Disturbance rejection in iISS feedback nonlinear systems: a sensitivity trade-off

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Abstract: *This note investigates the trade-off arising in disturbance attenuation for nonlinear feedback systems in the framework of integral input-to-state stability. Similarly to the linear case, we show that if a gain tuning on one subsystem is used to drastically reduce the effect of its exogenous disturbances, then the other subsystem's disturbance attenuation is qualitatively the same as in open loop.*

I. INTRODUCTION

The objective of the present paper is to provide some insights on how the well-known sensitivity / co-sensitivity trade-off arising for feedback linear time-invariant (LTI) systems extends to nonlinear plants. More precisely, consider two feedback nonlinear subsystems and assume that the nonlinear gain of one subsystem can be made smaller by a convenient control design. Then the nonlinear loop gain becomes smaller and the small-gain stability criterion is satisfied with a larger margin. A natural question is then whether this induces stronger robustness to disturbances for the overall feedback system. We give an answer to this question in the dissipative formulation for input-to-state stability (ISS, [18]) and integral ISS (iISS, [20]) systems.

The results presented along this paper rely on small gain arguments. More precisely, we make use of recent results on Lyapunov-based small gain theorems for iISS [10], which include ISS as a special case. Compared to other nonlinear small gains existing in the literature such as [11], [12], [21], [1], [4], this result allows both to deal with not necessarily ISS systems, and to provide an explicit construction of a Lyapunov function for the overall interconnection in the presence of exogenous inputs, which are two helpful features for this work.

Instead of relying on the exact knowledge of differential equation models, we employ iISS dissipation inequalities to describe nonlinear systems in feedback loop. Compared to the frequency analysis for LTI systems (cf. classical textbooks such as [5]), iISS dissipation inequalities do not provide an equality between the input and its response, but rather an *inequality* that provides only a "worst-case" estimate (sometimes not very tight) of the input influence on the overall system: no distinction can be made between systems that are strongly sensitive to inputs, and those for which the dissipation inequality is simply too loose.

In order to overpass this difficulty, we proceed in two different manners. The first one (Section IV) consists in building, for a given pair (α, γ) of iISS supply rates, an iISS system $\dot{x} = f(x, d)$ for which these estimates are tight, in the sense that *all* disturbances that may act on that system have a negative impact on the system's performance and that this effect is not compensated by a dissipation rate stronger than the prescribed one. Roughly speaking, this is done by imposing (at least in some relevant state regions)

$$\frac{\partial V}{\partial x}(x) f(x, d) = -\alpha(|x|) + \gamma(|d|) \,,$$

where V denotes a given Lyapunov function candidate. The equality sign in this equation guarantees the sought tightness of the estimates. We show that, given an iISS supply pair (α, γ), Lyapunov-based small gain arguments always authorize the existence of such a system and consequently the non-rejection of some disturbances. Of course, this first approach is of purely theoretical interest, as the constructed system has typically no practical relevance. The second approach (Section V) demonstrates this trade-off without introducing such fictitious subsystems. Assuming that one subsystem admits a bounded disturbance that does have a negative effect on its performance, we show that, in feedback, this effect cannot be attenuated by the gain tuning of the other subsystem.

Notation. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a set $\mathcal{A} \subset \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z|$. Given a constant $\delta > 0$, $\mathcal{B}_\delta := \{x \in \mathbb{R}^n : |x| \leq \delta\}$. Given a set $A \subset \mathbb{R}$ and a constant $a \in \mathbb{R}$, $A_{\geq a} := \{s \in A : s \geq a\}$. $\text{sat}_\delta : \mathbb{R}^n \to \mathbb{R}^n$ is defined, for all $x \in \mathbb{R}^n$, by $\text{sat}_\delta(x) := (\delta \text{sat}(x_1/\delta), \ldots, \delta \text{sat}(x_n/\delta))^T$, where $\text{sat}(s) := \min(|s|, 1)\text{sign}(s)$ for all $s \in \mathbb{R}$. Given a function $\sigma : \mathbb{R}^m \to \mathbb{R}^n$, $\ker(\sigma) := \{x \in \mathbb{R}^m : \sigma(x) = 0\}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{PD}$ if it is positive definite. It is said to be in class $\mathcal{K}$ if, in addition, it is increasing. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for any fixed $t \geq 0$ and $\beta(s, \cdot)$ is continuous non increasing and tends to zero at infinity for any fixed $s \geq 0$. Given $\alpha \in \mathcal{K}$, $\alpha(\infty) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined as $\lim_{s \to \infty} \alpha(s)$. Given $\alpha, \gamma \in \mathcal{K}$, $\alpha(\infty) > \gamma(\infty)$ means that either $\alpha \in \mathcal{K}_\infty$, or $\alpha(\infty) = c_\alpha \in \mathbb{R}_{\geq 0}$ and

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$\gamma(\infty) = c_\gamma \in \mathbb{R}_{\geq 0}$ with $c_\alpha > c_\gamma$. $\mathcal{U}^m$ is the set of measurable locally essentially bounded signals $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. Given $u \in \mathcal{U}^m$, $\|u\| := \text{ess}\sup_{t\geq 0} |u(t)|$. Given $\Delta \geq 0$, $\mathcal{U}^m_{\leq\Delta} := \{u \in \mathcal{U}^m : \|u\| \leq \Delta\}$. $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a Lyapunov function candidate if it is C^1, positive definite and radially unbounded.

II. Problem statement

We consider two dynamical systems Σ_1 and Σ_2 interconnected in a feedback configuration through their outputs y_1 and y_2, and subject to exogenous disturbances d_1 and d_2, cf. Fig. 1.

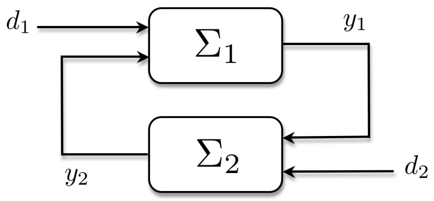

Fig. 1. Feedback interconnection.

It is well known that, when Σ_1 and Σ_2 are LTI, the sensitivity / co-sensitivity tradeoff impedes the disturbance rejection of both d_1 and d_2 at the same frequency. To sketch out this tradeoff, consider single input - single output systems and let H_i denote the transfer function of Σ_i, $i \in \{1, 2\}$. If H_1 is tuned in such a way that $H_2H_1(1-H_2H_1)^{-1} \to 0$ at a given frequency, then one cannnot avoid $(1-H_1H_2)^{-1} \to 1$. This results in $H_2(1-H_1H_2)^{-1} \to H_2$, meaning that the d_1-rejection imposes that the effect of d_2 is similar to the open-loop. This fundamental obstruction to control design was first studied in [3]. It imposes, in particular, a compromise between precision / output disturbance rejection and sensor noise attenuation. See [8], [15], [17], [7] for an in-depth analysis. The aim of this paper is to analyze to what extent this result can be adapted to nonlinear plants. The feedback interconnection considered in this note is

$$\dot{x}_1 = f_1(x_1, x_2, d_1, \theta) \tag{1a}$$
$$\dot{x}_2 = f_2(x_2, x_1, d_2)\,, \tag{1b}$$

where $(x_1^T, x_2^T)^T =: x \in \mathbb{R}^{n_1+n_2}$ denote the sate of each subsystem, $(d_1^T, d_2^T)^T =: d \in \mathcal{U}^{m_1+m_2}$ are exogenous disturbances, and $\theta \in \Theta \subset \mathbb{R}^p$ is a free parameter as, for instance, a vector of tuning gains. We stress that this structure does not necessarily require that the subsystems (1a) and (1b) are connected through their whole states, but rather authorizes output feedback interconnection as f_1 (resp. f_2) may involve only part of x_2 (resp. x_1) or a function of its entries.

While the above LTI reasoning does not require any stability assumption on Σ_1 and Σ_2 when considered individually, the small-gain approach we follow in this note imposes that each subsystem is iISS with a class $\mathcal{K}$ dissipiation rate [20].

Assumption 1 *There exist $\alpha_1, \gamma_1, \varphi_1 \in \mathcal{K}$, $\underline{\alpha}_1, \overline{\alpha}_1 \in \mathcal{K}_\infty$, and a C^1 function $V_1 : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ satisfying $\underline{\alpha}_1(|x_1|) \leq V_1(x_1) \leq \overline{\alpha}_1(|x_1|)$ with the property that, given any $\lambda > 1$, there exist $\theta \in \Theta$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ and all $d_1 \in \mathbb{R}^{m_1}$,*

$$\frac{\partial V_1}{\partial x_1} f_1 \leq -\alpha_1(|x_1|) + \frac{1}{\lambda}\left[\gamma_1(|x_2|) + \varphi_1(|d_1|)\right]. \tag{2}$$

This first assumption not only guarantees iISS for the x_1-subsystem (1a), but also that the disturbance rejection for this subsystem can be tuned at will by a convenient choice of the parameter θ. More precisely, considering $u_1 := (x_2^T, d_1^T)^T$ as the exogenous input of (1a) and relying on classical reasonings for iISS systems (cf. [2, Corollary IV.3]), Assumption 1 naturally yields the following trajectory estimate for (1a):

$$|x_1(t)| \leq \beta(\left|x_1^0\right|, t) + \eta\left(\frac{1}{\lambda}\int_0^t \tilde{\gamma}_1(|u_1(\tau)|)d\tau\right) \tag{3}$$

where $x_1(\cdot) := x_1(\cdot; x_1^0, x_2, d_1, \theta)$, $x_2(\cdot) := x_2(\cdot; x_2^0, x_1, d_2)$, $\tilde{\gamma}_1(\cdot) := 2\max\{\gamma_1(\cdot), \varphi_1(\cdot)\}$ and η and β denote respectively class $\mathcal{K}$ and $\mathcal{KL}$ functions. Thus, once the exogenous signals x_2 and d_1 are given, the above estimate illustrates the possibility to arbitrarily reject their effect on the behavior of the x_1-subsystem by conveniently tuning θ (*i.e.* by increasing λ). Note that the dissipation rate α_1 is assumed to belong to class $\mathcal{K}$ rather than simply $\mathcal{PD}$ as in [2]. This is motivated by the small gain argument [10] we invoke in the sequel. Hence, Assumption 1 actually imposes iISS plus ISS with respect to small inputs[1] with an assignable supply rate. Even though Assumption 1 may be hard to achieve in practice, this note aims precisely at showing that, despite such a strong stabilizability assumption, disturbance rejection cannot be expected to be arbitrary in a feedback interconnection. Nonetheless, we stress that, under specific matching conditions, Assumption 1 can be ensured by control designs available in the literature such as [16, Lemma 3]. The results in [22], [14] may also be inspiring. For instance, the following result easily follows from [16, Lemma 3].

Proposition 1 (Actuation errors) *Let $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ and $g_i : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$, $i \in \{1, \dots, p\}$, be locally Lipschitz functions and assume that the system*

$$\dot{x}_1 = f(x_1) + \sum_{i=1}^{p} g_i(x_1)u_i$$

is globally asymptotically stabilizable by a continuous state feedback $u = \kappa^\circ : \mathbb{R}^{n_1} \to \mathbb{R}^p$, satisfying $\kappa^\circ(0) = 0$, with associated C^1 Lyapunov function V_1 satisfying, for each $i \in \{1, \dots, p\}$,

$$\frac{\partial V_1}{\partial x_1}(0)g_i(0) = 0\,.$$

Then there exists a continuous state feedback $\kappa : \mathbb{R}^{n_1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ such that the system

$$\dot{x}_1 = f(x_1) + \sum_{i=1}^{p} g_i(x)(\kappa(x_1, \theta) + x_2 + d_1)\,,$$

[1] This combination is sometimes referred to as Strong iISS.

where θ denotes a scalar gain, satisfies Assumption 1 with this function V_1.

On the other hand, the x_2-subsystem is assumed to be iISS, with a fixed supply rate.

Assumption 2 *There exist α_2, $\gamma_2, \varphi_2 \in \mathcal{K}$, $\underline{\alpha}_2, \overline{\alpha}_2 \in \mathcal{K}_\infty$, and a C^1 function $V_2 : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ and all $d_2 \in \mathbb{R}^{m_2}$,*

$$\underline{\alpha}_2(|x_2|) \leq V_2(x_2) \leq \overline{\alpha}_2(|x_2|) \tag{4}$$

$$\frac{\partial V_2}{\partial x_2} f_2(x_2, x_1, d_2) \leq -\alpha_2(|x_2|) + \gamma_2(|x_1|) + \varphi_2(|d_2|). \tag{5}$$

Here also the dissipation rate is assumed to be of class $\mathcal{K}$ rather than $\mathcal{PD}$ [2]. This assumption is necessary for the application of the small gain theorem [10] on which we base this study.

III. TUNING FOR d_1-REJECTION

We state the following result, which formally shows that, as expected, the tuning of θ allows for arbitrary attenuation of d_1.

Proposition 2 *Let Assumptions 1 and 2 hold and assume that the following implication holds true for each $i \in \{1, 2\}$:*

$$\gamma_{3-i} \in \mathcal{K}_\infty \quad \Rightarrow \quad \alpha_i \in \mathcal{K}_\infty \,. \tag{6}$$

Assume also that the small gain condition[2]

$$c_2\gamma_2 \circ \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\gamma_1(s) \leq \alpha_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)\,, \quad \forall s \geq 0 \tag{7}$$

holds for some constants $c_1 > 0$ and $c_2 > 1$. Then, there exist $\beta \in \mathcal{KL}$, $\underline{\alpha}, \gamma, \zeta \in \mathcal{K}_\infty$, and $\Delta > 0$ and, given any $\ell > 1$, there exist $\theta \in \Theta$ such that, for all $x^0 \in \mathbb{R}^{n_1+n_2}$, all $d_1 \in \mathcal{U}^{m_1}$ and all $d_2 \in \mathcal{U}^{m_2}$, the feedback interconnection (1) is iISS and ISS with respect to small inputs, and its solution satisfies

$$\underline{\alpha}(|x(t)|) \leq \beta(\left|x^0\right|, t) + \int_0^t \gamma\left(|d_1(\tau)|/\ell\right) d\tau + \int_0^t \gamma(|d_2(\tau)|) d\tau\,, \quad \forall t \geq 0\,. \tag{8}$$

and, for all $d_1 \in \mathcal{U}^{m_1}_{\leq \ell\Delta}$ and all $d_2 \in \mathcal{U}^{m_2}_{\leq \Delta}$,

$$|x(t)| \leq \beta(\left|x^0\right|, t) + \zeta(\|d_1\|/\ell) + \zeta(\|d_2\|). \tag{9}$$

It is worth noting that the upper and lower bounds on V_i (namely, $\underline{\alpha}_i$ and $\overline{\alpha}_i$), $i \in \{1, 2\}$ involved in (7) could be removed if (2) and (5) were replaced by dissipation inequalities involving only V_i rather than x_i. We keep the original small-gain condition (7) of [10] as the bounds (2) and (5) are usually easier to establish in practice.

We also stress that small-gain condition in [10] requires both c_1 and c_2 to be greater than 1. Relaxing to only $c_1 > 0$ in (7) is made possible by the fact that, in the context of the present article, the constant λ multiplying the supply rate γ_1 is tunable at will through the parameter θ (cf. Assumption 1).

Apart from these details, the iISS and ISS with respect to small inputs of the feedback interconnection (1) under (6)-(7) directly follows from previous results of the second author [10]. See Section VII-A for the complete proof. Let us recall that the small-gain condition (7) is not symmetric. We have chosen to assume (7) rather than its counterpart:

$$c_1\gamma_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\gamma_2(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s),$$

in order to allow for the interconnection of not necessarily ISS subsystems. See [10] for details.

Compared to [10], the novelty of Proposition 2 stands in the explicit estimate of the disturbance attenuation allowed by the tuning gain θ. Indeed, since the functions $\underline{\alpha}$, β, γ and ζ in (8)-(9) are independent of ℓ, Proposition 2 guarantees that the effect of the exogenous disturbance d_1 over the solutions' behavior can be made arbitrarily small provided a convenient tuning of θ (*i.e.*, corresponding to sufficiently large λ and ℓ). In addition, since (9) ensures ISS with respect to all d_1 of amplitude smaller than $\ell\Delta$, with Δ independent of ℓ, the class of ISS-tolerated disturbances can be enlarged at will. These constitute two interesting features for the rejection of the d_1 disturbance. However, no such d_2-disturbance attenuation appears in the trajectory estimates (8)-(9). This fact could either be due to an intrinsic property of feedback interconnections, or simply to the looseness of the upper bounds (8)-(9). The rest of the paper shows that this property is indeed intrinsic and that no such d_2-attenuation can be expected in general.

IV. SENSITIVITY TO d_2: A "WORST CASE" SYSTEM

In contrast to the previous section, we now show that the increase of λ, by a convenient tuning of the gain θ, is in general of no help in reducing the influence of d_2 over x_2. The proof of this result is provided in Section VII-C.

Theorem 1 *Let Assumption 1 hold, let $d_2^{min} < d_2^{max}$ be two positive constants, and let $\alpha_2, \gamma_2, \varphi_2$ denote some given $\mathcal{K}$ functions. Let $V_2 : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ be any Lyapunov function candidate satisfying*

$$\frac{\partial V_2}{\partial x_2}(x_2) \neq 0\,, \quad \forall x_2 \neq 0\,. \tag{10}$$

Then one can always find class $\mathcal{K}$ functions ν_2 and η_2, and a vector field $f_2 : \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{n_2}$, continuous on $\mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times (\mathbb{R}^{m_2} \setminus \{0\})$, satisfying Assumption 2 with these prescribed functions α_2, γ_2 and φ_2, and such that, given any $\theta \in \Theta$, any initial state $x_2^0 \in \mathbb{R}^{n_2}$ and any disturbance $d_1 \in \mathcal{U}^{m_1}$ and $d_2 \in \mathcal{U}^{m_2}$ satisfying

$$d_2^{min} \leq \|d_2(t)\| \leq d_2^{max}\,, \tag{11}$$

[2] Condition (7) requires in particular that either $\gamma_1(\infty)$ is finite or $\alpha_1 \in \mathcal{K}_\infty$. In both cases, Assumption 1 guarantees that a convenient tuning of θ makes (1a) ISS with respect to x_2. More details can be found in [10].

all forward complete solutions of (1) starting with $\left|x_2^0\right| \geq \eta_2\left(\|d_2\|\right)$ *satisfy*

$$|x_2(t)| \geq \nu_2(\text{ess inf}_{\tau\geq 0}|d_2(\tau)|),\ \forall t \geq 0. \qquad (12)$$

Theorem 1 shows that the only knowledge of the dissipation inequality associated to each subsystem cannot guarantee, in general, an arbitrary d_2-disturbance attenuation even when control gains can be tuned in order to decrease the sensitivity of the x_1-subsystem with respect to its inputs. Indeed, it guarantees that such an interconnection may always yield, for some particular systems, the existence of an incompressible lower bound (12) whose size is somewhat "proportional" to the minimal value of $|d_2|$, for solutions starting sufficiently far from the origin. The crucial point is that this lower bound holds regardless of the chosen gain θ. It is therefore hopeless to expect arbitrary d_2-disturbance rejection for this system by relying only on the associated dissipation inequalities.

Remark 1 *If in addition to the assumptions of Theorem 1, the small gain condition (7) holds, then the assumptions of Proposition 2 are satisfied and consequently the feedback interconnection (1) is iISS and ISS with respect to small inputs (cf. (8)-(9)) if* λ *is made small enough by a convenient choice of* θ*. In particular, (1) results forward complete and the lower bound (12) holds at all times.*

The property stated as Theorem 1 is quite intuitive once the inequality (5) is sufficiently tight. The contribution of this result is, in fact, to show that such a dissipation inequality is always tight for some particular systems. More precisely, the proof of Theorem 1 relies on the following lemma, that may have interest on its own. It is similar in spirit to [10, Lemma 1], but applies to any given Lyapunov function candidate. Its proof is provided in Section VII-B.

Lemma 1 *Given* $m, n \in \mathbb{N}_{\geq 1}$*, let* $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *be any continuous function satisfying*

$$|x| \leq \sigma(u) \quad \Rightarrow \quad \varphi(x, u) \geq 0\,, \qquad (13)$$

for some continuous function $\sigma : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$*. Consider any Lyapunov function candidate* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *satisfying*

$$\frac{\partial V}{\partial x}(x) \neq 0\,, \quad \forall x \neq 0\,. \qquad (14)$$

Then, there exists a vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$*, continuous on* $\mathbb{R}^n \times (\mathbb{R}^m \setminus ker(\sigma))$*, such that, for all* $x \in \mathbb{R}^n$ *and all* $u \in \mathbb{R}^m$*,*

$$\frac{\partial V}{\partial x}(x) f(x, u) \leq \varphi(x, u) \qquad (15)$$

$$|x| \geq \sigma(u) \;\Rightarrow\; \frac{\partial V}{\partial x}(x) f(x, u) = \varphi(x, u)\,. \qquad (16)$$

This lemma shows that, under mild assumptions, the dissipation inequality (5) is always "tight" for what we refer to as a *worst-case* system. In other words, any Lyapunov function candidate constitutes a tight iISS/ISS estimate of the behavior of these systems. This can be seen by taking φ as an iISS or ISS supply pair for this system. Here we refer to a "worst case" situation as, for this system, the application of *any* input signal works against the convergence of the associated Lyapunov function, and that it can be compensated by no greater dissipation rate than $\alpha(|x|)$.

Remark 2 *The right-hand side* f *of the constructed system may not be locally Lipschitz. However, depending on the choice of the function* φ*, the existence of solutions may be guaranteed at all time. For instance, the application of the comparison lemma guarantees forward completeness for any function* φ *satisfying, at least for large* $|x|$*,*

$$\varphi(x, u) \leq cV(x) + \eta(|u|)\,,$$

where $c \in \mathbb{R}$ *and* $\eta : \mathbb{R}^m \to \mathbb{R}$ *denotes a continuous function. See [9] for further discussions on how forward completeness of feedback systems can be guaranteed. Also, the fact that* f *is not necessarily continuous in* $u = 0$ *is not a crucial issue as Lemma 1 will typically be used for inputs lower-bounded away from zero.*

V. Sensitivity to d_2: impeding disturbances

In most situations, exogenous inputs do not systematically work against the convergence of the associated Lyapunov function, as opposed to the worst-case systems developed in Section IV. For instance, for the scalar system $\dot{x} = -x + d$, any positive signal d tends to slowing down the convergence of x to zero for positive values of the initial state x^0, but it actually speeds it up if $x^0 \leq 0$. This observation suggests that no tight Lyapunov function, in the sense of Lemma 1, exists for most dynamical systems of practical relevance, nor can a Lyapunov function candidate W satisfying

$$\dot{W} \geq -\alpha(|x|) + \gamma(|u|)\,, \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m,$$

with $\alpha, \gamma \in \mathcal{K}$, be expected in general. On the other hand, in many cases, disturbances do induce an increase of the associated Lyapunov function at least in some regions of the state space. It is also reasonable to assume that their size is bounded for bounded states. This motivates the following assumption, which can be seen as a destabilizing counterpart of the small control property, cf. *e.g.* [19], [6].

Assumption 3 *There exists a Lyapunov function candidate* $W_2 : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$*, a* $\mathcal{K}$ *function* Υ_2 *and a continuous*[3] *function* $d_2 : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{m_2}$ *such that, given any* $x = (x_1^T, x_2^T)^T \in \mathbb{R}^{n_1+n_2}$*,*

$$|d_2(x)| \leq \Upsilon_2(|x|) \qquad (17)$$

[3] The continuity requirement on d_2 may probably be relaxed by relying on Arstein-type constructions [19] to get a continuous destablizing feedback. Since such a construction is of little interest in the context of this note, we assume continuity of d_2 for simplicity.

$$\frac{\partial W_2}{\partial x_2}(x_2) f_2(x_2, x_1, d_2(x)) > 0 \,.$$

This assumption ensures that at least one disturbance, whose size is somewhat "proportional" to the norm of the state, tends to destabilize the x_2-subsystem with x_1 as an input.

For feedback systems satisfying Assumption 3, the following result shows that the tuning of the gain θ cannot be expected to induce arbitrary d_2-disturbance rejection.

Theorem 2 *Let Assumption 3 hold. Then there exists $\Upsilon \in \mathcal{K}$ such that, given any $\delta > 0$, there exists a signal $d_2^\star \in \mathcal{U}^{m_2}$ satisfying*

$$\|d_2^\star\| \leq \Upsilon(\delta) \tag{18}$$

such that, given any $\theta \in \Theta$ and any $d_1 \in \mathcal{U}^{m_1}$, the set $\mathbb{R}^n \backslash \mathcal{B}_\delta$ is globally attractive for the feedback interconnection (1) (i.e., $\liminf_{t\to\infty} \left|x(t; x^0, d)\right| \geq \delta$) if the latter is forward complete.

The above result establishes that, for all systems satisfying Assumption 3, either the resulting interconnection is not forward complete (in which case disturbance rejection is obviously not achieved), or any ball centered at the origin can be made repellent for the overall interconnection, regardless of the choice of the tuning gain θ, by a bounded disturbance $d_2^\star$ whose amplitude is "proportional" to the size of the chosen ball. This means that the maximum disturbance rejection is purely a function of the applied disturbance $d_2^\star$ and that the tuning of θ has no effect on it. We stress that, in the above result, the larger the upper bound in (18) is, the further away from origin solutions will asymptotically go to (as $\mathcal{B}_\delta$ grows larger).

Remark 3 *If, in addition, the vector fields f_1 and f_2 are chosen according to Assumptions 1 and 2 and the small gain condition (7) holds, then Proposition 2 ensures that the overall system is iISS (hence, forward complete).*

VI. Conclusion

Motivated by the observation that the smaller the loop gain is, the larger the internal stability margin is for a feedback system, this paper has investigated the effect of decreasing the loop gain on external stability, and established a natural trade-off between rejection of disturbances entering in different places in the feedback loop. If one subsystem's parameters are tuned to reduce the effects of its disturbances, then the other subsystem eventually has been shown to behave as if it were in open-loop. While this trade-off is quite natural, the dissipation formulation of this paper enables to confirm the property for nonlinear systems, thus without relying on transfer functions. This iISS framework employed in this paper also allows to encompass subsystems whose solutions are not necessarily bounded for bounded inputs.The extension to the interconnection of more than two subsystems cas be envisioned based on large-scale small gain theorems such as [4].

VII. Proofs

We start by recalling the following lemma, whose proof can be found along the lines of [10].

Lemma 2 *For each $i \in \{1, 2\}$, let $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ be a C^1 function satisfying, for all $x_i \in \mathbb{R}^{n_i}$, $\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|)$ with $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, and assume that there exist $\alpha_i, \gamma_i, \varphi_i \in \mathcal{K}$ such that (6) holds and, for all $s \geq 0$,*

$$c_2\gamma_2 \circ \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\gamma_1(s) \leq \alpha_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s) \tag{19}$$

with $c_1, c_2 > 1$. Then there exist $\rho_1, \rho_2, \alpha, \gamma \in \mathcal{K}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\begin{aligned}\textstyle\sum_{i=1}^2 \rho_i(V_i(x_i)) \Big[-\alpha_i(|x_i|) + \gamma_i(|x_{3-i}|) + \varphi_i(|d_i|) \Big] \\ \leq -\alpha(|x|) + \gamma(|d|) \,.\end{aligned}$$

This lemma provides an explicit construction of a Lyapunov function for the feedback interconnection of iISS systems under the small gain condition (19). It is instrumental for the proof of Proposition 2.

A. Proof of Proposition 2

First notice that φ_1 can be assumed to be of class $\mathcal{K}_\infty$ without loss of generality (if it is not, just consider any $\mathcal{K}_\infty$ function greater than φ_1). In view of [20, Corollary 10], there exist $\mathcal{K}_\infty$ functions φ_0 and $\tilde{\varphi}_1$ such that $\varphi_1(rs) \leq \tilde{\varphi}_1(r)\varphi_0(s)$ for all $r, s \geq 0$. Given any $d_1 \in \mathbb{R}^{n_1}$ and any $\lambda > 0$, pick $r = |d_1| / \varphi_0^{-1}(\lambda)$ and $s = \varphi_0^{-1}(\lambda)$. We then conclude from the above expression that

$$\frac{\varphi_1(|d_1|)}{\lambda} \leq \tilde{\varphi}_1 \left(\frac{|d_1|}{\varphi_0^{-1}(\lambda)} \right) . \tag{20}$$

Given any arbitrary $\ell > 1$, pick any λ satisfying

$$\lambda > \max \{\varphi_0(\ell); 1\} \,. \tag{21}$$

Observing that the supply rate for (1a) with respect to x_2 is $\frac{1}{\lambda}\gamma_1$, (7) ensures that the small gain condition (19) in Lemma 2 is fulfilled for some $c_1 > 0$. Assumptions 1 and 2 are then enough to apply Lemma 2 and conclude the existence of $\rho_1, \rho_2, \alpha \in \mathcal{K}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$ such that, given any $x = (x_1^T, x_2^T)^T \in \mathbb{R}^{n_1+n_2}$ and any $d = (d_1^T, d_2^T)^T \in \mathbb{R}^{m_1+m_2}$,

$$\begin{aligned}\textstyle\sum_{i=1}^2 \rho_i(V_i(x_i)) \Big[-\alpha_i(|x_i|) + \gamma_i(|x_{3-i}|) + \tilde{\varphi}_i(|d_i|) \Big] \\ \leq -\alpha(|x|) + \tilde{\gamma}(|d|) \,,\end{aligned} \tag{22}$$

where, for notation homogeneity, $\tilde{\varphi}_2 := \varphi_2$. Similarly to [10], consider the function V defined as

$$V(x) := \int_0^{V_1(x_1)} \rho_1(s)ds + \int_0^{V_2(x_2)} \rho_2(s)ds \,. \tag{23}$$

It easily follows from Assumptions 1 and 2 that its derivative along the trajectories of (1) yields

$$\begin{aligned}\dot V \;&\leq\; \rho_1(V_1)\left(-\alpha_1(|x_1|)+\frac{\gamma_1(|x_2|)}{\lambda}+\frac{\varphi_1(|d_1|)}{\lambda}\right)\\&\quad+\rho_2(V_2)\big(-\alpha_2(|x_2|)+\gamma_2(|x_1|)+\varphi_2(|d_2|)\big)\,.\end{aligned}$$

By (20) and (21), it then follows that

$$\begin{aligned}\dot V \;&\leq\; \rho_1(V_1)\left(-\alpha_1(|x_1|)+\gamma_1(|x_2|)+\tilde\varphi_1\left(|d_1|/\ell\right)\right)\\&\quad+\rho_2(V_2)\big(-\alpha_2(|x_2|)+\gamma_2(|x_1|)+\tilde\varphi_2(|d_2|)\big)\,,\end{aligned}$$

which, in view of (22), implies that

$$\dot V \leq -\alpha(|x|)+\tilde\gamma\left(\left|\begin{pmatrix} d_1/\ell \\ d_2\end{pmatrix}\right|\right)\,. \tag{24}$$

Using the fact that $\tilde\gamma(a+b)\leq\tilde\gamma(2a)+\tilde\gamma(2b)$ for all $a,b\geq 0$ (since $\tilde\gamma\in\mathcal{K}_\infty$) and integrating this inequality with [2, Corollary IV.3] along the solutions of (1) yields

$$\begin{aligned}V(x(t)) \;&\leq\; \tilde\beta(V(x(0)),t)+2\int_0^t\tilde\gamma\left(2\left|d_1(\tau)\right|/\ell\right)d\tau\\&\quad+2\int_0^T\tilde\gamma(2\left|d_2(\tau)\right|)d\tau\,,\quad\forall t\geq 0\,,\end{aligned}$$

where $\tilde\beta$ denotes a $\mathcal{KL}$ function. Moreover, exploiting (23) and the bounds on V_1 and V_2 guaranteed by Assumptions 1 and 2, V results a Lyapunov function candidate, which guarantees the existence of $\mathcal{K}_\infty$ functions $\underline{\alpha},\overline{\alpha}\in\mathcal{K}_\infty$ such that, for all $x\in\mathbb{R}^n$, $\underline{\alpha}(|x|)\leq V(x)\leq\overline{\alpha}(|x|)$. We obtain that

$$\begin{aligned}\underline{\alpha}(|x(t)|)\leq\bar\beta(\left|x^0\right|,t)\quad&+\quad 2\int_0^t\tilde\gamma\left(2\left|d_1(\tau)\right|/\ell\right)d\tau\\&+\quad 2\int_0^t\tilde\gamma(2\left|d_2(\tau)\right|)d\tau\,,\end{aligned}$$

where β is the $\mathcal{KL}$ function defined as $\bar\beta(s,t):=\tilde\beta(\overline{\alpha}(s),t)$ for all $s,t\geq 0$. This establishes (8) with $\gamma(s):=2\tilde\gamma(2s)$ for all $s\geq 0$. Finally, let $\Delta>0$ be any constant satisfying $\Delta\leq\tilde\gamma(\infty)$. Then, recalling that $|(d_1/\ell,d_2)|\geq\max\left\{|d_1/\ell|,|d_2|\right\}$, the quantity $\alpha^{-1}\circ\frac{1}{2}\tilde\gamma(|(d_1/\ell,d_2)|)$ is finite for all $|d_1|\leq\ell\Delta$ and all $|d_2|\leq\Delta$, and it holds from (24) that

$$|x|\geq\alpha^{-1}\circ\frac{1}{2}\tilde\gamma\left(\left|\begin{matrix} d_1/\ell \\ d_2\end{matrix}\right|\right)\Rightarrow\dot V\leq-\frac{1}{2}\alpha(|x|).$$

Classical ISS reasonings then ensures ISS with respect to all $d_1\in\mathcal{U}^{m_1}_{\leq\ell\Delta}$ and all $d_2\in\mathcal{U}^{m_2}_{\leq\Delta}$, and establishes (9) for some $\mathcal{KL}$ function $\hat\beta$ and some $\mathcal{K}_\infty$ function ζ. The conclusion follows by noticing that $\bar\beta$ and $\hat\beta$ can be taken the same by considering the $\mathcal{KL}$ function $\beta(s,t):=\max\left\{\bar\beta(s,t),\hat\beta(s,t)\right\}$ for all $s,t\geq 0$.

B. *Proof of Lemma 1*

Let $c:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ denote the function defined for all $s\geq 0$ as

$$c(s):=\min_{|x|=s}\left|\frac{\partial V}{\partial x}(x)\right|\,. \tag{25}$$

Since V is C^1, c is continuous and, in view of (14), it is positive definite. Now, let $\zeta:\mathbb{R}_{\geq 0}\times\mathbb{R}^m\to\mathbb{R}_{\geq 0}$ denote the function defined, for all $s\geq 0$ and all $u\in\mathbb{R}^m$, as

$$\zeta(s,u):=\begin{cases} sc(s)^2 & \text{if } s\leq\sigma_u\\ \sigma(u)c(\sigma(u))^2 & \text{if } s>\sigma_u\,.\end{cases} \tag{26}$$

Note that ζ is continuous on $\mathbb{R}_{\geq 0}\times\mathbb{R}^m$ and that $\zeta(s,u)>0$ for all $s\neq 0$ and all $u\notin\ker(\sigma)$. We claim that the result holds with the vector field f defined as $f(0,u):=0$ and, for all $x\neq 0$,

$$f(x,u):=\varphi(x,u)\frac{\xi(x,u)}{\left|\frac{\partial V}{\partial x}(x)\right|^2}\left(\frac{\partial V}{\partial x}(x)\right)^T\,,$$

where $\xi:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}_{\geq 0}$ denotes the function defined as

$$\xi(x,u):=\begin{cases} 1 & \text{if } |x|\geq\sigma(u)\\ a(u)\,|x|+b(u) & \text{if } |x|\in[\sigma(u)/2;\sigma(u))\\ \mathrm{sat}(\zeta(|x|,u)) & \text{if } |x|<\sigma(u)/2\,,\end{cases}$$

with

$$\begin{aligned}a(u)\;&:=\;\frac{1-\mathrm{sat}(\zeta(\sigma(u)/2,u))}{\sigma(u)/2}\\ b(u)\;&:=\;2\,\mathrm{sat}(\zeta(\sigma(u)/2),u)-1\,.\end{aligned}$$

Note that, with this choice of a and b and recalling the properties of ζ, the function ξ is continuous on $\mathbb{R}^n\times(\mathbb{R}^m\setminus\ker(\sigma))$ and satisfies

$$0\leq\xi(x,u)\leq 1\,,\quad\forall x\in\mathbb{R}^n,\,u\in\mathbb{R}^m\,. \tag{27}$$

Moreover, in view of (25) and (26), given any $u\in\mathbb{R}^m\setminus\ker(\sigma)$, it satisfies

$$\begin{aligned}\limsup_{x\to 0}\frac{\xi(x,u)}{\left|\frac{\partial V}{\partial x}(x)\right|^2}\;&=\;\limsup_{x\to 0}\frac{\mathrm{sat}(\zeta(|x|,u))}{\left|\frac{\partial V}{\partial x}(x)\right|^2}\\&\leq\;\limsup_{x\to 0}|x|\frac{\left|\frac{\partial V}{\partial x}(x)\right|^2}{\left|\frac{\partial V}{\partial x}(x)\right|^2}\\&\leq\;0\,.\end{aligned}$$

Since ξ is a nonnegative function, it follows that, given any $u\in\mathbb{R}^m\setminus\ker(\sigma)$,

$$\lim_{x\to 0}\frac{\xi(x,u)}{\left|\frac{\partial V}{\partial x}(x)\right|^2}=0\,,$$

and consequently, for each $u^\star\in\mathbb{R}^m\setminus\ker(\sigma)$,

$$\lim_{(x,u)\to(0,u^\star)}f(x,u)=0=f(0,u^\star)\,.$$

Exploiting the obvious continuity of f in any $x^\star\neq 0$ for any $u^\star\in\mathbb{R}^m\setminus\ker(\sigma)$, it follows that f is continuous on $\mathbb{R}^n\times(\mathbb{R}^m\setminus\ker(\sigma))$.

Furthermore, for any $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ satisfying $|x| \geq \sigma(u)$, it holds that

$$f(x,u) = \varphi(x,u) \frac{1}{\left|\frac{\partial V}{\partial x}(x)\right|^2} \left(\frac{\partial V}{\partial x}(x)\right)^T .$$

Consequently

$$\begin{aligned} \frac{\partial V}{\partial x} f(x,u) &= \frac{\partial V}{\partial x}(x) \frac{\varphi(x,u)}{\left|\frac{\partial V}{\partial x}(x)\right|^2} \left(\frac{\partial V}{\partial x}(x)\right)^T \\ &= \varphi(x,u) , \end{aligned}$$

which establishes (16). Finally, if $|x| < \sigma(|u|)$, then (13) guarantees that $\varphi(x,u) \geq 0$. In view of (27), it follows that

$$\begin{aligned} \frac{\partial V}{\partial x}(x) f(x,u) &\leq \frac{\varphi(x,u)}{\left|\frac{\partial V}{\partial x}(x)\right|^2} \left|\frac{\partial V}{\partial x}(x)\right|^2 \\ &\leq \varphi(x,u) , \end{aligned}$$

which, together with (16) establishes (15) and thus ends the proof.

C. Proof of Theorem 1

First of all, notice that, since V_2 is a Lyapunov function candidate, (4) holds for some $\underline{\alpha}_2, \overline{\alpha}_2 \in \mathcal{K}_\infty$. Let $u_2 := (x_1^T, d_2^T)^T$ and consider

$$\varphi(x_2, u_2) = -\alpha_2(|x_2|) + \gamma_2(|x_1|) + \tilde{\varphi}_2(|d_2|) ,$$

where $\tilde{\varphi}_2$ is the class $\mathcal{K}$ function defined as

$$\tilde{\varphi}_2(s) := \frac{1}{2} \min\left\{\varphi_2(s); \alpha_2(s)\right\} , \quad \forall s \geq 0 .$$

This construction of $\tilde{\varphi}_2$ ensures that the function $\alpha_2^{-1} \circ \tilde{\varphi}_2$ is well defined over $\mathbb{R}_{\geq 0}$. Also, this function satisfies (13) for any continuous nonnegative function σ such that, for all $u_2 \in \mathbb{R}^{n_1+m_2}$, $\sigma(u_2) \leq \alpha_2^{-1} \circ \tilde{\varphi}_2(|d_2|)$. In particular, this condition is fulfilled with $\sigma(u_2) = \sigma_2(|d_2|)$, if σ_2 is the $\mathcal{K}$ function defined as

$$\sigma_2(s) := \alpha_2^{-1} \circ \tilde{\varphi}_2 \left(\frac{d_2^{min} s}{2 d_2^{max}}\right), \forall s \geq 0. \tag{28}$$

Applying Lemma 1 to V_2 with the above functions φ and σ ensures the existence of a vector field f_2 such that $\dot{V}_2 := \frac{\partial V_2}{\partial x_2}(x_2) f_2(x_2, x_1, d_2) \leq -\alpha_2(|x_2|) + \gamma_2(|x_1|) + \tilde{\varphi}_2(|d_2|)$, for all $x \in \mathbb{R}^n$ and all $d_2 \in \mathbb{R}^{n_2}$. This makes Assumption 2 fulfilled by noticing that $\tilde{\varphi}_2(s) \leq \varphi_2(s)$ for all $s \in \mathbb{R}_{\geq 0}$. Lemma 1 also guarantees that, for all x and d_2 satisfying $|x_2| \geq \sigma_2(|d_2|)$,

$$\begin{aligned} \dot{V}_2 &= -\alpha_2(|x_2|) + \gamma_2(|x_1|) + \tilde{\varphi}_2(|d_2|) \\ &\geq -\alpha_2(|x_2|) + \tilde{\varphi}_2(|d_2|) . \end{aligned} \tag{29}$$

Note that, since $\sigma(u_2) = \sigma_2(|d_2|)$ and $\sigma_2 \in \mathcal{K}$, $\ker(\sigma) = \mathbb{R}^{n_1} \times (\mathbb{R}^{m_2} \setminus \{0\})$. Lemma 1 thus ensures that f_2 is continuous over $\mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times (\mathbb{R}^{m_2} \setminus \{0\})$. Now, consider any disturbance $d_2 \in \mathcal{U}^{m_2}$ satisfying (11) and let $\underline{d}_2 := \operatorname{ess\,inf}_{\tau \geq 0} |d_2(\tau)|$. Note that it holds that

$$\underline{d}_2 \geq d_2^{min} , \quad \|d_2\| \leq d_2^{max} . \tag{30}$$

Let $\theta \in \Theta$ be any arbitrary tuning gain, let $d_1 \in \mathcal{U}^{m_1}$, and consider any forward complete solution of (1) starting with an initial condition $x^0 = (x_1^{0T}, x_2^{0T})^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying

$$\left|x_2^0\right| \geq \alpha_2^{-1} \circ \tilde{\varphi}_2(\overline{d}_2) . \tag{31}$$

In view of (28), this ensures in particular that

$$\left|x_2^0\right| > \sigma_2(\|d_2\|) . \tag{32}$$

Let $t_1 \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ be defined as

$$t_1 := \sup\left\{t \geq 0 \,:\, |x_2(\tau)| \geq \sigma_2(\|d_2\|) \,\forall \tau \in [0,t)\right\} . \tag{33}$$

In view of (32) and invoking the continuity of solutions, it holds that $t_1 \in \mathbb{R}_{>0} \cup \{\infty\}$ and, for all $t \in [0, t_1)$, it holds from (4) and (29) that

$$\begin{aligned} \dot{v}_2(t) &\geq -\alpha_2(|x_2(t)|) + \tilde{\varphi}_2(|d_2(t)|) \\ &\geq -\alpha_2 \circ \underline{\alpha}_2^{-1}(v_2(t)) + \tilde{\varphi}_2(\underline{d}_2) , \end{aligned} \tag{34}$$

where $v_2(\cdot) := V_2(x_2(\cdot))$. We then rely on the following lemma, proved in Section VII-E.

Lemma 3 *Let α be a class $\mathcal{K}$ locally Lipschitz function and let $a \in \mathbb{R}_{\geq 0}$. Let $[0, \bar{t}) \subset \mathbb{R}_{\geq 0}$ be the maximum interval of existence of a diffentiable function v whose derivative satisfies $\dot{v}(t) \geq -\alpha(v(t)) + a$ for all $t \in [0, \bar{t})$. Then the following implication holds:*

$$\alpha(v(0)) \geq a \,\Rightarrow\, \alpha(v(t)) \geq a, \forall t \in [0, \bar{t}).$$

Recalling that the function $\alpha_2 \circ \underline{\alpha}_2^{-1}$ is invertible over $[0, \tilde{\varphi}_2(\underline{d}_2)]$ by construction of $\tilde{\varphi}_2$, Equation (34) together with Lemma 3 ensure that

$$\begin{gathered} v_2(0) \geq \underline{\alpha}_2 \circ \alpha_2^{-1} \circ \tilde{\varphi}_2(\underline{d}_2) \,\Rightarrow \\ v_2(t) \geq \underline{\alpha}_2 \circ \alpha_2^{-1} \circ \tilde{\varphi}_2(\underline{d}_2) , \;\forall t \in [0, t_1) , \end{gathered}$$

which yields, in view of (4),

$$\left|x_2^0\right| \geq \eta_2(\underline{d}_2) \;\Rightarrow\; |x_2(t)| \geq \nu_2(\underline{d}_2) ,$$

where the functions $\eta_2, \nu_2 \in \mathcal{K}$ are defined as

$$\begin{aligned} \eta_2 &:= \alpha_2^{-1} \circ \tilde{\varphi}_2 \\ \nu_2 &:= \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2 \circ \alpha_2^{-1} \circ \tilde{\varphi}_2 . \end{aligned} \tag{35}$$

Equation (31) guarantees that the left-hand side of this implication holds true. Hence

$$|x_2(t)| \geq \nu_2(\underline{d}_2) , \quad \forall t \in [0, t_1) . \tag{36}$$

In other words, Theorem 1 is proved if we show that $t_1 = +\infty$. If it were not the case, then it would mean, in view of (33), that

$$|x_2(t_1)| = \sigma_2(\|d_2\|) . \tag{37}$$

Consider the greatest time $t_2 \geq 0$ for which

$$|x_2(t)| \geq \nu_2(\underline{d}_2) , \quad \forall t \in [0, t_2] . \tag{38}$$

In view of (36), we necessarily have that $t_2 \geq t_1$. But (28) and (35) ensure that $\sigma_2(\overline{d}_2) < \nu_2(\underline{d}_2)$. The continuity of solutions together with (33), (37) and (38) then impose that $t_2 < t_1$, which induces a contradiction. Thus, t_1 is infinite, which makes (36) valid for all $t \geq 0$ and concludes the proof.

D. Proof of Theorem 2

Let Υ_2, W_2 and d_2 be generated by Assumption 3. Since W_2 is a Lyapunov function candidate, there exist $\underline{a}_2, \overline{a}_2 \in \mathcal{K}_\infty$ such that, for all $x_2 \in \mathbb{R}^{n_2}$,

$$\underline{a}_2(|x_2|) \leq W_2(x_2) \leq \overline{a}_2(|x_2|)\,. \quad (39)$$

Given any $\delta \geq 0$, let $\bar{\delta} := \underline{a}_2^{-1} \circ \overline{a}_2(\delta)$. Note that $\bar{\delta} \geq \delta$. Define also

$$d_2'(x) := \mathrm{sat}_{\Upsilon_2(\bar{\delta})}\big(d_2(x)\big) \quad \forall x \in \mathbb{R}^{n_1+n_2}. \quad (40)$$

Given any $\theta \in \Theta$, any $d_1 \in \mathbb{R}^{n_1}$ and any $x^0 \in \mathbb{R}^n$, let $x(\cdot) := x(\cdot; x^0, d_1, d_2', \theta)$ denote the solution of (1) and let $d_2^\star(t) := d_2'(x(t))$ for all $t \geq 0$. Note that, if the system (1) is forward complete, then $d_2^\star(t)$ exists for all $t \geq 0$. Also, in view of (17) and (40) and recalling that $\underline{a}_2^{-1} \circ \overline{a}_2(s) \geq s$ for all $s \geq 0$, $d_2^\star$ satisfies (18) with $\Upsilon = \Upsilon_2$. In addition,

$$|x(t)| \leq \bar{\delta} \quad \Rightarrow \quad d_2^\star(t) = d_2(x(t))\,.$$

In view of Assumption 3, the derivative of W_2 along the solutions of (1) then satisfies

$$|x(t)| \leq \bar{\delta} \quad \Rightarrow \quad \dot{W}_2(x_2(t)) > 0\,. \quad (41)$$

Hence, with (39) and the continuity of $x(\cdot)$,

$$|x(s)| \leq \bar{\delta},\, \forall s \in [0,t) \;\Rightarrow\; |x_2(t)| > \overline{a}_2^{-1} \circ \underline{a}_2(|x_2^0|). \quad (42)$$

We prove the following two facts:

$$|x_2^0| > \bar{\delta} \quad \Rightarrow \quad |x_2(t)| \geq \delta,\, \forall t \geq 0 \quad (43)$$

$$|x_2^0| \leq \bar{\delta} \;\Rightarrow\; \exists t \geq 0 \,:\, |x_2(t)| \geq \bar{\delta}\,. \quad (44)$$

These combined properties establish that, in any cases, the solution $x_2(\cdot)$ (and consequently $x(\cdot)$) eventually leaves the ball $\mathcal{B}_\delta$ and never goes back into it, *i.e.*, for all $x^0 \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\lim_{t\to\infty} |x_2(t)|_{\mathbb{R}^{n_2}\setminus\mathcal{B}_\delta} = 0\,,$$

which indeed establishes the result. In order to establish (43)-(44), consider any $x_2^0 \in \mathcal{B}_{\bar{\delta}}$ and let

$$\varepsilon := \min\left\{\dot{W}_2(z) \,:\, \overline{a}_2^{-1} \circ \underline{a}_2(|x_2^0|) \leq |z| \leq \bar{\delta}\right\}\,.$$

Note that such an ε exists as $\dot{W}_2$ is continuous and the considered set is compact and non empty (since $\overline{a}_2^{-1} \circ \underline{a}_2(\bar{\delta}) \leq \bar{\delta}$). In view of (41), it holds that $\varepsilon > 0$ and $\dot{W}_2(x_2(t)) \geq \varepsilon$ as long as $|x(t)| \leq \bar{\delta}$. Integrating this inequality and exploiting (39) then guarantees that, as long as $x_2(\cdot)$ remains inside $\mathcal{B}_{\bar{\delta}}$,

$$|x_2(t)| \geq \overline{a}_2^{-1}\left(\underline{a}_2(|x_2^0|) + \varepsilon t\right)\,.$$

This establishes (44). Finally, for $|x_2^0| > \bar{\delta}$, either $|x_2(t)| > \bar{\delta}$ at all times in which case the right-hand side of (43) holds, or there exists a time $t^\star$ at which $|x_2(t^\star)| = \bar{\delta}$. Exploiting the above inequality by considering $x_2(t^\star)$ as the initial condition then shows that, as long as $x_2(\cdot)$ remains inside $\mathcal{B}_{\bar{\delta}}$, $|x_2(t)| \geq \overline{a}_2^{-1}\left(\underline{a}_2(\bar{\delta}) + \varepsilon t\right) \geq \delta$, which establishes (43) and thus ends the proof.

E. Proof of Lemma 3

We distintiguish between two cases.

Case 1: $a < \alpha(\infty)$. Consider the differential equation $\dot{y} = -\alpha(y) + a$. Then letting $z := y - \alpha^{-1}(a)$ yields $\dot{z} = -\tilde{\alpha}(z)$ where $\tilde{\alpha}$ is the locally Lipschitz class $\mathcal{K}$ function defined as $\tilde{\alpha}(s) := \alpha(s + \alpha^{-1}(a)) - a$. By [13, Lemma 4.4], $z(\cdot)$ exists over $\mathbb{R}_{\geq 0}$ and satisfies $z(t) = \beta(z(0), t)$, where $\beta \in \mathcal{KL}$, for all $z(0) \geq 0$, and all $t \geq 0$. In terms of y, this reads $y(t) = \beta(y(0) - \alpha^{-1}(a), t) + \alpha^{-1}(a)$ for all $y(0) \geq \alpha^{-1}(a)$. But [13, Lemma 3.3] guarantees that, if $v(0) \geq y(0)$, then $v(t) \geq y(t)$ for all $t \in [0, \bar{t})$. It follows that, for all $v(0) \geq \alpha^{-1}(a)$, $v(t) \geq \alpha^{-1}(a)$ for all $t \in [0, \bar{t})$.

Case 2: $a \geq \alpha(\infty)$. In this case, $\dot{v}(t) \geq 0$ for all $t \in [0, \bar{t})$, which makes the claim trivial.

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References

[1] D. Angeli and A. Astolfi. A tight small gain theorem for not necessarily ISS systems. In *Proc. 44th. IEEE Conf. Decision Contr.*, Sevilla, Spain, 2005.

[2] D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input to state stability. *IEEE Trans. on Automat. Contr.*, 45:1082–1097, 2000.

[3] H.W. Bode. *Network analysis and feedback amplifier design*. Van Nostrand, USA, 1945.

[4] S. Dashkovskiy, B. Rueffer, and F. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010.

[5] G.F. Franklin, J.D. Powell, and A. Emami-Naeini. *Feedback Control of Dynamic Systems*. Prentice-Hall, 2010.

[6] R.A. Freeman and P.V. Kokotović. *Robust Nonlinear control design: State-space and Lyapunov control techniques*. Birkhäuser, Boston, 1996.

[7] J. Freudenberg, C. Hollot, R. Middleton, and V. Toochinda. Fundamental design limitations of the general control configuration. *IEEE Trans. on Automat. Contr.*, 48(8):1355–1370, 2003.

[8] J. Freudenberg and D. Looze. *Frequency domain properties of scalar and multivariable feedback systems*. Springer-Verlag, Berlin, 1988.

[9] H. Ito, R. Freeman, and A. Chaillet. Towards uniform linear time-invariant stabilization of systems with persistency of excitation. In *Proc. 49th IEEE Conf. on Dec. and Contr.*, Atlanta, USA, December 2010.

[10] H. Ito and Z.P. Jiang. Necessary and sufficient small gain conditions for integral input-to-state stable systems: A Lyapunov perspective. *IEEE Trans. on Automat. Contr.*, 54:2389–2404, 2009.

[11] Z. P. Jiang, A. Teel, and L. Praly. Small gain theorems for ISS systems and applications. *Math. of Cont. Sign. and Syst.*, 7:95–120, 1994.

[12] Z.P. Jiang, I.M.Y. Mareels, and Y. Wang. A Lyapunov formulation of nonlinear small gain theorem for interconnected systems. *Automatica*, 32:1211–1215, 1996.

[13] H. Khalil. *Nonlinear systems*. Prentice Hall, 3rd ed., New York, 2001.

[14] D. Liberzon, E.D. Sontag, and Y. Wang. Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation. *Syst. & Contr. Letters*, 46:111–127, 2002.

[15] J.M. Maciejowski. *Multivariable feedback design*. Addison-Wesley, 1989.

[16] L. Praly and Y. Wang. Stabilization in spite of matched unmodelled dynamics and an equivalent definition of input-to-state stability. *Math. of Cont. Sign. and Syst.*, 9:1–33, 1996.

[17] M. Seron, J. Braslavsky, and G.C. Goodwin. *Fundamental limitations in filtering and control*. Springer Verlag, 1997.

[18] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. on Automat. Contr.*, 34(4):435–443, 1989.

[19] E.D. Sontag. A ”universal” construction of Artstein’s theorem on nonlinear stabilization. *Syst. & Contr. Letters*, 13(2):117–123, 1989.
[20] E.D. Sontag. Comments on integral variants of ISS. *Syst. & Contr. Letters*, 34:93–100, 1998.
[21] A.R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Trans. on Automat. Contr.*, 40:1256–1270, 1996.
[22] A.R. Teel and L. Praly. On assigning the derivative of a disturbance attenuation Control Lyapunov Function. *Math. of Cont., Sign. and Syst.*, 13:95–124, 2000.