

Error estimate and unfolding for periodic homogenization

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Abstract. This paper deals with the error estimate in problems of periodic homogenization. The methods used are those of the periodic unfolding. We give the upper bound of the distance between the unfolded gradient of a function belonging to $H^1(\Omega)$ and the space $\nabla_x H^1(\Omega) \oplus \nabla_y L^2(\Omega; H^1_{per}(Y))$. These distances are obtained thanks to a technical result presented in Theorem 2.3 : the periodic defect of a harmonic function belonging to $H^1(Y)$ is written with the help of the norms $H^{1/2}$ of its traces differences on the opposite faces of the cell Y . The error estimate is obtained without any supplementary hypothesis of regularity on correctors.

1. Introduction

The error estimate in periodic homogenization problems was presented for the first time in Bensoussan, Lions and Papanicolaou [2]. It can also be found in Oleinik, Shamaev and Yosifian [8], and more recently in Cioranescu and Donato [5]. In all these books, the result is proved under the assumption that the correctors belong to $W^{1,\infty}(Y)$ ($Y =]0, 1[^n$ being the reference cell). The estimate is of order $\varepsilon^{1/2}$. The additional regularity of the correctors holds true when the coefficients of the operator are very regular, which is not necessarily the situation in homogenization. In [6] we obtained an error estimate without any regularity hypothesis on the correctors but we supposed that the solution of the homogenized problem belonged to $W^{2,p}(\Omega)$ ($p > n$). The exponent of ε in the error estimate is inferior to $1/2$ and depends on n and p .

The aim of this work is to give further error estimates with again minimal hypotheses on the correctors and the homogenized problem. In all this study we will make use of the notation of [4].

The paper is organized as follows. In paragraph 2 we prove some technical results related to periodic defect. In Theorem 2.1 we give an estimate of the distance between a function ϕ belonging to $W^{1,p}(Y)$ and the space of periodic functions $W^{1,p}_{per}(Y)$. This distance depends on the $W^{1-\frac{1}{p},p}$ norms of the differences of the traces of ϕ on opposite faces of Y . Theorem 2.1 is a consequence of Lemma 2.2. In this lemma we proved that the distance between a function and the space of periodic functions with respect to the first k variables is isomorphic to the direct sum of the spaces of the differences of the traces on the opposite faces Y_j and $\bar{e}_j + Y_j$. (i.e., on $y_j = 0$ and $y_j = 1$), $1 \leq j \leq k$. This lemma is proved by an explicit lifting of the traces from the faces of Y .

In Theorem 2.3 we show that the $H^{1/2}$ periodic defect of an harmonic function on Y with values in a separable Hilbert space X is equivalent to its H^1 norm. The orthogonal of space $H^1_{per}(Y; X)$ is in fact isomorphic to the direct sum of the spaces of the differences of the traces on the opposite faces of cell Y .

Paragraph 3 is dedicated to Theorem 3.4 which is the essential tool to obtain estimates. This theorem is related to the periodic unfolding method (see [4]). We show that for any ϕ in $H^1(\Omega)$, where Ω is an open bounded set of \mathbb{R}^n with Lipschitz boundary, there exists a function $\hat{\phi}_\varepsilon$ in $H^1_{per}(Y; L^2(\Omega))$, such that the distance between the unfolded $\mathcal{T}_\varepsilon(\nabla_x \phi)$ and $\nabla_x \phi + \nabla_y \hat{\phi}_\varepsilon$ is of order of ε in the space $[L^2(Y; H^{-1}(\Omega))]^n$.

Theorems 4.1, 4.2 and 4.5 give an estimate of the error without any hypothesis on the regularity of the correctors, but with different hypotheses on the boundary of Ω . They require that the right hand side of the homogenized problem be in $L^2(\Omega)$.

In this article, the constants appearing in the estimates will be independent from ε .

2. Periodicity defect

We denote $Y =]0, 1[^n$ the unit cell of \mathbb{R}^n and we put $Y_j = \{y \in \bar{Y} \mid y_j = 0\}$, $j \in \{1, \dots, n\}$.

Theorem 2.1 : For any $\phi \in W^{1,p}(Y)$, $p \in]1, \infty]$, there exists $\widehat{\phi} \in W_{per}^{1,p}(Y)$ such that

$$\|\phi - \widehat{\phi}\|_{W^{1,p}(Y)} \leq C \sum_{j=1}^n \|\phi|_{\bar{\varepsilon}_j + Y_j} - \phi|_{Y_j}\|_{W^{1-\frac{1}{p},p}(Y_j)}$$

The constant depends only on n .

The proof of the theorem is based on Lemma 2.2. We introduce the following spaces:

$$W_0 = W^{1,p}(Y), \quad W_k = \left\{ \phi \in W^{1,p}(Y) \mid \phi(\cdot) = \phi(\cdot + \bar{\varepsilon}_i), \quad i \in \{1, \dots, k\} \right\}, \quad k \in \{1, \dots, n\}$$

Lemma 2.2 : For any $\phi \in W^{1,p}(Y)$ and for any $k \in \{1, \dots, n\}$, there exists $\widehat{\phi}_k \in W_k$ such that

$$\|\phi - \widehat{\phi}_k\|_{W^{1,p}(Y)} \leq C \sum_{j=1}^k \|\phi|_{\bar{\varepsilon}_j + Y_j} - \phi|_{Y_j}\|_{W^{1-\frac{1}{p},p}(Y_j)}$$

The constant depends on n .

Proof : The lemma is proved by a finite induction. We choose a function θ belonging to $\mathcal{D}(-1/2, 1/2)$ equal to 1 in the neighborhood of zero. We recall (see [1]) that for any $k \in \{0, \dots, n-1\}$, there exists a continuous lifting \tilde{r}_k in W_k of the traces on Y_{k+1} of the W_k elements.

Let ϕ be in $W^{1,p}(Y)$. We put $\widehat{\phi}_0 = \phi$. We suppose the lemma proved for k , $k \in \{0, \dots, n-1\}$. There exists $\widehat{\phi}_k \in W_k$ such that

$$\|\phi - \widehat{\phi}_k\|_{W^{1,p}(Y)} \leq C \sum_{j=1}^k \|\phi|_{\bar{\varepsilon}_j + Y_j} - \phi|_{Y_j}\|_{W^{1-\frac{1}{p},p}(Y_j)}$$

Of course if $k = 0$ the right hand side of the above inequality is equal to zero. We define $\widehat{\phi}_{k+1}$ by

$$\widehat{\phi}_{k+1} = \widehat{\phi}_k + \frac{1}{2} \left\{ \theta(y_{k+1}) - \theta(1 - y_{k+1}) \right\} \tilde{r}_k \left(\widehat{\phi}_k|_{\bar{\varepsilon}_{k+1} + Y_{k+1}} - \widehat{\phi}_k|_{Y_{k+1}} \right)$$

The function $\widehat{\phi}_{k+1}$ belongs to W_k and verifies

$$\widehat{\phi}_{k+1}|_{\bar{\varepsilon}_{k+1} + Y_{k+1}} = \frac{1}{2} \left\{ \widehat{\phi}_k|_{\bar{\varepsilon}_{k+1} + Y_{k+1}} + \widehat{\phi}_k|_{Y_{k+1}} \right\} = \widehat{\phi}_{k+1}|_{Y_{k+1}}$$

Hence it belongs to W_{k+1} . We have

$$\begin{aligned} \|\phi - \widehat{\phi}_{k+1}\|_{W^{1,p}(Y)} &\leq \|\phi - \widehat{\phi}_k\|_{W^{1,p}(Y)} + \|\widehat{\phi}_k - \widehat{\phi}_{k+1}\|_{W^{1,p}(Y)} \\ &\leq C \sum_{j=1}^k \|\phi|_{\bar{\varepsilon}_j + Y_j} - \phi|_{Y_j}\|_{W^{1-\frac{1}{p},p}(Y_j)} + C \|\widehat{\phi}_k|_{\bar{\varepsilon}_{k+1} + Y_{k+1}} - \widehat{\phi}_k|_{Y_{k+1}}\|_{W^{1-\frac{1}{p},p}(Y_{k+1})} \end{aligned}$$

Besides, we have

$$\begin{aligned} \|\widehat{\phi}_k|_{\bar{\varepsilon}_{k+1} + Y_{k+1}} - \widehat{\phi}_k|_{Y_{k+1}}\|_{W^{1-\frac{1}{p},p}(Y_{k+1})} &\leq \|(\phi - \widehat{\phi}_k)|_{\bar{\varepsilon}_{k+1} + Y_{k+1}} - (\phi - \widehat{\phi}_k)|_{Y_{k+1}}\|_{W^{1-\frac{1}{p},p}(Y_{k+1})} \\ &\quad + \|\phi|_{\bar{\varepsilon}_{k+1} + Y_{k+1}} - \phi|_{Y_{k+1}}\|_{W^{1-\frac{1}{p},p}(Y_{k+1})} \\ &\leq C \|\phi - \widehat{\phi}_k\|_{W^{1,p}(Y)} + \|\phi|_{\bar{\varepsilon}_{k+1} + Y_{k+1}} - \phi|_{Y_{k+1}}\|_{W^{1-\frac{1}{p},p}(Y_{k+1})} \end{aligned}$$

Hence we obtain the result for $k + 1$ and the lemma is proved. \square

Proof of Theorem 2.1 : We have $W^n = W_{per}^{1,p}(Y)$. Thanks to Lemma 2.2 Theorem 2.1 is proved by taking $k = n$. \square

Remark 1 : Let X be a Banach space. We can prove as in Lemma 2.2 and Theorem 2.1 that for any $\Phi \in W^{1,p}(Y; X)$ there exists $\widehat{\Phi} \in W_{per}^{1,p}(Y; X)$ such that

$$\|\Phi - \widehat{\Phi}\|_{W^{1,p}(Y; X)} \leq C \sum_{j=1}^n \|\Phi|_{\varepsilon_j + Y_j} - \Phi|_{Y_j}\|_{W^{1-\frac{1}{p},p}(Y_j; X)}$$

The constant depends only on n . \square

Let X be a separable Hilbert space. We equip $H^1(Y; X)$ with the inner product

$$\langle \phi, \psi \rangle = \int_Y \nabla \phi \cdot \nabla \psi + \left(\int_Y \phi \right) \cdot \left(\int_Y \psi \right)$$

where \cdot is the inner product in X . The norm associated to this scalar product is equivalent to the norm of $H^1(Y; X)$.

Theorem 2.3 : For any $\phi \in H^1(Y; X)$ there exists a unique $\widehat{\phi} \in H_{per}^1(Y; X)$ such that

$$\phi - \widehat{\phi} \in (H_{per}^1(Y; X))^\perp \quad \|\widehat{\phi}\|_{H^1(Y; X)} \leq \|\phi\|_{H^1(Y; X)} \quad \|\phi - \widehat{\phi}\|_{H^1(Y; X)} \leq C \sum_{j=1}^n \|\phi|_{\varepsilon_j + Y_j} - \phi|_{Y_j}\|_{H^{1/2}(Y_j; X)}$$

The constant depends only on n . The function $\widehat{\phi}$ verifies

$$\int_Y \phi = \int_Y \widehat{\phi}, \quad \forall \psi \in (H_{per}^1(Y; \mathbb{R}))^\perp \quad \int_Y \nabla(\phi - \widehat{\phi}) \nabla \psi = 0 \quad \text{in } X.$$

Proof : We take a Hilbert basis $(x_n)_{n \in \mathbb{N}}$ of X . Any element ϕ belonging to $H^1(Y; X)$ is decomposed into a series $\phi = \sum_{n=0}^{\infty} \phi_n x_n$, where ϕ_n belongs to $H^1(Y)$. We apply Theorem 2.1 to each component ϕ_n and then by orthogonal projection we obtain Theorem 2.3. \square

Corollary : If \mathcal{X} is a Hilbert space continuously embedded in X then for any $\phi \in H^1(Y; \mathcal{X})$ there exists $\widehat{\phi} \in H_{per}^1(Y; \mathcal{X})$ such that

$$\begin{aligned} \|\phi - \widehat{\phi}\|_{H^1(Y; \mathcal{X})} &\leq C \sum_{j=1}^n \|\phi|_{\varepsilon_j + Y_j} - \phi|_{Y_j}\|_{H^{1/2}(Y_j; \mathcal{X})} \\ \|\phi - \widehat{\phi}\|_{H^1(Y; X)} &\leq C \sum_{j=1}^n \|\phi|_{\varepsilon_j + Y_j} - \phi|_{Y_j}\|_{H^{1/2}(Y_j; X)} \end{aligned}$$

The constant depends only on n .

3. Approximation and periodic unfolding

Let Ω be a bounded domain in \mathbb{R}^n with lipschitzian boundary. We put

$$\begin{aligned} \widetilde{\Omega}_{\varepsilon, k} &= \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < k\sqrt{n}\varepsilon\}, \quad k \in \{1, 2\}, \\ \Omega_\varepsilon &= \text{Int}\left(\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\right), \quad \Xi_\varepsilon = \left\{\xi \in \mathbb{Z}^n \mid \varepsilon(\xi + \overline{Y}) \cap \Omega \neq \emptyset\right\} \end{aligned}$$

We have

$$\Omega \subset \Omega_\varepsilon \subset \tilde{\Omega}_{\varepsilon,1}, \quad \text{and} \quad \forall i \in \{1, \dots, n\}, \quad \Omega_\varepsilon + \varepsilon \vec{e}_i \subset \tilde{\Omega}_{\varepsilon,2}.$$

We recall that there exists a linear and continuous extension operator \mathcal{P} from $H^1(\Omega)$ into $H^1(\tilde{\Omega}_{\varepsilon,2})$, such that for any $\phi \in H^1(\Omega)$, $\mathcal{P}(\phi)$ belongs to $H^1(\tilde{\Omega}_{\varepsilon,2})$ and verifies

$$\begin{aligned} \mathcal{P}(\phi)|_\Omega &= \phi, & \|\nabla_x \mathcal{P}(\phi)\|_{[L^2(\tilde{\Omega}_{\varepsilon,2})]^n} &\leq C \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \\ \|\mathcal{P}(\phi)\|_{L^2(\tilde{\Omega}_{\varepsilon,2})} &\leq C \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \} \end{aligned}$$

More precisely, we have

$$(3.1) \quad \|\mathcal{P}(\phi)\|_{L^2(\tilde{\Omega}_{\varepsilon,2})} + \varepsilon \|\nabla_x \mathcal{P}(\phi)\|_{[L^2(\tilde{\Omega}_{\varepsilon,2})]^n} \leq C \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \}$$

In the rest of this paragraph, without having to specify it every time, any function belonging to $H^1(\Omega)$ is extended to $\tilde{\Omega}_{\varepsilon,2}$, the extension verifying (3.1). In order to simplify the notation, we will still denote by ϕ its extension.

In the sequel, we will make use of definitions and results from [4] concerning the periodic unfolding method. For almost every x belonging to \mathbb{R}^n , there exists a unique element in \mathbb{Z}^n denoted $[x]$ such that

$$x = [x] + \{x\}, \quad \{x\} \in Y.$$

Let us now recall the definition of the unfolding operator \mathcal{T}_ε which to each function $\phi \in L^1(\Omega_\varepsilon)$ associates a function $\mathcal{T}_\varepsilon(\phi) \in L^1(\Omega \times Y)$,

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \quad \text{for } x \in \Omega \text{ and } y \in Y.$$

We have

$$\left| \int_\Omega \phi - \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\phi) \right| \leq \|\phi\|_{L^1(\{x \in \Omega_\varepsilon \mid \text{dist}(x, \partial\Omega) < \sqrt{n}\varepsilon\})}$$

For the other properties of \mathcal{T}_ε , we refer the reader to [4].

Now, for any $\phi \in L^2(\Omega_\varepsilon)$ we define the operator ‘‘mean in the cells’’ M_Y^ε by setting

$$M_Y^\varepsilon(\phi)(x) = \int_Y \mathcal{T}_\varepsilon(\phi)(x, y) dy = \frac{1}{\varepsilon^n} \int_{\{\varepsilon [\frac{x}{\varepsilon}] + \varepsilon Y\}} \phi(z) dz, \quad x \in \Omega.$$

Function $M_Y^\varepsilon(\phi)$ belongs to $L^2(\Omega)$ and verifies

$$\|M_Y^\varepsilon(\phi)\|_{L^2(\Omega)} \leq \|\phi\|_{L^2(\Omega_\varepsilon)}.$$

Proposition 3.1 : For any ϕ belonging to $H^1(\Omega)$ we have

$$(3.2) \quad \|\phi - M_Y^\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}.$$

Proof : Let $\phi \in H^1(\Omega)$. We apply the Poincaré-Wirtinger inequality to the restrictions $x \longrightarrow \phi|_{\varepsilon(\xi+Y)}(x) - M_Y^\varepsilon(\phi)(\varepsilon\xi)$ belonging to $H^1(\varepsilon(\xi+Y))$

$$\|\phi - M_Y^\varepsilon(\phi)(\varepsilon\xi)\|_{L^2(\varepsilon(\xi+Y))}^2 \leq C\varepsilon^2 \|\nabla_x \phi\|_{[L^2(\varepsilon(\xi+Y))]^n}^2, \quad \varepsilon(\xi+Y) \subset \Omega_\varepsilon$$

We add all these inequalities and obtain (3.2). \square

We recall the definition of the scale-splitting operator \mathcal{Q}_ε . The function $\mathcal{Q}_\varepsilon(\phi)$ is the restriction to Ω of Q_1 -interpolate of the discrete function $M_Y^\varepsilon(\phi)$.

Corollary : For any $\phi \in L^2(\Omega_\varepsilon)$ we have

$$(3.3) \quad \|\phi - M_Y^\varepsilon(\phi)\|_{H^{-1}(\Omega)} \leq C\varepsilon \|\phi\|_{L^2(\Omega_\varepsilon)}.$$

For any $\phi \in H^1(\Omega)$ we have

$$(3.4) \quad \begin{cases} \|\phi - \mathcal{T}_\varepsilon(\phi)\|_{L^2(\Omega \times Y)} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \\ \|\mathcal{Q}_\varepsilon(\phi) - M_Y^\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}. \end{cases}$$

Proof : If $\psi \in H_0^1(\Omega)$, we immediately have

$$\int_{\Omega} (\phi - M_Y^\varepsilon(\phi))\psi = \int_{\Omega_\varepsilon} (\phi - M_Y^\varepsilon(\phi))\psi = \int_{\Omega_\varepsilon} \phi(\psi - M_Y^\varepsilon(\psi)) \leq C\varepsilon \|\phi\|_{L^2(\Omega_\varepsilon)} \|\nabla_x \psi\|_{[L^2(\Omega)]^n}$$

hence inequality (3.3).

We have (see [4]): if $\phi \in L^2(\Omega_\varepsilon)$ then

$$\|\mathcal{T}_\varepsilon(\phi - M_Y^\varepsilon(\phi))\|_{L^2(\Omega \times Y)} \leq \|\phi - M_Y^\varepsilon(\phi)\|_{L^2(\Omega_\varepsilon)}$$

and moreover, $\mathcal{T}_\varepsilon \circ M_Y^\varepsilon(\phi) = M_Y^\varepsilon(\phi)$. We eliminate the mean function $M_Y^\varepsilon(\phi)$ with (3.2) to obtain (3.4). We also have (see [4]) $\|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}$ and according to (3.2) we obtain the second inequality of (3.4). \square

Proposition 3.2 : For any ϕ belonging to $L^2(\tilde{\Omega}_{\varepsilon,2})$ and any ψ belonging to $L^2(Y)$, we have

$$(3.5) \quad \|\mathcal{Q}_\varepsilon(\phi)\psi(\{\frac{\cdot}{\varepsilon}\})\|_{L^2(\Omega)} \leq C \|\phi\|_{L^2(\tilde{\Omega}_{\varepsilon,2})} \|\psi\|_{L^2(Y)}$$

The constant depends only on n .

Proof : We set for $i = (i_1, \dots, i_n) \in \{0, 1\}^n$,

$$x \in \varepsilon(\xi + Y), \quad \bar{x}_{i,\xi}^{i_k} = \begin{cases} \frac{x_k - \varepsilon\xi_k}{\varepsilon} & \text{if } i_k = 1 \\ 1 - \frac{x_k - \varepsilon\xi_k}{\varepsilon} & \text{if } i_k = 0 \end{cases}$$

From the definition of $\mathcal{Q}_\varepsilon(\phi)$ (see [4]) it results that

$$x \in \Omega_\varepsilon, \quad \mathcal{Q}_\varepsilon(\phi)(x) = \sum_{i_1, \dots, i_n} M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon i) \bar{x}_{1,\xi}^{i_1} \dots \bar{x}_{n,\xi}^{i_n}, \quad \xi = \left\lfloor \frac{x}{\varepsilon} \right\rfloor$$

hence

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} |\mathcal{Q}_\varepsilon(\phi)|^2 |\psi(\{\frac{\cdot}{\varepsilon}\})|^2 &\leq 2^n \sum_{i_1, \dots, i_n} |M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon i)|^2 \int_{\varepsilon(\xi+Y)} |\psi(\{\frac{\cdot}{\varepsilon}\})|^2 \\ &= 2^n \sum_{i_1, \dots, i_n} |M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon i)|^2 \varepsilon^n \|\psi\|_{L^2(Y)}^2 \end{aligned}$$

For any ξ we have $|M_Y^\varepsilon(\phi)(\varepsilon\xi)|^2 \leq \frac{1}{\varepsilon^n |Y|} \int_{\varepsilon(\xi+Y)} |\phi|^2$. We add the above inequalities for all $\xi \in \Xi_\varepsilon$ and we obtain

$$\int_{\Omega} |\mathcal{Q}_\varepsilon(\phi)|^2 |\psi(\{\frac{\cdot}{\varepsilon}\})|^2 \leq 4^n \|\phi\|_{L^2(\tilde{\Omega}_{\varepsilon,2})}^2 \|\psi\|_{L^2(Y)}^2$$

□

Proposition 3.3 : For any ϕ belonging to $H^1(\Omega)$, there exists $\widehat{\psi}_\varepsilon$ belonging to $H_{per}^1(Y; L^2(\Omega))$ ⁽¹⁾ such that

$$(3.6) \quad \begin{cases} \|\widehat{\psi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \} \\ \|\mathcal{T}_\varepsilon(\phi) - \widehat{\psi}_\varepsilon\|_{H^1(Y; H^{-1}(\Omega))} \leq C\varepsilon \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \} \end{cases}$$

Proof : Proposition 3.3 is proved in two steps. We begin with constructing a new unfolding operator which for any $\phi \in H^1(\Omega)$ allows us to estimate in $L^2(Y; H^{-1}(\Omega))$, the difference between the restrictions to two neighboring cells of the unfolded of ϕ . Then, we evaluate the periodic defect of the functions $y \rightarrow \mathcal{T}_\varepsilon(\phi)(\cdot, y)$ and conclude thanks to Theorem 2.3.

Let $K_i = \text{Int}(\overline{Y} \cup (\vec{e}_i + \overline{Y}))$, $i \in \{1, \dots, n\}$. For any x in Ω , $\varepsilon \left(\left[\frac{x}{\varepsilon} \right] + K_i \right)$ is included in $\tilde{\Omega}_{\varepsilon,2}$.

Step one. We define the unfolding operator $\mathcal{T}_{\varepsilon,i}$ from $L^2(\tilde{\Omega}_{\varepsilon,2})$ into $L^2(\Omega \times K_i)$ by

$$\forall \psi \in L^2(\tilde{\Omega}_{\varepsilon,2}), \quad \mathcal{T}_{\varepsilon,i}(\psi)(x, y) = \psi \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon y \right) \quad \text{for } x \in \Omega \text{ and a. e. } y \in K_i.$$

The restriction of $\mathcal{T}_{\varepsilon,i}(\psi)$ to $\Omega \times Y$ is equal to the unfolded $\mathcal{T}_\varepsilon(\psi)$. Moreover, we have the following equalities in $L^2(\Omega \times Y)$:

$$\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, \cdot + \vec{e}_i) = \mathcal{T}_\varepsilon(\psi)(\cdot + \varepsilon \vec{e}_i, \cdot), \quad i \in \{1, \dots, n\}$$

Let us take $\Psi \in H_0^1(\Omega)$, extended by 0 on $\mathbb{R}^n \setminus \Omega$. A linear change of variables and the above relations give

$$\begin{aligned} \text{for a. e. } y \in Y, \quad \int_{\Omega} \mathcal{T}_{\varepsilon,i}(\psi)(x, y + \vec{e}_i) \Psi(x) dx &= \int_{\Omega} \mathcal{T}_{\varepsilon,i}(\psi)(x + \varepsilon \vec{e}_i, y) \Psi(x) dx \\ &= \int_{\Omega + \varepsilon \vec{e}_i} \mathcal{T}_{\varepsilon,i}(\psi)(x, y) \Psi(x - \varepsilon \vec{e}_i) dx \end{aligned}$$

We deduce

$$\begin{aligned} & \left| \int_{\Omega} \{ \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \} \Psi - \int_{\Omega} \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \{ \Psi(\cdot - \varepsilon \vec{e}_i) - \Psi \} \right| \\ & \leq C \|\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y)\|_{L^2(\tilde{\Omega}_{\varepsilon,1})} \|\Psi\|_{L^2(\Omega \Delta \{\Omega + \varepsilon \vec{e}_i\})} \end{aligned}$$

where $\Omega \Delta \{\Omega + \varepsilon \vec{e}_i\} = (\Omega \setminus \{\Omega + \varepsilon \vec{e}_i\}) \cup (\{\Omega + \varepsilon \vec{e}_i\} \setminus \Omega)$; Ω is a bounded domain with lipschitzian boundary and Ψ belongs to $H_0^1(\Omega)$, we thus have

$$\begin{aligned} \|\Psi\|_{L^2(\Omega \Delta \{\Omega + \varepsilon \vec{e}_i\})} &\leq C\varepsilon \|\nabla_x \Psi\|_{[L^2(\Omega)]^n}, \\ \|\Psi(\cdot - \varepsilon \vec{e}_i) - \Psi\|_{L^2(\Omega)} &\leq C\varepsilon \left\| \frac{\partial \Psi}{\partial x_i} \right\|_{L^2(\Omega)}, \quad i \in \{1, \dots, n\}, \end{aligned}$$

hence

$$\begin{aligned} & \langle \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y), \Psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \int_{\Omega} \{ \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y) \} \Psi \\ &\leq C\varepsilon \|\nabla_x \Psi\|_{[L^2(\Omega)]^n} \|\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y)\|_{L^2(\tilde{\Omega}_{\varepsilon,1})} \leq C\varepsilon \|\Psi\|_{H_0^1(\Omega)} \|\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y)\|_{L^2(\tilde{\Omega}_{\varepsilon,1})}. \end{aligned}$$

⁽¹⁾ Of course $H_{per}^1(Y; L^2(\Omega))$ is the same space as $L^2(\Omega; H_{per}^1(Y))$. The same remark holds for all other spaces appearing in the sequel.

We deduce that

$$\|\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y)\|_{H^{-1}(\Omega)} \leq C\varepsilon \|\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, y)\|_{L^2(\tilde{\Omega}_{\varepsilon,1})},$$

which leads to the following estimate of the difference between $\mathcal{T}_{\varepsilon,i}(\psi)|_{\Omega \times Y}$ and one of its translated:

$$(3.7) \quad \|\mathcal{T}_{\varepsilon,i}(\psi)(\cdot, \cdot + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\psi)\|_{L^2(Y; H^{-1}(\Omega))} \leq C\varepsilon \|\psi\|_{L^2(\tilde{\Omega}_{\varepsilon,2})}$$

The constant depends only on the boundary of Ω .

Step two. Let $\phi \in H^1(\Omega)$. The estimate (3.7) applied to ϕ and its partial derivatives gives

$$\begin{aligned} \|\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \cdot + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)\|_{L^2(Y; H^{-1}(\Omega))} &\leq C\varepsilon \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \} \\ \|\mathcal{T}_{\varepsilon,i}(\nabla_x \phi)(\cdot, \cdot + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\nabla_x \phi)\|_{[L^2(Y; H^{-1}(\Omega))]^n} &\leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \end{aligned}$$

We recall (see [4]) that $\nabla_y(\mathcal{T}_{\varepsilon,i}(\phi)) = \varepsilon \mathcal{T}_{\varepsilon,i}(\nabla_x \phi)$. The above estimates can also be written:

$$\|\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \cdot + \vec{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)\|_{H^1(Y; H^{-1}(\Omega))} \leq C\varepsilon \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \}$$

From these inequalities, for any $i \in \{1, \dots, n\}$, we deduce the estimate of the difference of the traces of $y \rightarrow \mathcal{T}_{\varepsilon}(\phi)(\cdot, y)$ on the faces Y_i and $\vec{e}_i + Y_i$

$$(3.8) \quad \|\mathcal{T}_{\varepsilon}(\phi)(\cdot, \cdot + \vec{e}_i) - \mathcal{T}_{\varepsilon}(\phi)\|_{H^{1/2}(Y_i; H^{-1}(\Omega))} \leq C\varepsilon \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \}$$

which measures the periodic defect of $y \rightarrow \mathcal{T}_{\varepsilon}(\phi)(\cdot, y)$. Thanks to Theorem 2.3 we decompose $\mathcal{T}_{\varepsilon}(\phi)$ in the sum of an element $\widehat{\psi}_{\varepsilon}$ belonging to $H_{per}^1(Y; L^2(\Omega))$ and an element $\bar{\phi}_{\varepsilon}$ belonging to $(H^1(Y; L^2(\Omega)))^{\perp}$ such that

$$(3.9) \quad \begin{cases} \|\bar{\phi}_{\varepsilon}\|_{H^1(Y; H^{-1}(\Omega))} \leq C \sum_{j=1}^n \|\mathcal{T}_{\varepsilon}(\phi)(\cdot, \cdot + \vec{e}_j) - \mathcal{T}_{\varepsilon}(\phi)\|_{H^{1/2}(Y_j; H^{-1}(\Omega))} \\ \leq C\varepsilon \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \} \\ \|\bar{\phi}_{\varepsilon}\|_{H^1(Y; L^2(\Omega))} \leq C \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \} \end{cases}$$

The constants do not depend on ε . □

Theorem 3.4 : For any $\phi \in H^1(\Omega)$, there exists $\widehat{\phi}_{\varepsilon} \in H_{per}^1(Y; L^2(\Omega))$ such that

$$(3.10) \quad \begin{cases} \|\widehat{\phi}_{\varepsilon}\|_{H^1(Y; L^2(\Omega))} \leq C \|\nabla_x \phi\|_{[L^2(\Omega)]^n}, \\ \|\mathcal{T}_{\varepsilon}(\nabla_x \phi) - \nabla_x \phi - \nabla_y \widehat{\phi}_{\varepsilon}\|_{[L^2(Y; H^{-1}(\Omega))]^n} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}. \end{cases}$$

The constants depend only on n and $\partial\Omega$.

Proof : Let $\phi \in H^1(\Omega)$. The function ϕ is decomposed

$$\phi = \Phi + \varepsilon \underline{\phi}, \quad \text{where } \Phi = \mathcal{Q}_{\varepsilon}(\phi) \quad \text{and} \quad \underline{\phi} = \frac{1}{\varepsilon} \mathcal{R}_{\varepsilon}(\phi), \quad \mathcal{R}_{\varepsilon}(\phi) = \phi - \mathcal{Q}_{\varepsilon}(\phi),$$

with the following estimate (see [4]):

$$(3.11) \quad \|\nabla_x \Phi\|_{[L^2(\Omega)]^n} + \|\underline{\phi}\|_{L^2(\Omega)} + \varepsilon \|\nabla_x \underline{\phi}\|_{[L^2(\Omega)]^n} \leq C \|\nabla_x \phi\|_{[L^2(\Omega)]^n}.$$

Proposition 3.3 applied to $\underline{\phi}$ gives us the existence of an element $\widehat{\phi}_\varepsilon$ in $H_{per}^1(Y; L^2(\Omega))$ such that

$$(3.12) \quad \begin{cases} \|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C \|\nabla_x \phi\|_{[L^2(\Omega)]^n}, \\ \|\mathcal{T}_\varepsilon(\underline{\phi}) - \widehat{\phi}_\varepsilon\|_{H^1(Y; H^{-1}(\Omega))} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}. \end{cases}$$

We evaluate $\|\mathcal{T}_\varepsilon(\nabla_x \Phi) - \nabla_x \Phi\|_{[L^2(Y; H^{-1}(\Omega))]^n}$.

From the inequality (3.2), applied to each partial derivative of Φ , it follows

$$(3.13) \quad \left\| \frac{\partial \Phi}{\partial x_i} - M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \right\|_{H^{-1}(\Omega)} \leq C\varepsilon \|\nabla_x \Phi\|_{[L^2(\Omega)]^n} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}$$

There results, from the definition of Φ , that $y \rightarrow \mathcal{T}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) (\cdot, y)$ is linear with respect to each variable. For any $\psi \in H_0^1(\Omega)$, we have

$$\begin{aligned} \langle \mathcal{T}_\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right), \psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= \int_\Omega \left\{ \mathcal{T}_\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) \right\} \psi \\ &= \int_{\Omega_\varepsilon} \left\{ \mathcal{T}_\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) \right\} M_Y^\varepsilon(\psi) \end{aligned}$$

Set for $i = (i_1, \dots, i_n) \in \{0, 1\}^n$,

$$\overline{y}_i^{i_k} = \begin{cases} y_k & \text{if } i_k = 1 \\ 1 - y_k & \text{if } i_k = 0 \end{cases}$$

We have

$$\mathcal{T}_\varepsilon(\Phi)(\varepsilon\xi, y) = \sum_{i_1, \dots, i_n} M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon i) \overline{y}_1^{i_1} \dots \overline{y}_n^{i_n}, \quad \xi = \left[\frac{x}{\varepsilon} \right]$$

hence

$$\begin{aligned} \mathcal{T}_\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\varepsilon\xi, y) &= \sum_{i_2, \dots, i_n} \frac{M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon(1, i_2, \dots, i_n)) - M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon(0, i_2, \dots, i_n))}{\varepsilon} \overline{y}_2^{i_2} \dots \overline{y}_n^{i_n} \\ M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\varepsilon\xi) &= \frac{1}{2^{n-1}} \sum_{i_2, \dots, i_n} \frac{M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon(1, i_2, \dots, i_n)) - M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon(0, i_2, \dots, i_n))}{\varepsilon} \end{aligned}$$

We deduce that

$$\begin{aligned} &\int_{\Omega_\varepsilon} \left\{ \mathcal{T}_\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) \right\} M_Y^\varepsilon(\Psi) = \\ &\varepsilon^n \sum_\xi \sum_{i_2, \dots, i_n} \left(\frac{M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon(1, i_2, \dots, i_n)) - M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon(0, i_2, \dots, i_n))}{\varepsilon} - M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\varepsilon\xi) \right) \\ &\quad \times \overline{y}_2^{i_2} \dots \overline{y}_n^{i_n} M_Y^\varepsilon(\psi)(\varepsilon\xi) \end{aligned}$$

The above integral is equal to

$$\varepsilon^n \sum_\xi \frac{M_Y^\varepsilon(\phi)(\varepsilon\xi + \varepsilon\vec{e}_1) - M_Y^\varepsilon(\phi)(\varepsilon\xi)}{\varepsilon} \sum_{i_2, \dots, i_n} (M_Y^\varepsilon(\psi)(\varepsilon\xi - \varepsilon(0, i_2, \dots, i_n)) - \overline{M_Y^\varepsilon(\psi)}(\varepsilon\xi)) \overline{y}_2^{i_2} \dots \overline{y}_n^{i_n}$$

where

$$\overline{M_Y^\varepsilon(\psi)}(\varepsilon\xi) = \frac{1}{2^{n-1}} \sum_{i_2, \dots, i_n} M_Y^\varepsilon(\psi)(\varepsilon\xi - \varepsilon(0, i_2, \dots, i_n))$$

which gives the following inequality

$$\langle \mathcal{T}_\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right), \psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \leq C\varepsilon |y_2^{i_2} \dots y_n^{i_n}| \|\nabla_x \phi\|_{[L^2(\Omega)]^n} \|\psi\|_{H_0^1(\Omega)}$$

and

$$\forall y \in Y, \quad \left\| \mathcal{T}_\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) (\cdot, y) - M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_1} \right) \right\|_{H^{-1}(\Omega)} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}.$$

Considering (3.13) and all the partial derivatives, we obtain

$$\|\mathcal{T}_\varepsilon(\nabla_x \Phi) - \nabla_x \Phi\|_{[L^2(Y; H^{-1}(\Omega))]^n} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}$$

Thanks to (3.12), and to the above inequality and, moreover, to

$$\|\varepsilon \nabla_x \underline{\phi}\|_{[H^{-1}(\Omega)]^n} \leq C\varepsilon \|\underline{\phi}\|_{L^2(\Omega)} \leq C\varepsilon \|\nabla_x \phi\|_{[L^2(\Omega)]^n}$$

the second estimate of (3.10) is proved. \square

4. Error estimate

We consider the following homogenization problem: find $\phi^\varepsilon \in H_{\Gamma_0}^1(\Omega)$ such that

$$(4.1) \quad \begin{cases} \forall \psi \in H_{\Gamma_0}^1(\Omega) = \{\phi \in H^1(\Omega) \mid \phi = 0 \text{ on } \Gamma_0\}, \\ \int_{\Omega} A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \nabla \phi^\varepsilon \cdot \nabla \psi = \int_{\Omega} f \psi, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n with lipschitzian boundary, Γ_0 is a part of $\partial\Omega$ whose measure is nonnull or empty, f belongs to $L^p(\Omega)$, $p > \frac{2n}{n+2}$, (if $\Gamma_0 = \emptyset$, we suppose that $\int_{\Omega} f = 0$) and A is a square matrix of elements belonging to $L_{per}^\infty(Y)$, verifying the condition of uniform ellipticity $c|\xi|^2 \leq A(y)\xi \cdot \xi \leq C|\xi|^2$ a.e. $y \in Y$, with c and C strictly positive constants.

We have shown, see [4], that $\nabla_x \phi^\varepsilon - \nabla_x \Phi - \mathcal{U}_\varepsilon(\nabla_y \widehat{\phi})$ strongly converges towards 0 in $[L^2(\Omega)]^n$, where \mathcal{U}_ε is the averaging operator defined by

$$\Psi \in L^2(\Omega \times Y) \quad \mathcal{U}_\varepsilon(\Psi)(x) = \int_Y \Psi \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon z, \left\{ \frac{x}{\varepsilon} \right\} \right) dz, \quad \mathcal{U}_\varepsilon(\Psi) \in L^2(\Omega),$$

and where

$$(\Phi, \widehat{\phi}) \in H_{\Gamma_0}^1(\Omega) \times L^2(\Omega; H_{per}^1(Y)/\mathbb{R})$$

is the solution of the limit problem of unfolding homogenization

$$(4.2) \quad \begin{cases} \forall (\Psi, \widehat{\psi}) \in H_{\Gamma_0}^1(\Omega) \times L^2(\Omega; H_{per}^1(Y)/\mathbb{R}) \\ \int_{\Omega} \int_Y A\{\nabla_x \Phi + \nabla_y \widehat{\phi}\} \cdot \{\nabla_x \Psi + \nabla_y \widehat{\psi}\} = \int_{\Omega} f \Psi. \end{cases}$$

If $\Gamma_0 = \emptyset$, we take $\int_{\Omega} \phi^\varepsilon = \int_{\Omega} \Phi = 0$.

We recall that the correctors χ_i , $i \in \{1, \dots, n\}$, are the solutions of the following variational problems

$$\chi_i \in H_{per}^1(Y), \quad \int_Y A(y) \nabla_y (\chi_i(y) + y_i) \nabla_y \psi(y) dy = 0, \quad \forall \psi \in H_{per}^1(Y)$$

They allow us to express $\widehat{\phi}$ in terms of $\nabla_x \Phi$

$$\widehat{\phi} = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \chi_i.$$

In Theorem 3 of [6] our hypothesis was that the solution Φ of the homogenized problem belonged to $W^{2,p}(\Omega)$ ($p > n$) and we gave the following error estimate :

$$\|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla_x \phi^\varepsilon - \nabla_x \Phi - \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{[L^2(\Omega)]^n} \leq C \varepsilon^{\inf\{1/2, 1-n/p\}},$$

the constant depends on $n, p, A, \|\Phi\|_{W^{2,p}(\Omega)}$ and $\partial\Omega$. Then in Theorem 4 from [6] we obtained, by an interpolation method, the error estimate in the case where $\Gamma_0 = \partial\Omega$, and where the boundary of Ω is of class $C^{1,1}$ and where f belongs to $L^p(\Omega)$ ($p > \frac{2n}{n+2}$).

$$\|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla_x \phi^\varepsilon - \nabla_x \Phi - \mathcal{U}_\varepsilon(\nabla_y \widehat{\phi})\|_{[L^2(\Omega)]^n} \leq C \varepsilon^{\inf\{1/2, \frac{1/2+1/n-1/p}{1+1/n}\}} \|f\|_{L^p(\Omega)},$$

the constant depends on n, p, A and $\partial\Omega$.

If Φ belongs to $H^2(\Omega)$ the function $\sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)$ does not generally belong to $H^1(\Omega)$. However if Φ belongs only to $H^1(\Omega)$ then from its definition the function $\mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right)$ belongs to $W^{1,\infty}(\Omega)$. Hence function $\mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)$ belongs to $H^1(\Omega)$ and thanks to (3.5) it verifies

$$\begin{aligned} \|\mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{L^2(\Omega)} &\leq C \|\nabla_x \Phi\|_{[L^2(\Omega)]^n} \|\chi_i\|_{L^2(Y)} \leq C \|\nabla_x \Phi\|_{[L^2(\Omega)]^n} \\ \|\mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{H^1(\Omega)} &\leq \frac{C}{\varepsilon} \|\nabla_x \Phi\|_{[H^1(\Omega)]^n} \|\chi_i\|_{H^1(Y)} \leq \frac{C}{\varepsilon} \|\nabla_x \Phi\|_{[H^1(\Omega)]^n} \end{aligned}$$

This is the reason why in the approximate solution we replace $\frac{\partial \Phi}{\partial x_i}$ with $\mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right)$. In the following theorems we are going to obtain estimates that are better than those obtained in [6], with weaker hypotheses.

4.1 First case : Homogeneous Dirichlet or Neumann condition and boundary of class $C^{1,1}$.

Theorem 4.1 : *We suppose that Ω is a $C^{1,1}$ bounded domain in \mathbb{R}^n , $\Gamma_0 = \partial\Omega$ and $f \in L^2(\Omega)$. Then we have*

$$(4.3) \quad \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla_x \phi^\varepsilon - \nabla_x \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{[L^2(\Omega)]^n} \leq C \varepsilon^{1/2} \|f\|_{L^2(\Omega)}.$$

The constant depends on n, A and $\partial\Omega$.

Theorem 4.2 : *We suppose that Ω is a $C^{1,1}$ bounded domain in \mathbb{R}^n , $\Gamma_0 = \emptyset$, $f \in L^2(\Omega)$. Then we have*

$$(4.4) \quad \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla_x \phi^\varepsilon - \nabla_x \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{[L^2(\Omega)]^n} \leq C \varepsilon^{1/2} \|f\|_{L^2(\Omega)}$$

The constant depends on n, A and $\partial\Omega$.

The proof of Theorems 4.1 and 4.2 is based on the following proposition.

Proposition 4.3 : *We suppose that the solution Φ of the unfolded problem belongs to $H^2(\Omega)$. Therefore we have*

$$(4.5) \quad \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla_x \phi^\varepsilon - \nabla_x \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{[L^2(\Omega)]^n} \leq C\varepsilon^{1/2}$$

The constant depends on A , n , $\|\Phi\|_{H^2(\Omega)}$ and $\partial\Omega$.

Proof : We denote by $\rho(x) = \text{dist}(x, \partial\Omega)$ the distance between $x \in \Omega$ and the boundary of Ω .

We show that if $(\Phi, \hat{\phi})$ is the solution of the unfolded problem, then $\Phi + \sum_{i=1}^n \varepsilon \rho_\varepsilon \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)$ is an approximate solution to the homogenization problem (4.1); $\rho_\varepsilon(\cdot) = \inf \left\{ \frac{\rho(\cdot)}{\varepsilon}, 1 \right\}$. The presence of the function ρ_ε in the sum guarantees the nullity of the approximate solution on Γ_0 .

Step one. We present some estimates of ρ_ε , $\nabla_x \Phi$ and $\chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)$ on the neighborhood $\hat{\Omega}_\varepsilon = \{x \in \Omega \mid \rho(x) < \varepsilon\}$ of the boundary of Ω . We have

$$(4.6) \quad \begin{cases} \|\nabla_x \rho_\varepsilon\|_{[L^\infty(\Omega)]^n} = \|\nabla_x \rho_\varepsilon\|_{[L^\infty(\hat{\Omega}_\varepsilon)]^n} = \varepsilon^{-1}, \\ \|\nabla_x \Phi\|_{[L^2(\hat{\Omega}_\varepsilon)]^n} \leq C\varepsilon^{1/2} \|\Phi\|_{H^2(\Omega)} \\ \implies \|\mathcal{Q}_\varepsilon(\nabla_x \Phi)\|_{[L^2(\hat{\Omega}_\varepsilon)]^n} + \|M_Y^\varepsilon(\nabla_x \Phi)\|_{[L^2(\hat{\Omega}_\varepsilon)]^n} \leq C\varepsilon^{1/2} \|\Phi\|_{H^2(\Omega)}, \\ \|\chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{L^2(\hat{\Omega}_\varepsilon)} + \|\nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{[L^2(\hat{\Omega}_\varepsilon)]^n} \leq C\varepsilon^{1/2} \|\nabla_y \chi_i\|_{[L^2(Y)]^n} \leq C\varepsilon^{1/2}. \end{cases}$$

The estimate of ρ_ε follows from its definition. The estimate of $\nabla_x \Phi$ in $[L^2(\hat{\Omega}_\varepsilon)]^n$ comes from the gradient belonging to $H^2(\Omega)$. The number of cells covering $\hat{\Omega}_\varepsilon$ is of order of ε^{1-n} , hence we obtain the estimates of $\nabla_y \chi_i$ and χ_i on the neighborhood of the boundary of Ω . We will note for the rest of the demonstration that the support of $1 - \rho_\varepsilon$ is contained in $\Omega \setminus \hat{\Omega}_\varepsilon$.

Step two. Let $\Psi \in H_{\Gamma_0}^1(\Omega)$. Thanks to Theorem 3.4, there exists $\hat{\psi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega))$ verifying the estimates (3.10). We take the couple $(\Psi, \hat{\psi}_\varepsilon)$ as a test-function in the unfolded problem (4.2) and we introduce ρ_ε . The gradient of Φ belongs to $[H^1(\Omega)]^n$, and according to (4.6)

$$(4.7) \quad \|(1 - \rho_\varepsilon) \nabla_x \Phi\|_{[L^2(\Omega)]^n} \leq \|\nabla_x \Phi\|_{[L^2(\hat{\Omega}_\varepsilon)]^n} \leq C\varepsilon^{1/2} \|\Phi\|_{H^2(\Omega)},$$

which gives us

$$\left| \int_\Omega f \Psi - \int_{\Omega \times Y} A(y) \rho_\varepsilon(x) \left\{ \nabla_x \Phi(x) + \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(x) \nabla_y \chi_i(y) \right\} (\nabla_x \Psi + \nabla_y \hat{\psi}_\varepsilon) \right| \leq C\varepsilon^{1/2} \|\Psi\|_{H^1(\Omega)}.$$

In the integral on $\Omega \times Y$ we replace $\nabla_x \Psi + \nabla_y \hat{\psi}_\varepsilon$ by $\mathcal{T}_\varepsilon(\nabla_x \Psi)$, thanks to (3.10) of Theorem 3.4. The function $\rho_\varepsilon \nabla_x \Phi$ belongs to $[H_0^1(\Omega)]^n$ and verifies $\|\rho_\varepsilon \nabla_x \Phi\|_{[H^1(\Omega)]^n} \leq C\varepsilon^{-1/2} \|\Phi\|_{H^2(\Omega)}$ for

$$\begin{aligned} \left\| \nabla_x \left\{ \rho_\varepsilon \frac{\partial \Phi}{\partial x_i} \right\} \right\|_{[L^2(\Omega)]^n} &\leq \left\| \nabla_x \rho_\varepsilon \frac{\partial \Phi}{\partial x_i} \right\|_{[L^2(\Omega)]^n} + \left\| \rho_\varepsilon \nabla_x \left\{ \frac{\partial \Phi}{\partial x_i} \right\} \right\|_{[L^2(\Omega)]^n} \\ &\leq \|\nabla_x \rho_\varepsilon\|_{[L^\infty(\hat{\Omega}_\varepsilon)]^n} \left\| \frac{\partial \Phi}{\partial x_i} \right\|_{L^2(\hat{\Omega}_\varepsilon)} + \left\| \nabla_x \left\{ \frac{\partial \Phi}{\partial x_i} \right\} \right\|_{[L^2(\Omega)]^n} \leq C\varepsilon^{-1/2} \|\Phi\|_{H^2(\Omega)}. \end{aligned}$$

Then we remove ρ_ε in the products $\rho_\varepsilon(x) \nabla_x \Phi(x)$ and $\rho_\varepsilon(x) \frac{\partial \Phi}{\partial x_i}(x) \nabla_y \chi_i(y)$ by using (4.7) again. And then we replace $\nabla_x \Phi$ with $M_Y^\varepsilon(\nabla_x \Phi)$ and in the sum we replace $\frac{\partial \Phi}{\partial x_i}$ with $M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right)$. Thanks to (3.2), we obtain

$$\left| \int_\Omega f \Psi - \frac{1}{|Y|} \int_{\Omega \times Y} A(\cdot) \left\{ M_Y^\varepsilon(\nabla_x \Phi) + \sum_{i=1}^n M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i(\cdot) \right\} \mathcal{T}_\varepsilon(\nabla_x \Psi) \right| \leq C\varepsilon^{1/2} \|\Psi\|_{H^1(\Omega)}$$

By inverse unfolding we transform the integral on $\Omega \times Y$ into an integral on Ω . Then we replace $M_Y^\varepsilon(\nabla_x \Phi)$ with $\nabla_x \Phi$ and we reintroduce ρ_ε in front of $M_Y^\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)$

$$(4.8) \quad \left\| (1 - \rho_\varepsilon) M_Y^\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right\|_{L^2(\Omega)} \leq \left\| M_Y^\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \right\|_{L^2(\widehat{\Omega}_\varepsilon)} \left\| \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right\|_{[L^2(\widehat{\Omega}_\varepsilon)]^n} \leq C\varepsilon \|\Phi\|_{H^2(\Omega)}.$$

This done, we have

$$\left| \int_\Omega f \Psi - \int_\Omega A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \left\{ \nabla_x \Phi + \sum_{i=1}^n \rho_\varepsilon M_Y^\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right\} \nabla_x \Psi \right| \leq C\varepsilon^{1/2} \|\Psi\|_{H^1(\Omega)}.$$

From (3.4) we obtain

$$\left\| \left\{ M_Y^\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) - \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \right\} \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right\|_{L^2(\Omega)} \leq C\varepsilon \|\Phi\|_{H^2(\Omega)}$$

hence

$$\left| \int_\Omega f \Psi - \int_\Omega A\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \left\{ \nabla_x \Phi + \sum_{i=1}^n \rho_\varepsilon \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \nabla_y \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right\} \nabla_x \Psi \right| \leq C\varepsilon^{1/2} \|\Psi\|_{H^1(\Omega)}.$$

We now estimate the terms which appear in the calculation of the gradient of the approximate solution but do not appear in the above expression. Thanks to (4.6) and (3.5) we have

$$(4.9) \quad \left\{ \begin{array}{l} \left\| \varepsilon \frac{\partial \rho_\varepsilon}{\partial x_j} \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right\|_{L^2(\Omega)} \leq \left\| \varepsilon \frac{\partial \rho_\varepsilon}{\partial x_j} \right\|_{L^\infty(\widehat{\Omega}_\varepsilon)} \left\| \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \right\|_{L^2(\widehat{\Omega}_\varepsilon)} \|\chi_i\|_{L^2(Y)} \\ \leq C\varepsilon^{1/2} \|\Phi\|_{H^2(\Omega)}, \\ \left\| \varepsilon \rho_\varepsilon \frac{\partial}{\partial x_j} \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right\|_{L^2(\Omega)} \leq \varepsilon \|\rho_\varepsilon\|_{L^\infty(\Omega)} \left\| \nabla \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \right\|_{L^2(\Omega)} \|\chi_i\|_{L^2(Y)} \\ \leq C\varepsilon \|\Phi\|_{H^2(\Omega)}. \end{array} \right.$$

Now we use the equality

$$\int_\Omega f \Psi = \int_\Omega A\left(\left\{\frac{x}{\varepsilon}\right\}\right) \nabla_x \phi^\varepsilon(x) \nabla_x \Psi(x),$$

and we take as a test function

$$\Psi = \phi^\varepsilon - \left[\Phi + \sum_{i=1}^n \varepsilon \rho_\varepsilon \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right],$$

to obtain

$$\left\| \nabla_x \phi^\varepsilon - \nabla_x \left[\Phi + \sum_{i=1}^n \varepsilon \rho_\varepsilon \mathcal{Q}_\varepsilon\left(\frac{\partial \Phi}{\partial x_i}\right) \chi_i\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \right] \right\|_{[L^2(\Omega)]^n} \leq C\varepsilon^{1/2}$$

This gives the estimate (4.5) thanks to the Poincaré inequality or the Poincaré-Wirtinger inequality and (4.7) and (4.9). \square

Proofs of Theorems 4.1 and 4.2 : The boundary of Ω is of class $\mathcal{C}^{1,1}$, then for $f \in L^2(\Omega)$, the solution Φ from (4.2) with $\Gamma_0 = \partial\Omega$ or $\Gamma_0 = \emptyset$, belongs to $H^2(\Omega)$ and verifies $\|\Phi\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$. \square

Corollary of Theorem 4.1 : When the correctors belong to $W^{1,\infty}(Y)$ we obtain the classical error estimate (see [2], [5] and [8]). \square

4.2 Second case : lipschitzian boundary.

Proposition 4.4 : We suppose that solution Φ of the unfolded problem (4.2) belongs to $H_{loc}^2(\Omega) \cap W^{1,q}(\Omega)$ ($q > 2$) and verifies

$$(4.10) \quad \|\rho \nabla_x \Phi\|_{[H^1(\Omega)]^n} < +\infty,$$

Then we have

$$(4.11) \quad \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla_x \phi^\varepsilon - \nabla_x \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{[L^2(\Omega)]^n} \leq C \varepsilon^{\frac{q-2}{3q-2}}$$

The constant depends on $A, n, q, \|\Phi\|_{W^{1,q}(\Omega)} + \|\rho \nabla_x \Phi\|_{[H^1(\Omega)]^n}$ and $\partial\Omega$.

Proof : We equip $W^{1,q}(\Omega) \cap H_{loc}^2(\Omega)$ with the norm

$$\|\Psi\| = \|\Psi\|_{W^{1,q}(\Omega)} + \|\rho \nabla_x \Psi\|_{[H^1(\Omega)]^n}$$

As in proposition 7, we show that if $(\Phi, \hat{\phi})$ is the solution of (4.2), then

$$\Phi + \sum_{i=1}^n \varepsilon \rho_{\varepsilon,\alpha} \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)$$

is an approximate solution of problem (4.1), where $\rho_{\varepsilon,\alpha}(\cdot) = \inf \left\{ \frac{d(\cdot)}{\varepsilon^\alpha}, 1 \right\}$, α belongs to interval $]0, 1]$ and will be fixed later.

Step one. We present some estimates of $\rho_{\varepsilon,\alpha}$ and Φ on the neighborhood $\hat{\Omega}_{\varepsilon,\alpha} = \{x \in \Omega ; \rho(x) < \varepsilon^\alpha\}$ of the boundary of Ω . We have

$$(4.12) \quad \begin{cases} \|\nabla_x \rho_{\varepsilon,\alpha}\|_{[L^\infty(\Omega)]^n} = \|\nabla_x \rho_{\varepsilon,\alpha}\|_{[L^\infty(\hat{\Omega}_{\varepsilon,\alpha})]^n} = \varepsilon^{-\alpha}, \\ \|\nabla_x \Phi\|_{[L^2(\hat{\Omega}_{\varepsilon,\alpha})]^n} \leq C \varepsilon^{\alpha(\frac{1}{2}-\frac{1}{q})} \|\nabla_x \Phi\|_{[L^q(\Omega)]^n}, \\ \|\rho_{\varepsilon,\alpha} \nabla_x \Phi\|_{[H^1(\Omega)]^n} \leq C \varepsilon^{-\alpha} \|\Phi\|. \end{cases}$$

Step two. Let $\Psi \in H_{\Gamma_0}^1(\Omega)$. Thanks to Theorem 3.4, there exists $\hat{\psi}_\varepsilon \in L^2(\Omega; H_{per}^1(Y))$ verifying the estimates (3.10). We take the couple $(\Psi, \hat{\psi}_\varepsilon)$ as test-function in the unfolded problem (4.2) and we introduce $\rho_{\varepsilon,\alpha}$. The gradient of Φ verifies

$$(4.13) \quad \|(1 - \rho_{\varepsilon,\alpha}) \nabla_x \Phi\|_{[L^2(\Omega)]^n} \leq \|\nabla_x \Phi\|_{[L^2(\hat{\Omega}_{\varepsilon,\alpha})]^n} \leq C \varepsilon^{\alpha\{\frac{1}{2}-\frac{1}{q}\}} \|\Phi\|,$$

according to (4.12). This gives us

$$\left| \int_{\Omega} f \Psi - \int_{\Omega \times Y} A \rho_{\varepsilon,\alpha} \left\{ \nabla_x \Phi + \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \nabla_y \chi_i \right\} (\nabla_x \Psi + \nabla_y \hat{\psi}_\varepsilon) \right| \leq C \varepsilon^{\alpha\{\frac{1}{2}-\frac{1}{q}\}} \|\Psi\|_{H^1(\Omega)}$$

In the integral on $\Omega \times Y$ we replace $\nabla_x \Psi + \nabla_y \widehat{\psi}_\varepsilon$ with $\mathcal{T}_\varepsilon(\nabla_x \Psi)$, thanks to (3.10) from Theorem 3.4 and to (4.12). Function $\rho_{\varepsilon,\alpha} \nabla_x \Phi$ belongs to $[H_0^1(\Omega)]^n$ and thanks to (3.2), (4.12) and (4.13) we get

$$\left\| \rho_{\varepsilon,\alpha} \frac{\partial \Phi}{\partial x_i} - \rho_{\varepsilon,\alpha} M_Y^\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \right\|_{L^2(\Omega)} \leq C \varepsilon^{\inf\{\alpha(\frac{1}{2}-\frac{1}{q}), 1-\alpha\}} \|\Phi\|$$

We also have $\|\mathcal{T}_\varepsilon(\rho_{\alpha,\varepsilon}) - \rho_{\varepsilon,\alpha}\|_{L^\infty(\Omega \times Y)} \leq C \varepsilon^{1-\alpha}$. Now we proceed as in Proposition 4.3 to obtain

$$\left| \int_\Omega f \Psi - \int_\Omega A \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right) \left\{ \nabla_x \Phi + \sum_{i=1}^n \rho_{\varepsilon,\alpha} \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right) \right\} \nabla_x \Psi \right| \leq C \varepsilon^{\inf\{\alpha(\frac{1}{2}-\frac{1}{q}), 1-\alpha\}} \|\Psi\|_{H^1(\Omega)}.$$

We choose $\alpha = \frac{2q}{3q-2}$. We estimate the terms that appear in the calculation of the gradient of the approximate solution but do not appear in the above expression thanks to (4.12). We now use the equality

$$\int_\Omega f \Psi = \int_\Omega A \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right) \nabla_x \phi^\varepsilon(x) \nabla_x \Psi(x)$$

and we take

$$\Psi = \phi^\varepsilon - \left(\Phi + \sum_{i=1}^n \varepsilon \rho_{\varepsilon,\alpha} \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right) \right)$$

as test-function, to obtain

$$\|\nabla_x \phi^\varepsilon - \nabla_x \left(\Phi + \sum_{i=1}^n \varepsilon \rho_{\varepsilon,\alpha} \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right) \right)\|_{[L^2(\Omega)]^n} \leq C \varepsilon^{\frac{q-2}{3q-2}}.$$

□

Theorem 4.5 : *We suppose that Ω is a bounded domain in \mathbb{R}^n with lipschitzian boundary and Γ_0 is a union of connected components of $\partial\Omega$. Then, there exists γ in the interval $\left]0, \frac{1}{3}\right]$ depending on A , n and $\partial\Omega$ such that for any $f \in L^2(\Omega)$*

$$(4.14) \quad \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla_x \phi^\varepsilon - \nabla_x \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{[L^2(\Omega)]^n} \leq C \varepsilon^\gamma \|f\|_{L^2(\Omega)}$$

The constant C depends on n , A and $\partial\Omega$.

Proof :

Step one. We denote \mathcal{A} the square matrix associated to the homogenized operator (see [5]). Let $R > 0$ such that $\Omega \subset B(O; R)$ and $w \in H_0^1(B(O; R))$ the solution of the variational problem

$$\int_{B(O; R)} \mathcal{A} \nabla_x w \nabla_x v = \int_\Omega f v \quad \forall v \in H_0^1(B(O; R))$$

We have $w \in H^2(B(O; R))$ and $\|w\|_{H^2(B(O; R))} \leq C \|f\|_{L^2(\Omega)}$. The function Φ is solution of the homogenized problem (see [5])

$$\int_\Omega \mathcal{A} \nabla_x \Phi \nabla_x v = \int_\Omega f v \quad \forall v \in H_{\Gamma_0}^1(\Omega)$$

Hence $-\text{div}(\mathcal{A}(\nabla_x w - \nabla_x \Phi)) = 0$ in $H^{-1}(\Omega)$ and $w - \Phi$ belongs to $\mathcal{C}^\infty(\Omega)$. We also have

$$\forall i \in \{1, \dots, n\} \quad -\text{div} \left(\mathcal{A} \left(\nabla_x \frac{\partial w}{\partial x_i} - \nabla_x \frac{\partial \Phi}{\partial x_i} \right) \right) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega)$$

Lemma 2.2 of [7] gives us

$$\forall i \in \{1, \dots, n\} \quad \int_{\Omega} \rho^2 |\nabla_x (\frac{\partial w}{\partial x_i} - \frac{\partial \Phi}{\partial x_i})|^2 \leq C \int_{\Omega} |\frac{\partial w}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}|^2$$

From the estimates of w and $\Phi - w$, it follows: $\|\Phi\|_{H^1(\Omega)} + \|\rho \nabla_x \Phi\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}$.

Step two. Theorem A.3 of [3] asserts the existence of a real $q > 2$, depending on A and $\partial\Omega$, such that Φ belongs to $W^{1,q}(\Omega)$. Thanks to Proposition 4.4 we obtain Theorem 4.5. \square

Comments : In Theorems 4.1 and 4.2, if $\nabla\Phi = 0$ on $\partial\Omega$, the error estimate is of order ε .

In Theorem 4.1 if we replace $f \in L^2(\Omega)$ by $f \in L^p(\Omega)$ ($\frac{2n}{n+2} < p \leq 2$), then we prove that

$$\|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} + \|\nabla_x \phi^\varepsilon - \nabla_x \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\} \right)\|_{[L^2(\Omega)]^n} \leq C \varepsilon^{\inf\{\frac{1}{2}, n(\frac{1}{n} + \frac{1}{2} - \frac{1}{p})\}} \|f\|_{L^p(\Omega)}.$$

The constant depends on n, p, A and $\partial\Omega$.

The estimates obtained in Theorems 4.1, 4.2 and 4.5 remain true if we suppose that the coefficients of the square matrix A of problem (4.1) belong to $W^{1,\infty}(\Omega; L^\infty_{per}(Y))$. \square

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