Frobenius Algebras and Classical Proof Nets

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The semantics of proofs for classical logic is a very recent discipline; the construction of proofs semantics that are completely faithful to the natural symmetries of classical logic is even more recent. In this paper we present a theory of proof nets which is related to those in [LS05,Hyl04,FP05], but which differs from them in its ability to take account of *resources*, in the sense of linear logic. It also has the interesting property (like [Hyl04]) of being based on a topological foundation.

This work originated as an investigation in the denotational semantics of classical logic [LN09], furthering the work in [Lam07]. As it often happens here, it involved the construction of bialgebras, in this particular case in the category of posets and bimodules. The fact that these bialgebras were actually Frobenius algebras was noticed, but it took some time for the extreme interest of this property to sink in.

Definition 1 (Frobenius algebra). *Let* $(\mathbf{C}, \otimes, \mathbf{1})$ *be a symmetric monoidal category (SMC), and* A *an object of it. A* Frobenius algebra *is a sextuple* $(A, \Delta, \Pi, \nabla, \amalg)$ *where* $(A, \nabla, \amalg)$ *is a commutative monoid,* (A, Δ, Π) *a co-commutative comonoid, where the following diagram commutes:*

$$
\begin{array}{ccccc}
A \otimes A & = & A \otimes A & = & A \otimes A \\
{\scriptstyle \Delta\otimes \mathrm{Id}}\downarrow & & {\scriptstyle \nabla}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes\Delta} \\
A \otimes A \otimes A & & A & & A \otimes A \otimes A \\
{\scriptstyle \mathrm{Id}\otimes\nabla}\downarrow & & {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \nabla\otimes\mathrm{Id}} \\
A \otimes A & = & A \otimes A & = & A \otimes A
\end{array}
$$

A Frobenius algebra is thin *if* $\Pi \circ \amalg$ *is the identity.*

The following is well-known.

Proposition 1. *The tensor of two Frobenius algebras is also a Frobenius algebra, where the monoid and comonoid operations are defined as usual in an SMC. It is thin if both factors are.*

Definition 2. *A Frobenius category* $\mathbf{C}$ *is a symmetrical monoidal category where every object* A *is equipped with a* thin *Frobenius algebra structure* $(A, \nabla_A, \amalg_A, \Delta_A, \Pi)$ *and such that the algebra on the tensor of two objects is the usual tensor algebra, as above.*

Frobenius algebras have gained a lot of attention after they were found to be closely related to 2-dimensional Topologica Quantum Field Theories (TQFTs). The main result was achieved by several people independently [Dij89,Koc04], and can be stated as follows. We present a slightly modified version of the standard result, which better fits our purposes and is an easy corollary of it.

Theorem 1. *The free Frobenius category* $\mathbf{F}$ *on one object generator is equivalent to the two following categories.*

1. *Take finite disjoint unions of* m *circles as an object* m*. A map* $m \to n$ *is a Riemann surface (with boundary) whose boundary is the disjoint sum* $m+n$ *(and would be orientable if the circles were extended to discs), such that every connected component has a nonempty boundary, where two surfaces are identified modulo homeomorphism. Composition of two maps* $m \to n, n \to p$ *is gluing, forgetting the boundaries in the middle, and dropping the components that do not touch the resulting boundary* $m + p$*.*
2. *Take finite sets* $[m] = \{0, 1, \ldots, m-1\}$ *as objects, seen as discrete topological spaces. A map* $[m] \to [n]$ *is a topological graph* G *(i.e. a CW-complex of dimension one), equipped with an injective function* $[m+n] \to G$ *such that every connected components of* G *is in the image of that function, with two graphs being identified if they are equivalent modulo homology. Composition is also gluing and dropping the components that are left out of the resulting set of endpoints.*

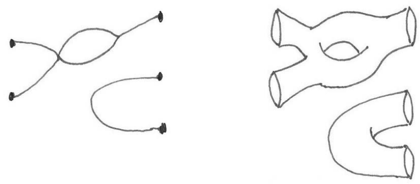


Fig 1. A map $2 \to 3$ shown in the two equivalent characterizations of the free Frobenius category. Objects are seen as distinguished end-points in the left or as circles to the right. One of the connected components has genus 1, the other 0. In both cases the map is determined by grouping of the atoms in a partition and an assignment of genera to the classes of the partition.

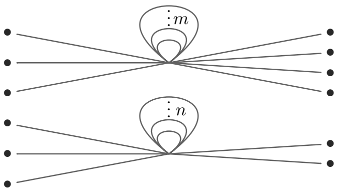
Since we are dealing with the universal algebra of categories, a free Frobenius category is defined only up to equivalence of categories, with the standard universal property associated to that situation. The two characterizations in Theorem 1 happen to be skeletal categories and are isomorphic. Our nonstandard notion of Frobenius category requires thinness; maps in the standard, non-thin free Frobenius category can contain several "floating" components that do not touch the border.

Since homology is much more technical than homotopy, we prefer to replace the second result above with:

2'. *Objects are sets of the form* $[m]$*. A map* $[m] \to [n]$ *is a topological graph* G *equipped with an injective function* $[m+n] \to G$ *that touches all connected components of* G*, where two such things are identified if they are homotopy equivalent in the co-slice* *category* $(m+n)/Top$*, where homotopies are defined to be constant on the base* $[m+n]$.

This allows for a treatment which is at the same time well-formalized and accessible to many more readers.

Theorem 2. *Every map in* $\mathbf{F}$ *can be represented by a graph* G *of the following form, where every connected component is a "star" whose central node has* n *loops attached to it, with* $n \geqslant 0$.



This prompts the following definition

Definition 3 (Linking). *We define a* linking *to be a triple*

$$P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$$

where

- P *is a finite set*
- $\mathcal{C}omp_P$ *is the set of classes of a* partition *of the set* P. *Its elements are called* components.
- *the function* $\mathcal{G}en_P : \mathcal{C}omp_P \to \mathbb{N}$ *(called* genus*) assigns a natural number to each component in* $\mathcal{C}omp_P$

Notice the abuse of notation, where a single letter P can be the full thing above or just its underlying set.

It should be obvious that a map $m \to n$ in $\mathbf{F}$ can be described as a linking on the set $m+n$. Naturally a formal definition of composition in terms of linkings is a bit trickier.

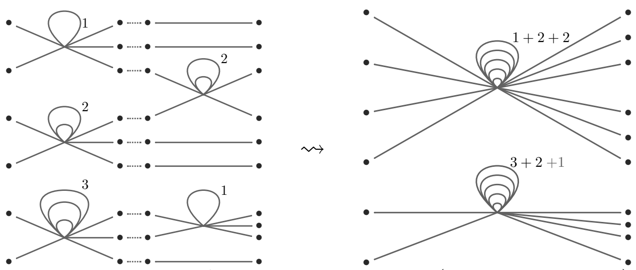


Fig. 2. Maps in a free Frobenius category (drawn horizontally) seen as topological graphs with object generators for nodes, and the bouquet of circles determining the genus. Composition of the maps amounts to glueing of graphs along nodes, and is determined by the homotopy type of the new graph as depicted.

Proposition 2. *The category* $\mathbf{F}$ *is compact-closed, the dual of an object being itself.*

This is easy to see, since given a map $m \to n$ stuff in m can be transfered to the right side by a purely formal manipulation, and vice-versa. More generally, *any* Frobenius category is compact-closed, but a proof of this requires some real algebra.

The relevance of the "Frobenius equations" for proof theory is due to the fact that they address the contraction-against-contraction case in cut elimination, seen for example in the proof to the right in Figure 3.

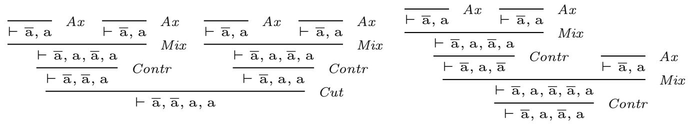


Fig. 3. Two proofs identified by Frobenius equations

We introduce a standard language for classical propositional logic, with atoms $\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots$, negatoms $\overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{c}}, \ldots$ and conjunction $\wedge$, disjunction $\vee$. We call something which is either an atom ot a negatom a *literal*. Negation of a compound formula is defined by de Morgan duality. Sequents are defined as usual, and given a formula A or a sequent Γ we denote by $\mathcal{L}it(A)$, $\mathcal{L}it(\Gamma)$ their sets of occurences of literals.

Definition 4 (F-prenet). *We define an* F-prenet *to be a pair*

$$P \triangleright \Gamma$$

consisting of a sequent Γ, *and a linking* $(P, \mathcal{C}omp_P, \mathcal{G}en_P)$ *where the underlying set* P *is* $\mathcal{L}it(\Gamma)$, *and every class in* $\mathcal{C}omp_P$ *contains only atoms of the same type and their negation.*

When we say that P "is" the set of literal occurences of Γ, we mean actually that P is an arbitrary set, equipped with a bijection with the actual literal occurences in Γ. The point is that this bijection never has to be made explicit in practice, while working directly with atom occurences would force ugly contortions.

Several deductive systems can be used with F-prenets. The first one is just the ordinary one-sided sequent calculus for classical logic, with the Mix rule (of linear logic) added. It is presented in full in [LS05], under the name CL. In general, a sequent calculus can be used to define a theory of proof nets is every n-ary introduction rule of the calculus

$$\frac{\vdash \Gamma_1 \quad \vdash \Gamma_2 \quad \cdots \quad \vdash \Gamma_n}{\vdash \Gamma}$$

can be transformed into a family of n morphisms $P_i \triangleright \Gamma_i \to Q \triangleright \Gamma$ in the following *syntactic category.*

Definition 5 (Syntactic Category). *Let* $\mathcal{F}Synt$ *have F-prenets for objects, where a map*

$$f : P \triangleright \Gamma \to Q \triangleright \Delta$$

is given by an ordinary function on the underlying set of literals

$$f : P \to Q \qquad \big(= \mathcal{L}it(\Gamma) \to \mathcal{L}it(\Delta)\big)$$

such that

1. *for every formula* A, f *maps* $\mathcal{L}it(A)$ *to a subset of* $\mathcal{L}it(\Delta)$ *which defines a subformula of a formula in* Δ, *while preserving the syntactic left-right order on literals.*

2. *for every $C \in \mathcal{C}omp_P$, one has that $f(C) \subseteq \mathcal{L}it(\Delta)$ is contained in a component $C' \in \mathcal{C}omp_Q$, with $\mathcal{G}en_P(C) \leq \mathcal{G}en_Q(C')$.*

The procedure to obtain an F-prenet $P \triangleright \Gamma$ from a proof of a sequent $\vdash \Gamma$ is absolutely straightforward. The cases that are worth mentioning specifically are Weakening and Contraction. Assuming we have constructed $P \triangleright \Gamma$ from a proof, then adding the formula A through weakening gives us a linking on the disjoint union $P \uplus \mathcal{L}it(\mathrm{A})$ where every added component is a singleton with associated genus 0. For contraction, if the two visible occurences of A in $P \triangleright \Gamma, \mathrm{A}, \mathrm{A}$ are contracted, we get an F-prenet $P(\mathrm{A} \curlyvee \mathrm{A}) \triangleright \Gamma, \mathrm{A}$ by connecting the ith literal of the first instance of A and the ith literal in the second instance to a single "terminal", where i ranges over the number of literals in A.

This $(- \curlyvee -)$ operation can be iterated, and can be applied to subformulas and subsequents as well as formulas. In what follows we use superscripts to disambiguate occurences when we feel it is useful.

Definition 6. *In the category $\mathcal{F}Synt$, we define the families of cospans Mix and $\wedge$ to be*

$$\begin{array}{ccc} P_l \triangleright \Gamma & & P_r \triangleright \Gamma \\ & \searrow^{Mix:l} \quad {}^{Mix:r}\swarrow & \\ & P_l \uplus P_r(\Gamma \curlyvee \Gamma) \triangleright \Gamma & \end{array}$$

and

$$\begin{array}{ccc} P_l \triangleright \Gamma, \mathrm{A}^1 \wedge \mathrm{B}^1, \mathrm{A}^2 & & P_r \triangleright \mathrm{B}^2, \mathrm{A}^3 \wedge \mathrm{B}^3, \Gamma \\ & \searrow^{\wedge:l} \quad {}^{\wedge:r}\swarrow & \\ & Q \triangleright \Gamma, \mathrm{A} \wedge \mathrm{B} & \end{array}$$

where Q is $P_l \uplus P_r\big(\Gamma \curlyvee \Gamma, (\mathrm{A}^1 \curlyvee \mathrm{A}^2) \curlyvee \mathrm{A}^3, (\mathrm{B}^1 \curlyvee \mathrm{B}^2) \curlyvee \mathrm{B}^3\big)$.

Definition 7. *An* anodyne map $P \triangleright \Gamma \xrightarrow{\;\ominus\;} Q \triangleright \Delta$ *is a syntactic map that can be decomposed*

$$P \triangleright \Gamma \xrightarrow{\sim} Q \triangleright \Delta_1 \xrightarrow{\vee} \cdots \xrightarrow{\vee} Q \triangleright \Delta_n = \Delta$$

as an isomorphism followed by a sequence of $\vee$-introduction maps (which do not affect the linking, only the sequent).

There is an important anodyne map, which corresponds to the removal of all outer disjuctions: We write

$$[P \triangleright \Gamma] \xrightarrow{\;\boxminus\;} P \triangleright \Gamma$$

to denote the anodyne map whose domain is the sequent where all outer disjunctions have been removed.

Definition 8 (Correct F-nets). *An F-prenet $P \triangleright \Gamma$ is a* CL-correct F-net, *(or simply an* F-net*) if it is at the root of a* correctness diagram *$\mathcal{T} \to \mathcal{F}Synt$, meaning a diagram for which:*

1. *$\mathcal{T}$ is a poset which is an inverted tree (i.e. the root is the top, the leaves are minimal), with $P \triangleright \Gamma$ at its root;*
2. *maps of the diagram $\mathcal{T}$ are either anodyne, or belong to a $\wedge$- or Mix-cospan;*
3. *the only branchings are $\wedge$- and Mix-cospans ;*
4. *every leaf of the tree is an F-prenet $Q \triangleright \Delta$ with $\mathcal{C}omp_Q = \{\{\mathrm{a}, \bar{\mathrm{a}}\}, \{x_1\}, \ldots, \{x_m\}\}$ and a map $\mathcal{G}en_Q$ which is 0 everywhere, i.e, an axiom with weakenings.*

This can be strengthened by forcing the anodyne maps always to be $\Box$-maps, and to have an alternation between these and maps from cospans. We show

Theorem 3 (Sequentialization). *Correct F-nets are precisely those F-prenets that come from* CL *without Cut.*

Given a linking P let $|P|$ be stand for the size of its underlying set, $|\mathcal{C}omp_P|$ for the number of components, and $|\mathcal{G}en_P|$ for the sum of all genera in P, i.e. $|\mathcal{G}en_P| = \sum_{C \in \mathcal{C}omp_P} \mathcal{G}en_P(C)$. The following observation is crucial to the proof:

Lemma 1 (Counting axiom links in an F-prenet). *If an F-prenet $P \triangleright \Gamma$ corresponds to a* CL *proof, then*

$$|Ax| = |P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|,$$

where $|Ax|$ is the number of axioms in the proof (corollary: any correctness diagram for this proof will have the same number of leaves).

This lemma, along with some additional analysis of proofs guarantees finiteness of the search space:

Theorem 4. *Given an F-prenet, its* CL*-correctness (*CL*-sequentializability) can be checked in finite time, i.e. the* CL*-correctness criterion yields a decision procedure for* CL*-correct F-nets.*

We have strong evidence that the procedure is NP-complete, actually.

When Cut comes into play, things change a bit. First of all, we define a *cut formula* to be $\mathrm{A} \diamondplus \overline{\mathrm{A}}$, where $- \diamondplus -$ is a new binary connective that is only allowed to appear as a root in a sequent.

Our original goal is to normalize these prenets with cuts by means of composition in **F** (remember it is compact-closed). This use of Frobenius algebras in classical logic is quite different from the one proposed by Hyland [Hyl04]. It more resembles the work in [LS05], where the equivalent to the category **F** there is obtained from an "interaction category" construction [Hyl04, Section 3] on sets and relations, where composition is defined by the means of a trace operator.

Cut elimination defined that way immediately causes problems. Look at the right part of Figure 4. For the resulting F-prenet to come from a proof we need the singleton component to come from a weakening, but this cannot happen according to our interpretation since its genus is > 0.

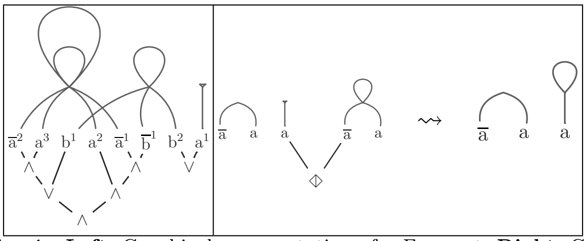


Fig. 4. Left: Graphical representation of a F-prenet. **Right:** Cut elimination performed on two CL-correct F-nets that results in a F-prenet not corresponding to a CL proof.

These issues can be dealt with by changing the deductive system and we define a new sound and complete calculus for classical logic, FL.

$$\frac{}{\vdash \overline{a}, a\ ;}\ Ax \qquad \frac{\vdash \Gamma\ ;\ \Delta}{\vdash \Gamma\ ;\ \Delta, a, a, \ldots, a, \overline{a}, \overline{a}, \ldots, \overline{a}}\ MulWeak$$

$$\frac{\vdash \Gamma, A, B\ ;\ \Delta}{\vdash \Gamma, A \vee B\ ;\ \Delta}\ \vee_l \qquad \frac{\vdash \Gamma, A\ ;\ \Delta, B}{\vdash \Gamma, A \vee B\ ;\ \Delta}\ \vee_c \qquad \frac{\vdash \Gamma\ ;\ \Delta, A, B}{\vdash \Gamma\ ;\ \Delta, A \vee B}\ \vee_r$$

$$\frac{\vdash \Gamma_1, A\ ;\ \Delta_1 \quad \vdash B, \Gamma_2\ ;\ \Delta_2}{\vdash \Gamma_1, A \wedge B, \Gamma_2\ ;\ \Delta_1, \Delta_2}\ \wedge_l \qquad \frac{\vdash \Gamma\ ;\ \Delta, A, B}{\vdash \Gamma\ ;\ \Delta, A \wedge B}\ \wedge_r$$

$$\frac{\vdash \Gamma_1, A\ ;\ \Delta_1 \quad \vdash \Gamma_2\ ;\ B, \Delta_2}{\vdash \Gamma_2\ ;\ A \wedge B, \Gamma_1, \Delta_1, \Delta_2}\ \wedge_c$$

$$\frac{\vdash \Gamma, A, A\ ;\ \Delta}{\vdash \Gamma, A\ ;\ \Delta}\ Contr_l \qquad \frac{\vdash \Gamma\ ;\ \Delta, A, A}{\vdash \Gamma\ ;\ \Delta, A}\ Contr_r$$

$$\frac{\vdash \Gamma, A\ ;\ \Delta, A}{\vdash \Gamma, A\ ;\ \Delta}\ Contr_c$$

$$\frac{\vdash \Gamma\ ;\ \Delta_1 \quad \vdash \Delta\ ;\ \Delta_2}{\vdash \Gamma, \Delta\ ;\ \Delta_1, \Delta_2}\ Mix$$

$$\frac{\vdash \Gamma, A\ ;\ \Delta_1 \quad \vdash \overline{A}, \Delta\ ;\ \Delta_2}{\vdash \Gamma, \Delta\ ;\ \Delta_1, \Delta_2}\ Cut_l \qquad \frac{\vdash \Gamma\ ;\ \Delta, A\overline{A}}{\vdash \Gamma\ ;\ \Delta}\ Cut_r$$

$$\frac{\vdash \Gamma, A\ ;\ \Delta_1 \quad \vdash \Delta\ ;\ \overline{A}, \Delta_2}{\vdash \Delta\ ;\ \Gamma, \Delta_1, \Delta_2}\ Cut_c$$

Fig. 5. System FL.

The purpose of the stoup is to keep track the part that is sure to come from weakening, and also to allow the introduction of arbitrary linking configurations through weakening. This is because *MulWeak* is interpreted by adding to the linking a set $\{a, a, \ldots, a, \overline{a}, \overline{a}, \ldots, \overline{a}\}$, which contains a *single* component of genus zero. The definition of correctness for FL needing to accommodate the new connective for cut, we introduce another cospan in the syntactic category of F-prenets $\mathcal{F}Synt$. We also relax the definition of anodyne map to allow functions that are injective but not bijective, to take account of the new Weakening rule. With these modifications, Theorem 3 and Theorem 4 can be restated, with one marked difference: this time, for FL-correct net we have $|Ax| \leq |P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|$.

While problems like the counterexample above are solved, in general we still cannot eliminate the cuts on an FL-correct net and always get one which is also FL-correct. Thus we still do not have a category. This calls for a little more analysis. First notice that F-prenets do form a category themselves. It is easy to see that this category is equivalent to the free Frobenius category generated by the set of literal types (where an atom and its negation have the same "type"). And thus we can consider FL-correct (and CL-correct) nets to be a class of maps in that category, which is not closed under composition. But this large category (as usual objects are formulas and a map $A \rightarrow B$ is a $P \rhd \overline{A}, B$) has *two* order enrichments.

Definition 9. *Let $P \rhd \Gamma, Q \rhd \Gamma$ be two linkings over the same sequent. We write*

- *$P \leqslant Q$ if $\mathcal{C}omp_P = \mathcal{C}omp_Q$ and $\mathcal{G}en_P \leqslant \mathcal{G}en_Q$, i.e, the genus functions are ordered pointwise.*
- *$P \preceq Q$ if $\mathcal{C}omp_P$ is a finer partition than $\mathcal{C}omp_Q$ and the genus of every component in $\mathcal{C}omp_Q$ is greater than the sum of the genera of the components of $\mathcal{C}omp_P$ it contains.*

These order structures do define enrichments when they are considered as being defined on morphisms, as above. Both have their interests, but we don't have much space left. So we just state one of several corollarie of that analysis:

Theorem 5. *Let $P \rhd \Gamma$ be the result of eliminating the cuts on an FL-correct net. Then there exists an FL-correct linking $Q \geqslant P$.*

So we can obtain a category by cheating on our original goal and define a composition that "fattens" the one given by ordinary Frobenius categories, which we will describe in the full paper.

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