BAD BOUNDARY BEHAVIOR IN STAR INVARIANT SUBSPACES II

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ABSTRACT. We continue our study begun in [HR11] concerning the radial growth of functions in the model spaces $(IH^2)^\perp$.

1. INTRODUCTION

Suppose $I = BS_\mu$ is an inner function with Blaschke factor B, with zeros $\{\lambda_n\}_{n\geq 1}$ in the open unit disk $\mathbb{D}$ repeated according to multiplicity, and singular inner factor S_μ with associated positive singular measure μ on the unit circle $\mathbb{T}$. The following result was shown by Frostman in 1942 for Blaschke products (see [Fro42] or [CL66]) and by Ahern-Clark for general inner functions [AC71, Lemma 3].

Theorem 1.1 (Frostman, 1942; Ahern-Clark, 1971). *Let $\zeta \in \mathbb{T}$ and I be inner with $\mu(\{\zeta\}) = 0$. Then the following assertions are equivalent.*

(1) *Every divisor of I has a radial limit of modulus one at ζ.*
(2) *Every divisor of I has a radial limit at ζ.*
(3) *The following condition holds*

$$\sum_{n\geq 1} \frac{1-|\lambda_n|}{|\zeta-\lambda_n|} + \int_{\mathbb{T}} \frac{1}{|\zeta - e^{it}|} d\mu(e^{it}) < \infty. \tag{1.2}$$

Based on a stronger condition than the above, Ahern and Clark [AC70] were able to characterize "good" non-tangential boundary behavior of functions in the model spaces $(IH^2)^\perp$ of the classical Hardy space H^2 (see [Nik86] for a very complete treatment of model spaces).

Theorem 1.3 ([AC70]). *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n\geq 1}$ and associated singular measure μ. For $\zeta \in \mathbb{T}$, the following are equivalent:*

(1) *Every $f \in (IH^2)^\perp$ has a radial limit at ζ.*
(2) *The following condition holds*

$$\sum_{n\geq 1} \frac{1-|\lambda_n|}{|\zeta-\lambda_n|^2} + \int_{\mathbb{T}} \frac{1}{|\zeta - e^{it}|^2} d\mu(e^{it}) < \infty. \tag{1.4}$$

In this paper, we will study what happens when we are somewhere in between the Frostman condition (1.2) and the Ahern-Clark condition (1.4). In order to do so we will introduce an auxiliary function. Let $\varphi : (0, +\infty) \to \mathbb{R}^+$ be a positive increasing function such that

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(1) $x \to \dfrac{\varphi(x)}{x}$ is bounded,

(2) $x \longmapsto \dfrac{\varphi(x)}{x^2}$ is decreasing,

(3) $\varphi(x) \asymp \varphi(x + o(x))$, $x \downarrow 0$.

Such a function φ will be called *admissible*. One can check that functions like $\varphi(x) = x^p, 1 \leq p < 2$, and $\varphi(x) = x^p \log(1/x)$, $1 < p < 2$, are admissible. Our main result is the following.

Theorem 1.5. *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n\geq 1}$ and associated singular measure μ, φ an admissible function, and $\zeta \in \mathbb{T}$. If*

$$\sum_{n\geq 1} \frac{1 - |\lambda_n|}{\varphi(|\zeta - \lambda_n|)} + \int_{\mathbb{T}} \frac{1}{\varphi(|\zeta - e^{it}|)} d\mu(e^{it}) < \infty, \tag{1.6}$$

then every $f \in (IH^2)^\perp$ satisfies

$$|f(r\zeta)| \lesssim \frac{\sqrt{\varphi(1-r)}}{1-r}. \tag{1.7}$$

When $\varphi(x) = x$ then we are in the Frostman situation (1.2) and no restriction is given for the growth of f since generic functions in H^2 satisfy the growth condition

$$|f(r\zeta)| = o\Big(\frac{1}{\sqrt{1-r}}\Big)$$

On the other hand, when $\varphi(x) = x^2$ we reach the Ahern-Clark situation (1.4) . For other φ such as $\varphi(x) = x^{3/2}$ or perhaps $\varphi(x) = x^2 \log(1/x)$ we get that even though functions in $(IH^2)^\perp$ can be poorly behaved (as in the title of this paper), the growth is controlled.

There is some history behind these types of problems. When $\varphi(x) = x^{2N+2}$, where $N = 0, 1, 2, \cdots$, Ahern and Clark [AC70] showed that (1.6) is equivalent to the condition that $f^{(j)}$, $0 \leq j \leq N$, have radial limits at ζ for every $f \in (IH^2)^\perp$. When $\varphi(x) = x^p$, $p \in (1, \infty)$, Cohn [Coh86] showed that (1.6) is equivalent to the condition that every $f \in H^q \cap I\overline{H_0^q}$, where $q = p(p-1)^{-1}$, has a finite radial limit at ζ.

Why did we write this second paper? In [HR11] we discussed controlled growth of functions from $(BH^2)^\perp$, where B is a Blaschke product not satisfying the condition (1.4) of the Ahern-Clark theorem. We have a general result but stated in very different terms, and using very different techniques, than the paper here. In particular, in [HR11] we obtain two-sided estimates for the reproducing kernels which yields more precise results. The results presented here are one-sided estimates but are for general inner functions and not just Blaschke products.

2. Proof of the main result

It is well known that $(IH^2)^\perp$ is a reproducing kernel Hilbert space with kernel function

$$k_\lambda^I(z) := \frac{1 - \overline{I(\lambda)}I(z)}{1 - \overline{\lambda}z}.$$

It suffices to prove Theorem 1.5 for $\zeta = 1$. If $\|\cdot\|$ denotes the norm in H^2, the estimate in (1.5) follows from the following result along with the obvious estimate

$$|f(r)| \leq \|f\|\|k_r^I\|, \quad f \in (IH^2)^\perp, \quad r \in (0,1).$$

Theorem 2.1. *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n\geq 1}$ and associated singular measure μ and φ be an admissible function. If*

$$\sum_{n\geq 1} \frac{1-|\lambda_n|}{\varphi(|1-\lambda_n|)} + \int_{\mathbb{T}} \frac{1}{\varphi(|1-e^{it}|)} d\mu(e^{it}) < \infty, \tag{2.2}$$

then

$$\|k_r^I\|^2 \lesssim \frac{\varphi(1-r)}{(1-r)^2}. \tag{2.3}$$

Proof. Our first observation is that since $x \longmapsto \varphi(x)/x$ is bounded, (2.2) implies condition (1.2). By Theorem 1.1 this implies that $\lim_{r\to 1^-} |B(r)| = \lim_{r\to 1^-} |S_\mu(r)| = 1$. Hence

$$\|k_r^I\|^2 = \frac{1-|I(r)|^2}{1-r^2} = \frac{1-\exp(\log(|I(r)|^2))}{1-r^2} = \frac{1-\exp(\log(|B(r)|^2 + \log|S_\mu(r)|^2))}{1-r^2},$$

and since $\log|B(r)| \to 0$ and $\log|S_\mu(r)| \to 0$ when $r \to 1$, we get

$$\begin{aligned} \|k_r^I\|^2 &= \frac{1-\exp(\log|B(r)|^2 + \log|S_\mu(r)|^2)}{1-r^2} \\ &= \frac{1-\Big(1+\Big(\log|B(r)|^2 + \log|S_\mu(r)|^2\Big) + o\Big(\log|B(r)|^2 + \log|S_\mu(r)|^2\Big)\Big)}{1-r^2} \\ &\sim \frac{\log|B(r)|^{-2} + \log|S_\mu(r)|^{-2}}{1-r^2}. \end{aligned}$$

Thus to prove the estimate in (2.3) we need to prove

$$\frac{\log|B(r)|^{-2}}{1-r^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2} \tag{2.4}$$

and

$$\frac{\log|S_\mu(r)|^{-2}}{1-r^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}. \tag{2.5}$$

Case 1: the Blaschke product B.

First note that from the Frostman condition (1.2) we get

$$\frac{1-|\lambda_n|}{|1-\lambda_n|} \longrightarrow 0. \tag{2.6}$$

In particular, from a certain index n_0 on the points λ_n, $n \geq n_0$, will be pseudohyperbolically far from the radius $[0,1)$, i.e., there is a δ such that for every $n \geq n_0$ and $r \in [0,1)$,

$$|b_{\lambda_n}(r)| \geq \delta.$$

This implies

$$\log \frac{1}{|b_{\lambda_n}(r)|^2} \asymp 1 - |b_{\lambda_n}(r)|^2.$$

A well known calculation shows that

$$1-|b_{\lambda_n}(r)|^2 = \frac{(1-r^2)(1-|\lambda_n|^2)}{|1-r\overline{\lambda_n}|^2}.$$

Thus

$$\frac{\log|B(r)|^{-2}}{1-r^2} = \frac{1}{1-r^2}\sum_{n\geq 1}\log\frac{1}{|b_{\lambda_n}(z)|^2} \asymp \sum_{n\geq 1}\frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}r|^2}. \tag{2.7}$$

Now let $\lambda_n = r_n e^{i\theta_n}$. We need the following two easy estimates:

$$|1-\rho e^{i\theta}|^2 \asymp (1-\rho)^2+\theta^2, \quad \rho\approx 1, \theta\approx 0. \tag{2.8}$$

$$(|z|^2+|w|^2)^{1/2} \asymp |z|+|w|, \quad z,w\in\mathbb{C}. \tag{2.9}$$

In particular, $|1-\lambda_n|^2 \asymp (1-r_n)^2+\theta_n^2$. We now remember condition (2.6) which implies that $1-r_n = 1-|\lambda_n| = o(|1-\lambda_n|) = o((1-r_n)+\theta_n)$ so that necessarily $1-r_n = o(\theta_n)$. Hence

$$|1-\overline{\lambda}_n r|^2 \asymp (1-r_n r)^2+\theta_n^2 = (1-r_n+r_n(1-r))^2+\theta_n^2 \asymp (1-r)^2+\theta_n^2.$$

The estimate in (2.7) yields

$$\begin{aligned}\frac{\log|B(r)|^{-2}}{1-r^2} &\asymp \sum_{n\geq 1}\frac{1-|\lambda_n|^2}{|1-\overline{\lambda}_n r|^2} \asymp \sum_{n\geq 1}\frac{1-r_n}{(1-r)^2+\theta_n^2} \asymp \sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^2} + \sum_{\{n:1-r\geq\theta_n\}}\frac{1-r_n}{(1-r)^2}\\ &= \sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^2} + \frac{1}{(1-r)^2}\sum_{\{n:1-r\geq\theta_n\}}(1-r_n).\end{aligned} \tag{2.10}$$

Let us discuss each summand in (2.10) individually. For the first, we use the fact that φ is admissible and so $\varphi(\theta)\asymp\varphi(|1-e^{i\theta}|)$ to get

$$\begin{aligned}\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^2} &= \sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\sqrt{\varphi(\theta_n)}\theta_n^2/\sqrt{\varphi(\theta_n)}}\\ &\leq \underbrace{\left(\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\varphi(\theta_n)}\right)^{1/2}}_{\text{bounded by assumption}}\left(\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^4/\varphi(\theta_n)}\right)^{1/2}\\ &\lesssim \left(\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\varphi(\theta_n)(\theta_n^2/\varphi(\theta_n))^2}\right)^{1/2}.\end{aligned}$$

By assumption, $x\to\varphi(x)/x^2$ is decreasing. Hence we can bound $\theta_n^2/\varphi(\theta_n)$ below in this last sum by $(1-r)^2/\varphi(1-r)$. Hence

$$\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}\left(\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\varphi(\theta_n)}\right)^{1/2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}.$$

For the second sum in (2.10) we have

$$\begin{aligned}\sum_{\{n:1-r\geq\theta_n\}}(1-r_n) &= \sum_{\{n:1-r\geq\theta_n\}}(1-r_n)\frac{\sqrt{\varphi(\theta_n)}}{\sqrt{\varphi(\theta_n)}}\\ &\leq \underbrace{\left(\sum_{\{n:1-r\geq\theta_n\}}\frac{(1-r_n)}{\varphi(\theta_n)}\right)^{1/2}}_{\text{bounded by assumption}}\left(\sum_{\{n:1-r\geq\theta_n\}}(1-r_n)\varphi(\theta_n)\right)^{1/2}\\ &\lesssim \sqrt{\varphi(1-r)}\left(\sum_{\{n:1-r\geq\theta_n\}}(1-r_n)\right)^{1/2},\end{aligned}$$

where we have used the fact that φ is increasing. Dividing through the square root of the sum, this last inequality (and then squaring) implies

$$\sum_{\{n:1-r\geq\theta_n\}}(1-r_n)\lesssim\varphi(1-r).$$

This verifies (2.4).

<u>Case 2: the singular inner factor S_μ.</u>

This case is very similar to the first case. Indeed,

$$\frac{\log|S_\mu(r)|^{-2}}{1-r^2}=2\int_{\mathbb{T}}\frac{1}{|1-re^{i\theta}|^2}d\mu(e^{i\theta})\asymp\int_{\mathbb{T}}\frac{1}{(1-r)^2+\theta^2}d\mu(e^{i\theta})$$

where we have again used (2.8). As in the Blaschke situation we split the integral into two parts depending on which term in the denominator dominates:

$$\begin{aligned}\frac{\log|S_\mu(r)|^{-2}}{1-r^2} &\lesssim \int_{\{\theta:1-r\leq\theta\}}\frac{1}{(1-r)^2+\theta^2}d\mu(e^{i\theta})+\int_{\{\theta:1-r\geq\theta\}}\frac{1}{(1-r)^2+\theta^2}d\mu(e^{i\theta})\\ &\asymp \int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^2}d\mu(e^{i\theta})+\frac{1}{(1-r)^2}\int_{\{\theta:1-r\geq\theta\}}d\mu(e^{i\theta}).\end{aligned}\tag{2.11}$$

Let us consider the first integral.

$$\begin{aligned}\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^2}d\mu(e^{i\theta}) &= \int_{\{\theta:1-r\leq\theta\}}\frac{1}{\sqrt{\varphi(\theta)}\theta^2/\sqrt{\varphi(\theta)}}d\mu(e^{i\theta})\\ &\leq \left(\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\varphi(\theta)}d\mu(e^{i\theta})\right)^{1/2}\left(\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^4/\varphi(\theta)}d\mu(e^{i\theta})\right)^{1/2}.\end{aligned}$$

Note that $|1-e^{i\theta}|\asymp\theta$. Then using the hypothesis of admissibility we have $\varphi(\theta)\asymp\varphi(|1-e^{i\theta}|)$ and so

$$\int\frac{1}{\varphi(\theta)}d\mu(e^{i\theta})\asymp\int\frac{1}{\varphi(|1-e^{i\theta}|)}d\mu(e^{i\theta})$$

which is bounded by assumption. Hence, by the Cauchy-Schwarz inequality,

$$\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^2}d\mu(e^{i\theta})\lesssim\left(\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^4/\varphi(\theta)}d\mu(e^{i\theta})\right)^{1/2}=\left(\int_{\{\theta:1-r\leq\theta\}}\frac{\varphi^2(\theta)}{\varphi(\theta)\theta^4}d\mu(e^{i\theta})\right)^{1/2}.$$

Now using the fact that $x \longrightarrow \varphi(x)/x^2$ is decreasing we obtain $\varphi^2(\theta)/\theta^4 \leq (\varphi(1-r))^2/(1-r)^4$. Hence

$$\int_{\{\theta:1-r\leq\theta\}} \frac{1}{\theta^2} d\mu(e^{i\theta}) \lesssim \frac{\varphi(1-r)}{(1-r)^2}\left(\int_{\{\theta:1-r\leq\theta\}} \frac{1}{\varphi(\theta)} d\mu(e^{i\theta})\right)^{1/2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}.$$

We turn to the second integral in (2.11) to get

$$\begin{aligned}\int_{\{\theta:1-r\geq\theta\}} d\mu(e^{i\theta}) &= \int_{\{\theta:1-r\geq\theta\}} \frac{\sqrt{\varphi(\theta)}}{\sqrt{\varphi(\theta)}} d\mu(e^{i\theta}) \\ &\leq \left(\int_{\{\theta:1-r\geq\theta\}} \varphi(\theta) d\mu(e^{i\theta})\right)^{1/2}\left(\int_{\{\theta:1-r\geq\theta\}} \frac{1}{\varphi(\theta)} d\mu(e^{i\theta})\right)^{1/2}.\end{aligned}$$

We have already seen above that the second factor above is bounded by assumption. Using the fact that φ is increasing we get

$$\int_{\{\theta:1-r\geq\theta\}} d\mu(e^{i\theta}) \lesssim \left(\int_{\{\theta:1-r\geq\theta\}} \varphi(\theta) d\mu(e^{i\theta})\right)^{1/2} \leq \sqrt{\varphi(1-r)}\left(\int_{\{\theta:1-r\geq\theta\}} d\mu(e^{i\theta})\right)^{1/2}.$$

Dividing through by the integral (and then squaring), we obtain

$$\int_{\{\theta:1-r\geq\theta\}} d\mu(e^{i\theta}) \lesssim \varphi(1-r),$$

which verifies (2.5). ■

3. AN EXAMPLE

The Blaschke situation was discussed in [HR11] where we obtained two-sided estimates for the reproducing kernels. It can be shown with concrete examples that the estimates from Theorem 2.1 are in general weaker than those obtained in [HR11] for Blaschke products.

Let us discuss the simplest case, in fact close enough to a Blaschke product, that a singular inner function S_μ with a discrete measure μ. Let

$$\mu = \sum_{n\geq 1} \alpha_n \delta_{\zeta_n},$$

where $\delta_{\zeta_n} \in \mathbb{T}$ and α_n are positive numbers with $\sum_n \alpha_n < \infty$ guaranteeing that μ is a finite measure on $\mathbb{T}$. Let us fix

$$\zeta_n = e^{i\theta_n} = e^{i/n}, \quad \alpha_n = \frac{1}{n^{1+\varepsilon}}, \quad n = 1, 2, \ldots.$$

Also let $\varphi(t) = t^\gamma$ which defines an admissible function for $1 < \gamma < 2$. In order to have condition (2.2) it is necessary and sufficient to have

$$\sum_n \alpha_n \frac{1}{\varphi(|1-e^{i\theta_n}|)} \simeq \sum_n \frac{1}{n^{1+\varepsilon}} \frac{1}{\varphi(1/n)} \simeq \sum_n \frac{n^\gamma}{n^{1+\varepsilon}} = \sum_n \frac{1}{n^{1+\varepsilon-\gamma}} < \infty$$

which is equivalent to $\gamma < \varepsilon$. We suppose that

$$1 < \varepsilon < 2. \tag{3.1}$$

By Theorem 2.1 we deduce that

$$\|k_r^I\|^2 \lesssim \frac{\varphi(1-r)}{(1-r)^2} = \left(\frac{1}{1-r}\right)^{2-\gamma}.$$

In this situation we have

$$|f(r)| \lesssim \frac{1}{(1-r)^{1-\gamma/2}}, \quad f \in (S_\mu H^2)^\perp,$$

which is slower growth than the standard estimate

$$|f(r)| \lesssim \frac{1}{(1-r)^{1/2}}, \quad f \in H^2.$$

In this situation, it is actually possible to get a double-sided estimate for the reproducing kernel: since φ is admissible, Theorem 1.1 implies that $I(r) \longrightarrow \eta \in \mathbb{T}$ when $r \to 1^-$. In particular for $r \in (0,1)$, this implies that

$$|I(r)| = \exp\left(-\sum_n \alpha_n \frac{1-r^2}{|\zeta_n - r|^2}\right) \sim 1 - \sum_n \alpha_n \frac{1-r^2}{|\zeta_n - r|^2}.$$

Let us consider the reproducing kernel of $(S_\mu H^2)^\perp$ at $r = \rho_N = 1 - 2^{-N}$. Indeed,

$$\begin{aligned}
\|k^I_{\rho_N}\|^2 &= \frac{1-|I(\rho_N)|^2}{1-\rho_N^2} \asymp 2^N \left(1 - \exp\left(-\sum_n \alpha_n \frac{1-\rho_N^2}{|\zeta_n - \rho_N|^2}\right)\right) \\
&\asymp 2^N \left(1 - \left(1 - \sum_n \alpha_n \frac{1/2^N}{|\zeta_n - \rho_N|^2}\right)\right) \\
&\asymp \sum_n \frac{\alpha_n}{|\zeta_n - \rho_N|^2}.
\end{aligned}$$

Now using (2.8)

$$|\zeta_n - \rho_N|^2 \asymp \frac{1}{n^2} + \frac{1}{2^{2N}},$$

and so

$$\begin{aligned}
\|k^I_{\rho_N}\|^2 &\asymp \sum_n \frac{\alpha_n}{1/n^2 + 1/2^{2N}} = \sum_{n \le 2^N} \frac{\alpha_n}{1/n^2} + \sum_{n > 2^N} \frac{\alpha_n}{1/2^{2N}} \\
&\asymp \sum_{n \le 2^N} \frac{n^2}{n^{1+\varepsilon}} + 2^{2N} \sum_{n > 2^N} \frac{1}{n^{1+\varepsilon}} \asymp 2^{(2-\varepsilon)N} \\
&= \left(\frac{1}{1-\rho_N}\right)^{2-\varepsilon}
\end{aligned}$$

or, equivalently,

$$\|k^I_{\rho_N}\| \asymp \left(\frac{1}{1-\rho_N}\right)^{1-\varepsilon/2} \tag{3.2}$$

(the estimate extends to the whole radius). As a consequence, the estimate from Theorem 2.1 is not optimal, though it is possible to come closer to it by choosing e.g., $\varphi(t) = t^\varepsilon / \log^{1+\gamma}(1/t)$, $\gamma > 0$.

4. A LOWER ESTIMATE

We finish the paper with a construction of an $f \in (S_\mu H^2)^\perp$, with μ the discrete measure discussed in the previous section, getting close to the growth given by the norm of the reproducing kernels thoughout the whole radius $(0,1)$. As in [HR11] our construction will be based on unconditional sequences. We need to recall some material on generalized interpolation in Hardy spaces for which we refer the reader to [Nik02, Section C3]. Let $I = \prod_n I_n$ be a factorization of an inner function I into inner functions I_n, $n \in \mathbb{N}$. The sequence $\{I_n\}_{n\geq 1}$ satisfies the generalized Carleson condition, sometimes called the Carleson-Vasyunin condition, which we will write $\{I_n\}_{n\geq 1} \in (CV)$, if there is a $\delta > 0$ such that

$$|I(z)| \geq \delta \inf_{n\geq 1} |I_n(z)|, \quad z \in \mathbb{D}. \tag{4.1}$$

In the special case of a Blaschke product $B = B_\Lambda$ with simple zeros $\Lambda = \{\lambda_n\}_{n\geq 1}$ and $I_n = b_{\lambda_n}$, this is equivalent to the well-known Carleson condition $\inf_n |B_{\Lambda\setminus\{\lambda_n\}}(\lambda_n)| \geq \delta > 0$.

If $\{I_n\}_{n\geq 1} \in (CV)$ then $\{(I_n H^2)^\perp\}_{n\geq 1}$ is an unconditional basis for $(IH^2)^\perp$ meaning that every $f \in (IH^2)^\perp$ can be written uniquely as

$$f = \sum_{n\geq 1} f_n, \quad f_n \in (I_n H^2)^\perp,$$

with

$$\|f\|^2 \asymp \sum_{n\geq 1} \|f_n\|^2.$$

In our situation we have $I = S_\mu$ and

$$I_n = e^{\alpha_n \frac{z+\zeta_n}{z-\zeta_n}}.$$

The corresponding spaces $(I_n H^2)^\perp$ are known to be isometrically isomorphic to the Paley-Wiener space of analytic functions of exponential type $\alpha_n/2$ and square integrable on the real axis. In this situation a sufficient condition for (4.1) is known:

$$\sup_{n\geq 1} \sum_{k\neq n} \frac{\mu(\{\zeta_n\})\mu(\{\zeta_k\})}{|\zeta_n - \zeta_k|^2} < \infty$$

(see [Nik86, Corollary 6, p. 247]). So, since $\varepsilon > 1$ by (3.1), we have

$$\sup_{n\geq 1} \sum_{k\neq n} \frac{1/n^{1+\varepsilon} 1/k^{1+\varepsilon}}{|1/n - 1/k|^2} = \sup_{n\geq 1} \sum_{k\neq n} \frac{1/n^{\varepsilon-1} 1/k^{\varepsilon-1}}{|n-k|^2} \leq \frac{\pi^2}{3} < \infty.$$

Hence $(IH^2)^\perp$ is an ℓ^2-sum of Paley-Wiener spaces (each of which possesses for instance the harmonic unconditional basis). In particular, picking

$$\lambda_n := r_n \zeta_n = r_n e^{i/n}, \quad r_n = 1 - \frac{1}{n},$$

the sequence $\{K_n\}_{n\geq 1}$, where

$$K_n = \frac{k_{\lambda_n}^{I_n}}{\|k_{\lambda_n}^{I_n}\|} \in (I_n H^2)^\perp,$$

is an unconditional sequence in $(IH^2)^\perp$. Observe that $\Lambda = \{\lambda_n\}_{n\geq 1}$ is *not* a Blaschke sequence. We can introduce the family of functions

$$f_\beta := \sum_{n\geq 1} \beta_n K_n$$

where $\|f_\beta\|^2 \asymp \sum_{n\geq 1} |\beta_n|^2 < \infty$. Let us estimate the norms $\|k_{\lambda_n}^{I_n}\|$. First observe that

$$\alpha_n \frac{\lambda_n + \zeta_n}{\lambda_n - \zeta_n} = \alpha_n \frac{r_n + 1}{r_n - 1} = \frac{1}{n^{1+\varepsilon}} \frac{2 - 1/n}{-1/n} = -\frac{2-1/n}{n^\varepsilon} \longrightarrow 0, \quad n \to \infty.$$

Hence

$$\begin{aligned} \|k_{\lambda_n}^{I_n}\|^2 &= \frac{1 - |I_n(\lambda_n)|^2}{1 - r_n^2} \asymp \frac{1 - |I_n(\lambda_n)|}{1 - r_n} = \frac{1 - \exp\Big(\log|I_n(\lambda_n)|\Big)}{1 - r_n} \\ &= \frac{1 - \exp\Big(\alpha_n \frac{\lambda_n + \zeta_n}{\lambda_n - \zeta_n}\Big)}{1 - r_n} \sim \frac{1 - \Big(1 + \alpha_n \frac{r_n+1}{r_n-1}\Big)}{1 - r_n} \\ &\sim \frac{2\alpha_n}{(1 - r_n)^2}, \end{aligned}$$

so that

$$\|k_{\lambda_n}^{I_n}\| \asymp \sqrt{\frac{\alpha_n}{(1-r_n)^2}} = \frac{\sqrt{n^{-(1+\varepsilon)}}}{1/n} = n^{1-1/2-\varepsilon/2} = n^{(1-\varepsilon)/2}.$$

Observe now that the λ_n's belong to a Stolz domain with vertex at 1. Indeed,

$$1 - |\lambda_n| = 1 - r_n = 1/n \simeq |1 - \zeta_n| \asymp |1 - \lambda_n|$$

(this follows from (2.8)). For fixed $\beta = \{\beta_n\}_{n\geq 1}$ with $\beta_n \geq 0$ we compute

$$\text{Re} f_\beta(\lambda_N) \simeq \sum_{n\geq 1} \beta_n n^{(\varepsilon-1)/2} \text{Re} \frac{1 - \overline{I_n(\lambda_n)} I_n(\lambda_N)}{1 - \overline{\lambda_n}\lambda_N}.$$

We have already seen that $\mathbb{R} \ni I_n(\lambda_n) \longrightarrow 1$, $n \to \infty$, and

$$I_n(\lambda_n) \sim 1 - \alpha_n \frac{1 + r_n}{1 - r_n} \sim 1 - \frac{2}{n^\varepsilon}.$$

We have to consider

$$\alpha_n \frac{\lambda_N + \zeta_n}{\lambda_N - \zeta_n}.$$

For n or N big enough, $\text{Re}(\lambda_N + \zeta_n) \asymp \text{Im}(\lambda_N + \zeta_n) \asymp |\lambda_N + \zeta_n| \asymp 1$. We thus have to consider the denominator. We observe that by Lemma 2.8

$$(4.2) \quad |\lambda_N - \zeta_n| = |1 - \overline{\zeta_n}\lambda_N| \asymp (1 - r_N) + \left|\frac{1}{n} - \frac{1}{N}\right| = \frac{1}{N} + \left|\frac{1}{n} - \frac{1}{N}\right| \asymp \begin{cases} \frac{1}{n} & \text{if } n \leq N \\ \frac{1}{N} & \text{if } n > N \end{cases}$$

As a consequence,

$$\alpha_n \frac{\lambda_N + \zeta_n}{\lambda_N - \zeta_n} \longrightarrow 0, \quad n \to \infty.$$

Again:

$$I_n(\lambda_N) \sim 1 + \alpha_n \frac{\lambda_N + \zeta_n}{\lambda_N - \zeta_n}.$$

Hence

$$\begin{aligned} 1-\overline{I_n(\lambda_n)}I_n(\lambda_N) &\sim 1-\left(1+\alpha_n\frac{r_n+1}{r_n-1}\right)\left(1+\alpha_n\frac{\lambda_N+\zeta_n}{\lambda_N-\zeta_n}\right)\sim\alpha_n\frac{1+r_n}{1-r_n}+\alpha_n\frac{\zeta_n+\lambda_N}{\zeta_n-\lambda_N}\\ &= \alpha_n\left(\frac{1+r_n}{1-r_n}+\frac{\zeta_n+\lambda_N}{\zeta_n-\lambda_N}\right)=\alpha_n\frac{(1+r_n)(\zeta_n-\lambda_N)+(1-r_n)(\zeta_n+\lambda_N)}{(1-r_n)(\zeta_n-\lambda_N)}\\ &= 2\alpha_n\frac{\zeta_n-r_n\lambda_N}{(1-r_n)(\zeta_n-\lambda_N)}=2\alpha_n\zeta_n\frac{1-\overline{\zeta_n}r_n\lambda_N}{(1-r_n)(\zeta_n-\lambda_N)}\\ &= 2\alpha_n\zeta_n\frac{1-\overline{\lambda_n}\lambda_N}{(1-r_n)(\zeta_n-\lambda_N)}. \end{aligned}$$

From here we have

$$\frac{1-\overline{I_n(\lambda_n)}I_n(\lambda_N)}{1-\overline{\lambda_n}\lambda_N}\sim\frac{2\alpha_n\zeta_n}{(1-r_n)(\zeta_n-\lambda_N)}=\frac{2}{n^\varepsilon}\frac{\zeta_n}{\zeta_n-\lambda_N}. \tag{4.3}$$

We claim that at least for $n\geq 2N$,

$$\left|\frac{\zeta_n}{\zeta_n-\lambda_N}\right|\asymp \mathrm{Re}\frac{\zeta_n}{\zeta_n-\lambda_N}.$$

Indeed,

$$\frac{\zeta_n}{\zeta_n-\lambda_N}=\frac{1-\zeta_n\overline{\lambda}_N}{|\zeta_n-\lambda_N|^2},$$

so that for the claim to hold it is sufficient to check that

$$|1-\zeta_n\overline{\lambda}_N|\asymp \mathrm{Re}(1-\zeta_n\overline{\lambda}_N)$$

for $n\geq 2N$. We have already seen in (4.2) that

$$|1-\zeta_n\overline{\lambda}_N|\asymp\frac{1}{N},\quad n\geq 2N.$$

Now

$$\mathrm{Re}(1-\zeta_n\overline{\lambda}_N)=1-r_N\mathrm{Re}\left(e^{i(1/n-1/N)}\right)=1-\left(1-\frac{1}{N}\right)\left(\cos\left(\frac{1}{n}-\frac{1}{N}\right)\right)\asymp\frac{1}{N},\quad n\geq 2N,$$

which proves the claim. We thus can pass in (4.3) to real parts so that for $n\geq 2N$

$$\begin{aligned} \mathrm{Re}\left(\frac{1-\overline{I_n(\lambda_n)}I_n(\lambda_N)}{1-\overline{\lambda_n}\lambda_N}\right) &\sim \mathrm{Re}\left(\frac{2}{n^\varepsilon}\frac{\zeta_n}{\zeta_n-\lambda_N}\right)\sim\frac{2}{n^\varepsilon}\mathrm{Re}\left(\frac{1-\zeta_n\overline{\lambda}_N}{|\zeta_n-\lambda_N|^2}\right)\\ &\asymp \frac{2}{n^\varepsilon}\frac{1/N}{1/n^2+(1/n-1/N)^2}\asymp\frac{2}{n^\varepsilon}\frac{1/N}{(1/N)^2}\\ &\asymp \frac{N}{n^\varepsilon},\quad \text{when } n\geq 2N. \end{aligned}$$

Hence

$$\mathrm{Re}f_\beta(\lambda_N)\ \gtrsim\ \sum_{n\geq 1}\beta_n\frac{1}{n^{(1-\varepsilon)/2}}\frac{\mathrm{Re}(1-\zeta_n\overline{\lambda}_N)}{|\zeta_n-\lambda_N|^2}\gtrsim N\sum_{n\geq 2N}\frac{\beta_n}{n^{(1+\varepsilon)/2}}.$$

Pick for instance $\beta_n = n^{-(1+\eta)/2}$, where $\eta > 0$ is arbitrary, so that obvioulsy $\beta_n \geq 0$ and $\beta \in \ell^2$. Then

$$\mathrm{Re} f_\beta(\lambda_N) \gtrsim N \sum_{n \geq 2N} \frac{1}{n^{1+(\varepsilon+\eta)/2}} \sim N \frac{1}{N^{(\varepsilon+\eta)/2}} = N^{1-\varepsilon/2-\eta/2} \asymp \left(\frac{1}{1-|\lambda_N|} \right)^{1-\varepsilon/2-\eta/2}$$

where $\eta > 0$ is arbitrarily small. Compare this with the estimate of the reproducing kernel (3.2). With better choices of β it is of course clear that we can come closer to the maximal growth given by the reproducing kernel.

Finally, we point out that when $I(z) \longmapsto 1$ when $z \to 1$ in a fixed Stolz domain, it is, in general, particularly difficult to decide whether or not a sequence of reproducing kernels for $(IH^2)^\perp$, with the parameter in a Stolz domain with vertex at 1, is an unconditional basis or not. Even when $\sup_n |I(\lambda_n)| < 1$, there is a characterization known for unconditional basis which is, in general, difficult to check.

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