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Exponential convergence of an observer based on partial field measurements for the wave equation

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Abstract: We analyze an observer strategy based on partial – i.e. in a subdomain – measurements of the solution of a wave equation, in order to compensate for unknown initial conditions. We prove the exponential convergence of this observer under a non-standard observability condition, whereas using measurements of the time-derivative of the solution would lead to a standard observability condition arising in stabilization and exact controlability. Nevertheless, we directly relate our specific condition to the classical geometric control condition.

Key-words: observer, filtering, nudging, observability condition, exponential stability, wave equation, second-order evolution equation

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Convergence exponentielle d'un observateur utilisant des mesures partielles pour l'équation des ondes

Résumé : Nous analysons un observateur formulé pour l'équation des ondes et utilisant des mesures de la solution dans un sous-domaine afin de compenser des conditions initiales inconnues. Nous démontrons la convergence exponentielle de cet observateur moyennant une condition d'observabilité non-standard, alors qu'une mesure de la dérivée en temps de la solution conduirait à la forme standard de la condition d'observabilité apparaissant en stabilisation et contrôlabilité exacte. Néanmoins, nous établissons un lien direct entre notre condition spécifique et la condition de contrôle géométrique classique.

Mots-clés : observateur, filtrage, nudging, condition d'observabilité, stabilité exponentielle, équation des ondes, équation du second ordre en temps

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ $(1 \leq n \leq 3)$ with a Lipschitz boundary $\partial\Omega$. We consider the following wave equation

$$\begin{cases} \ddot{u}(x,t) - \Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{1}$$

and we recall that if $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$ then $(u, \dot{u}) \in C((0,T); H_0^1(\Omega)) \times C((0,T); L^2(\Omega))$ for any $T > 0$, see e.g. [4].

In the observer framework of this paper, we assume that the initial condition (u_0, v_0) is unknown, albeit that we have a perfect measurement of the solution in $\omega \subset \Omega$ an open and non-empty subset of Ω. Our objective is to propose an observer which approximates the system (1) using this partial measurement

$$z(t) = u(\cdot,t)|_\omega, \qquad t \geq 0. \tag{2}$$

To that purpose we define the operator

$$\mathcal{L}_\omega : H^1(\omega) \to H_0^1(\Omega), \qquad \mathcal{L}_\omega \phi = \psi, \tag{3}$$

where ψ is the solution of the following elliptic equation

$$\begin{cases} \Delta\psi = 0, & \text{in } \Omega\backslash\omega \\ \psi = 0, & \text{on } \partial\Omega \\ \psi = \phi, & \text{in } \overline{\omega} \end{cases} \tag{4}$$

namely, a harmonic lifting operator. As a candidate observer, we will consider the following first-order system

$$\begin{cases} \dot{\hat{u}}(x,t) = \hat{v}(x,t) + \gamma\, \mathcal{L}_\omega\big(z(t) - \hat{u}(\cdot,t)|_\omega\big), & (x,t) \in \Omega \times (0,\infty) \\ \dot{\hat{v}}(x,t) - \Delta\hat{u}(x,t) = 0, & (x,t) \in \Omega \times (0,\infty) \\ \hat{u}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ \hat{u}(x,0) = \hat{v}(x,0) = 0, & x \in \Omega \end{cases} \tag{5}$$

with $\gamma > 0$ a gain parameter. Note that this system amounts to a modification of the wave equation written in a first-order form, with a correction term based on the discrepancy between the measurement and the observer primary variable. This strategy is the direct adaptation to the wave equation of the "Schur Displacement Feedback" filtering methodology originally proposed in [8] for elasticity-like formulations. Of course this type of approach can be categorized as a Luenberger observer [6] – also sometimes referred to as "nudging" [1] – applied on an evolution PDE.

2 Exponential convergence of the observer system

In order to establish that (5) is an adequate observer for (1), we should study the decay of the error

$$\begin{pmatrix} \tilde{u}(\cdot,t) \\ \tilde{v}(\cdot,t) \end{pmatrix} = \begin{pmatrix} u(\cdot,t) - \hat{u}(\cdot,t) \\ \dot{u}(\cdot,t) - \hat{v}(\cdot,t) \end{pmatrix}, \qquad t \geq 0.$$

It is easy to see that this error satisfies the following system

$$\begin{cases} \dot{\tilde{u}}(x,t) = \tilde{v}(x,t) - \gamma\,\mathcal{L}_\omega(\tilde{u}|_\omega)(x,t), & (x,t) \in \Omega \times (0,\infty) \\ \dot{\tilde{v}}(x,t) = \Delta\tilde{u}(x,t), & (x,t) \in \Omega \times (0,\infty) \\ \tilde{u}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ \tilde{u}(x,0) = u_0(x), \quad \tilde{v}(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{6}$$

and we will prove the exponential decay of the associated energy, namely,

$$\tilde{E}(t) = \frac{1}{2}\left(\int_\Omega |\nabla\tilde{u}(x,t)|^2 \,\mathrm{d}\Omega + \int_\Omega |\tilde{v}(x,t)|^2 \,\mathrm{d}\Omega\right). \tag{7}$$

Proposition 2.1. *Assume that we have the observability condition*

$$\int_0^T \|u(\cdot,t)\|^2_{H^1(\omega)}\,dt \geq C\left(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\right), \quad \forall (u_0,v_0) \in H^1_0(\Omega) \times L^2(\Omega), \tag{8}$$

for some $T > 0$, then there exist strictly positive constants M and μ such that

$$\tilde{E}(t) \leq M e^{-\mu t}\tilde{E}(0), \quad \forall t > 0. \tag{9}$$

Proof. Let us consider the space H^1_ω defined by $H^1(\omega)$ equipped with the norm

$$\|\phi\|^2_{H^1_\omega} = \langle \nabla(\mathcal{L}_\omega\phi),\, \nabla(\mathcal{L}_\omega\phi)\rangle_{L^2(\Omega)}.$$

It is straightforward to see that this norm is equivalent to the usual norm of $H^1(\omega)$, since $\mathcal{L}_\omega$ is a lifting operator with Dirichlet boundary conditions and we can then adapt the proof of the Poincaré inequality. We now define

$$B_0 : H^1_\omega \to H^1_0(\Omega), \qquad B_0\phi = \mathcal{L}_\omega\phi, \tag{10}$$

a bounded operator. The adjoint of B_0 (via Riesz representation) is then

$$B_0^* : H^1_0(\Omega) \to H^1_\omega, \qquad B_0^*\psi = \psi|_\omega, \tag{11}$$

as can be shown by a simple manipulation on the definition (10). We now introduce the state space operators

$$A : \mathcal{D}(\Delta) \times H^1_0(\Omega) \to H^1_0(\Omega) \times L^2(\Omega), \qquad A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \tag{12}$$

with $\mathcal{D}(\Delta) = \left\{\psi \in H^1_0(\Omega) \text{ s.t. } \Delta\psi \in L^2(\Omega)\right\}$, and

$$B : H^1_\omega \to H^1_0(\Omega) \times L^2(\Omega), \qquad B = \begin{pmatrix} B_0 \\ 0 \end{pmatrix}. \tag{13}$$

With these notations, the error system (6) can be rewritten as

$$\begin{cases} \begin{pmatrix} \dot{\tilde{u}}(t) \\ \dot{\tilde{v}}(t) \end{pmatrix} = (A - \gamma BB^*)\begin{pmatrix} \tilde{u}(t) \\ \tilde{v}(t) \end{pmatrix}, & t > 0 \\ \begin{pmatrix} \tilde{u}(0) \\ \tilde{v}(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \end{cases} \tag{14}$$

Therefore, the exponential decay of the energy (7) is equivalent to the following observability inequality, see e.g. Theorem 2.3 in [5]

$$\int_0^T \left\| B^* \binom{u(t)}{\dot{u}(t)} \right\|_{H^1_\omega}^2 \mathrm{d}t \geq C \left(\|u_0\|^2_{H^1_0(\Omega)} + \|v_0\|^2_{L^2(\Omega)} \right), \quad \forall (u_0, v_0) \in H^1_0(\Omega) \times L^2(\Omega), \tag{15}$$

where u is the solution of the original wave equation (1), and for some strictly positive constants T and C, and of course this condition is directly equivalent to (8). □

However, the observability condition (8) is somewhat non-standard, which has to do with the fact that we are considering an observer formulation instead of a stabilization (or exact controllability) problem. Nevertheless, the following proposition directly relates our observability condition with classical results.

Proposition 2.2. *Assume that the geometric control condition of Bardos, Lebeau and Rauch [2] is satisfied for some $\breve{\omega}$ strict subset of ω such that dist$(\Omega\backslash\omega, \breve{\omega}) > 0$. Then the observability condition* (8) *holds for some time $T > 0$.*

Proof. Since the geometric control condition holds, we have the classical observability condition [2]

$$\int_0^{\breve{T}} \int_{\breve{\omega}} |\dot{u}(x,t)|^2 \, \mathrm{d}\Omega \mathrm{d}t \geq C \left(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)} \right), \quad \forall (u_0, v_0) \in H^1_0(\Omega) \times L^2(\Omega), \tag{16}$$

for some time $\breve{T} > 0$. We will show that this entails (8) by an argument inspired from [9]. Let $\psi \in C_c^\infty(\overline{\Omega})$ be a cutoff function defined by

$$\psi(x) = \begin{cases} 0, & \text{if } x \in \Omega\backslash\omega \\ 1, & \text{if } x \in \breve{\omega} \end{cases}$$

and $0 \leq \psi(x) \leq 1$ for every $x \in \overline{\Omega}$. Denote also $\phi(t) = t^2(\breve{T} - t)^2$. Then, by repeated integrations by parts we obtain

$$\begin{aligned} 0 &= \int_0^{\breve{T}} \int_\omega \phi\psi(\ddot{u} - \Delta u) u \, \mathrm{d}\Omega \mathrm{d}t \\ &= \int_0^{\breve{T}} \int_\omega \ddot{\phi}\psi \frac{|u|^2}{2} \, \mathrm{d}\Omega \mathrm{d}t - \int_0^{\breve{T}} \int_\omega \phi\psi |\dot{u}|^2 \, \mathrm{d}\Omega \mathrm{d}t + \int_0^{\breve{T}} \int_{\partial\omega} \phi \frac{\partial\psi}{\partial n} \frac{|u|^2}{2} \, \mathrm{d}\Gamma \mathrm{d}t \\ &\qquad - \int_0^{\breve{T}} \int_\omega \phi \Delta\psi \frac{|u|^2}{2} \, \mathrm{d}\Omega \mathrm{d}t + \int_0^{\breve{T}} \int_\omega \phi\psi |\nabla u|^2 \, \mathrm{d}\Omega \mathrm{d}t. \end{aligned}$$

The definition of ψ entails $(\partial\psi/\partial n)|_{\partial\omega} = 0$, hence

$$\int_0^{\breve{T}} \int_\omega \phi\psi |\dot{u}|^2 \, \mathrm{d}\Omega \mathrm{d}t = \int_0^{\breve{T}} \int_\omega \phi\psi |\nabla u|^2 \, \mathrm{d}\Omega \mathrm{d}t + \int_0^{\breve{T}} \int_\omega \ddot{\phi}\psi \frac{|u|^2}{2} \, \mathrm{d}\Omega \mathrm{d}t - \int_0^{\breve{T}} \int_\omega \phi \Delta\psi \frac{|u|^2}{2} \, \mathrm{d}\Omega \mathrm{d}t.$$

The above identity combined with the properties of the cutoff functions ϕ and ψ provides, for any strictly positive ε, the existence of a constant $C > 0$ such that

$$C \int_{\varepsilon}^{\breve{T}-\varepsilon} \int_{\breve{\omega}} |\dot{u}|^2 \, \mathrm{d}\Omega \mathrm{d}t \leq \int_{0}^{\breve{T}} \int_{\omega} |\nabla u|^2 \, \mathrm{d}\Omega \mathrm{d}t + \int_{0}^{\breve{T}} \int_{\omega} |u|^2 \, \mathrm{d}\Omega \mathrm{d}t. \tag{17}$$

Substituting $\breve{T} + 2\varepsilon$ for $\breve{T}$ in all the above computations gives

$$C \int_{\varepsilon}^{\breve{T}+\varepsilon} \int_{\breve{\omega}} |\dot{u}|^2 \, \mathrm{d}\Omega \mathrm{d}t \leq \int_{0}^{\breve{T}+2\varepsilon} \|u(\cdot,t)\|^2_{H^1(\omega)} \, \mathrm{d}t.$$

We proceed by making the change of variable $\tau = t - \varepsilon$ in the left-hand side integral, yielding

$$C \int_{0}^{\breve{T}} \int_{\breve{\omega}} |\dot{u}(x,\tau+\varepsilon)|^2 \, \mathrm{d}\Omega \mathrm{d}\tau \leq \int_{0}^{\breve{T}+2\varepsilon} \|u\|^2_{H^1(\omega)} \, \mathrm{d}t. \tag{18}$$

Noting that $u(x, t+\varepsilon)$ satisfies the wave equation with initial data $(u(x,\varepsilon), \dot{u}(x,\varepsilon))$ and applying (16) with this shifted solution, we obtain

$$\int_{0}^{\breve{T}} \int_{\breve{\omega}} |\dot{u}(x,t+\varepsilon)|^2 \, \mathrm{d}\Omega \mathrm{d}t \geq C \left(\|u(\varepsilon)\|^2_{H^1(\Omega)} + \|\dot{u}(\varepsilon)\|^2_{L^2(\Omega)} \right). \tag{19}$$

Combining (18), (19) and the the fact that the energy of the solution of the wave equation is exactly conserved over time, we have the observability inequality (8) for any T strictly greater than $\breve{T}$. □

Note that this proof requires a geometric control condition slightly stronger than that directly associated with the measurement domain ω, since $\breve{\omega}$ is a strict subset. However, micro-local analysis – see [3], albeit based for our specific case on the H^1 norm – may provide an alternative approach to circumvent this restriction.

As a final remark, we emphasize that the standard observability condition (16) would be directly applicable to establish the exponential convergence of an observer formulation based on the measurement of the "velocity" $\dot{u}$ in the subset, as used in "Direct Velocity Feedback" Luenberger observer strategies [7]. In our case, the specific difficulty comes from the fact that we assume that we measure u itself, and that we do not wish to differentiate this measurement in time, typically to avoid amplifying the measurement errors which always arise in practice. Note that in our approach such errors – disregarded in this short note – would simply entail an additional source term in the observer equations.

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