

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 56/2010

DOI: 10.4171/OWR/2010/56

**Mini-Workshop: Wellposedness and Controllability of  
Evolution Equations**

Organised by  
Birgit Jacob, Wuppertal  
Jonathan Partington, Leeds  
Sandra Pott, Paderborn/Glasgow  
Hans Zwart, Enschede

December 12th – December 18th, 2010

*Mathematics Subject Classification (2000)*: Primary: 47D06, 93B05, 93C20. Secondary: 93B28,  
47A57, 47B35, 42B37.

**Preprint of the extended abstract of the talk by Luc Miller.**

The full report can be downloaded from the website of the MFO via  
"Scientific Programme/Annual Schedules/Mini-Workshops in 2010/Mini-Workshop:  
Wellposedness and Controllability of Evolution Equations/OWR\_2010\_56.pdf".

## Abstracts

### Fast control cost for heat-like semigroups: Lebeau-Robbiano strategy and Hautus test

LUC MILLER

(joint work with Thomas Duykaerts)

Since the seminal work of Russell and Weiss in [7], resolvent conditions for various notions of admissibility, observability and controllability, and for various notions of linear evolution equations have been studied intensively, sometimes under the name of infinite-dimensional Hautus test, cf. [8, 2]. This talk based on [1] investigates resolvent conditions for null-controllability in arbitrary time: necessary conditions for general semigroups, and sufficient conditions for analytic normal semigroups and semigroups with negative self-adjoint generators.

#### 1. INTRODUCTION

Let  $-A$  be the generator of a strongly continuous semigroup on a Hilbert space  $\mathcal{E}$ . Let  $C$  be a bounded operator from the domain  $D(A)$  with the graph norm to another Hilbert space  $\mathcal{F}$ . The norms in  $\mathcal{E}$  and  $\mathcal{F}$  are denoted  $\|\cdot\|$ . We refer to the monograph [8] for a full account of the control theory of semigroups.

Recall the usual *admissibility* condition (for some time  $T > 0$  hence all  $T > 0$ ),

$$(1) \quad \exists K_T > 0, \forall v \in D(A), \quad \int_0^T \|Ce^{-tA}v\|^2 dt \leq K_T \|v\|^2.$$

If  $C$  is admissible for  $A$  then null-controllability at time  $T$  is equivalent to *final-observability* at time  $T$  (cf. [8], i.e.

$$(2) \quad \exists \kappa_T > 0, \forall v \in \mathcal{E}, \quad \|e^{-TA}v\|^2 \leq \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt.$$

*The control property investigated here is (2) for all  $T > 0$ .*

**1.1. Control cost.** The coefficient  $\kappa_T$  in (2) is the *control cost*: it is the ratio of the size of the input over the size of the initial state which the input steers to the zero final state in a lapse of time  $T$ . It blows up as  $T \rightarrow 0$ . E.g. for the heat semigroup on a compact manifold  $M$  with Dirichlet boundary conditions observed from a subset  $\Omega$ :  $\kappa_T \leq c_0 \exp(2c/T)$ ,  $T \in (0, 1)$ , where  $c_0$  is a positive constant, implies  $c \geq d^2/4$  where  $d$  is the furthest a point of  $M$  can be from  $\Omega$ , and is implied by  $c > 3L^2/4$  where  $L$  is the length of the longest generalized geodesic in  $M$  which does not intersect  $\Omega$  ( $L < +\infty$  is known as the condition of Bardos-Lebeau-Rauch).

For many evolutions of parabolic type,  $\kappa_T$  is bounded by  $c_0 \exp(2c/T^\beta)$  where  $c$ ,  $c_0$  and  $\beta$  are positive constants. E.g. thermoelastic plates without rotatory inertia, the plate equation with square root damping, diffusions in discontinuous media or in a potential well, diffusions generated by the fractional Laplacian or non-selfadjoint elliptic generators, cf. references in [5].

1.2. **Resolvent conditions.** The resolvent condition:  $\exists M > 0$ ,

$$(3) \quad \|v\|^2 \leq \frac{M}{(\operatorname{Re} \lambda)^2} \|(A - \lambda)v\|^2 + \frac{M}{\operatorname{Re} \lambda} \|Cv\|^2, \quad v \in D(A), \quad \operatorname{Re} \lambda > 0.$$

was introduced in [7] as a necessary condition for exact observability in infinite time of exponentially stable semigroups.

When  $A$  is skew-adjoint (equivalently when the semigroup is a unitary group), it was proved in [3] that the following resolvent condition is necessary and sufficient for final-observability (hence exact observability) in some time  $T > 0$ :  $\exists M > 0$ ,

$$\|v\|^2 \leq M \|(iA - \lambda)v\|^2 + M \|Cv\|^2, \quad v \in D(A), \quad \lambda \in \mathbb{R}.$$

We refer to [6] for more background and references. This result was extended to some more general groups in [2, theorem 1.2].

When  $-A$  generates an exponentially stable normal semigroup, [2, theorem 1.3] proves that the resolvent condition (3) is sufficient for the weaker notion:

$$(4) \quad \exists T > 0, \exists \kappa_T > 0, \forall v \in \mathcal{E}, \|e^{-TA}v\|^2 \leq \kappa_T \int_0^\infty \|Ce^{-tA}v\|^2 dt.$$

In this framework (4) implies (2) for *some* time  $T$ .

But it seems that resolvent conditions for final-observability for *any*  $T > 0$  in (2) has not been investigated yet, although it is quite natural for heat-like semigroups.

## 2. RESULTS

2.1. **Necessary resolvent conditions for semigroups.** The proof mainly consists in changing  $i$  into  $-1$  in [3, lemma 5.2]. Cf. also the proof of [7, theorem 1.2].

**Theorem 1.** Let  $B_T = \sup_{t \in [0, T]} \|e^{-tA}\|$  be the semigroup bound up to time  $T$ .

If (1) and (2) hold then :  $\forall v \in D(A), \lambda \in \mathbb{C}, \operatorname{Re} \lambda > 0$ ,

$$\|v\|^2 \leq 2e^{2T \operatorname{Re} \lambda} \left( (B_T^2 + 2\kappa_T K_T) \frac{\|(A - \lambda)v\|^2}{(\operatorname{Re} \lambda)^2} + \kappa_T \frac{\|Cv\|^2}{\operatorname{Re} \lambda} \right),$$

**Theorem 2.** If final-observability (2) holds for all  $T \in (0, T_0]$  with the control cost  $\kappa_T = c_0 e^{\frac{2c}{T^\beta}}$  for some positive  $\beta, c$  and  $c_0$  then the resolvent condition

$$\|v\|^2 \leq a_0 e^{2a(\operatorname{Re} \lambda)^\alpha} (\|(A - \lambda)v\|^2 + \|Cv\|^2), \quad v \in D(A), \quad \operatorname{Re} \lambda > 0,$$

holds with power  $\alpha = \frac{\beta}{\beta+1}$  and rate  $a = c^{\frac{1}{\beta+1}} \frac{\beta+1}{\beta^\alpha}$ .

It still holds for  $\lambda \in \mathbb{C}$  if  $\operatorname{Re} \lambda$  is replaced by  $\operatorname{Re}_+ \lambda := \max\{\operatorname{Re} \lambda, 0\}$ .

2.2. **Sufficient resolvent conditions for an analytic normal semigroup.** The proof is based on the Lebeau-Robbiano strategy of [5]. N.b. (1) is not assumed.

**Theorem 3.** Assume that  $-A$  generates an analytic normal semigroup, hence there exists  $\omega \in \mathbb{R}$  and  $\theta \in [0, \frac{\pi}{2})$  such that  $\sigma(A) \subset \{z \in \mathbb{C} : \arg(z - \omega) \leq \theta\}$ .

The resolvent condition with  $\alpha \in (0, 1)$ ,  $\omega_0 < \omega$ ,  $\lambda_0 > \omega_0$ , positive  $a_0$  and  $a$ ,

$$\|v\|^2 \leq \frac{\cos^2 \theta}{(\lambda - \omega_0)^2} \|(A - \lambda)v\|^2 + a_0 e^{2a\lambda^\alpha} \|Cv\|^2, \quad v \in D(A), \quad \lambda \geq \lambda_0,$$

implies final-observability (2) for all time  $T > 0$  with the control cost estimate

$$\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T \leq 2a^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}, \quad \text{where } \beta = \frac{\alpha}{1-\alpha}.$$

### 2.3. Sufficient resolvent condition for a negative self-adjoint generator.

The proof combines the Lebeau-Robbiano strategy of [5], the control transmutation method of [4] (which deduces the final-observability of the heat-like equation  $\dot{v} + Av = 0$  from the exact observability of the wave-like equation  $\ddot{w} + Aw = 0$ ) and results on resolvent conditions from [6].

**Theorem 4.** *Assume that the positive self-adjoint operator  $A$  and the operator  $C$  bounded from  $D(\sqrt{A})$  with the graph norm to  $\mathcal{F}$  satisfy the admissibility and observability conditions with nonnegative powers  $\gamma$  and positive  $\delta$ ,  $L_*$  and  $M_*$ :*

$$\|Cv\|^2 \leq L_*\lambda^\gamma \left( \frac{1}{\lambda} \|(A-\lambda)v\|^2 + \|v\|^2 \right), \quad v \in D(A), \quad \lambda \geq \inf A,$$

$$\|v\|^2 \leq M_*\lambda^\delta \left( \frac{1}{\lambda} \|(A-\lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in D(A), \quad \lambda \geq \inf A.$$

If  $\gamma + \delta < 1$  then final-observability (2) holds for all  $T > 0$  with the cost estimate

$$\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T < +\infty, \quad \text{where } \beta = \frac{1+\gamma+\delta}{1-\gamma-\delta}.$$

The assumption of the control transmutation method corresponds to  $\gamma = \delta = 0$ . The Russell-Weiss condition (3) corresponds to  $\delta = -1$ .

A logarithmic improvement of this theorem is also proved in [1] thanks to this new variant of the Lebeau-Robbiano strategy of [5]:

**Theorem 5.** *Assume the admissibility condition (1) and that  $-A$  generates a normal semigroup. If the logarithmic observability condition on spectral subspaces*

$$\|v\|^2 \leq a_0 e^{2a\lambda/((\log(\log \lambda))^\alpha \log \lambda)} \|Cv\|^2, \quad v \in \mathbf{1}_{\operatorname{Re} A < \lambda} \mathcal{E}, \quad \lambda \geq \lambda_0.$$

holds with  $\alpha > 2$ ,  $\lambda_0$ ,  $a_0$ , a positive then final-observability (2) holds for all  $T > 0$ .

#### REFERENCES

- [1] T. Duyckaerts and L. Miller, *Resolvent conditions for the control of parabolic equations*, manuscript, 2010, soon available online at <http://hal.archives-ouvertes.fr>.
- [2] B. Jacob and H. Zwart, *On the Hautus test for exponentially stable  $C_0$ -groups*, SIAM J. Control Optim. **48** (2009), no. 3, 1275–1288.
- [3] L. Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal. **218** (2005), no. 2, 425–444.
- [4] L. Miller, *The control transmutation method and the cost of fast controls*, SIAM J. Control Optim. **45** (2006), no. 2, 762–772.
- [5] L. Miller, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*, Discrete Contin. Dyn. Syst. B **14** (2010), no. 4, 1465–1485.
- [6] L. Miller, *Resolvent conditions for the control of unitary groups*, manuscript, 2010.
- [7] D. L. Russell and G. Weiss, *A general necessary condition for exact observability*, SIAM J. Control Optim. **32** (1994), no. 1, 1–23.
- [8] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basel Textbooks, Birkhäuser Verlag, Basel, 2009.