

Nonquadratic Local Risk-Minimization for Hedging Contingent Claims in Incomplete Markets*

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Abstract. We introduce a new criterion to perform hedging of contingent claims in incomplete markets. Our approach is close to the one proposed by Schweizer [*Stochastic Process. Appl.*, 37 (1991), pp. 339–363] in that it uses the concept of locally risk-minimizing strategies. But we aim at being more general by defining the local risk as a general, nonnecessarily quadratic, convex function of the local cost process. We derive the corresponding optimal strategies and value function in both discrete and continuous time settings. Finally we give an application of our hedging method in the stochastic volatility case as well as in the jump diffusion case. We work with a single traded asset, but our approach may be generalized to deal with claims depending on multiple assets.

Key words. hedging, incomplete markets, local risk-minimization, derivatives

AMS subject classifications. 91G20, 91G80, 49J20, 93E20

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1. Motivations. Using a quadratic function to assess the risk associated with some cost process means that we will equally penalize the profits and losses generated by our trading strategy. Though this approach leads to some neat mathematical developments (see the important literature on quadratic local risk-minimization or on mean-variance hedging such as Föllmer and Sondermann [8], Föllmer and Schweizer [6], [7], or Heath, Platen, and Schweizer [9]) it probably does not perfectly reflect the way market practitioners assess their risk. In order to overcome this, we shall introduce a risk function f ,¹ $f \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ with $f(0) = f'(0) = 0$, strictly convex and with $\lim_{|x| \rightarrow \infty} f(x) = +\infty$, that will typically allow putting more weight on losses than on profits.

2. Discrete time setting. We first place ourselves in a multiperiod model where the evolution of the stock price is given by a stochastic process S_k ($k = 0, \dots, T$) on some probability space (Ω, \mathcal{F}, P) . Let \mathcal{F}_k denote the σ -field of events which are observable up to and including time k . We assume that S_k is adapted and square-integrable. In order to avoid complicated notations, we work with zero interest rates, but this is not a loss of generality, as it basically amounts to having discounted stock prices with the existence of a “nonrisky” asset.

A contingent claim is described by a square-integrable random variable $H \in L^2(P)$ that will be assumed of the form $H = \delta^H S_T + \beta^H$ with δ^H and β^H being \mathcal{F}_T -measurable ran-

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¹We will further assume that f has at most quadratic growth in order to circumvent integrability conditions by working in $L^2(P)$, where P is one chosen probability measure.

dom variables. Thus we are considering European-type options that can have a cash settled component and a deliverable one.

For example, a call option with strike K would correspond to $H = (S_T - K)^+ = 1_{S_T \geq K} S_T - K 1_{S_T \geq K}$, so that $\delta^H = 1_{S_T \geq K}$ and $\beta^H = -K 1_{S_T \geq K}$.

2.1. Trading strategies. A trading strategy Φ is given by two stochastic processes δ_k ($k = 0, \dots, T$) and β_k ($k = 0, \dots, T$). δ_k is the amount of stock held in period k , ($= [t_k, t_{k+1})$) and has to be fixed at the beginning of that period; i.e., we assume that δ_k is \mathcal{F}_k -measurable ($k = 0, \dots, T$).

The amount β_k held in the market account in period k is fixed at the beginning of this period too; i.e., we assume that β_k is \mathcal{F}_k -measurable ($k = 0, \dots, T$).

We further assume that both δ and β are in $L^2(P)$.

The reader familiar with the local risk-minimization framework will already notice the difference from the usual setting, where typically only the cash component β of the strategy Φ is allowed to be adapted. The relaxation of the predictability of the strategy in both its components allows us to consider both types of options (cash settled and deliverable) and will allow for a generalization to the case when there are liquidity costs.

For such a trading strategy, the theoretical value of the portfolio at time k is given by

$$V_k = \delta_k S_k + \beta_k \quad (k = 1, \dots, T),$$

and so it is its value just after applying the strategy. We will denote by *H-admissible* trading strategies which satisfy that each V_k is square-integrable and such that the contingent claim H is produced in the end; i.e., we require $V_T = H$, which can always be met with our measurability requirements upon taking $\delta_T = \delta_T^H$ and $\beta_T = \beta_T^H$.

2.2. Costs and risk processes. The costs ΔC_k incurred at time t_k , $k > 0$, from following strategy Φ , so from changing the stock amount that we hold from δ_{k-1} to δ_k and from changing the amount invested in the money market account from β_{k-1} to β_k , are given in the absence of transaction costs² by

$$\begin{aligned} \Delta C_k(\Phi) &= (\delta_k - \delta_{k-1})S_k + (\beta_k - \beta_{k-1}) \quad \forall k \in \{k = 1, \dots, T\} \\ \Leftrightarrow \Delta C_k(\Phi) &= V_k - V_{k-1} - \delta_{k-1}(S_k - S_{k-1}) \quad \forall k \in \{k = 1, \dots, T\}. \end{aligned}$$

We then define the local risk ΔR_k at time k associated with the costs incurred at time $k + 1$. It is

$$\Delta R_k(\Phi) = \mathbb{E}(f(\Delta C_{k+1}) | \mathcal{F}_k)$$

or with obvious notation

$$\Delta R_k(\Phi) = \mathbb{E}_k(f(\Delta C_{k+1})).$$

Our objective is now to find those trading strategies that will sequentially minimize the risk process. We make this statement precise in the next section.

²Fully incorporating transaction costs would introduce a dependence on the sign of the readjustment as well as a nonlinear dependence on the quantity; see [1] for the case where there are only liquidity costs.

2.3. Locally risk-minimizing strategies. The sequential minimization program runs backward in time; i.e., given $(\delta_{k+1}^*, \delta_{k+2}^*, \dots, \delta_T^*)$ and $(\beta_{k+1}^*, \beta_{k+2}^*, \dots, \beta_T^*)$ (or equivalently $(V_{k+1}^*, V_{k+2}^*, \dots, V_T^*)$), we look for δ_k^* and β_k^* (or V_k^*) such that ΔR_k is minimized.

Problem ()*. Given a contingent claim H , find a Φ^* , H -admissible strategy such that

$$\forall k \in (0, \dots, T - 1), \Delta R_k(\Phi) \geq \Delta R_k(\Phi^*) \quad \forall \Phi \text{ } H\text{-admissible,}$$

$$\text{with } \delta_{k+1} = \delta_{k+1}^* \quad \text{and} \quad \beta_{k+1} = \beta_{k+1}^*.$$

Given the conditions imposed on f , S_k , and β_k (or V_k), we have the existence and uniqueness of the optimal strategy, and it is characterized through the first-order optimality equations

$$(1) \quad \begin{cases} \mathbb{E}_k(f'(\Delta C_{k+1}(\Phi^*))) = 0, \\ \mathbb{E}_k(f'(\Delta C_{k+1}(\Phi^*))\Delta S_{k+1}) = 0. \end{cases}$$

The above equations will prove useful for further characterizing optimal strategies, particularly in continuous time, but it is important to notice that the existence and uniqueness of the solution of problem (*) hold even though f is not regular: all that is required is that it be strictly convex with suitable bounds as prescribed in the introduction. The characterizing set of equations (1), on the other hand, holds only when f is differentiable.

Theorem 1. *Problem (*) has a unique solution Φ^* whose components δ^* and β^* solve the set of equations (1).*

Proof. Let $h(x, y, \omega) \equiv E_k(f((U - x)S + (V - y))) (\omega)$ with U, V , and $S \in \mathcal{L}^2(P)$. First, we observe that for a fixed ω , h is a convex function of x and y so that it has a global minimum (x^*, y^*) if and only if (x^*, y^*) is a critical point of h , i.e., $\nabla h(x^*, y^*) = 0$. Second, we have $\lim_{\|(x,y)\| \rightarrow \infty} h(x, y, \omega) = +\infty$ so that h being continuous it has a global minimum P -a.s. Finally, we show that (x^*, y^*) is \mathcal{F}_k -measurable: letting $D_n = \{j2^{-n} | j \in \mathbb{Z}\}$ be the set of dyadic rational of order n , we define

$$(x_n(\omega), y_n(\omega)) = \inf\{(x, y) \in D_n \times D_n, h(x, y, \omega) \leq h(x', y', \omega) \forall (x', y') \in D_n \times D_n\}.$$

Since $\omega \mapsto h(x, y, \omega)$ is \mathcal{F}_k -measurable, (x_n, y_n) is also \mathcal{F}_k -measurable. As (x_n, y_n) is bounded in n P -a.e. and h is continuous in (x, y) , $(\tilde{x}, \tilde{y}) = \liminf_{n \rightarrow \infty} (x_n, y_n)$ is a \mathcal{F}_k -measurable minimizer of h and by uniqueness it is equal to (x^*, y^*) . ■

The set of equations (1) is equivalent to the property that the process $(C_k^f)_k$ with $C_k^f = \sum_{i=0}^k f'(\Delta C_{i+1})$ —which we will refer to as the f -cost process—is a martingale strongly orthogonal to S_k . This property, which we name pseudo-optimality as in Schweizer [12], will be the main ingredient of the extensions to the continuous time setting.

3. Continuous time setting. Now let (Ω, \mathcal{F}, P) be a probability space with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right-continuity and completeness. $T \in \mathbb{R}^+$ denotes a fixed and finite time horizon. Furthermore, we assume that \mathcal{F}_0 is trivial and that $\mathcal{F}_T = \mathcal{F}$. Let $S = (S_t)_{0 \leq t \leq T}$ be a semimartingale with a decomposition

$$S = S_0 + M + A$$

such that $M = (M_t)_{0 \leq t \leq T}$ is a square-integrable martingale with $M_0 = 0$ and $A = (A_t)_{0 \leq t \leq T}$ is a continuous and adapted process of finite variation $|A|$ with $A_0 = 0$. Throughout the article, we use a right-continuous version of S .

3.1. Trading strategies. A trading strategy Φ is a pair of processes $\delta = (\delta_t)_{0 \leq t \leq T}$, $\beta = (\beta_t)_{0 \leq t \leq T}$ satisfying the following conditions:

$$\begin{cases} \delta \text{ is càdlàg and adapted,} \\ \beta \text{ is càdlàg and adapted.} \end{cases}$$

An option is again described by a square-integrable random variable $H \in L^2(P)$, with $H = \delta^H S_T + \beta^H$, δ^H and β^H being \mathcal{F}_T -measurable random variables.

So as to make precise what strategies can be considered, we first need to introduce some classical notations and definitions. Let (X_t) and (Y_t) be two stochastic processes. We will also refer to the portfolio value V_t at time t as the quantity $V_t = \delta_t S_t + \beta_t$.

If \mathcal{P} is a partition of $[0, T]$ such that

$$\mathcal{P} = \{0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_n = T\},$$

then we recall that the quadratic variation process along \mathcal{P} is

$$[X, X]_t^{\mathcal{P}} = \sum_{k=1}^n (X_{\tau_k \wedge t} - X_{\tau_{k-1} \wedge t})^2.$$

And the quadratic covariation process along \mathcal{P} is

$$[X, Y]_t^{\mathcal{P}} = \sum_{k=1}^n (X_{\tau_k \wedge t} - X_{\tau_{k-1} \wedge t})(Y_{\tau_k \wedge t} - Y_{\tau_{k-1} \wedge t}).$$

We will work with sequences of Riemann partitions \mathcal{P}_n which satisfy $|\mathcal{P}_n| \equiv \sup_k |\tau_k^n - \tau_{k-1}^n|$ tends to 0 as $n \rightarrow \infty$.

Let us finally recall that X is said to have finite quadratic variation on $[0, T]$ if $\lim_{n \rightarrow \infty} [X, X]_T^{\mathcal{P}_n}$ exists in the topology of uniform convergence in probability for any sequence \mathcal{P}_n of Riemann partitions.

Now we are in a position to introduce some restrictions on our strategies so that the optimality conditions are well defined. We shall concentrate on strategies which are *H-admissible* in the sense that

$$\begin{cases} \delta_T = \delta^H \text{ P-a.s.,} \\ \beta_T = \beta^H \text{ P-a.s.,} \\ \delta \text{ has finite and integrable quadratic variation,} \\ \beta \text{ has finite and integrable quadratic variation,} \\ \delta \text{ and } \beta \text{ have finite and integrable quadratic covariation.} \end{cases}$$

The justification for considering such strategies will be given in the next section.

3.2. Local risk-minimization and the *f*-cost process. This section is devoted to the two main concepts that will allow us to find optimal strategies and the relationship with them in a particular framework where S is driven by an Itô process. We start by introducing the local risk-minimization criterion and then define the *f*-cost process needed to derive pseudo-optimal strategies.

Local risk-minimization. In order to extend the idea of local risk-minimization already seen in discrete time to our continuous time framework, we first introduce the concept of small perturbations and then characterize the optimal strategies as the ones that minimize the local risk, at the limit, with respect to these small perturbations.

Small perturbations.

Definition. A small perturbation is a bounded admissible³ strategy $\phi = (\beta, \delta)$ such that $\beta_T = 0$ and $\delta_T = 0$.

Local risk along a partition. We start with an H -admissible strategy Φ , and we want to study the increase of risk when the strategy is perturbed at some discrete times. To do so, given a partition τ of $[0, T]$, where $\tau = \{0 = t_0, t_1, \dots, t_k = T\}$, and a small perturbation Δ , we define a process r_f^τ in the following way:

$$r_f^\tau[\Phi, \phi](t, \omega) = \sum_{t_i, t_{i+1} \in \tau} \frac{\Delta R_{t_i}(\Phi + \phi|_{[t_i, t_{i+1}]}) (\omega) - \Delta R_{t_i}(\Phi)(\omega)}{t_{i+1} - t_i} \mathbf{1}_{[t_i, t_{i+1}]}(t)$$

with $\Delta R_{t_i}(\Phi) = \mathbb{E}(f(\Delta C_{t_{i+1}}) | \mathcal{F}_{t_i})$.

Now we can define the local risk-minimization in the same way as we did for the discrete time setting.

Definition. An H -admissible strategy Φ is called locally risk-minimizing for the option H if for every small perturbation ϕ and every increasing sequence of partitions $(\tau_n)_{n \in \mathbb{N}}$ tending to the identity we have

$$\liminf_{n \rightarrow \infty} r^{\tau_n}[\Phi, \phi] \geq 0 \quad \mathcal{P}\text{-a.e.}$$

As a matter of fact, this definition naturally extends the notion of local minimization of a local risk. However, the associated optimality conditions are not readily derived in the general semimartingale case. In the following section, we will introduce the concept of a pseudo-optimal strategy similar to the discrete time setting. In the case where the asset follows an Itô process, one can show that optimal and pseudo-optimal strategies are the same. This equivalence will be worked out in detail for the particularly relevant situations of stochastic volatility models in the last section of this article.

The f -cost process. Now we proceed with defining the process f -cost process which will allow us to characterize pseudo-optimal strategies by analogy with discrete time.

Definition. For an H -admissible strategy Φ we define the f -cost process $C_t(\Phi)$ as the following limit, whenever it exists:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{l_n} f'(\beta^{\tau_k^n} - \beta^{\tau_{k-1}^n} + (\delta^{\tau_k^n} - \delta^{\tau_{k-1}^n}) S^{\tau_k^n}),$$

where convergence is required in ucp topology, for any sequences \mathcal{P}_n of Riemann partitions of $[0, T]$ of length l_n , and where we used the notation X^T for the process stopped at T .

³Admissible means that it satisfies the same regularity requirements as an H -admissible strategy with different terminal conditions.

We now focus on an H -admissible strategy Φ and state a theorem relative to the existence of the f -cost process.⁴

Theorem 2. *The f -cost process of an H -admissible strategy Φ is well defined and is given by the following formula:*

$$\begin{aligned}
 C_t(\Phi) &= f''(0) \left(V_t - V_0 - \int_{0+}^t \delta_{s-} dS_s \right) \\
 &\quad + \frac{f^{(3)}(0)}{2} \left([\beta, \beta]_t^c + 2 \int_{0+}^t S_{s-} d[\beta, \delta]_s^c + \int_{0+}^t S_{s-}^2 d[\delta, \delta]_t^c \right) \\
 (2) \quad &\quad + \sum_{0 < s \leq t} f'(\Delta\beta_s + \Delta\delta_s S_s) - f''(0)(\Delta\beta_s + \Delta\delta_s S_s)
 \end{aligned}$$

with notation $[X, Y]^c$ standing for the continuous part of the (càdlàg) quadratic covariation process.

Proof. We follow Protter [11] rather closely in his proof of the Itô formula for general semimartingales. Let \mathcal{P}_n be a refining sequence of Riemann partitions of $[0, T]$, $\mathcal{P}_n = \{0 = t_0^n \leq \dots \leq t_n^n = T\}$:

$$C_t^{\mathcal{P}_n}(\Phi) = \sum_{k=1}^n f'(\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k}).$$

Since β , δ , and S are càdlàg processes, and $\sum_s (\Delta\beta_s)^2$, $\sum_s (\Delta\delta_s)^2$, and $\sum_s (\Delta S_s)^2$ are (absolutely) convergent series, given $\epsilon > 0$, we can find two sets A and B such that A and B are disjoint and $A \cup B$ exhausts the jump times of β , δ , and S on $(0, T]$, A being a set of jump times that β , δ , and S have a.s. a finite number of times and B being such that $\sum_{0 < s \leq t} (\Delta\beta)^2 \leq \epsilon^2$, $\sum_{0 < s \leq t} (\Delta\delta)^2 \leq \epsilon^2$, and $\sum_{0 < s \leq t} (\Delta S)^2 \leq \epsilon^2$.

Thus we have

$$C_t^{\mathcal{P}}(\Phi) = \sum_{k,A} f'(\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k}) + \sum_{k,B} f'(\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k}),$$

where $\sum_{k,A}$ denotes $\sum_k 1_{\{A \cap (t_{k-1}, t_k] \neq \emptyset\}}$ and $\sum_{k,B}$ denotes $\sum_k 1_{\{B \cap (t_{k-1}, t_k] \neq \emptyset\}}$. ■

The first sum converges to $\sum_{s \in A} f'(\Delta\beta_s + \Delta\delta_s S_s)$.

In the second sum we apply Taylor's theorem, which says

$$f'(x) = f''(0)x + \frac{1}{2}f^{(3)}(0)x^2 + R(x),$$

where $|R(x)| \leq r(x)x^2$ such that $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function with $\lim_{u \downarrow 0} r(u) = 0$.

⁴The authors came across a similar result in the work of Diop [4].

Thus we have

$$\begin{aligned}
 & \sum_{k,B} f'(\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k}) \\
 (3) \quad & = f''(0) \sum_{k,B} (\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k}) \\
 (4) \quad & + \frac{1}{2} f^{(3)}(0) \sum_{k,B} (\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k})^2 \\
 (5) \quad & + \sum_{k,B} R(\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k}).
 \end{aligned}$$

The first sum (3) is equal to

$$\sum_k (\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k}) - \sum_{k \in A} (\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}})S_{t_k}),$$

which converges in ucp topology to

$$\left(V_t - V_0 - \int_{0+}^t \delta_s - dS_s \right) - \sum_{s \in A} (\Delta\beta_s + \Delta\delta_s S_s).$$

The second sum (4), after developing and switching to less cumbersome notations, is equal to

$$\sum_{k,B} (\beta_k - \beta_{k-1})^2 + 2S_k(\beta_k - \beta_{k-1})(\delta_k - \delta_{k-1}) + S_k^2(\delta_k - \delta_{k-1})^2.$$

$\sum_{k,B} (\beta_k - \beta_{k-1})^2 = \sum_k (\beta_k - \beta_{k-1})^2 - \sum_{k,A} (\beta_k - \beta_{k-1})^2$, and the first sum converges to $[\beta, \beta]_t$ while $\sum_{k \in A} (\beta_k - \beta_{k-1})^2$ converges to $\sum_{s \in A} \Delta\beta_s^2$.

Now $\sum_{k,B} 2S_k(\beta_k - \beta_{k-1})(\delta_k - \delta_{k-1}) = \sum_{k,B} 2S_{k-1}(\beta_k - \beta_{k-1})(\delta_k - \delta_{k-1}) + \sum_{k,B} (S_k - S_{k-1})(\beta_k - \beta_{k-1})(\delta_k - \delta_{k-1})$.

The first term is equal to $\sum_k 2S_{k-1}(\beta_k - \beta_{k-1})(\delta_k - \delta_{k-1}) - \sum_{k,A} 2S_{k-1}(\beta_k - \beta_{k-1})(\delta_k - \delta_{k-1})$ and converges to $2 \int_{0+}^t S_s - d[\beta, \delta]_s - 2 \sum_{s \in A} S_s - \Delta\beta_s \Delta\delta_s$.

The second term is less than $\sup_{k,B} |(S_k - S_{k-1})| \sum_{k,B} |\beta_k - \beta_{k-1}| |\delta_k - \delta_{k-1}|$ and again less than $\sup_{k,B} |(S_k - S_{k-1})| (\sum_k (\beta_k - \beta_{k-1})^2 + \sum_k (\delta_k - \delta_{k-1})^2)$. Taking the limit when $n \rightarrow \infty$ we find that $|\sum_{k,B} (S_k - S_{k-1})(\beta_k - \beta_{k-1})(\delta_k - \delta_{k-1})| \leq \epsilon \sqrt{[\delta, \delta]_t} \sqrt{[\beta, \beta]_t}$.

Finally

$$\begin{aligned}
 \sum_{k,B} S_k^2(\delta_k - \delta_{k-1})^2 & = \sum_{k,B} S_{k-1}^2(\delta_k - \delta_{k-1})^2 \\
 & + 2 \sum_{k,B} S_{k-1}(S_k - S_{k-1})(\delta_k - \delta_{k-1})^2 + \sum_{k,B} (S_k - S_{k-1})^2(\delta_k - \delta_{k-1})^2.
 \end{aligned}$$

The first term is equal to $\sum_k S_{k-1}^2(\delta_k - \delta_{k-1})^2 - \sum_{k,A} S_{k-1}^2(\delta_k - \delta_{k-1})^2$ and converges to $\int_{0+}^t S_s^2 - d[\delta, \delta]_s - \sum_{s \in A} S_s^2 - (\Delta\delta_s)^2$.

The second term is less than $\sup_{k,B} |S_k| \sup_{k,B} |(S_k - S_{k-1})| (\sum_k (\delta_k - \delta_{k-1})^2)$, and if we assume for now that $S \leq K < \infty$ uniformly in t , then we have $|\sum_{k,B} S_{k-1} (S_k - S_{k-1}) (\delta_k - \delta_{k-1})^2| \leq K \epsilon [\delta, \delta]_t$.

The last term is less than $\epsilon^2 [\delta, \delta]_t$ by following the same reasoning.

Now we turn to the last term (5) of Taylor’s development:

$$(6) \quad \left| \sum_{k,B} R(\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}}) S_{t_k}) \right| \leq \sum_{k,B} r(|\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}}) S_{t_k}|) (\beta_{t_k} - \beta_{t_{k-1}} + (\delta_{t_k} - \delta_{t_{k-1}}) S_{t_k})^2.$$

Again, assuming that $\sup S \leq K \leq \infty$ over $[0, T]$, we have (6) $\leq \sup r((K + 1)\epsilon) [\delta, \delta]_t$.

We are now ready to take the limit when ϵ goes to zero. The last term tends to zero from the property of r , and it remains to prove that the series $\sum_{s \in A}$ are absolutely convergent. We next proceed by localization, as in Protter [11], by considering first $V_K = \inf\{t > 0, |\delta| > K\}$, $W_K = \inf\{t > 0, |\beta| > K\}$, and $Z_K = \inf\{t > 0, |S| > K\}$ so that $1_{[0, V_K]} \delta$, $1_{[0, W_K]} \beta$, and $1_{[0, Z_K]} S$ are $[-K, K]$ -valued. Therefore we have that $|f'(x) - f''(0)x| \leq Cx^2$ for some constant C . This allows us to write

$$\left| \sum_{s \in A} f'(\Delta\beta_s + \Delta\delta S_s) - f''(0) \left(\sum_{s \in A} \Delta\beta_s + \Delta\delta S_s \right) \right| \leq C \sum_{s \in A} \Delta\beta_s^2 + 2\Delta\beta_s \Delta\delta S_s + \Delta\delta_s^2 S_s^2 \leq C([\beta, \beta]_t + 2K[\delta, \beta]_t + K^2[\delta, \delta]_t) < \infty.$$

And the series are absolutely convergent, which completes the proof.

Corollary 1. *The f -cost process of an H -admissible strategy Φ can also be expressed in terms of the portfolio value V :*

$$(7) \quad C_t(\Phi) = f''(0) \left(V_t - V_0 - \int_{0+}^t \delta_{s-} dS_s \right) + \frac{f^{(3)}(0)}{2} \left([V, V]_t^c - 2 \int_{0+}^t \delta_{s-} d[V, S]_s^c + \int_{0+}^t \delta_{s-}^2 d[S, S]_s^c \right) + \sum_{0 < s \leq t} f'(\Delta V_s - \delta_{s-} \Delta S_s) - f''(0) (\Delta V_s - \delta_{s-} \Delta S_s).$$

Proof. The proof follows easily from applying the definition of V in the formula (2) and straightforward calculations using quadratic variation properties. ■

With the f -cost process well defined for strategies of interest in continuous time, we can now state the criteria which will characterize pseudo-optimal strategies, by analogy with the discrete time case.

Definition. An H -admissible strategy Φ will be called pseudo-optimal for the local risk-minimization if its f -cost process is a martingale strongly orthogonal to the martingale part M of the process S .

In the next section, we will derive the corresponding set of equations that pseudo-optimal strategies have to solve in two different Markovian frameworks.

4. Application to stochastic volatility models. In this section we will assume further hypotheses on the trading strategies so that we can derive an explicit formula for the f -cost process and completely characterize the pseudo-optimal strategies for the local risk-minimization.

We model the evolution of S through a set of SDEs with stochastic volatility:

$$(8) \quad dS_t = \mu_t dt + \sigma_t dW_t^1,$$

$$(9) \quad d\sigma_t = \gamma_t dt + \Sigma_t dW_t^2$$

with smooth μ_t , γ_t , and Σ_t , W^1 and W^2 being standard Wiener processes under \mathcal{P} with constant instantaneous correlation ρ , i.e., $d\langle W^1, W^2 \rangle_t = \rho dt$.

We shall also assume that appropriate conditions hold for the functions μ_t , γ_t , and Σ_t so that the system of SDEs (8), (9) admits a unique strong continuous solution for S and σ , with $S > 0$ and $\sigma > 0$. With these diffusion assumptions we will now place ourselves in a Markovian framework and look for the optimal strategy Φ as a smooth function of the state variables

$$\begin{aligned} \delta_t &= \delta(t, S_t, \sigma_t), \\ V_t &= V(t, S_t, \sigma_t). \end{aligned}$$

4.1. PDE formulation. We first derive a PDE formulation. For that purpose let us express the cost process as a function of the diffusion parameters and the strategy

$$\begin{aligned} C_t(\Phi) &= \int_0^t \left[f''(0) \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_u \mu_u \right) \right. \\ &\quad + \frac{f^{(3)}(0)}{2} \left(\left(\frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left(\frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ &\quad \left. - f^{(3)}(0) \delta_u \left(\frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_u^2 \sigma_u^2 \right] du \\ &\quad + \int_0^t f''(0) \left(\frac{\partial V}{\partial S} - \delta_u \right) \sigma_u dW_u^1 + \int_0^t f''(0) \frac{\partial V}{\partial \sigma} \Sigma_u dW_u^2 \end{aligned}$$

which we have obtained from (7).

Now, applying to the strategy Φ the first pseudo-optimality criterion, i.e., that C must be a martingale under the measure P , we find the PDE satisfied by the portfolio value V :

$$\begin{aligned} f''(0) & \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_u \mu_u \right) \\ & + \frac{f^{(3)}(0)}{2} \left(\left(\frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left(\frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ & - f^{(3)}(0) \delta_u \left(\frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_u^2 \sigma_u^2 = 0 \end{aligned}$$

with the terminal conditions corresponding to $V_T = \delta^H S_T + \beta^H$.

Applying to the strategy Φ the second pseudo-optimality criterion, i.e., that the martingale C must be orthogonal to S , we find the equation satisfied by the optimal hedge δ :

$$\left(\frac{\partial V}{\partial S} - \delta_u \right) \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u = 0.$$

Complete markets case. The case of complete markets allows us to recover the celebrated Black and Scholes formula [3], [10] regardless of the choice for the function f . Indeed, by taking Σ the volatility of volatility equal to zero, the optimality equations reduce to

$$(10) \quad \delta_u = \frac{\partial V}{\partial S},$$

$$(11) \quad \frac{\partial V}{\partial u} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 = 0.$$

Equation (10) gives the perfect hedging strategy in that context, since upon suitable boundary conditions it is well known that the PDE (11) has a unique solution. Of course one can verify that the f -cost process is then identically zero, which amounts to having a self-financing strategy that perfectly replicates the contingent claim H .

4.2. Link with FBSDE. Given the two equations we found for the optimal portfolio, we can now relate the pseudo-optimal strategies for the local risk-minimization with the solution of an FBSDE associated with the diffusion process of the stock price S . This is based on the generalization of the Feynman–Kac formula (see the survey paper on BSDEs in finance from El Karoui, Peng and Quenez [5], for instance), which links quasi-linear PDEs with BSDEs.

We begin by rewriting the two equations satisfied by the theoretical portfolio value V_t and the stock quantity δ_t as

$$(12) \quad \begin{aligned} & \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u \\ & = \delta_u \mu_u + \alpha \left(\left(\frac{\partial V}{\partial S} \sigma_u + \frac{\partial V}{\partial \sigma} \rho \Sigma_u - \delta_u \sigma_u \right)^2 + (1 - \rho^2) \left(\frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 \right), \end{aligned}$$

$$(13) \quad \frac{\partial V}{\partial S} \sigma_u + \frac{\partial V}{\partial \sigma} \rho \Sigma_u - \delta_u \sigma_u = 0,$$

where $\alpha = -\frac{1}{2} \frac{f^{(3)}(0)}{f''(0)}$. Inserting (13) in (12), we find

$$(14) \quad \frac{\partial V}{\partial u} + \Lambda V = \frac{\mu}{\sigma} \left(\frac{\partial V}{\partial S} \sigma + \frac{\partial V}{\partial \sigma} \rho \Sigma \right) + \alpha \left(\sqrt{1 - \rho^2} \frac{\partial V}{\partial \sigma} \Sigma \right)^2,$$

$$(15) \quad \delta \sigma = \frac{\partial V}{\partial S} \sigma + \frac{\partial V}{\partial \sigma} \rho \Sigma,$$

where Λ is the infinitesimal generator corresponding to the diffusion equations (8) and (9) under the measure \mathcal{P} :

$$\Lambda V_u = \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u.$$

So we are now in a position to state the main result of this section, effectively relating a pseudo-optimal strategy with the solution of an FBSDE.

Theorem 3. *Any pseudo-optimal strategy $\Phi = (V, \delta)$ for the local risk-minimization yields a solution to the following FBSDE:*

$$\begin{aligned} dS_t &= \mu_t dt + \sigma_t dW_t^1, \\ d\sigma_t &= \gamma_t dt + \Sigma_t \left(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right), \\ -dY_s &= g(s, S_s, \sigma_s, Y_s, Z_s^1, Z_s^2) ds - Z_s^1 dW_s^1 - Z_s^2 dW_s^2, \\ Y_T &= \beta^H + \delta^H S_T, \end{aligned}$$

where $W = (W^1, W^2)$ is a standard two-dimensional Brownian motion and $g(s, S, \sigma, Y, Z^1, Z^2) = -\frac{\mu}{\sigma} Z^1 - \alpha (Z^2)^2$, with $Y = V$ and $Z = (\delta \sigma, \frac{\partial V}{\partial \sigma} \Sigma \sqrt{1 - \rho^2})$.

Proof. The result follows from application of the Itô formula to the pseudo-optimal strategy $\Phi^* = (V^*, \delta^*)$, which then solve (14) and (15). We get

$$\begin{aligned} dV_t^* &= \frac{\partial V}{\partial t} + \Lambda V_t + \frac{\partial V}{\partial S} \sigma dW_s^1 + \frac{\partial V}{\partial \sigma} \Sigma (\rho dW_s^1 + \sqrt{1 - \rho^2} dW_s^2) \\ \Leftrightarrow dV_t^* &= -g \left(t, S_t, \sigma_t, V_t, \delta_t^*, \sqrt{1 - \rho^2} \frac{\partial V}{\partial \sigma} \Sigma \right) + \frac{\partial V}{\partial S} \sigma dW_s^1 + \frac{\partial V}{\partial \sigma} \Sigma (\rho dW_s^1 + \sqrt{1 - \rho^2} dW_s^2) \\ \Leftrightarrow -dV_t^* &= g \left(t, S_t, \sigma_t, V_t, \delta_t^*, \sqrt{1 - \rho^2} \frac{\partial V}{\partial \sigma} \Sigma \right) - \delta^* \sigma_s dW_s^1 - \frac{\partial V}{\partial \sigma} \Sigma \sqrt{1 - \rho^2} dW_s^2, \end{aligned}$$

which is the result announced with $Y = V$ and $Z = (\delta \sigma, \frac{\partial V}{\partial \sigma} \Sigma \sqrt{1 - \rho^2})$. ■

4.3. The minimization problem. After using the pseudo-optimal criteria to characterize strategies, we return to the original minimization problem to show that those identified strategies are actually optimal.

Given the smoothness of the risk function f , we can rewrite the process r_f^T by using a Taylor development around the nonperturbed strategy Φ . Let us fix $t \in [0, T]$; because of the definition of the process $r_f^T[\Phi, \phi]$ and as we work with increasing sequences of partitions, we may assume that t is one of the $t_{i(n)}^n$ (we will thereafter drop the superscript n and simply

write t_i instead), and we have

$$\begin{aligned} r_f^\tau[\Phi, \phi](t, \omega) &= \frac{\Delta R_{t_i}(\Phi + \phi|_{[t_i, t_{i+1}]}) (\omega) - \Delta R_{t_i}(\Phi) (\omega)}{t_{i+1} - t_i} \\ &= \frac{\mathbb{E}_{t_i} (f(\Delta C_{t_{i+1}}(\Phi + \phi|_{[t_i, t_{i+1}]})) (\omega) - \mathbb{E}_{t_i} (f(\Delta C_{t_{i+1}}(\Phi))) (\omega)}{t_{i+1} - t_i}. \end{aligned}$$

Applying Taylor's formula with remainder term to $g : (x, y) \mapsto f(x + y)$ in the expectation, we have that

$$\begin{aligned} f(\Delta C_{t_{i+1}}(\Phi + \phi|_{[t_i, t_{i+1}]})) &= f(\Delta C_{t_{i+1}}(\Phi)) - \beta_{t_i} f'(\Delta C_{t_{i+1}}(\Phi)) \\ &\quad - \delta_{t_i} S_{t_{i+1}} f'(\Delta C_{t_{i+1}}(\Phi)) + \frac{1}{2} (\beta_{t_i} + \delta_{t_i} S_{t_{i+1}})^2 g(\tilde{\phi}), \end{aligned}$$

where $g(\tilde{\phi}) = f''(\Delta C(\tilde{\phi}))$ with $\tilde{\phi} = (\tilde{\beta}, \tilde{\delta})$ such that $|\tilde{\beta}| \leq \beta$ and $|\tilde{\delta}| \leq \delta$.

Rearranging and simplifying we get

$$\begin{aligned} r_f^\tau[\phi, \Delta](t, \omega) &= \beta_{t_i} \frac{\mathbb{E}_{t_i} (f'(\Delta C_{t_{i+1}}(\phi))) (\omega)}{t_{i+1} - t_i} + \delta_{t_i} \frac{\mathbb{E}_{t_i} (S_{t_{i+1}} f'(\Delta C_{t_{i+1}}(\phi))) (\omega)}{t_{i+1} - t_i} \\ &\quad + \frac{\mathbb{E}_{t_i} \left((\beta_{t_i} + \frac{1}{2} \delta_{t_i} S_{t_{i+1}})^2 g(\tilde{\phi}) \right) (\omega)}{t_{i+1} - t_i}. \end{aligned}$$

Because we work with Itô processes, the following stand:

$$\begin{aligned} \lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (f'(\Delta C_{t_{i+1}}(\phi))) (\omega)}{t_{i+1} - t_i} &= \Lambda (f' \circ \Delta)_{t_i}, \\ \lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (S_{t_{i+1}} f'(\Delta C_{t_{i+1}}(\phi))) (\omega)}{t_{i+1} - t_i} &= \Lambda (S \cdot f' \circ \Delta)_{t_i} \end{aligned}$$

and

$$\begin{aligned} \lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (g(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} &= \Lambda g_{t_i}, \\ \lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (S_{t_{i+1}} g(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} &= \Lambda (S \cdot g)_{t_i}, \\ \lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (S_{t_{i+1}}^2 g(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} &= \Lambda (S^2 \cdot g)_{t_i}, \end{aligned}$$

where Λ is the infinitesimal generator associated with the diffusion:

$$\Lambda h = \frac{\partial h}{\partial S} \mu + \frac{\partial h}{\partial \sigma} \gamma + \frac{1}{2} \frac{\partial^2 h}{\partial S^2} \sigma^2 + \frac{1}{2} \frac{\partial^2 h}{\partial \sigma^2} \Sigma^2 + \frac{\partial^2 h}{\partial S \partial \sigma} \rho \sigma \Sigma.$$

Finally the process r_f^τ evaluated in t is worth

$$r_f^\tau[\phi, \Delta](t, \omega) = \beta_t \Lambda (f' \circ \Delta C)_t + \delta_t \Lambda (S \cdot f' \circ \Delta C)_t + \frac{1}{2} (\beta_t^2 \Lambda g_t + 2\beta_t \delta_t \Lambda (S \cdot g)_t + \delta_t^2 \Lambda (S^2 \cdot g)_t).$$

Now we first take the component δ of the perturbation equal to zero, that is, we perturb only β , so that we have the following first condition for the strategy ϕ to be locally risk-minimizing:

$$\beta_t \Lambda (f' \circ \Delta C)_t + \frac{1}{2} \beta_t^2 \Lambda g_t \geq 0 \text{ } \mathcal{P}\text{-a.e. } \forall \beta_t.$$

As a consequence we must have $\Lambda (f' \circ \Delta C)_t = 0$.

Likewise we take the component β equal to zero, and we get the following second condition for the strategy ϕ to be locally risk-minimizing:

$$\delta_t \Lambda (S \cdot f' \circ \Delta C)_t + \frac{1}{2} \delta_t^2 \Lambda (S^2 \cdot g)_t \geq 0 \text{ } \mathcal{P}\text{-a.e. } \forall \delta_t.$$

Therefore we must have $\Lambda (S \cdot f' \circ \Delta C)_t = 0$. But we observe that

$$\begin{cases} \Lambda (f' \circ \Delta C)_t = 0, \\ \Lambda (S \cdot f' \circ \Delta C)_t = 0 \end{cases} \Leftrightarrow \begin{cases} f''(0) \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} \mu_t + \frac{\partial V}{\partial \sigma} \gamma_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_t^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_t^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_t \Sigma_t - \delta_t \mu_t \right), \\ \quad + \frac{f^{(3)}(0)}{2} \left(\left(\frac{\partial V}{\partial S} \right)^2 \sigma_t^2 + \left(\frac{\partial V}{\partial \sigma} \right)^2 \Sigma_t^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_t \Sigma_t \right), \\ \quad - f^{(3)}(0) \delta_t \left(\frac{\partial V}{\partial S} \sigma_t^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_t \Sigma_t \right) + \frac{f^{(3)}(0)}{2} \delta_t^2 \sigma_t^2 = 0, \\ \quad \left(\frac{\partial V}{\partial S} - \delta_t \right) \sigma_t^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_t \Sigma_t = 0. \end{cases}$$

Finally, as announced in the beginning of this section, we see that in this context of stochastic volatility model, the optimal strategies with respect to the local risk-minimization problem are the same as the pseudo-optimal strategies. As one can see in the derivation, the only requirement to obtain this result is the existence of the infinitesimal generator and its expression in terms of the parameters of the SDE driving the process so as to identify the sets of two equations, which therefore allows for a straightforward generalization to Itô processes.

5. Application to stochastic volatility/jump diffusion models. In this section, we want to provide an example of a situation where the nonquadratic risk definitely implies a different hedging strategy, not only through the Taylor expansion around zero of the risk function. We therefore model the evolution of S through an SDE with stochastic volatility and Poisson jumps in the vein of the Bates model [2]:

$$\begin{aligned} dS_t &= \mu_t dt + \sigma_t dW_t^1 + kdN_t, \\ d\sigma_t &= \gamma_t dt + \Sigma_t dW_t^2 \end{aligned}$$

with W^1 and W^2 Wiener processes under \mathcal{P} and $d(W^1, W^2)_t = \rho dt$, N_t a Poisson process of intensity λ , and the amplitude of the jumps k having probability distribution K . We also assume that W_t , N_t , and k are independent. As in the case of stochastic volatility we shall assume that appropriate conditions hold on the adapted processes μ , σ , γ , and Σ so that the set of SDEs has a unique strong solution.

With these assumptions we will again place ourselves in a Markovian framework and look for the optimal strategy Φ as a smooth function of the state variables

$$\begin{aligned} \delta_t &= \delta(t, S_t, \sigma_t), \\ V_t &= V(t, S_t, \sigma_t). \end{aligned}$$

5.1. PIDE formulation. We first derive a PIDE formulation. For that purpose let us express the cost process as a function of the diffusion parameters and the strategy

$$\begin{aligned} C_t(\Phi) &= \int_0^t \left(f''(0) \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_{u-} \mu_u \right) \right. \\ &\quad + \frac{f^{(3)}(0)}{2} \left(\left(\frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left(\frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ &\quad \left. - f^{(3)}(0) \delta_{u-} \left(\frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_{u-}^2 \sigma_u^2 \right) du \\ &\quad + \int_0^t f''(0) \left(\frac{\partial V}{\partial S} - \delta_{u-} \right) \sigma_u dW_u^1 \\ &\quad + \int_0^t f''(0) \frac{\partial V}{\partial \sigma} \Sigma_u dW_u^2 \\ &\quad + \int_0^t \int_{\mathbb{R}} f'(\Delta V_u - \delta_{u-} \Delta S_u) K(k) dk dN_u \end{aligned}$$

which we have again obtained from (7) and with ΔV_u the jump in V when there is a jump ΔS_u of size k on S at time u being equal to $V(u-, S_{u-} + k, \sigma_{u-}) - V(u-, S_{u-}, \sigma_{u-})$.

Now, applying to the strategy Φ the first pseudo-optimality criterion, i.e., that C must be a martingale under the measure P , we find the PIDE satisfied by the portfolio value V :

$$\begin{aligned} f''(0) \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_{u-} \mu_u \right) \\ + \frac{f^{(3)}(0)}{2} \left(\left(\frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left(\frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ - f^{(3)}(0) \delta_{u-} \left(\frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_{u-}^2 \sigma_u^2 \\ + \int_{\mathbb{R}} f'(\Delta V_u - \delta_{u-} \Delta S_u) K(k) dk \lambda_u = 0 \end{aligned}$$

with the terminal conditions corresponding to $V_T = \delta^H S_T + \beta^H$.

Applying to the strategy Φ the second pseudo-optimality criterion, i.e., that the martingale C must be orthogonal to S , we find the equation satisfied by the optimal hedge δ :

$$\left(\frac{\partial V}{\partial S} - \delta_{u-}\right) \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u + \int_{\mathbb{R}} f'(\Delta V_u - \delta_{u-} \Delta S_u) k K(k) dk \lambda_u = 0.$$

Contrarily to the stochastic volatility case, where only the local behavior of the risk function f in 0 mattered, the optimal strategy in the jump-diffusion model requires the knowledge of the risk function f on its whole support.

Conclusion. We have introduced in this article a new methodology in order to hedge contingent claims in incomplete markets. This methodology gives more flexibility in the choice of the function used to assess the risk of the strategy, and we believe it can fit better with market practice. Furthermore, being a local minimization of a local risk, it is expected to be more robust to fluctuations of market conditions so that traders will be able to adapt their hedging strategy at lower costs than with a more global approach. Finally, since the methodology defines the risk as a function of the trading costs, transaction costs can be included in the setting rather seamlessly. This will be shown in the case of liquidity costs in a forthcoming paper [1]. Further extensions include multidimensional semimartingales setting to describe the stock price processes and therefore deal with multiasset options.

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