à mon père

SEMICLASSICAL APPROXIMATION AND NON-COMMUTATIVE GEOMETRY

T. PAUL
CNRS AND CMLS, ECOLE POLYTECHNIQUE

Contents

Abstract. We consider the long time semiclassical evolution for the linear Schrödinger equation. We show that, in the case of chaotic underlying classical dynamics and for times up to $\hbar^{-2+\epsilon}$, $\epsilon > 0$ the symbol of a propagated observable by the corresponding von Neumann-Heisenberg equation, in a sense made precise below, is precisely obtained by the push-forward of the symbol of the observable at time $t = 0$. The corresponding definition of the symbol calls upon a kind of Toeplitz quantization framework, and the symbol itself is an element of the noncommutative algebra of the (strong) unstable foliation of the underlying dynamics.

1. Introduction

In this note we consider the long time semiclassical evolution through the linear Schrödinger equation, or more precisely to the associated von Neumann equation

$$i\hbar \frac{d}{dt} O^t = [O^t, H], \tag{1.1}$$

where H is a Schrödinger operator $H = -\hbar^2\Delta + V$ with smooth confining V ($V(x) \to +\infty$ as $|x| \to \infty$)- or a more general semiclassical pseudodifferential operator of principal symbol h, elliptic and selfadjoint on the Hilbert space $L^2(\mathcal{M})$, where $\mathcal{M}$ is a manifold of dimension $n + 1$.

It is well known [3, 4] that, for times smaller than $C \log \frac{1}{\hbar}$, C small enough, O^t is still a Weyl (semiclassical) pseudodifferential operator and that its principal symbol is the push-forward of the initial one by the Hamiltonian flow associated to the principal symbol h of

H. It is easy to get convinced [6] that this is already not true for large values of C (greater than $\frac{2}{3}$ times the natural Liapunov exponent of the flow.

Through this paper we will suppose that the Hamiltonian flow generated by h is Anosov, and moreover that there exists a smooth action of $\mathbb{R}^{2(n+1)}$ on $T^*\mathcal{M}$, $(\mu,\nu;s,p)\in\mathbb{R}^{2(n+1)}\to\Xi^\nu\circ\Lambda^\mu\circ T^{s,p}$, satisfying

$$\Xi^\nu\circ\Lambda^\mu\circ T^{s,p}\circ\Phi^t=\Phi^t\circ\Xi^{e^{-\lambda t}\nu}\circ\Lambda^{e^{\lambda t}\mu}\circ T^{s+tp,p},\ \lambda>0. \tag{1.2}$$

(1.2) is obviously mimicked on the case of the geodesic flow on surface of constant curvature (Λ^μ (resp. Ξ^ν) is the (resp. anti-)horocyclic flow, $T^{s,0}$ is the geodesic one and $T^{o,p}$ correspond to shifting the energy), but we will not suppose we are in this case and we'll use only (1.2). Moreover we will restrict this note to the bidimensional situation $n=1$ (the extension to any n through (1.2) is straightforward) and take $\mathcal{M}=\mathbb{R}^{n+1}$. The results are local and therefore are easily adaptable to the non flat situation using the results of [7]. Finally we could extend some of our result to the case of variable Liapunov exponents.

We will suppose that $O^{t=0}$ is a semiclassical pseudodifferential operator with smooth symbol supported in $h^{-1}(I)$ for some interval $I\subset\mathbb{R}$ such that $h^{-1}(I)$ is compact.

We first define, associated to $a\in\mathcal{D}(\mathbb{R}^{n+1})$, $\|a\|_{L^2}=1$ and the family of so-called (Gaussian) coherent states $\varphi_{(p,q)}(x):=(\pi\hbar)^{-\frac{n+1}{4}}e^{-\frac{(x-q)^2}{2\hbar}}e^{i\frac{px}{\hbar}}$, $(p,q)=z\in\mathbb{R}^{2(n+1)}$, the family of Lagrangian states:

$$\psi_z^a:=\int e^{\frac{i}{\hbar}\int_z^{\Lambda^\mu\circ T^{s,0}(z)}\eta}a(\mu,s)\varphi_{\Lambda^\mu\circ T^{s,0}(z)}\frac{d\mu ds}{\hbar^{\frac{n+1}{2}}},\ \eta:=\xi.dx \text{ (the symplectic potential on } T^*\mathbb{R}^2). \tag{1.3}$$

It is easy to see that, microlocally in the interior of I and for a support of a small enough, the operator defined by $\int_{h^{-1}(I)}|\psi_z^a\rangle\langle\psi_z^a|\frac{d^{n+1}z}{\hbar^{n+1}}$ (here we denote by $|\psi\rangle\langle\psi|$ the orthogonal projector on the vector ψ) is equal to the identity modulo $\hbar^\infty$.

The key idea of this paper will be to write any pseudodifferential operator in the form

$$O=\int_{h^{-1}(I)}|\psi_z^{O_z a}\rangle\langle\psi_z^a|\frac{d^{n+1}z}{\hbar^{n+1}} \tag{1.4}$$

for a suitable family of bounded pseudodifferential operators O_z.

The interest of such a formulation will be the fact that it is preserved by the evolution through (1.2). More precisely we prove in theorem 2.1 that, for $n=1$ and any $0\leq t\leq C\hbar^{-2+\epsilon}$, $\epsilon>0$, there exists a bounded operator O_z^t such that the solution $O(t)$ of (1.2) microlocalized on $h^{-1}(I')$ satisfies

$$\|\int_{h^{-1}(I)}|\psi_z^{O_z^t a}\rangle\langle\psi_z^a|\frac{d^{n+1}z}{\hbar^{n+1}}-O(t)\|_{\mathcal{B}(L^2)}=O(\hbar^\infty) \tag{1.5}$$

(valid also for $O=$identity). This suggests to call O_z^t the symbol of $O(t)$ at the point z.

In fact we will identify the symbol of $O(t)$ as a noncommutative object related to the space of leaves of the unstable foliation of the dynamics generated by the principal symbol h of H. Let us give the motivation behind this identification.

The classical limit of the equation (**??**) is the well known Liouville equation $\dot{\mathcal{O}}=\{\mathcal{O},h\}$, where $\{.,.\}$ is the Poisson bracket on $T^*\mathcal{M}$, solved by the push forward of the initial condition by the Hamiltonian flow Φ^t associated to h. Though the flow is defined for all times t, the limit as $t \to -\infty$ doesn't have any meaning as a flow, being nevertheless the key ingredient of the theory of chaotic behaviour. In fact, if such a limit flow would exist it would be natural to say that it would be constant on the strong unstable manifold associated to any point $z \in T^*\mathcal{M}$ which, in our case, is the set of points $\Lambda_z = \{\Lambda^\mu \circ T^{s,0}(z), (\mu, s) \in \mathbb{R}^{n+1}\}$. Therefore the pushforward of a smooth initial condition $\mathcal{O}$ by "$\Phi^{-\infty}$ should be constant of each Λ_z, that is to say it should be a "function" on the space of leaves of the unstable foliation, orbits of the action of $\mathbb{R}^{n+1}$, $(\mu, s) \to \Lambda^\mu \circ T^{s,0}$. The leaves Λ_z being usually dense on the energy shell, any (non constant) such function couldn't have any regularity property (trace of the shearing off of the flow for long values of the time). The noncommutative geometry develops a topological theory for such singular spaces by, roughly speaking, replacing the algebra of continuous functions by a noncommutative one which, in the case of space of orbits of the action of a locally compact group, reduces to the crossed product of the algebra of continuous functions on the ambient manifold by the group. Let us note that this change of paradigm is invisible by the classical dynamics which is purely local.

One of the main result of the present paper is to show that, approaching the limit $t \to -\infty$ on the time evolution of the classical dynamics by a correlated semiclassical limit of the Schrödinger equation $\hbar \to 0$, $0 << t < \hbar^{-2+\epsilon}$, one recovers a dynamics based on the noncommutative algebra of the strong unstable manifold, that is the "space" of the invariants of the local classical theory.

The noncommutative algebra of the unstable foliation is the geometrical setting of the classical limit of the long time quantum evolution.

Let us remark finally that long time quantum evolution creates oscillations even in the symbols of observables. Therefore on expects a microlocalization of the symbol of the observable to be necessary. At the same time these oscillations are, at each point of $T^*\mathcal{M}$ along the unstable manifold, a highly non-linear object. It is then natural to expect that the good geometrical setting is not the cotangent bundle over $T^*\mathcal{M}$ but precisely the unstable foliation, which is not a fibration in general, an object handleable by noncommutative geometry.

2. Propagation

Let O be a pseudodifferential operator whose symbol $\mathcal{O}$ is smooth and compactly supported inside $h^{-1}(I)$.

Theorem 2.1. *Let us take $n = 1$. There exists bounded smooth and explicitly computable functions on $\mathbb{R}^4$, $\mathcal{O}_z^t \sim \mathcal{O} + \sum_{j=1}^\infty \mathcal{O}_j^t \hbar^j$, such that, uniformly for $0 \leq t \leq \hbar^{-2+\epsilon}$,*

$$\|e^{-i\frac{tH}{\hbar}} O e^{+i\frac{tH}{\hbar}} - \int_{h^{-1}(I)} |\psi_z^{\widetilde{O_z}^t a}\rangle\langle\psi_z^a| \frac{dz}{\hbar^2}\|_{\mathcal{B}(\mathcal{H})} + O(\hbar^\infty),$$ *where $\widetilde{O_z}^t$ has total Weyl symbol*

$$\widetilde{\mathcal{O}_z^t}(\nu, \mu, s, p) := \mathcal{O}^t(\Xi^{e^{\lambda t}\xi} \circ \Lambda^{e^{-\lambda t}x} \circ T^{s+tp,p} \circ \Phi^t(z)). \tag{2.1}$$

Sketch of the proof: the proof consist in several steps.

- we first prove the result for $t=0$. In order to do that we first show that, for a with small enough support and microlocally on $h^{-1}(I')$ for $I'\subset I$, $\int_{h^{-1}(I)}|\psi_z^a\rangle\langle\psi_z^a|\frac{dz}{\hbar^n}=\mathbb{I}+O(\hbar^\infty)$, where $\mathbb{I}$ is the identity. Since ψ_z^a is a Lagrangian distribution, $O\psi_z^a=\psi_z^{a'}+O(\hbar^\infty)$ where a' is obtained by the action of differentiable operators (transport equation).
- a having a frequency set included in the null section, one proves that $a'=\widetilde{O}_z^0a+O(\hbar^\infty)$ where the Weyl symbol of $\widetilde{O}^0$ has the form $\mathcal{O}_z(\nu,\mu,p,s)\sim\sum\hbar^j\mathcal{O}_j(\Xi^{e^{\lambda t}\xi}\circ\Lambda^{e^{-\lambda t}x}\circ T^{s,p}(z)))$.
- the next step is the heart of the proof. We want to show that $e^{-i\frac{tH}{\hbar}}Oe^{+i\frac{tH}{\hbar}}\psi_z^a=\psi_z^{a^t}$ for some a^t satisfying an equation that we derive, thanks to the main hypothesis. In fact $a^t=\widetilde{O}_z^ta$ where :

(2.2)
$\dot{\widetilde{O}}_z^t=[\widetilde{O}_z^t,H_2+H_3]$, H_2 with a quadratic symbol and H_3 differential operator of third order.

  (2.2) can be solved at any order, H^2 being a quadratic operator (therefore giving an explicit solution) and H_3 being treated by perturbation methods, after microlocalizing near the zeroth section.
- thanks to this inoffensive microlocalization, we show that the preceding solution is valid for with an error term of the form

$$e^{-i\frac{tH}{\hbar}}Oe^{+i\frac{tH}{\hbar}}\psi_z^a=\psi_z^{a_k^t}+O((t\hbar^{-2})^{k+1}\|a\|_{H^k(\mathbb{R}^n)}). \tag{2.3}$$

- taking $k>n$ and $t\leq\hbar^{-2+\epsilon}$, $\epsilon>0$ we get, since $h^{-1}(I)$ is compact, that $i\hbar\partial_tO_k^t=[O_k^t,H]+O_{\mathcal{B}(L^2)}(\hbar^{k\epsilon})$ where $O_k^t:=\int_{h^{-1}(I)}|\psi_z^{\widetilde{a_k^t}}\rangle\langle\psi_z^a|\frac{dz}{\hbar^n}$, from which we deduce, by unitary of the propagator, $O^t=O_k^t+O(t\hbar^{k\epsilon-1})$ and, taking k arbitrary, the result (2.1).

Remark 2.2. As a Corollary of the proof of Theorem 2.1 it is easy to prove that a similar result is still valid when we replace the functions a by a $\hbar$-dependant ones of the form $a_\hbar(\dot{)}=\hbar^{-n\epsilon'/2}a(\hbar^{-\epsilon'}\dot{)}$ for $\epsilon'\geq 0$ small enough (see (2.3)). More precisely, taking $\epsilon'<\epsilon$ (2.1) is still valid, and (**??**) is valid by replacing ϵ by $\epsilon-\epsilon'$.

Let us mention another Corollary of the proof of Theorem 2.1.

Proposition 2.3. *There exist* $a^t\sim\sum_{j=0}^\infty a_j^t\hbar^j$ *such that for* $0\leq t\leq\hbar^{-2+\epsilon}$, $\epsilon>0$,

$$e^{+i\frac{tH}{\hbar}}\psi_z^a=e^{i\int_0^t pdq}\psi_{\Phi^t(z)}^{\widetilde{a^t}}+O(\hbar^\infty)\ \textit{with}\ \widetilde{a^t}(\mu,s)=e^{-it\hbar\partial_s^2}a^t(e^{\lambda t}\mu,s).\ \textit{In particular}$$

$$e^{+i\frac{tH}{\hbar}}\psi_z^a=e^{i\int_0^t pdq}\psi_{\Phi^t(z)}^{\widehat{a^t}}+O(\hbar^\epsilon)\ \textit{with}\ \widehat{a^t}(\mu,s)=e^{-it\hbar\partial_s^2}a(e^{\lambda t}\mu,s)$$

3. Noncommutative geometry interpretation

We first prove the following Lemma.

Lemma 3.1. *Let us define*

$$\sigma_{\mathcal{O}^t}(z,z'):=F(z_1,x,s;x'.s') \tag{3.1}$$

where $z=\Lambda^xT^{s,0}(z_1)$, $z'=\Lambda^{x'}T^{s',0}(z_1)$ *and* $F(z_1,.,.)$ *is the integral symbol of an operator of Weyl symbol given by (2.1). Then* $\sigma_{\mathcal{O}^t}(z,z')$ *doesn't depend on* z_1.

The Lemma is easily proven by the translation invariance properties of the Weyl quantization procedure.

We want to identify $\sigma_{\mathcal{O}^t}$ as an element of the crossed product $\mathcal{A}$ of the algebra $\mathcal{C}_I$ of continuous functions on $h^{-1}(I)$ by the group $\mathbb{R}^{n+1}$ under the action (1.2). A function $\sigma(z,z'),\ z'\in\Lambda_z$ cam be seen as a continuous function from G to $\mathcal{A}$ by

$$f(\mu,t)(z)=\sigma(z,z'),\quad z'=\Lambda^\mu\circ T^t(z). \tag{3.2}$$

Moreover we get an action of G on $\mathcal{C}_I$ by, $\forall g\in G$,

$$\alpha_{(\mu,t)}h(z)=h(\Lambda^\mu\circ T^t(z)). \tag{3.3}$$

The algebra structure on $\mathcal{C}_I\rtimes_\alpha G$ is given by the $\star$-product $(f_1\star f_2)(g)=\int f_1(g_1)\alpha_{g_1}(f_2(g_1^{-1}g))dg_1$.

An easy computation, using Theorem 2.1 and the symbolic property of Weyl quantization shows easily that, at leading order and for all $0\leq t_1,t_2\leq \hbar^{-2+\epsilon}$,

$$\sigma_{\mathcal{O}^{t_1}\mathcal{O}^{t_2}}\sim\sigma_{\mathcal{O}^{t_1}}\star\sigma_{\mathcal{O}^{t_2}}. \tag{3.4}$$

Moreover the norm $|||.|||$ on $\mathcal{A}$ is equal to the supremum over z of the operator norm on $L^2(\mathbb{R}^{n+1})$ of the operator of integral kernel $\sigma(z,z')$ (more precisely $\mathcal{A}\rtimes_\alpha G$ is the completion of the algebra of compactly supported kernels $\sigma(z,z')$ with respect to the norm $|||.|||$.

We can also give a corresponding interpretation of the vectors ψ_z^a. Let us define $\alpha\in\mathcal{A}$ by, for $z'=\Lambda^\mu\circ T^{s,0}(z)$, $\alpha(z,z'):=a(\mu,s)$. Then $\psi_z^a=\psi^\alpha:=\int_{\Lambda_z}e^{\frac{i}{\hbar}\int_z^{z'}\eta}\alpha(z',z)\varphi_{z'}dz'$.

We associate to any element γ of $\mathcal{A}$ an operator $T(\gamma)$ on $L^2(\mathbb{R}^{n+1})$ defined by

$$T(\gamma):=\int_{h^{-1}(I)}|\psi^{\gamma\star\alpha}\rangle\langle\psi^\alpha|\frac{dz}{\hbar^{n+1}}. \tag{3.5}$$

In particular a bounded pseudodifferential operator is such an operator (with $\gamma\sim\sum_{j=0}^\infty\gamma_j\hbar^j$). Moreover, by definition of the norm $|||.|||$, $T(\gamma)$ is a bounded operator for all $\gamma\in\mathcal{A}\rtimes_\alpha G$ and it is easy to see, using arguments of the proof of theorem 2.1, that $T(gamma)$ is bounded *uniformly with* $\hbar\in[0,1]$ for γ compactly supported. Noting that (3.5) is a way of writing (1.4) we get:

Theorem 3.2. *For $0\leq t\leq\hbar^{-2+\epsilon}$ there exist Γ^t of symbol $\gamma^t\sim\sum_{j=0}^\infty\gamma_j^t\hbar^j\in\mathcal{A}$ such that*

$$T(\gamma)^t:=e^{-i\frac{tH}{\hbar}}T(\gamma)e^{+i\frac{tH}{\hbar}}=T(\gamma^t)+O(\hbar^\infty)\qquad \textit{with}\qquad \gamma_0^t=\Phi^t\#\gamma_0+O(\hbar^\epsilon). \tag{3.6}$$

Moreover the leading order symbol of $T(\gamma)^{t_1}T(\gamma)^{t_2}$ is $\gamma^{t_1}\star\gamma^{t_2}$.

Sketch of the proof:

- the fact that Theorem 2.1 is valid also for operators defined by (3.5) is contained in the proof of Theorem 2.1 itself.
- the fact that the symbol of $T(\gamma^t)$ is in the completion by the norm $|||.|||$ is obtained by the Calderon-Vaillancourt theorem, since $\mathcal{O}(\Xi^{e^{\lambda t}\xi}\circ\Lambda^{e^{-\lambda t}x}\circ T^{s+tp+t,p}(z))$ is bounded and smooth, therefore defines a bounded (non semiclassical) pseudodifferential operator.
- the product formula of principal symbols is nothing but (3.4).

Let us remark also that an extension on the lines of Remark 2.2 is also valid in this framework.

4. Semiclassical measures

In the same way that one associates to a vector ψ (or density matrix) the quantity $\langle\psi,\varphi_z\rangle|^2$ considered as a measure by the formula $\langle\psi, O\psi\rangle = \int \mathcal{O}_T(z,\bar z)|\langle\psi,\varphi_z\rangle|^2 dz$, where $\mathcal{O}_T$ is the Toeplitz symbol of O, on can associate to ψ (or a density matrix) a sort of "off-diagonal" version by the quantity $R_\psi(z,\bar{z'}) := \langle\varphi_z,\psi\rangle\langle\psi,\varphi_{z'}\rangle$ for $z'\in\Lambda_z$.

R_ψ can be considered as an element of the dual of a (dense) subalgebra of $\mathcal{A}$ and will have better properties of semiclassical propagation. For sake of shortness we express the result in the case of eigenvectors of the Hamiltonian H, leaving the straightforward derivation for $R_{e^{i\frac{tH}{\hbar}}\psi}$ in the same topology.

Theorem 4.1. *Let us define for I compact interval of $\mathbb{R}$, $\mathcal{D}_I \sim \mathcal{D}(h^{-1}(I)\times\mathbb{R}^n)$ the subalgebra of smooth compactly supported elements of $\mathcal{A}$. Let ψ be an eigenfunction of H.*

Then, restricted to $z\in h^{-1}(I)$, $R_\psi(z,\bar{z'})$ considered as a function on $h^{-1}(I)\times\mathbb{R}^n$, belongs to $\mathcal{D}'_I$. Moreover, in the weak- topology and, $\forall\epsilon>0$, uniformly for $0\le t\le\hbar^{-2+\epsilon}$*

$$\Phi^t \# R(z,\bar{z'}) = R(z,\bar{z'}) + O(\hbar^\epsilon). \tag{4.1}$$

The proof consists in using Γ, a quantization using $a_\hbar$ as in Remark 2.2 of a symbol $\gamma(z,z')$ compactly supported in z'. Writing the formula for $\langle\varphi_z,\Gamma\varphi_{z'}\rangle$ we get that $R_\psi\in\mathcal{D}'_I$, and it is easy to see that Theorem 2.1 applies to Γ, out of which we derive (4.1).

5. Perspectives

Other situations with a noncommutative semiclassical limit can be treated, e.g. the integrable cases.

In this paper we presented only preliminary results concerning the quantization of algebra of the unstable foliation. In particular more symbolic results can be obtained in full generality.

The construction of Section 4 is of course possible for $t\to-\infty$ verbatim by replacing the unstable by the stable foliation, and the flow Λ^μ by Ξ^ν. We believe that it is possible to construct operators whose (noncommutative) symbols will be concentrated on the intersection of the two foliation, and to derive a result similar to Theorem 4.1 by some invariance property along homoclinic trajectories. All these works are in progress.

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