green function  $G(r, r', 1)$  of the operator  $T_0$  defined by (3.14).  $G(r, r', 1)$  is the solution of

$$((T_0 + 1)u)(r) = \delta_{r'}(r), \quad u(0) = u(1) = 0. \quad (5.1)$$

We have

$$G(r, r', 1) = A_1 u_1(r) + A_2 u_2(r) \quad r \leq r'$$

$$G(r, r', 1) = B_1 u_1(r) + B_2 u_2(r) \quad r > r',$$

where  $u_1(r) = \sqrt{r}I_0(r)$  and  $u_2(r) = \sqrt{r}K_0(r)$  are independent solutions of the related homogeneous equation, ( $I_0$  and  $K_0$  are the modified Bessel functions).

The coefficients depend of  $r'$  but we omit the indices for the sake of clarity. Due to the boundary conditions and to the fact that the derivative (with respect to  $r$ ) of  $G(r, r', 1)$  has the discontinuity in  $r'$  of a Heaviside function, they satisfy :

$$A_1 u_1(0) + A_2 u_2(0) = 0 \quad B_1 u_1(1) + B_2 u_2(1) = 0$$

$$B_1 - A_1 = \frac{-u_2(r')}{W(r')} \quad B_2 - A_2 = \frac{u_1(r')}{W(r')}$$

where  $W(r')$  is the value of the Wronskian of  $u_1$  and  $u_2$  taken at the point  $r'$ .

The first equation is always satisfied since  $u_1(0) = u_2(0) = 0$ . Let us set  $A_2 = 0$ . We have  $W(r') = u_1'(r')u_2(r') - u_1(r')u_2'(r') = r'\hat{W}(r')$  where  $\hat{W}(r')$  is the Wronskian of the modified Bessel functions  $I_0$  and  $K_0$ . As  $r'\hat{W}(r') = 1$  (see [2]), we get after solving the above system, and doing  $r = r'$  :

$$\begin{aligned} G(r, r, 1) &= u_1(r) \left[ -u_1(r) \frac{u_2(1)}{u_1(1)} + u_2(r) \right] \\ &= rI_0(r) \left[ -I_0(r) \frac{K_0(1)}{I_0(1)} + K_0(r) \right]. \end{aligned}$$

Using again the properties of the modified Bessel functions (see [2]) we can write

$$G(r, r, 1) \leq rI_0(r)K_0(r).$$

The function

$$g(r) = \frac{I_0(r)K_0(r)}{1 + |\log r|}$$

has a limit at  $r = 0$  equal to 1 (see [2]), so

$$c_0 = \max_{[0,1[} \frac{I_0(r)K_0(r)}{1 + |\log r|} \quad (5.2)$$

exists and

$$G(r, r, 1) \leq c_0 r(1 + |\log r|), \quad r \in [0, 1[.$$

Numerics suggest that  $g$  is decreasing on  $[0, 1]$ , so that one should have  $c_0 = 1$ . In next subsection, we give the proof of this result, which can not be found to our knowledge in the literature, and has been communicated to the author by J.P. Truc [20] :

**Proposition 5.2**  $\forall r \in ]0, 1] : \frac{I_0(r)K_0(r)}{1 - \log r} \leq 1.$

### 5.3 Proof of Proposition 5.2

The modified Bessel function  $I_0$  can be written as

$$I_0(r) = \sum_{k=0}^{+\infty} \frac{\left(\frac{r^2}{4}\right)^k}{k!^2} = 1 + \frac{r^2}{4} + \dots \quad (5.3)$$

Therefore we have

$$1 \leq I_0(r) \leq \sum_{k=0}^{+\infty} \frac{\left(\frac{r^2}{4}\right)^k}{k!} = e^{\frac{r^2}{4}}$$

and

$$\forall r \in ]0, 1] : 1 \leq I_0(r) \leq e^{\frac{1}{4}}. \quad (5.4)$$

According to the expression of the modified Bessel function  $K_0$

$$K_0(r) = -\left(\log(r/2) + \gamma\right)I_0(r) + \sum_{k=1}^{+\infty} \left(\sum_{j=1}^k \frac{1}{j}\right) \frac{\left(\frac{r^2}{4}\right)^k}{k!^2} \quad (5.5)$$

where  $\gamma$  denotes the Euler constant, we compute that

$$K_0(r)I_0(r) - (1 - \log r) = \delta(r) - 1, \quad (5.6)$$

where  $\delta(r)$  denotes the following function :

$$\delta(r) = (1 - I_0(r)^2) \log r - \left(-\log 2 + \gamma\right)I_0(r)^2 + I_0(r) \sum_{k=1}^{+\infty} \left(\sum_{j=1}^k \frac{1}{j}\right) \frac{\left(\frac{r^2}{4}\right)^k}{k!^2}. \quad (5.7)$$

Proposition 5.2 is then a straightforward consequence of the following Lemma

#### Lemma 5.3

$$\forall r \in ]0, 1] : \delta(r) \leq 1.$$

*Proof.*–

The function  $\delta(r)$  splits into 3 positive parts, which we study separately .

- An upper bound for  $(1 - I_0(r)^2) \log r.$

From (5.4) we deduce  $1 - I_0(r)^2 \geq 1 - e^{\frac{r^2}{2}}$ , and :

$$\forall r \in ]0, 1] : 0 \leq (1 - I_0(r)^2) \log r \leq \left(e^{\frac{r^2}{2}} - 1\right)(-\log r) \leq 0, 11.$$

- An upper bound for  $(-\log 2 + \gamma)I_0(r)^2$ .

A straightforward computation gives  $-\gamma + \log 2 \leq 0.12$  so using that  $I_0(r) \leq e^{\frac{1}{4}}$  we get

$$(-\log 2 + \gamma)I_0(r)^2 \leq 0.16.$$

- An upper bound for  $I_0(r) \sum_{k=1}^{+\infty} \left( \sum_{j=1}^k \frac{1}{j} \right) \frac{(r^2/4)^k}{k!^2}$ .

For  $k \in \mathbb{N}^*$ , we set  $s_k = \sum_{j=1}^k \frac{1}{j}$ . We have  $s_1 = 1$ . For  $k \geq 2$ , according to the inequality

$$\frac{1}{k} \leq \int_{k-1}^k \frac{dt}{t} = \log k - \log(k-1).$$

we get that:

$$\sum_{j=2}^k \frac{1}{j} \leq \log k$$

and for any integer  $k$ ,  $s_k \leq 1 + \log k$ . Thus

$$\sum_{k=1}^{+\infty} \left( \sum_{j=1}^k \frac{1}{j} \right) \frac{(r^2/4)^k}{k!^2} \leq \sum_{k=1}^{+\infty} \left( \frac{1 + \log k}{k!} \right) \frac{(r^2/4)^k}{k!}.$$

Noticing that, for any integer  $k \geq 1$

$$0 \leq \frac{1 + \log k}{k!} \leq \frac{1 + \log k}{k} \leq 1,$$

we can write,  $\forall r \in ]0, 1]$  :

$$\sum_{k=1}^{+\infty} \left( \sum_{j=1}^k \frac{1}{j} \right) \frac{(r^2/4)^k}{k!^2} \leq \sum_{k=1}^{+\infty} \frac{(r^2/4)^k}{k!} = e^{\frac{r^2}{4}} - 1 \leq e^{\frac{1}{4}} - 1.$$

Finally we have, for any  $r \in ]0, 1]$

$$I_0(r) \sum_{k=1}^{+\infty} \left( \sum_{j=1}^k \frac{1}{j} \right) \frac{(r^2/4)^k}{k!^2} \leq e^{\frac{1}{4}} (e^{\frac{1}{4}} - 1) \simeq 0.364.$$

Summing the 3 previous estimates one gets :  $\forall r \in ]0, 1]$  :  $\delta(r) \leq 0.11 + 0.16 + 0.37 \leq 1$ .

□

The optimality of the value  $c_0 = 1$  is due to the fact that

$$\lim_{r \rightarrow 0^+} \frac{K_0(r)I_0(r)}{1 - \ln r} = 1.$$

## 5.4 The Green function $G(r, r', k^2)$ of the operator $T_0$

We now compute the diagonal element for the Green function  $G(r, r', k^2)$  of the operator  $T_0$  defined by (3.14).  $G(r, r', k^2)$  is the solution of

$$((T_0 + k^2)u)(r) = \delta_{r'}(r), \quad u(0) = u(1) = 0. \quad (5.8)$$

We have, as previously

$$G(r, r, k^2) = u_1(r) \left[ -u_1(r) \frac{u_2(1)}{u_1(1)} + u_2(r) \right]$$

where  $u_1(r) = \sqrt{r}I_0(kr)$  and  $u_2(r) = \sqrt{r}K_0(kr)$  are independent solutions of the related homogeneous equation. This leads to

$$G(r, r, k^2) = rI_0(kr) \left[ -I_0(kr) \frac{K_0(k)}{I_0(k)} + K_0(kr) \right] \leq rI_0(kr)K_0(kr) \leq r(1 + |\log(kr)|).$$

## References

- [1] J. Avron, I. Herbst & B. Simon. *Schrödinger operators with magnetic fields*, Duke. Math. J. **45** (1978), 847-883.
- [2] M. Abramovitz, I. Stegun. *Handbook of mathematical functions*, National Bureau of Standards (1964)
- [3] A. Balinski, W. Evans & R. Lewis. *On the number of negative eigenvalues of Schrödinger operators with an Aharonov-Bohm magnetic field*, R. Soc. Lon. Proc. Ser. A Math. Phys. Eng. Sci. **457** (2001), 2481-2489.
- [4] Y. Colin de Verdière. *L'asymptotique de Weyl pour les bouteilles magnétiques*, Commun. Math. Phys. **105** (1986), 327-335.
- [5] Y. Colin de Verdière & F. Truc. *Confining particles in a purely magnetic field*, Ann Inst Fourier (to appear).
- [6] M. Cwickel. *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. of Math. **106** (1977), 93-100.
- [7] L. Erdős & J.P. Solovej. *Magnetic Lieb-Thirring inequalities with optimal dependence on the field strength*, Journ. Stat. Phys. **116** (2004), 475-506.
- [8] L. Erdős & J.P. Solovej. *Magnetic Lieb-Thirring inequalities for the 3-dimensional Pauli operator with a strong nonhomogeneous magnetic fields*, Ann. Henri Poincaré **5** (2004), 671-745.

- hal-00588164, version 3 - 9 Sep 2011
- [9] R. Frank. *Remarks on eigenvalue estimates and semi-group domination*, Spectral and Scattering Theory for Quantum Magnetic Systems, P. Briet et al (eds), 63-86, Contemp. Math. 500, Amer. Soc., Providence, RI, 2009
  - [10] R. Frank & R. Olofsson. *Eigenvalue bounds for Schrödinger operators with a homogeneous magnetic field*, arXiv 1102.0329
  - [11] D.Hundertmarkt, E.H.Lieb & L.E.Thomas. *A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator*, Adv. Theor. Math. Phys. **2** (1998), 719-731.
  - [12] H. Kovarik. *Eigenvalue bounds for two-dimensional magnetic Schrödinger operators*, arXiv 1103.5194
  - [13] A. Laptev. *The negative spectrum of a class of two-dimensional Schrödinger operators with spherically symmetric potentials*, Func. Anal. Appl. **34** (2000), 305-307.
  - [14] A. Laptev. *Spectral Inequalities for Partial Differential Equations and their applications* to appear in Proceedings of the 5th ICCM Beijing 2010
  - [15] A. Laptev, T. Weidl. *Recent results on Lieb-Thirring inequalities*, Journées Equations aux dérivées partielles, St Jean de Monts (2000).
  - [16] E.H. Lieb. *Bound states of the Laplace and Schrödinger operators with spherically symmetric potentials*, Bull. Amer. Math. Soc. **82** (1976), 751-753.
  - [17] E.H. Lieb, W. Thirring. *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Studies in Mathematical Physics, Princeton Univ. Press, Princeton, NJ (1976), 269-303.
  - [18] G.V. Rozenblum. *Distribution of the discrete spectrum of singular differential operators*, Soviet. Math. Soc. **20** (1976), 63-71.
  - [19] F. Truc. *Semi-classical asymptotics for magnetic bottles*, Asympt. Anal. **15** (1997), 385-395.
  - [20] J.P. Truc. *Une estimation sur les fonctions de Bessel*, Private communication.