Rings whose indecomposable modules are pure-projective or pure-injective

Francois Couchot

RINGS WHOSE INDECOMPOSABLE MODULES ARE PURE-PROJECTIVE OR PURE-INJECTIVE

FRANÇOIS COUCHOT

Abstract. It is proven each ring R for which every indecomposable right module is pure-projective is right pure-semisimple. Each commutative ring R for which every indecomposable module is pure-injective is a clean ring and for each maximal ideal P, R_P is a maximal valuation ring. Complete discrete valuation domain of rank one are examples of non-artinian semi-perfect rings with pure-injective indecomposable modules.

In this short note we give a partial answer to the following question posed by D. Simson in [5, Problem 3.2]:

"Give a characterization of rings R for which every indecomposable right R-module is pure-projective or pure-injective. Is every such a semi-perfect ring R right artinian or right pure-semisimple?"

From a result of Stenström we easily deduce that each ring R whose indecomposable right modules are pure-projective is right pure-semisimple.

Theorem 1. *Let R be a ring for which each indecomposable right module is pure-projective. Then R is right pure-semisimple.*

Proof. Let M be a non-zero right R-module. By [1, Proposition 1.13] M is pure-projective if each pure submodule N for which M/N is indecomposable is a direct summand. This last property holds since each indecomposable right R-module is pure-projective. Hence M is pure-projective and R is right pure-semisimple. □

Now we give an example of a non-artinian commutative semi-perfect ring whose indecomposable modules are pure-injective.

Example 2. Let R be a complete discrete valuation domain of rank one and let Q be its quotient field. By [4, Corollary 2 p.52] each indecomposable module is cyclic or isomorphic to a factor of Q and these modules are pure-injective. Clearly R is not artinian.

Theorem 3. *Let R be a commutative ring for which each indecomposable module is pure-injective. Then R is a clean ring and R_P is a maximal valuation ring for each maximal ideal P. Moreover, each indecomposable R-module is uniserial.*

Proof. By [6, Theorem 9(3)] each indecomposable pure-injective module has a local endomorphism ring. So, if each indecomposable module is pure-injective, then, by [2, Theorem III.7], R is a clean ring and R_P is a valuation ring for each maximal

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ideal P. Since R_P is indecomposable for each maximal ideal P, it is pure-injective. It follows that R_P is maximal for each maximal ideal P.

Let M be an indecomposable R-module. By [2, Proposition III.1], there exists a unique maximal ideal P such that $M = M_P$ and its structures of R-module and R_P-module coincide. So, we may assume that R is local. By [3, Theorem 5.4] M is either the pure-injective hull of an uniserial module or an essential extension of a cyclic module. Since R is maximal, it follows that M uniserial. □

Remark 4. It remains to characterize maximal valuation rings for which every indecomposable module is uniserial.

Example 5. If R is the ring defined in [2, Example III.8] or if R satisfies the equivalent conditions of [2, Theorem III.4] then R is a ring for which each indecomposable module is pure-injective and uniserial.

Corollary 6. *Let R be a perfect commutative ring for which each indecomposable module is pure-injective. Then R is pure-semisimple.*

Proof. R is a finite product of artinian valuation rings by Theorem 3. Hence R is pure-semisimple. □

Now we give an example of ring R whose indecomposable right modules are pure-injective and for which there exists an indecomposable left module which is not pure-injective.

Example 7. Let K be a field, let V be a vector space over K which is not of finite dimension, let $S = \mathrm{End}_K(V)$, let J be the set of finite rank elements of S and let R be the K-subalgebra of S generated by J. Then, R is Von Neumann regular, V and ${}_RR/J \cong K$ are the sole types of simple left modules, and $V^* = \mathrm{Hom}_K(V, K)$ and $R_R/J \cong K$ are the sole types of simple right modules. It is easy to check that V^* and R_R/J are injective, whence R is a right V-ring. On the other hand each right R-module contains a simple module. So, every indecomposable right module is simple. It follows that every indecomposable right module is pure-injective (injective). It is well known that V is FP-injective, V^{**} is the injective hull of V and $V \subset V^{**}$. So, V is not pure-injective.

References

[1] G. Azumaya. Countabled generatedness version of rings of pure global dimension 0. volume 168 of *Lecture Notes Series*, pages 43–79. Cambridge University Press, (1992).

[2] F. Couchot. Indecomposable modules and Gelfand rings. *Comm. Algebra*, 35(1):231–241, (2007).

[3] A. Facchini. Relative injectivity and pure-injective modules over Prüfer rings. *J. Algebra*, 110:380–406, (1987).

[4] I. Kaplansky. *Infinite Abelian Groups.* the University of Michigan Press, Ann Arbor, (1969).

[5] D. Simson. Dualities and pure semisimple rings. In *Abelian groups, module theory and topology*, volume 201 of *Lecture Notes in Pure and Appl. Math.*, pages 381–388. Marcel Dekker, (1998).

[6] B. Zimmermann-Huisgen and W. Zimmermann. Algebraically compact rings and modules. *Math. Z*, 161:81–93, (1978).

Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139, Département de mathématiques et mécanique, 14032 Caen cedex, France

E-mail address: francois.couchot@unicaen.fr