

# What makes the ARC-PRESERVING SUBSEQUENCE problem hard?\*

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**Abstract.** Given two arc-annotated sequences  $(S, P)$  and  $(T, Q)$  representing RNA structures, the ARC-PRESERVING SUBSEQUENCE (APS) problem asks whether  $(T, Q)$  can be obtained from  $(S, P)$  by deleting some of its bases (together with their incident arcs, if any). In previous studies [3, 6], this problem has been naturally divided into subproblems reflecting intrinsic complexity of arc structures. We show that APS(CROSSING, PLAIN) is NP-complete, thereby answering an open problem [6]. Furthermore, to get more insight into where actual border of APS hardness is, we refine APS classical subproblems in much the same way as in [11] and give a complete categorization among various restrictions of APS problem complexity.

**Keywords:** RNA structures, Arc-Preserving Subsequence, Computational complexity.

## 1 Introduction

At a molecular state, the understanding of biological mechanisms is subordinated to RNA functions discovery and study. Indeed, it is established that the conformation of a single-stranded RNA molecule (a linear sequence composed of ribonucleotides  $A$ ,  $U$ ,  $C$  and  $G$ , also called *primary structure*) partly determines the molecule function. This conformation results from the folding process due to local pairings between complementary bases ( $A-U$  and  $C-G$ ). The RNA secondary structure is a collection of folding patterns that occur in it.

RNA secondary structure comparison is important in many contexts, such as (i) identification of highly conserved structures during evolution which suggest a significant common function for the studied RNA molecules [9], (ii) RNA classification of various species (phylogeny)[2], (iii) RNA folding prediction by considering a set of already known secondary structures [13].

Structure comparison for RNA has thus become a central computational problem bearing many challenging computer science questions. At a theoretical level, RNA structure is often modelled as an *arc-annotated sequence*, that is a pair  $(S, P)$  where  $S$  is a sequence of ribonucleotides and  $P$  represents hydrogen bonds between pairs of elements of  $S$ . Different pattern matching and motif search problems have been investigated in the context of arc-annotated sequences among which we can mention ARC-PRESERVING SUBSEQUENCE (APS) problem, EDIT DISTANCE problem, ARC-SUBSTRUCTURE (AST) problem and LONGEST ARC-PRESERVING SUBSEQUENCE (LAPCS) problem (see for instance [3, 8, 7, 6, 1]). For other related studies concerning algorithmic aspects of (protein) structure comparison using *contact maps*, refer to [5, 10].

In this paper, we focus on APS problem: given two arc-annotated sequences  $(S, P)$  and  $(T, Q)$ , this problem asks whether  $(T, Q)$  can be exactly obtained from  $(S, P)$  by deleting some of its bases together with their incident arcs, if any. This problem is commonly encountered when one is searching for a given RNA pattern in an RNA database [7]. Moreover, from a theoretical point of view, APS problem can be seen as a restricted version of LAPCS problem, and hence has applications in

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structural comparison of RNA and protein sequences [3, 5, 12]. APS problem has been extensively studied in the past few years [6, 7, 3]. Of course, different restrictions on arc-annotation alter APS computational complexity, and hence this problem has been naturally divided into subproblems reflecting the complexity of the arc structure of both  $(S, P)$  and  $(T, Q)$ : PLAIN, CHAIN, NESTED, CROSSING or UNLIMITED (see Section 2 for details). All of them but one have been classified as to whether they are polynomial time solvable or **NP**-complete. The problem of the existence of a polynomial time algorithm for APS(CROSSING, PLAIN) problem was mentioned in [6] as the last open problem in the context of arc-preserving subsequences. Unfortunately, as we shall prove in Section 4, APS(CROSSING, PLAIN) is **NP**-complete even for restricted special cases.

In analyzing the computational complexity of a problem, we are often trying to define a precise boundary between polynomial and **NP**-complete cases. Therefore, as another step towards establishing the precise complexity landscape of APS problem, we consider that it is of great interest to subdivide existing cases into more precise ones, that is to refine classical complexity levels of APS problem, for determining more precisely what makes the problem hard. For that purpose, we use the framework introduced by Vialette [11] in the context of 2-intervals (a simple abstract structure for modelling RNA secondary structures). As a consequence, the number of complexity levels rises from 4 to 8, and all the entries of this new complexity table need to be filled. Previous known results concerning APS problem, along with our **NP**-completeness proofs, allow us to fill all the entries of this new table, therefore determining what exactly makes the APS problem hard.

The paper is organized as follows. Provided with notations and definitions (Section 2), in Section 3 we introduce and explain new refinements of the complexity levels we are going to study. In Section 4, we show that APS( $\{\square, \emptyset\}, \emptyset$ ) is **NP**-complete thereby proving that *classical* APS(CROSSING, PLAIN) is **NP**-complete as well. As another refinement to that result, we prove that APS( $\{<, \emptyset\}, \emptyset$ ) is **NP**-complete. Finally, in Section 5, we give new polynomial time solvable algorithms for restricted instances of APS(CROSSING, PLAIN).

## 2 Preliminaries

An RNA structure is commonly represented as an arc-annotated sequence  $(S, P)$  where  $S$  is a sequence of ribonucleotides (or bases) and  $P$  is a set of arcs connecting pairs of bases in  $S$ . Let  $(S, P)$  and  $(T, Q)$  be two arc-annotated sequences such that  $|S| \geq |T|$  (in the following,  $n = |S|$  and  $m = |T|$ ). APS problem asks whether  $(T, Q)$  can be exactly obtained from  $(S, P)$  by deleting some of its bases together with their incident arcs, if any.

Since the general problem is easily seen to be intractable [3], the arc structure must be restricted. Evans [3] proposed four possible restrictions on  $P$  (resp.  $Q$ ) which were largely reused in subsequent literature: (1) there is no base incident to more than one arc, (2) there are no arcs crossing, (3) there is no arc contained in another and (4) there is no arc.

These restrictions are used progressively and inclusively to produce five different levels of allowed arc structure: UNLIMITED - general problem with no restrictions, CROSSING - restriction (1); NESTED - restrictions (1) and (2); CHAIN - restrictions (1), (2) and (3); PLAIN - restriction (4).

Guo proved in [7] that APS(CROSSING, CHAIN) is **NP**-complete. Guo *et al.* observed in [6] that **NP**-completeness of APS(CROSSING, CROSSING) and APS(UNLIMITED, PLAIN) easily follows from results of Evans [3] concerning LAPCS problem. Furthermore, they gave a  $O(nm)$  time algorithm for APS(NESTED, NESTED). This algorithm can be applied to easier problems such as APS(NESTED,  $\alpha$ ) and APS(CHAIN,  $\alpha$ ) with  $\alpha \in \{\text{CHAIN, PLAIN}\}$ . Finally, Guo *et al.* mentioned in [6] that APS(CHAIN, PLAIN) can be solved in  $O(n + m)$  time. Observe that UNLIMITED level has no restrictions, and hence is of limited interest in our study. Consequently, from now on we will not be concerned anymore with that level. Until now, the question of the existence of an exact polynomial algorithm for APS(CROSSING, PLAIN) remained open. We will show in the present paper that APS(CROSSING, PLAIN) is **NP**-complete.

## 3 Refinement of APS problem

In this section, we propose a refinement of APS problem. We first state formally our approach and explain why such a refinement is relevant for both theoretical and experimental studies.

### 3.1 Splitting the levels.

As we will show soon,  $\text{APS}(\text{CROSSING}, \text{PLAIN})$  is **NP**-complete. That result answers the last open problem concerning APS computational complexity with respect to classical complexity levels. However, we are mainly interested in the elaboration of a precise border between **NP**-complete and polynomially solvable cases. Indeed, both theorists and practitioners might naturally ask for more information concerning APS hard cases in order to get valuable insight into what makes the problem difficult. As a next step towards better understanding what makes APS problem hard, we propose to refine classically models used for classifying arc-annotated sequences. Our refinement consists in splitting those models of arc-annotated sequences into more precise relations between arcs. For example, such a refinement provides a general framework for investigating polynomial time solvable and hard restricted instances of  $\text{APS}(\text{CROSSING}, \text{PLAIN})$ , thereby refining in many ways Theorem 1 (see Section 5).

We use three relations first introduced by Vialette [11] in the context of 2-intervals. Actually, his definition of 2-intervals could almost apply in this paper (the main difference lies in the fact that 2-intervals are used for representing sets of contiguous arcs). Vialette defined three possible relations between 2-intervals that can be used for arc-annotated sequences as-well. They are the following: for any two arcs  $A_1 = (i, j)$  and  $A_2 = (k, l)$  in  $P$ , we will write  $A_1 < A_2$  if  $i < j < k < l$  (*precedence* relation),  $A_1 \sqsubset A_2$  if  $k < i < j < l$  (*nested* relation) and  $A_1 \bowtie A_2$  if  $i < k < j < l$  (*crossing* relation). Two arcs  $A_1$  and  $A_2$  are  $\tau$ -comparable for some  $\tau \in \{<, \sqsubset, \bowtie\}$  if  $A_1 \tau A_2$  or  $A_2 \tau A_1$ . Let  $\mathcal{P}$  be a set of arcs and  $R$  be a non-empty subset of  $\{<, \sqsubset, \bowtie\}$ . The set  $\mathcal{P}$  is said to be *R-comparable* if any two distinct arcs of  $\mathcal{P}$  are  $\tau$ -comparable for some  $\tau \in R$ . An arc-annotated sequence  $(P, A)$  is said to be an *R-arc-annotated sequence* for some non-empty subset  $R$  of  $\{<, \sqsubset, \bowtie\}$  if  $A$  is *R-comparable*. By abuse of notation, we write  $R = \emptyset$  in case  $A = \emptyset$ . Observe that our model cannot deal with arc-annotated sequences which contain only one arc. However, having only one arc or none can not really affect the problem computational complexity. Just one guess reduces from one case to the other.

As a straightforward illustration of above definitions, classical complexity levels for APS problem can be expressed in terms of combinations of our new relations: **PLAIN** is fully described by  $R = \emptyset$ , **CHAIN** by  $R = \{<\}$ , **NESTED** by  $R = \{<, \sqsubset\}$  and **CROSSING** by  $R = \{<, \sqsubset, \bowtie\}$ . The key point is to observe that our refinement allows us to consider new structures for arc-annotated sequences, namely  $R \in \{\{\sqsubset\}, \{\bowtie\}, \{<, \bowtie\}, \{\sqsubset, \bowtie\}\}$ , which could not be considered using classical complexity levels. Although other refinements may be possible (in particular well-suited for parameterized complexity analysis), we do believe that such an approach allows a more precise analysis of APS complexity.

Of course one might object that some of these subdivisions are unlikely to appear in RNA secondary structures. However, it is of great interest to answer, at least partly, the following question: Where is the precise boundary between polynomial and **NP**-complete cases ? Indeed, such a question is relevant for both theoretical and experimental studies.

### 3.2 Immediate results

First, observe that we only have to consider cases of  $\text{APS}(R_1, R_2)$  where  $R_1$  and  $R_2$  are compatible, *i.e.*  $R_2 \subseteq R_1$ . Indeed, if this is not the case, we can immediately answer negatively since there exists two arcs in  $T$  which satisfy a relation in  $R_2$  which is not in  $R_1$ , and hence  $T$  simply cannot be obtained from  $S$  by deleting bases of  $S$ . Those useless cases are simply denoted by hatched areas in Table 1.

Some known results allow us to fill many entries of the new complexity table derived from our refinement. The remainder of this subsection is devoted to detailing these first easy statements. We begin with an easy observation concerning complexity propagation properties of APS problem.

**Observation 1** *Let  $R_1, R_2, R'_1$  and  $R'_2$  be four subsets of  $\{<, \sqsubset, \bowtie\}$  such that  $R'_2 \subseteq R_2 \subseteq R_1$  and  $R'_2 \subseteq R'_1 \subseteq R_1$ . If  $\text{APS}(R'_1, R'_2)$  is **NP**-complete (resp.  $\text{APS}(R_1, R_2)$  is polynomial time solvable) then so is  $\text{APS}(R_1, R_2)$  (resp.  $\text{APS}(R'_1, R'_2)$ ).*

On the positive side, Guo *et al.* have shown that  $\text{APS}(\text{NESTED}, \text{NESTED})$  is solvable in  $O(nm)$  time [6]. Another way of stating this is to say that  $\text{APS}(\{<, \sqsubset\}, \{<, \sqsubset\})$  is solvable in  $O(mn)$  time. That result together with Observation 1 may be summarized by saying that  $\text{APS}(R_1, R_2)$  for any compatible  $R_1$  and  $R_2$  such that  $\emptyset \notin R_1$  and  $\emptyset \notin R_2$  is polynomial time solvable.

Conversely, Evans has proved that  $\text{APS}(\text{CROSSING}, \text{CROSSING})$  is **NP**-complete [3]. A simple reading shows that her proof is concerned with  $\{<, \sqsubset, \emptyset\}$ -arc-annotated sequences, and hence she actually proved that  $\text{APS}(\{<, \sqsubset, \emptyset\}, \{<, \sqsubset, \emptyset\})$  is **NP**-complete. Similarly, in proving that  $\text{APS}(\text{CROSSING}, \text{CHAIN})$  is **NP**-complete [7], Guo actually proved that  $\text{APS}(\{<, \sqsubset, \emptyset\}, \{<\})$  is **NP**-complete. Note that according to Observation 1, this latter result implies that  $\text{APS}(\{<, \sqsubset, \emptyset\}, \{<, \sqsubset\})$  and  $\text{APS}(\{<, \sqsubset, \emptyset\}, \{<, \emptyset\})$  are **NP**-complete. Table 1 surveys known and new results for various types of our refined APS problem. Observe that this paper answers all questions concerning APS problem with respect to both classical and new complexity levels.

| APS                           |                               |                            |                    |                 |                    |                 |                 |               |
|-------------------------------|-------------------------------|----------------------------|--------------------|-----------------|--------------------|-----------------|-----------------|---------------|
| $R_1 \setminus R_2$           | $\{<, \sqsubset, \emptyset\}$ | $\{\sqsubset, \emptyset\}$ | $\{<, \emptyset\}$ | $\{\emptyset\}$ | $\{<, \sqsubset\}$ | $\{\sqsubset\}$ | $\{<\}$         | $\emptyset$   |
| $\{<, \sqsubset, \emptyset\}$ | <b>NP-C</b> [3]               | <b>NP-C</b> *              | <b>NP-C</b> [7]    | <b>NP-C</b> *   | <b>NP-C</b> [7]    | <b>NP-C</b> *   | <b>NP-C</b> [7] | <b>NP-C</b> * |
| $\{\sqsubset, \emptyset\}$    |                               | <b>NP-C</b> *              | ////               | <b>NP-C</b> *   | ////               | <b>NP-C</b> *   | ////            | <b>NP-C</b> * |
| $\{<, \emptyset\}$            |                               |                            | <b>NP-C</b> *      | <b>NP-C</b> *   | ////               | ////            | <b>NP-C</b> *   | <b>NP-C</b> * |
| $\{\emptyset\}$               |                               |                            |                    | $O(nm^2)$ *     | ////               | ////            | ////            | $O(nm^2)$ *   |
| $\{<, \sqsubset\}$            |                               |                            |                    |                 | $O(nm)$ [6]        | $O(nm)$ [6]     | $O(nm)$ [6]     | $O(nm)$ [6]   |
| $\{\sqsubset\}$               |                               |                            |                    |                 |                    | $O(nm)$ [6]     | ////            | $O(nm)$ [6]   |
| $\{<\}$                       |                               |                            |                    |                 |                    |                 | $O(nm)$ [6]     | $O(n+m)$ [6]  |
| $\emptyset$                   |                               |                            |                    |                 |                    |                 |                 | $O(n+m)$ [6]  |

**Table 1.** Complexity results after complexity levels refinement. ////: useless cases. \*: results from this paper.

#### 4 Hardness results

We show in this section that  $\text{APS}(\{\sqsubset, \emptyset\}, \emptyset)$  is **NP**-complete thereby proving that  $\text{APS}(\text{CROSSING}, \text{PLAIN})$  is **NP**-complete. That result answers an open problem posed by Gramm, Guo and Niedermeier in [6] which is the last open problem concerning the APS computational complexity with respect to classical complexity levels, *i.e.*, **PLAIN**, **CHAIN**, **NESTED** and **CROSSING**.

We provide a polynomial time reduction from the well known **NP**-complete 3-SAT problem [4]: Given a set  $\mathcal{V}_n$  of  $n$  variables and a set  $\mathcal{C}_q$  of  $q$  clauses (each composed of three literals) over  $\mathcal{V}_n$ , the problem asks to find a truth assignment for  $\mathcal{V}_n$  that satisfies all clauses of  $\mathcal{C}_q$ .

It is easily seen that  $\text{APS}(\{\sqsubset, \emptyset\}, \emptyset)$  is in **NP**. The remainder of the section is devoted to proving that it is also **NP**-hard. Let  $\mathcal{V}_n = \{x_1, x_2, \dots, x_n\}$  be a finite set of  $n$  variables and  $\mathcal{C}_q = \{c_1, c_2, \dots, c_q\}$  a collection of  $q$  clauses. Observe that there is no loss of generality in assuming that literals are left-right ordered, *i.e.*, if  $c_i = (x_j \vee x_k \vee x_l)$  then  $j < k < l$ . Let us first detail the construction of sequences  $S$  and  $T$ :

$$S = S_{x_1}^s A S_{x_1}^s S_{x_2}^s A S_{x_2}^s \dots S_{x_n}^s A S_{x_n}^s S_{c_1} S_{c_2} \dots S_{c_q} S_{x_1}^e S_{x_2}^e \dots S_{x_n}^e$$

$$T = T_{x_1}^s T_{x_2}^s \dots T_{x_n}^s T_{c_1} T_{c_2} \dots T_{c_q} T_{x_1}^e T_{x_2}^e \dots T_{x_n}^e$$

We now detail subsequences composing  $S$  and  $T$ . Let  $\gamma_m$  (resp.  $\gamma_{\overline{m}}$ ) be the number of occurrences of literal  $x_m$  (resp.  $\overline{x_m}$ ) in  $\mathcal{C}_q$ . For each variable  $x_m \in \mathcal{V}_n$ ,  $1 \leq m \leq n$ , we construct words  $S_{x_m}^s = AC^{k_m}$ ,  $S_{x_m}^s = C^{k_m} A$  and  $T_{x_m}^s = AC^{k_m} A$  where  $C^{k_m}$  represents a word of  $k_m = \max(\gamma_m, \gamma_{\overline{m}})$  consecutive bases  $C$ . For each clause  $c_i$  of  $\mathcal{C}_q$ ,  $1 \leq i \leq q$ , we construct words  $S_{c_i} = UGGGA$  and  $T_{c_i} = UGA$ . Finally, for each variable  $x_m \in \mathcal{V}_n$ ,  $1 \leq m \leq n$ , we construct words  $S_{x_m}^e = UUA$  and  $T_{x_m}^e = UA$ .

Having disposed of the two sequences, we now turn to defining the corresponding two arc structures (see Figure 1). In the following,  $\text{Seq}[i]$  will denote the  $i^{\text{th}}$  base of a sequence  $\text{Seq}$  and, for any  $1 \leq m \leq n$ ,  $l_m^s = |S_{x_m}^s|$ . For all  $1 \leq m \leq n$ , we create the two following arcs:  $(S_{x_m}^s[1], S_{x_m}^e[1])$  and  $(S_{x_m}^s[l_m^s], S_{x_m}^e[2])$ . For each clause  $c_i$  of  $\mathcal{C}_q$ ,  $1 \leq i \leq q$ , and for each  $1 \leq m \leq n$ , if the  $k^{\text{th}}$

(i.e. 1<sup>st</sup>, 2<sup>nd</sup> or 3<sup>rd</sup>) literal of  $c_i$  is  $x_m$  (resp.  $\overline{x_m}$ ) then we create an arc between any free (i.e. not already incident to an arc) base  $C$  of  $S_{x_m}^s$  (resp.  $S_{x_m}^s$ ) and the  $k^{th}$  base  $G$  of  $S_{c_i}$  (note that this is possible by definition of  $S_{x_m}^s$ ,  $S_{x_m}^s$  and  $S_{c_i}$ ). On the whole, the instance we have constructed is composed of  $3q + 2n$  arcs. We denote by APS-CP-construction any construction of this type. In the following, we will distinguish arcs between bases  $A$  and  $U$ , denoted by  $AU$ -arcs, from arcs between bases  $C$  and  $G$ , denoted by  $CG$ -arcs. An illustration of an APS-CP-construction is given in Figure 1. Clearly, our construction can be carried out in polynomial time. Moreover, the result of such a construction is indeed an instance of  $\text{APS}(\{\sqsubset, \emptyset\}, \emptyset)$ , since  $Q = \emptyset$  (no arc is added to  $T$ ) and  $P$  is a  $\{\sqsubset, \emptyset\}$ -comparable set (since there are no arcs  $\{\prec\}$ -comparable).

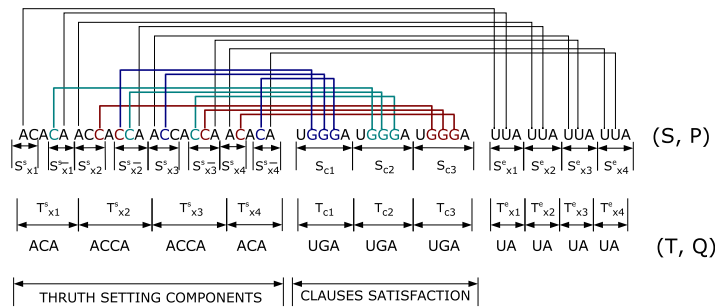
**Lemma 1.** *Let  $(S, P)$  and  $(T, Q)$  be any two arc-annotated sequences obtained from an APS-CP-construction. If  $(T, Q)$  can be obtained from  $(S, P)$  by deleting some of its bases together with their incident arcs, if any, then for each  $1 \leq i \leq q$  and  $1 \leq m \leq n$  : (i)  $T_{c_i}$  is obtained from  $S_{c_i}$  by deleting two of its three bases  $G$ , (ii)  $T_{x_m}^e$  is obtained from  $S_{x_m}^e$  by deleting one of its two bases  $U$ , (iii)  $T_{x_m}^s$  is obtained from  $S_{x_m}^s AS_{x_m}^s$  by deleting either  $S_{x_m}^s$  or  $S_{x_m}^s$ .*

*Proof.* Let  $(S, P)$  and  $(T, Q)$  be two arc-annotated sequences resulting from an APS-CP-construction.

(i) By construction, the first base  $U$  appearing in  $S$  (resp.  $T$ ) is  $S_{c_1}[1]$  (resp.  $T_{c_1}[1]$ ). Thus,  $T_{c_1}[1]$  is obtained from a base  $U$  of  $S$  at, or after,  $S_{c_1}[1]$ . Moreover, the number of bases  $A$  appearing after  $S_{c_1}[1]$  in  $S$  is equal to the number of bases  $A$  appearing after  $T_{c_1}[1]$  in  $T$ . Therefore, every base  $A$  appearing after  $S_{c_1}[1]$  and  $T_{c_1}[1]$  must be matched. That is, for each  $1 \leq i \leq q$ ,  $T_{c_i}[3]$  is matched to  $S_{c_i}[5]$ . In particular,  $T_{c_q}[3]$  is matched to  $S_{c_q}[5]$ . But since there are as many bases  $U$  between  $S_{c_1}[1]$  and  $S_{c_q}[5]$  as there are between  $T_{c_1}[1]$  and  $T_{c_q}[3]$ , any base  $U$  in this interval in  $S$  must be matched to any base  $U$  in this interval in  $T$ ; that is, for any  $1 \leq i \leq q$ ,  $T_{c_i}[1]$  is matched to  $S_{c_i}[1]$ . Thus, we conclude that for any  $1 \leq i \leq q$ ,  $T_{c_i}$  is obtained by deleting two of the three bases  $G$  of  $S_{c_i}$ .

(ii) By the above argument concerning the bases  $A$  appearing after  $S_{c_1}[1]$  and  $T_{c_1}[1]$ , we know that if  $(T, Q)$  can be obtained from  $(S, P)$ , then  $T_{x_m}^e[2]$  is matched to  $S_{x_m}^e[3]$  for any  $1 \leq m \leq n$ . Thus, for any  $1 \leq m \leq n$ ,  $T_{x_m}^e$  is obtained from  $S_{x_m}^e$ , and in particular  $T_{x_m}^e[1]$  is matched to either  $S_{x_m}^e[1]$  or  $S_{x_m}^e[2]$ .

(iii) By definition, as there is no arc incident to bases of  $T$ , at least one base incident to every arc of  $P$  has to be deleted. We just mentioned that  $T_{x_m}^e[1]$  is matched to either  $S_{x_m}^e[1]$  or  $S_{x_m}^e[2]$  for any  $1 \leq m \leq n$ . Thus, since by construction there is an arc between  $S_{x_m}^e[1]$  and  $S_{x_m}^s[l_m]$  (resp.  $S_{x_m}^e[2]$  and  $S_{x_m}^s[l_m]$ ), for any  $1 \leq m \leq n$  either  $S_{x_m}^s[l_m]$  or  $S_{x_m}^s[l_m]$  has to be deleted; and all these arcs connect a base  $A$  appearing before  $S_{c_1}[1]$  to a base  $U$  appearing after  $S_{c_q}[5]$ . Therefore, for any  $1 \leq m \leq n$  a base  $A$  appearing before  $S_{c_1}[1]$  in  $S$  is deleted. Originally, there are  $3n$  bases  $A$  appearing before  $S_{c_1}[1]$  in  $S$  and  $2n$  appearing before the first base of  $T_{c_1}[1]$  in  $T$ . Thus, the number of bases  $A$  not deleted in  $S$  and appearing before  $S_{c_1}[1]$  is equal to the number of bases  $A$  appearing before  $T_{c_1}[1]$  in  $T$ . But since, for each  $1 \leq m \leq n$ , a base  $A$  of either  $S_{x_m}^s$  or  $S_{x_m}^s$  is deleted, we conclude that for each  $1 \leq m \leq n$ ,  $T_{x_m}^s$  is obtained from  $S_{x_m}^s AS_{x_m}^s$ , by deleting either  $S_{x_m}^s$  or  $S_{x_m}^s$ .  $\square$



**Fig. 1.** Example of an APS-CP-construction with  $C_q = (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4})$ .

We now turn to proving that our construction is a polynomial time reduction from 3-SAT to APS(CROSSING, PLAIN).

**Lemma 2.** *Let  $I$  be an instance of 3-SAT problem with  $n$  variables and  $q$  clauses, and  $I'$  an instance  $((S, P); (T, Q))$  of  $\text{APS}(\{\square, \diamond\}, \emptyset)$  obtained by an APS-CP-construction from  $I$ . An assignment of the variables that satisfies the boolean formula of  $I$  exists iff  $T$  is an Arc-Preserving Subsequence of  $S$ .*

*Proof.* ( $\Rightarrow$ ) Suppose we have an assignment  $AS$  of the  $n$  variables that satisfies the boolean formula of  $I$ . By definition, for each clause there is at least one literal that satisfies it. In the following,  $j_i$  will define, for any  $1 \leq i \leq q$ , the smallest index of the literal of  $c_i$  (i.e. 1, 2 or 3) which, by its assignment, satisfies  $c_i$ . Let  $(S, P)$  and  $(T, Q)$  be two sequences obtained from an APS-CP-construction from  $I$ . We look for a set  $\mathcal{B}$  of bases to delete from  $S$  in order to obtain  $T$ .

For each variable  $x_m \in AS$  with  $1 \leq m \leq n$ , we define  $\mathcal{B}$  as follows: (i) if  $x_m = True$  then  $\mathcal{B}$  contains each base of  $S_{x_m}^s$  and  $S_{x_m}^e[1]$ , (ii) if  $x_m = False$  then  $\mathcal{B}$  contains each base of  $S_{x_m}^s$  and  $S_{x_m}^e[2]$ , (iii) if  $j_i = 1$  then  $\mathcal{B}$  contains  $S_{c_i}[3]$  and  $S_{c_i}[4]$ , (iv) if  $j_i = 2$  then  $\mathcal{B}$  contains  $S_{c_i}[2]$  and  $S_{c_i}[4]$ , (v) if  $j_i = 3$  then  $\mathcal{B}$  contains  $S_{c_i}[2]$  and  $S_{c_i}[3]$ .

Since a variable has a unique value (i.e. *True* or *False*), either each base of  $S_{x_m}^s$  and  $S_{x_m}^e[1]$  or each base of  $S_{x_m}^s$  and  $S_{x_m}^e[2]$  are in  $\mathcal{B}$  for all  $1 \leq m \leq n$ . Thus,  $\mathcal{B}$  contains at least one base in  $S$  of any  $AU$ -arc of  $P$ .

For any  $1 \leq i \leq q$ , two of the three bases  $G$  of  $S_{c_i}$  are in  $\mathcal{B}$ . Thus,  $\mathcal{B}$  contains at least one base in  $S$  of two thirds of  $CG$ -arcs of  $P$ . Moreover,  $S_{c_i}[j_i + 1]$  is the base  $G$  that is not in  $\mathcal{B}$ . We suppose in the following that the  $j_i^{th}$  literal of clause  $c_i$  is  $x_m$ , with  $1 \leq m \leq n$ . Thus, by the way we build the APS-CP-construction, there is an arc between a base  $C$  of  $S_{x_m}^s$  and  $S_{c_i}[j_i + 1]$  in  $P$ . By definition, if  $AS$  is an assignment of the  $n$  variables that satisfies the boolean formula,  $AS$  satisfies  $c_i$  and thus  $x_m = True$ . We mentioned, in the definition of  $\mathcal{B}$  that if  $x_m = True$  then each base of  $S_{x_m}^s$  is in  $\mathcal{B}$ . Thus, the base  $C$  of  $S_{x_m}^s$  incident to the  $CG$ -arc in  $P$  with  $S_{c_i}[j_i + 1]$  is in  $\mathcal{B}$ . A similar result can be found if the  $j_i^{th}$  literal of clause  $c_i$  is  $\overline{x_m}$ . Thus,  $\mathcal{B}$  contains at least one base in  $S$  of any  $CG$ -arc of  $P$ .

If  $S'$  is the sequence obtained from  $S$  by deleting all the bases of  $\mathcal{B}$  together with their incident arcs, if any, then there is no arc in  $S'$  (i.e. neither  $AU$ -arcs or  $CG$ -arcs). By the way we define  $\mathcal{B}$ ,  $S'$  is obtained from  $S$  by deleting all the bases of either  $S_{x_m}^s$  or  $S_{x_m}^e$ , two bases  $G$  of  $S_{c_i}$  and either  $S_{x_m}^e[1]$  or  $S_{x_m}^e[2]$ , for  $1 \leq i \leq q$  and  $1 \leq m \leq n$ . It is easily seen that sequence  $S'$  is similar to  $T$ .

( $\Leftarrow$ ) Let  $I$  be an instance of 3-SAT problem with  $n$  variables and  $q$  clauses. Let  $I'$  be an instance  $((S, P); (T, Q))$  of  $\text{APS}(\{\square, \diamond\}, \emptyset)$  obtained by an APS-CP-construction from  $I$  such that  $(T, Q)$  can be obtained from  $(S, P)$  by deleting some of its bases (i.e. a set of bases  $\mathcal{B}$ ) together with their incident arcs, if any. By Lemma 1, either all bases of  $S_{x_m}^s$  or all bases of  $S_{x_m}^e$  are in  $\mathcal{B}$ . Consequently, for  $1 \leq m \leq n$ , we define an assignment  $AS$  of the  $n$  variables of  $I$  as follows: (i) if all bases of  $S_{x_m}^s$  are in  $\mathcal{B}$  then  $x_m = True$ , (ii) if all bases of  $S_{x_m}^e$  are in  $\mathcal{B}$  then  $x_m = False$ .

Now, let us prove that for any  $1 \leq i \leq q$  clause  $c_i$  is satisfied by  $AS$ . By Lemma 1, for any  $1 \leq i \leq q$  there is a base  $G$  of segment  $S_{c_i}$  (say the  $j_i + 1^{th}$ ) that is not in  $\mathcal{B}$ . By the way we build the APS-CP-construction, there is a  $CG$ -arc in  $P$  between  $S_{c_i}[j_i + 1]$  and a base  $C$  of  $S_{x_m}^s$  (resp.  $S_{x_m}^e$ ) if the  $j_i^{th}$  literal of  $c_i$  is  $x_m$  (resp.  $\overline{x_m}$ ).

Suppose, *w.l.o.g.*, that the  $j_i^{th}$  literal of  $c_i$  is  $x_m$ . Since  $Q$  is an empty set, at least one base of any arc of  $P$  is in  $\mathcal{B}$ . Thus, the base  $C$  of  $S_{x_m}^s$  incident to the  $CG$ -arc in  $P$  with  $S_{c_i}[j_i + 1]$  is in  $\mathcal{B}$  (since  $S_{c_i}[j_i + 1] \notin \mathcal{B}$ ). Therefore, by Lemma 1, all the bases of  $S_{x_m}^s$  are in  $\mathcal{B}$ . By the way we define  $AS$ ,  $x_m = True$  and thus  $c_i$  is satisfied. A similar result can be obtained if the  $j_i^{th}$  literal of  $c_i$  is  $\overline{x_m}$ .  $\square$

We have thus proved the following.

**Theorem 1.**  $\text{APS}(\{\square, \diamond\}, \emptyset)$  is **NP**-complete.

It follows immediately from Theorem 1 that  $\text{APS}(\{\prec, \square, \diamond\}, \emptyset)$ , and hence  $\text{APS}(\text{CROSSING}, \text{PLAIN})$ , are **NP**-complete. One might naturally ask for more information concerning hard cases

of APS problem in order to get valuable insight into what makes the problem difficult. Another refinement of the hardness of APS(CROSSING,PLAIN) is given by the following theorem.

**Theorem 2.**  $\text{APS}(\{<, \bar{\bar{\}}\}, \emptyset)$  is **NP**-complete.

*Proof (Sketch of).* The proof is also by reduction from 3-SAT. Due to space considerations, the rather technical proof is deferred to the full version of the paper and only sketch main ideas behind Theorem 2. One of the reasons making the proof complicated is that all arcs have to be  $\{<, \bar{\bar{\}}\}$ -comparable, and hence we can not close an arc before closing all arcs which have been opened before. To overcome this problem we need pairs of strings gadgets which act as “signal repeaters” so that we can close and re-open a link carrying a truth value. Actually, repeaters unfortunately invert truth value and hence we need to deal with pairs of repeaters (*first kind* and *second kind* repeaters) where each repeater of the second kind is paired with a repeater of the first kind.  $\square$

## 5 Two polynomial time solvable APS problems

We prove in this section that  $\text{APS}(\{\bar{\bar{\}}\}, \emptyset)$  and  $\text{APS}(\{\bar{\bar{\}}\}, \{\bar{\bar{\}}\})$  are polynomial time solvable. In other words, relation  $\bar{\bar{\}}$  alone does not imply **NP**-completeness. We need the following notations. Sequences are the concatenation of zero or more elements from an alphabet. We use the period “.” as the concatenation operator, but frequently the two operands are simply put side by side. Let  $T = T[1]T[2] \dots T[m]$  be a sequence of length  $m$ . For all  $1 \leq i \leq j \leq m$ , we write  $T[i : j]$  to denote  $T[i]T[i+1] \dots T[j]$ . The *reverse* of  $T$  is the sequence  $T^R = T[m] \dots T[2]T[1]$ . A *factorization* of  $T$  is any decomposition  $T = x_1x_2 \dots x_q$  where  $x_1, x_2, \dots, x_q$  are (possibly empty) sequences. Let  $(T, A)$  be a  $\{\bar{\bar{\}}\}$ -arc-annotated sequence and  $(i, j) \in A$ ,  $i < j$ , be an arc. We call  $T[i]$  a *forward base* and  $T[j]$  a *backward base*. We will denote by  $\text{LF}_T$  the position of the last forward base in  $(T, A)$  and by  $\text{FB}_T$  the position of the first backward base in  $(T, A)$ , *i.e.*,  $\text{LF}_T = \max\{i : (i, j) \in A\}$  and  $\text{FB}_T = \min\{j : (i, j) \in A\}$ . By convention, we let  $\text{LF}_T = 0$  and  $\text{FB}_T = |T| + 1$  if  $A = \emptyset$ . Observe that  $\text{LF}_T < \text{FB}_T$ . We begin by proving a factorization result on  $\{\bar{\bar{\}}\}$ -arc-annotated sequences.

**Lemma 3.** *Let  $S$  and  $T$  be two  $\{\bar{\bar{\}}\}$ -arc-annotated sequences of length  $n$  and  $m$ , respectively. If  $T$  occurs as an arc preserving subsequence in  $S$ , then there exists a factorization (possibly trivial)  $T[\text{LF}_T + 1 : \text{FB}_T - 1] = xy$  such that  $T[1 : \text{LF}_T] \cdot x \cdot (y \cdot T[\text{FB}_T : m])^R$  occurs as an arc preserving subsequence in  $S[1 : \text{FB}_S - 1] \cdot S[\text{FB}_S : n]^R$ .*

*Proof.* Suppose that  $T$  occurs as an arc preserving subsequence in  $S$ . Since both  $S$  and  $T$  are  $\{\bar{\bar{\}}\}$ -arc-annotated sequences, then there exist two factorizations  $S[1 : \text{LF}_S] = uw$  and  $S[\text{FB}_S : n] = zv$  such that: (i)  $T[1 : \text{LF}_T]$  occurs in  $u$ , (ii)  $T[\text{LF}_T + 1 : \text{FB}_T - 1]$  occurs in  $w \cdot S[\text{LF}_S + 1 : \text{FB}_S - 1] \cdot z$  and (iii)  $T[\text{FB}_T : m]$  occurs in  $v$ . Then it follows that there exists a factorization  $T[\text{LF}_T + 1 : \text{FB}_T - 1] = xy$  such that  $x$  occurs in  $w \cdot S[\text{LF}_S + 1 : \text{FB}_S - 1]$  and  $y$  occurs in  $z$ , and hence  $T' = T[1 : \text{LF}_T] \cdot x \cdot (y \cdot T[\text{FB}_T : m])^R$  occurs as an arc preserving subsequence in  $S' = S[1 : \text{FB}_S - 1] \cdot S[\text{FB}_S : n]^R$   $\square$

**Theorem 3.**  $\text{APS}(\{\bar{\bar{\}}\}, \{\bar{\bar{\}}\})$  is solvable in  $O(nm^2)$  time.

*Proof.* The algorithm is as follows:

---

|   |   |
|---|---|
|   | <b>Data</b> : Two $\{\bar{\bar{\}}\}$ -arc-annotated sequences $S$ and $T$ of length $n$ and $m$ , respectively |
|   | <b>Result</b> : true iff $T$ occurs as an arc-preserving subsequence in $S$                                     |
|   | <b>begin</b>  |
| 1 | $S' = S[1 : \text{FB}_S - 1] \cdot S[\text{FB}_S : n]^R$  |
| 2 | <b>foreach</b> factorization $T[\text{LF}_T + 1 : \text{FB}_T - 1] = xy$ <b>do</b>                              |
| 3 | $T' = T[1 : \text{LF}_T] \cdot x \cdot (y \cdot T[\text{FB}_T : m])^R$  |
| 4 | <b>if</b> $T'$ occurs as an arc preserving subsequence in $S'$ <b>then</b>                                      |
| 5 | <b>return true</b>  |
| 6 | <b>return false</b>   |
|   | <b>end</b>  |

---

Correctness of the algorithm follows from Lemma 3. What is left is to prove the time complexity. Clearly,  $S' = S[1 : \text{FB}_S - 1] \cdot S[\text{FB}_S : n]^R$  is a  $\{\square\}$ -arc-annotated sequence. The key point is to note that, for any factorization  $T[\text{LF}_T + 1 : \text{FB}_T - 1] = xy$ , the obtained  $T' = T[1 : \text{LF}_T] \cdot x \cdot (y \cdot T[\text{FB}_T : m])^R$  is a  $\{\square\}$ -arc-annotated sequence as-well. Now let  $k$  be the number of arcs in  $T$ . So there are at most  $m - 2k$  iterations to go before eventually returning **false**. According to the above, Line 4 constitutes an instance of  $\text{APS}(\{\square\}, \{\square\})$ . But  $\text{APS}(\{\square\}, \{\square\})$  is a special case of  $\text{APS}(\{<, \square\}, \{<, \square\})$ , and hence is solvable in  $O(nm)$  time [6]. Then it follows that the algorithm as a whole runs in  $O(nm(m - 2k)) = O(nm^2)$  time.  $\square$

Clearly, the proof of Theorem 3 relies on an efficient algorithm for solving  $\text{APS}(\{\square\}, \{\square\})$ : the better the complexity for  $\text{APS}(\{\square\}, \{\square\})$ , the better the complexity for  $\text{APS}(\{\emptyset\}, \{\emptyset\})$ . We have used only the fact that  $\text{APS}(\{\square\}, \{\square\})$  is a special case of  $\text{APS}(\{<, \square\}, \{<, \square\})$ . It remains open, however, whether a better complexity can be achieved for  $\text{APS}(\{\square\}, \{\square\})$ . Theorem 3, combined with Observation 1, carries out easily to restricted versions.

**Corollary 1.**  $\text{APS}(\{\emptyset\}, \emptyset)$  is solvable in  $O(nm^2)$  time.

## 6 Conclusion

In this paper, we investigated the APS problem time complexity and gave a precise characterization of what makes the APS problem hard. We proved that  $\text{APS}(\text{CROSSING}, \text{PLAIN})$  is **NP**-complete thereby answering an open problem posed in [6]. Note that this result answers the last open problem concerning APS computational complexity with respect to classical complexity levels, *i.e.*, **PLAIN**, **CHAIN**, **NESTED** and **CROSSING**. Also, we refined the four above mentioned levels for exploring the border between polynomial time solvable and **NP**-complete problems. We proved that both  $\text{APS}(\{\square, \emptyset\}, \emptyset)$  and  $\text{APS}(\{<, \emptyset\}, \emptyset)$  are **NP**-complete and gave positive results by showing that  $\text{APS}(\{\emptyset\}, \emptyset)$  and  $\text{APS}(\{\emptyset\}, \{\emptyset\})$  are polynomial time solvable. Hence, the refinement we suggest shows that APS problem becomes hard when one considers sequences containing  $\{\emptyset, \alpha\}$ -comparable arcs with  $\alpha \neq \emptyset$ . Therefore, crossing arcs alone do not imply APS hardness. It is of course a challenging problem to further explore the complexity of the APS problem, and especially the parameterized views, by considering additional parameters such as the cutwidth or the depth of the arc structures.

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