Must Physics Be Constrained by the Archimedean Axiom? Relativity and Quanta with Scalars given by Reduced Power Algebras

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Dedicated to Marie-Louise Nykamp

Abstract

It is shown that, unknown to nearly everyone, modern theoretical Physics is significantly *constrained* by the tacit acceptance of the ancient Archimedean Axiom imposed upon Geometry by Euclid more than two millennia ago, an axiom which does not seem to have any modern physical motivation. By freeing oneself of this axiom a large variety of scalar fields and algebras other, and larger than the usual fields $\mathbb{R}$ and $\mathbb{C}$ of real, respectively complex numbers becomes available for mathematical modelling in theoretical Physics. This paper shows the validity of such modelling in Special Relativity and Quantum Mechanics, namely, in the case of the Lorentz transformations, the Heisenberg Uncertainty and the No Cloning property. The advantages in using such alternative scalars, specifically, given by *reduced power algebras* or *ultrapower fields*, are multiple. Among them, one can in a simple and direct way eliminate the so called "infinities in

physics". More generally, one can introduce many levels of precision in theoretical Physics. This is much unlike the present situation when, with the use of $\mathbb{R}$ as the only basic scalar field, there can exist only one single level of precision. Also, one can establish a *Second Relativity Principle* in which the covariance of the equations of Physics and of basic physical phenomena and properties is considered not only with respect to changes of reference frames, but also with changes of algebras or fields of scalars. In this regard, this paper shows that the Lorentz transformations, the Heisenberg Uncertainty and the No Cloning property are indeed covariant with respect to a large variety of scalars given by reduced power algebras, or in particular, ultrapower fields.

"History is written with the feet ..."

Ex-Chairman Mao, of the Long March fame ...

Science is not done scientifically, since it is mostly done by non-scientists ...

Anonymous

Physics is too important to be left to physicists ...

Anonymous

Is the claim about the validity of the so called "physical intuition" but a present day version of medieval claims about the sacro-sant validity of theological revelations ?

Anonymous

A “mathematical problem” ?
For sometime by now, American mathematicians
have decided to hide their date of birth
and not to mention it in their academic CV-s.
Why ?
Amusingly, Hollywood actors and actresses have their
birth date easily available on Wikipedia.
Can one, therefore, trust American
mathematicians ?
Why are they so blatantly against transparency ?
By the way, Hollywood movies have also for long
been hiding the date of their production ...

A bemused non-American mathematician

Part I : Special Relativity in Reduced Power Algebras

1. A Well Known Usual Deduction of the Lorentz Coordinate Transformations

Here, the Lorentz coordinate transformations, fundamental in Special Relativity, are extended to versions of Special Relativity that are reformulated in terms of scalars in reduced power algebras, instead of the usual real or complex scalars.

The following elementary way to obtain the Lorentz coordinate transformations is well known, [1]. Given two coordinate systems S and $S\,'$ with respective coordinates (x, t) and (x', t') in which the space x-axis and x'-axis are along the same line. We suppose that at time $t = t' = 0$ the origins O and O' of the two coordinate systems coincide, thus $x = x' = 0$. Let now S' move along the$x$-axis in the positive direction with the constant velocity v, and let two observers be respectively at O and O'.

In that setup, at the initial moment $t = t' = 0$ and when O and O' coincide, a light signal is emitted from O. Its propagation within S is then given by

$$(1.1) \quad x^2 = c^2 t^2$$

where $c > 0$ is the velocity of light.

Now, in view of the Principle of Constancy of the Velocity of Light, in the coordinate system S' the propagation of that light signal is according to

$$(1.2) \quad x'^{\,2} = c^2 t'^{\,2}$$

Consequently, one must have

$$(1.3) \quad x^2 - x'^{\,2} = c^2 (t^2 - t'^{\,2})$$

However, at least for small values of v, when compared with c, we must have

$$(1.4) \quad x' = k(c, v)(x - vt)$$

for some positive $k(c, v) \in \mathbb{R}$ that does not depend on x, t, x', t', and which in addition is such that

$$(1.5) \quad \lim_{v \to 0} k(c, v) = 1$$

since (1.4) and (1.5) are implied by the respective non-relativistic Galilean coordinate transformation.
Now in view of the Principle of Relativity of Motion, we can suppose that S' is fixed, and S is moving along the x'-axis and in the negative direction, with velocity $-v$. In that case, similar with (1.4), we obtain

$$(1.6) \quad x = k(c, v)(x' + vt')$$

By squaring (1.4) and (1.6), we obtain

$$(1.7) \quad x'^{\,2} + k(c,v)^2 x^2 - 2k(c,v)xx' = k(c,v)^2 v^2 t^2$$

$$(1.8) \quad x^2 + k(c,v)^2 x'^{\,2} - 2k(c,v)xx' = k(c,v)^2 v^2 t'^{\,2}$$

thus by subtracting the (1.7) from (1.8), it follows that

$$(1.9) \quad (x^2 - x'^{\,2})(k(c,v)^2 - 1) = k(c,v)^2 v^2 (t^2 - t'^{\,2})$$

and then in view of (1.3), we obtain

$$(1.10) \quad c^2(k(c,v)^2 - 1) = k(c,v)^2 v^2$$

or

$$(1.11) \quad (c^2 - v^2)k(c,v)^2 = c^2$$

In this way

$$(1.12) \quad k(c,v) = c/(c^2 - v^2)^{1/2} = 1/(1 - v^2/c^2)^{1/2}$$

which obviously satisfies (1.5).

The *space coordinate* Lorentz transformation results now from (1.4) and (1.12), namely

$$(1.13) \quad x' = (x - vt)/(1 - v^2/c^2)^{1/2}$$

In order to obtain the *time coordinate* Lorentz transformation, it will be convenient to proceed in full algebraic detail, and with a special attention to the operations of *division* and *square root* involved. For that purpose, we replace x' in (1.6) with its value from (1.4). The result is

$$(1.14) \quad \begin{aligned} x &= k(c,v)(k(c,v)(x - vt) + vt') = \\ &= k(c,v)^2 x - k(c,v)^2 vt + k(c,v)vt' \end{aligned}$$

or

$$(1.15) \quad k(c,v)vt' = k(c,v)^2 vt - (k(c,v)^2 - 1)x$$

Thus dividing by $k(c,v)v$, one has

$$(1.16) \quad t' = k(c,v)t - (k(c,v)^2 - 1)/(k(c,v)v))x$$

Dividing in (1.11) by $c^2 - v^2$, results that

$$(1.17) \quad k(c,v)^2 = c^2/(c^2 - v^2)$$

and then

$$(1.18) \quad k(c,v)^2 - 1 = v^2/(c^2 - v^2)$$

Now (1.16), (1.12) yield the desired *time coordinate* Lorentz transformation

$$(1.19) \quad t' = (t - vx/c^2)/(1 - v^2/c^2)^{1/2}$$

2. Extending the Lorentz Coordinate Transformations to Reduced Power Algebras

Let us consider instead of the field $\mathbb{R}$ of usual real numbers an arbitrary reduced power algebra $\mathbb{A}_{\mathcal{F}}$, see (A.1.4) in the Appendix. In other words, we shall model both space and time with such algebras $\mathbb{A}_{\mathcal{F}}$, instead of modelling them with the field $\mathbb{R}$ of usual real numbers. Here it is important to note that, in general, such algebras $\mathbb{A}_{\mathcal{F}}$ need *not* be linearly or totally ordered, see (A.4.1) - (A.4.4) in the Appendix. Furthermore, when they are not linearly or totally ordered, that is, when the respective filters $\mathcal{F}$ are not ultrafilters, then the corresponding algebras $\mathbb{A}_{\mathcal{F}}$ need *not* be one dimensional vector spaces, as is of course the case of $\mathbb{R}$.
It follows that the extension of the Lorentz coordinate transformations to reduced power algebras opens up a rather wide realm, one in which time, as much as each individual coordinate, may be *multi-*

dimensional, and in fact, even *infinite dimensional*.
Speculations regarding the possible meaning of such considerable extensions can, therefore, be diverse and rather numerous. One of them, coming from the multi-dimensionality of time, may be that it could possibly model *parallel universes* ...

And now, let us return to the aimed extension the Lorentz coordinate transformations to arbitrary reduced power algebras.

In this regard, it is sufficient to note that all the algebraic operations in section 1 above, operations leading to the usual Lorentz coordinate transformations in (1.13), (1.19), can automatically be replicated in all the reduced power algebras $\mathbb{A}_{\mathcal{F}}$, except when divisions and square roots are involved. Indeed, when divisions are involved in these algebras one has to consider the presence in them of *zero divisors* and *non-invertible* elements, see section A.2. in the Appendix. As for square roots, one has to proceed according to section A.5. in the Appendix.

3. Comments

3.1. Why Hold to the Archimedean Axiom ?

It is seldom realized, especially among physicists, that ever since ancient Egypt and the axiomatization of Geometry by Euclid, we keep holding to the Archimedean Axiom. This axiom, in simplest terms, such as of a partially ordered group G, for instance, means the following property

$$(3.1.1) \quad \exists \;\; u \in G, \; u \geq 0 \; : \; \forall \;\; x \in G \; : \; \exists \;\; n \in \mathbb{N} \; : \; x \leq nu$$

or in other words, there exists a "path length" u, so that every element x in the group can be "overtaken" by a finite number n of "steps" of "length" u. Clearly, if G is the set $\mathbb{R}$ of usual real numbers considered with the usual addition, then one can take as u any positive number. As is known, Geometry in ancient Egypt was important in connection with the yearly flood of the Nile and the subsequent need to redraw the boundaries of agricultural land. And for such a purpose, the Archimedean Axiom is obviously useful.

The question, however, is :

> Why hold to that axiom when dealing with such modern and highly non-intuitive theories of Physics, as Special and General Relativity, or Quantum Mechanics and Quantum Field Theory ?
> Is there any physical type reason in such modern theories for holding to the Archimedean Axiom ?

Indeed, one of the inevitable consequences of the Archimedean Axiom is that "infinity" is not a usual scalar, be it real or complex. Thus all usual algebraic and other operations do rather as a rule break down when reaching "infinity". And this elementary and inevitable fact leads to the long festering problem of the so called "infinities in Physics", a problem which is attempted to be dealt with by various "re-normalization" methods, or by what is an exceedingly complex and so far not yet successful venture, namely, String Theory.

On the other hand, the moment one simply frees oneself from the Archimedean Axiom, and starts to deal with scalars such as those given by various reduced power algebras, the mentioned troubles with "infinity" disappear. Indeed, since the Archimedean Axiom is no longer present in such algebras, these algebras have a rich structure of "infinitesimals" and "infinitely large" scalars, all of which are subjected to the usual algebraic and other operations, just as if they were usual real or complex numbers.

3.2. Two Alternatives When Freed From the Archimedean Axiom

The above way the Lorentz Coordinate Transformations have been extended to space-times built upon scalars given by reduced power algebras may at first seem to be both trivial and without interest. And the same appearance may arise with the extension to such space-times of the Heisenberg Uncertainty and No-Cloning, in [10], respectively, [11].

Here however, one should note the following.

First, even the multiplication in such reduced power algebras is no longer trivial. Indeed, such algebras can have zero divisors, see section A.2. in the Appendix. Consequently, it may easily happen that, although $c, v, x, t, k(c, v) \neq 0$, we will nevertheless have the products in which such quantities appear, and the respective products vanish, contrary to what happens in the usual case when scalars given by real numbers are employed. And clearly, such a vanishing of certain products may invalidate subsequent formulas, or at best, give them a different meaning from the usual one.
Also, mathematical expressions in various theories of Physics can contain operations other than mere multiplication, and such operations can have new properties and meanings, when performed in reduced power algebras.

Therefore, here, we may obviously face two rather different alternatives, namely

- the new properties and meanings in reduced power algebras do not correspond to any possible physical meaning,

or on the contrary

- such new properties and meanings which appear in reduced power algebras may possibly correspond to not yet explored physical realities.

We shall in the sequel mention several possible such new physical interpretations, if not in fact, possible realities.

3.3. Increased and Decreased Precision in Measurements

As a general issue, relating not only to Relativity or the Quanta, the presence of infinitesimal and infinitely large scalars in reduced power algebras may correspond to a new possibility of having no less than *two* radically different kind of measurements when it comes to their *relative precision.*

Namely, one has an *increased precision* in measurement, when measurement is done in terms of usual finite scalars, and one obtains as result some infinitesimal scalar in such algebras.

Alternatively, the presence of infinitely large scalars in such algebras may simply indicate that they were obtained in terms of finite scalars, and thus are but the result of a measurement with *decreased precision.*

In this regard, we can therefore have the following *relative* situations

- infinitesimal scalars are the result of increased precision measurements done in terms of finite or infinite scalars,
- finite scalars are the result of increased precision measurements done in terms of infinite scalars,
- finite or infinitely large scalars are the result of decreased precision measurements done in terms of infinitesimal scalars,
- infinitely large scalars are the result of decreased precision measurements done in terms of infinitesimal or finite scalars.

and surprisingly, one can also have the following *relative* situations

- infinitesimal scalars are the result of increased precision measurements done in terms of some less infinitesimal scalars,
- infinitesimal scalars are the result of decreased precision measurements done in terms of some more infinitesimal scalars,
- infinitely large scalars are the result of increased precision measurements done in terms of some more infinitely large scalars,
- infinitely large scalars are the result of decreased precision measurements done in terms of some less infinitely large scalars.

Indeed, one of the basic features of reduced power algebras is precisely their complicated and rich *self-similar* structure which distinguishes not only between infinitesimal, finite and infinitely large scalars, but also within the infinitely small scalars themselves, and similarly, within the infinitely large scalars. Specifically, infinitesimal scalars can be infinitely smaller, or on the contrary, infinitely larger than other infinitesimals. And similarly, infinitely large scalars can be infinitely smaller, or on the contrary, infinitely larger than other infinitely large scalars.

Here, however, we can note that such a possible interpretation of increased, or decreased precision which is *relative*, is in fact not new. Indeed, in terms of usual scalars, be they real or complex, there is a marked dichotomy between finite scalars, and on the other hand, the so called "infinities" which may on occasion arise from operations with finite scalars. And such simple "formulas" like $\infty + 1 = \infty$, are in fact expressing that fact. Namely, on one hand, from the point of view of "infinity", the finite number 1 has such an increased precision as to be irrelevant with respect to addition, while on the other hand, from the point of view of the finite number 1, the "infinity" has such a decreased precision as to alter completely the result when involved in addition.

3.4. The Issue of Universal Constants

Given the above possibilities in interpretation leading to relative precision measurement - be it as such an increased or a decreased one - one can reconsider the status of certain universal physical constants, such as for instance, the Planck constant h and the constant c giving the velocity of light in vacuum.
Indeed, when considered from our everyday macroscopic experience, h is supposed to be unusually small, while on the contrary, c is very large. Consequently, one may see h as a sort of "infinitesimal", while c then looks like "infinitely large".
The fact is that, within reduced power algebras, such an alternative view of h and c is possible. Therefore, one may find it appropriate to explore the possible physical meaning, or otherwise, that may possibly be associated with such an interpretation.

Appendix : Zero Divisors, Units and other Properties in Reduced Power Algebras

A.1. Construction of Reduced Power Algebras

The general construction of *reduced power algebras* goes as follows, [2-11]. Let Λ be any *infinite* set. Let $\mathcal{F}$ be any filter on Λ, such that

(A.1.1) $\quad \mathcal{F}_{re}(\Lambda) \subseteq \mathcal{F}$

where

(A.1.2) $\quad \mathcal{F}_{re}(\Lambda) = \{ \ I \subseteq \Lambda \ \ | \ \ \Lambda \setminus I \ \text{ is finite } \}$

is called the Frechèt filter on Λ.

We define on $\mathbb{R}^\Lambda$ the corresponding equivalence relation $\approx_\mathcal{F}$ by

(A.1.3) $\quad x \approx_\mathcal{U} y \quad \Longleftrightarrow \quad \{ \ \lambda \in \Lambda \mid x(\lambda) = y(\lambda) \ \} \in \mathcal{F}$

where $x, y \in \mathbb{R}^\Lambda$.

Then, through the usual quotient construction, we obtain the *reduced power algebra*

(A.1.4) $\quad \mathbb{A}_\mathcal{F} = \mathbb{R}^\Lambda / \approx_\mathcal{F}$

which has the following two properties.

The mapping

(A.1.5) $\quad \mathbb{R} \ni r \longmapsto (u_r)_\mathcal{F} \in \mathbb{A}_\mathcal{F}$

is an *embedding of algebras* in which $\mathbb{R}$ is a strict subset of $\mathbb{A}_\mathcal{F}$, where $u_r \in \mathbb{R}^\Lambda$ is defined by $u_r(\lambda) = r$, for $\lambda \in \Lambda$, while $(u_r)_\mathcal{F}$ is the coset of u_r with respect to the equivalence relation $\approx_\mathcal{F}$.

Further, on $\mathbb{A}_\mathcal{F}$ we have the *partial order* which is compatible with the algebra structure, namely

$$\text{(A.1.6)} \quad (x)_{\mathcal{F}} \leq (y)_{\mathcal{F}} \iff \{ \lambda \in \Lambda \mid x(\lambda) \leq y(\lambda) \} \in \mathcal{F}$$

where $x, y \in \mathbb{R}^\Lambda$.

As is well known

$$\text{(A.1.7)} \quad \mathbb{A}_{\mathcal{F}} \text{ is a field} \iff \mathcal{F} \text{ is an ultrafilter on } \Lambda$$

consequently

$$\text{(A.1.8)} \quad \mathbb{A}_{\mathcal{F}} \text{ has zero divisors} \iff \mathcal{F} \text{ is not an ultrafilter on } \Lambda$$

It will be useful to consider the *non-negative* elements in $\mathbb{A}_{\mathcal{F}}$, given by

$$\text{(A.1.9)} \quad \mathbb{A}^+_{\mathcal{F}} = \{ (x)_{\mathcal{F}} \mid x \in \mathbb{R}, \quad \{ \lambda \in \Lambda \mid x(\lambda) \geq 0 \} \in \mathcal{F} \}$$

A.2. Zero Divisors and Units in $\mathbb{A}_{\mathcal{F}}$

Let $\mathcal{F}$ be a filter on Λ which satisfies (A.1.1) and is not an ultrafilter on Λ. Given any $x \in \mathbb{R}^\Lambda$, we denote

$$\text{(A.2.1)} \quad Z(x) = \{ \lambda \in \Lambda \mid x(\lambda) = 0 \} \subseteq \Lambda$$

and obviously, we have the following four alternatives

$$\text{(A.2.2.1)} \quad Z(x) \in \mathcal{F}$$

$$\text{(A.2.2.2)} \quad Z(x) \notin \mathcal{F}$$

$$\text{(A.2.2.3)} \quad \Lambda \setminus Z(x) \in \mathcal{F}$$

$$\text{(A.2.2.4)} \quad \Lambda \setminus Z(x) \notin \mathcal{F}$$

Since $\mathcal{F}$ is not an ultrafilter, alternatives (A.2.2.1) and (A.2.2.3) are not incompatible. Therefore, the same applies to alternatives (A.2.2.2) and (A.2.2.4). It follows that we have the mutually exclusive four alternatives

(A.2.3.1) $\quad Z(x) \in \mathcal{F}$ and $\Lambda \setminus Z(x) \in \mathcal{F}$

(A.2.3.2) $\quad Z(x) \in \mathcal{F}$ and $\Lambda \setminus Z(x) \notin \mathcal{F}$

(A.2.3.3) $\quad Z(x) \notin \mathcal{F}$ and $\Lambda \setminus Z(x) \in \mathcal{F}$

(A.2.3.4) $\quad Z(x) \notin \mathcal{F}$ and $\Lambda \setminus Z(x) \notin \mathcal{F}$

Now in view of (A.1.3), we have

(A.2.4) $$Z(x) \in \mathcal{F} \quad \Longleftrightarrow \quad (x)_{\mathcal{F}} = 0 \in \mathbb{A}_{\mathcal{F}}$$

thus alternatives (A.2.3.1) and (A.2.3.2) are clarified in their consequence.

Let us now consider (A.2.3.3) and define $y \in \mathbb{R}^{\Lambda}$ by

(A.2.5) $$y(\lambda) \;=\; \left| \begin{array}{l} 1/x(\lambda) \;\text{ if }\; \lambda \in \Lambda \setminus Z(x) \\ \\ \text{arbitrary otherwise} \end{array} \right.$$

then (A.1.3), (A.2.4) give

(A.2.6) $$(x)_{\mathcal{F}},\; (y)_{\mathcal{F}} \neq 0 \in \mathbb{A}_{\mathcal{F}}, \quad (x)_{\mathcal{F}}\,(y)_{\mathcal{F}} = 1 \in \mathbb{A}_{\mathcal{F}}$$

thus $(x)_{\mathcal{F}}$ is an *invertible element*, or a *unit* in $\mathbb{A}_{\mathcal{F}}$, and $((x)_{\mathcal{F}})^{-1} = (y)_{\mathcal{F}}$.

In the case of (A.2.3.4), let us define $y \in \mathbb{R}^{\Lambda}$ by

(A.2.7) $$y(\lambda) \;=\; \left| \begin{array}{l} 0 \;\text{ if }\; \lambda \in \Lambda \setminus Z(x) \\ \\ 1 \;\text{ if }\; \lambda \in Z(x) \end{array} \right.$$

then (A.1.3), (A.2.4) give

$$(A.2.8) \quad (x)_{\mathcal{F}},\ (y)_{\mathcal{F}} \neq 0 \in \mathbb{A}_{\mathcal{F}}, \quad (x)_{\mathcal{F}}\,(y)_{\mathcal{F}} = 0 \in \mathbb{A}_{\mathcal{F}}$$

thus $(x)_{\mathcal{F}}$ is a *zero divisor* in $\mathbb{A}_{\mathcal{F}}$.

It follows that the set of *units*, or *invertible elements* in $\mathbb{A}_{\mathcal{F}}$ is given by

$$(A.2.9) \quad \mathbb{A}^{u}_{\mathcal{F}} = \{\ (x)_{\mathcal{F}} \mid x \in \mathbb{R}^{\Lambda},\ Z(x) \notin \mathcal{F},\ \Lambda \setminus Z(x) \in \mathcal{F}\ \}$$

while the set of *zero divisors* in $\mathbb{A}_{\mathcal{F}}$ is given by

$$(A.2.10) \quad \mathbb{A}^{zd}_{\mathcal{F}} = \{\ (x)_{\mathcal{F}} \mid x \in \mathbb{R}^{\Lambda},\ Z(x) \notin \mathcal{F},\ \Lambda \setminus Z(x) \notin \mathcal{F}\ \}$$

and clearly, we have the following partition in three disjoint subsets

$$(A.2.11) \quad \mathbb{A}_{\mathcal{F}} = \{0\} \bigcup \mathbb{A}^{zd}_{\mathcal{F}} \bigcup \mathbb{A}^{u}_{\mathcal{F}}$$

A.3. Infinitesimals and Infinitely Large Scalars

The reduced power algebras $\mathbb{A}_{\mathcal{F}}$ contain strictly as a subfield the field $\mathbb{R}$ of usual real numbers. In addition, the reduced power algebras $\mathbb{A}_{\mathcal{F}}$ contain vast amounts of *infinitesimal*, as well as *infinitely large* scalars.

In case in (A.1.3), and in the sequel, we replace $\mathbb{R}$ with $\mathbb{C}$, and thus $\mathbb{C}^{\Lambda}$ takes the place of $\mathbb{R}^{\Lambda}$, then we obtain reduced power algebras which contain strictly the field $\mathbb{C}$ of usual complex numbers. And again, the reduced power algebras will contain vast amounts of *infinitesimal*, as well as *infinitely large* scalars.

A.4. Reduced Power Fields

The following properties are equivalent :

(A.4.1) $\quad \mathcal{F}$ is an *ultrafilter* on Λ

$$(A.4.2) \quad \mathbb{A}^{zd}_{\mathcal{F}} = \phi, \quad \mathbb{A}_{\mathcal{F}} = \{0\} \bigcup \mathbb{A}^{u}_{\mathcal{F}} \ \text{ is a field}$$

(A.4.3) For every $x \in \mathbb{R}^\Lambda$, the four alternatives (A.2.3.1) - (A.2.3.4) reduce to the following two, namely, (A.2.3.2), (A.2.3.3), that is :

$$Z(x) \in \mathcal{F} \quad \text{and} \quad \Lambda \setminus Z(x) \notin \mathcal{F}$$

$$Z(x) \notin \mathcal{F} \quad \text{and} \quad \Lambda \setminus Z(x) \in \mathcal{F}$$

(A.4.4) The partial order $\leq_{\mathcal{F}}$ in (A.1.6) is a linear, or total order on the reduced power field $\mathbb{A}_{\mathcal{F}}$

A.5. Exponential Functions

In view of (A.1.9), one can obviously define the exponentiation

(A.5.1) $\quad \mathbb{A}^+_{\mathcal{F}} \times \mathbb{A}^+_{\mathcal{F}} \ni ((x)_{\mathcal{F}}, (y)_{\mathcal{F}}) \longmapsto (z)_{\mathcal{F}} = ((x)_{\mathcal{F}})^{((y)_{\mathcal{F}})} \in \mathbb{A}^+_{\mathcal{F}}$

by

(A.5.2) $\quad z(\lambda) = (x(\lambda))^{(y(\lambda))}, \quad \lambda \in I$

where $I \in \mathcal{F}$ is such that

(A.5.3) $\quad x(\lambda),\ y(\lambda) \geq 0, \quad \lambda \in I$

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Part II : Heisenberg Uncertainty in Reduced Power Algebras

1. Preliminaries

The Heisenberg uncertainty relation is known to be obtainable by a purely mathematical argument. Based on that fact, here it is shown that the Heisenberg uncertainty relation remains valid when Quantum Mechanics is re-formulated within far wider frameworks of *scalars*, namely, within one or the other of the infinitely many *reduced power* *algebras* which can replace the usual real numbers $\mathbb{R}$, or complex numbers $\mathbb{C}$. A major advantage of such a re-formulation is, among others, the disappearance of the well known and hard to deal with problem of the so called "infinities in Physics". The use of reduced power algebras also opens up a foundational question about the role, and in fact, about the very meaning and existence, of fundamental constants in Physics, such as Planck's constant h. A role, meaning, and existence which may, or on the contrary, may not be so objective as to be independent of the scalars used, be they the usual real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or scalars given by any of the infinitely many reduced power algebras, algebras which can so easily be constructed and used.

A remarkable feature of the Heisenberg uncertainty relation is that it can be obtained following a purely mathematical argument of a rather simple statistical nature, [1, pp. 67-70]. Based on that fact, here we shall show that the Heisenberg uncertainty relation remains valid when Quantum Mechanics is re-formulated within what appears to be a far more wide and appropriate framework of *scalars*, namely, those in any of the infinitely many algebras which belong to the class of *reduced power algebras*, [3-10].

As argued in [3-10], there are a number of important advantages in re-formulating the whole of Physics in terms of scalars given by reduced power algebras. Related to Quantum Theory, and specifically, to Quantum Field Theory, one of the major advantages of such a reformulation is the complete and automatic disappearance of the well known and hard to deal with problem of the so called "infinities in Physics".

In this regard, let us recall that in [3-10] it was shown how to construct in a simple way as *reduced powers* a large class of algebras which extend the field $\mathbb{R}$ of usual real numbers, or alternatively, the field $\mathbb{C}$ of usual complex numbers. A remarkable feature of these *reduced power algebras* is that they contain *infinitesimal*, as well as *infinitely large* elements, consequently, these algebras are *non-Archimedean*. Some of these algebras are in fact fields. Also, among them is the field $^*\mathbb{R}$ of nonstandard real numbers.

In [3-10] it was suggested and argued that much of present day Physics should be re-formulated in terms of such reduced power algebras, one of the main reasons for that being the considerably increased richness and complexity of their non-Archimedean self-similar mathematical structures, as opposed to the much simpler structures imposed on Physics by the Archimedean field $\mathbb{R}$ of the usual real numbers, and by the structures built upon it, such as the field $\mathbb{C}$ of complex numbers, various finite or infinite dimensional manifolds, Hilbert spaces, and so on. And in this regard, it was argued that one of the main advantages of such a re-formulation would be the automatic disappearance of the difficulties related to the so called "infinities in Physics", as a result of the presence of *infinitely large* elements in the reduced power algebras.

It was also pointed out that the present limitation to the exclusive use of scalars, vectors, etc., which belong to Archimedean mathematical structures is the result not of absolutely any kind of conscious and competent choice in Physics, but on the contrary, of the millennia long perpetuation of a mere historical accident, namely that in ancient Egypt the development of Geometry chose the Archimedean route, due to specific practical needs at those times, needs hardly at all related to those of modern Physics.

However, as not seldom happens in human affairs, accidentally acquired habits can become second nature. This may explain, even if not excuse as well, why for more than two millennia by now we have been so happily wallowing in the ancient Egyptian bondage, or rather slavery of Archimedean space-time structures ...

For convenience, we shall recall in a particular case the construction, [3-10], of *reduced power algebras*. Given any *filter* $\mathcal{F}$ on $\mathbb{N}$, we define

(1.1) $\quad \mathcal{I}_\mathcal{F} = \{ \ v = (v_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \ \mid \ \{ \ n \in \mathbb{N} \mid v_n = 0 \ \} \in \mathcal{F} \ \}$

which is a *proper ideal* in the algebra $\mathbb{R}^\mathbb{N}$. Thus we obtain the *reduced power algebra* associated to $\mathcal{F}$ as the quotient algebra

(1.2) $\quad \mathbb{R}_\mathcal{F} = \mathbb{R}^\mathbb{N} / \mathcal{I}_\mathcal{F}$

Furthermore, this algebra which is commutative, is also a strict extension of the field $\mathbb{R}$ of the usual real numbers, according to the embedding of algebras

$$(1.3) \quad \mathbb{R} \ni x \longmapsto (x, x, x, \ldots) + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} = \mathbb{R}^{\mathbb{N}} / \mathcal{I}_{\mathcal{F}}$$

In a similar manner one can obtain reduced power algebras extending the field $\mathbb{C}$ of the usual complex numbers. Namely, let us denote by

$$(1.4) \quad \mathcal{J}_{\mathcal{F}} = \{ \ w = (w_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \ \mid \ \{ \ n \in \mathbb{N} \mid w_n = 0 \ \} \in \mathcal{F} \ \}$$

which is a *proper ideal* in the algebra $\mathbb{C}^{\mathbb{N}}$. Thus we obtain the *reduced power algebra* associated to $\mathcal{F}$ as the quotient algebra

$$(1.5) \quad \mathbb{C}_{\mathcal{F}} = \mathbb{C}^{\mathbb{N}} / \mathcal{J}_{\mathcal{F}}$$

Furthermore, this algebra which is commutative, is also a strict extension of the field $\mathbb{C}$ of the usual complex numbers, according to the embedding of algebras

$$(1.6) \quad \mathbb{C} \ni z \longmapsto (z, z, z, \ldots) + \mathcal{I}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}} = \mathbb{C}^{\mathbb{N}} / \mathcal{J}_{\mathcal{F}}$$

We now establish a natural connection between the algebras $\mathbb{R}_{\mathcal{F}}$ and $\mathbb{C}_{\mathcal{F}}$.

In this regard, we note the following connection between the ideals $\mathcal{I}_{\mathcal{F}}$ and $\mathcal{J}_{\mathcal{F}}$. Namely

$$(1.7) \quad \begin{aligned} & w = (w_n = u_n + i v_n)_{n \in \mathbb{N}} \in \mathcal{J}_{\mathcal{F}} \iff \\ & \qquad \iff u = (u_n)_{n \in \mathbb{N}}, \ v = (v_n)_{n \in \mathbb{N}} \in \mathcal{I}_{\mathcal{F}} \end{aligned}$$

where $u_n, v_n \in \mathbb{R}$. It follows that we have the algebra homomorphisms

$$(1.8) \quad \begin{aligned} & Re : \mathbb{C}_{\mathcal{F}} \ni w = (w_n = u_n + i v_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \longmapsto \\ & \qquad \longmapsto u = (u_n)_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} \end{aligned}$$

$$(1.9) \quad \begin{aligned} Im : \mathbb{C}_{\mathcal{F}} \ni w = (w_n = u_n + iv_n)_{n\in\mathbb{N}} + \mathcal{J}_{\mathcal{F}} \longmapsto \\ \longmapsto v = (v_n)_{n\in\mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} \end{aligned}$$

as well as the algebra embeddings

$$(1.10) \quad \mathbb{R}_{\mathcal{F}} \ni u = (u_n)_{n\in\mathbb{N}} + \mathcal{I}_{\mathcal{F}} \longmapsto u = (u_n)_{n\in\mathbb{N}} + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}}$$

$$(1.11) \quad \mathbb{R}_{\mathcal{F}} \ni v = (v_n)_{n\in\mathbb{N}} + \mathcal{I}_{\mathcal{F}} \longmapsto iv = (iv_n)_{n\in\mathbb{N}} + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}}$$

Let us also define the surjective linear mapping

$$(1.12) \quad \begin{aligned} \mathbb{C}_{\mathcal{F}} \ni w = (w_n = u_n + iv_n)_{n\in\mathbb{N}} + \mathcal{J}_{\mathcal{F}} \longmapsto \\ \longmapsto \overline{w} = (\overline{w_n} = u_n - iv_n)_{n\in\mathbb{N}} + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}} \end{aligned}$$

As a consequence, we obtain

$$(1.13) \quad w = (w_n = u_n + iv_n)_{n\in\mathbb{N}} + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}}, \quad \overline{w} = w \quad \Longrightarrow \quad w \in \mathbb{R}_{\mathcal{F}}$$

Lastly, we can define the *absolute value* on $\mathbb{C}_{\mathcal{F}}$, by the mapping

$$(1.14) \quad \begin{aligned} \mathbb{C}_{\mathcal{F}} \ni z = (w_n = u_n + iv_n)_{n\in\mathbb{N}} + \mathcal{J}_{\mathcal{F}} \longmapsto \\ \longmapsto |z| = (|w_n| = \sqrt{(}u_n^2 + v_n^2))_{n\in\mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} \end{aligned}$$

Let us denote

$$(1.15) \quad \mathbb{R}^+_{\mathcal{F}} = \{ \ u = (u_n)_{n\in\mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} \ \mid \ \{ \ n \in \mathbb{N} \mid u_n \geq 0 \ \} \in \mathcal{F} \ \}$$

then we obtain the surjective mapping

$$(1.16) \quad \mathbb{C}_{\mathcal{F}} \ni z \longmapsto |z| \in \mathbb{R}^+_{\mathcal{F}}$$

and for $z \in \mathbb{C}_{\mathcal{F}}$, we have

$$(1.17) \quad |z| = 0 \quad \Longleftrightarrow \quad z = 0$$

Now, in view of (1.8), (1.9), (1.14), we have for $z \in \mathbb{C}_{\mathcal{F}}$ the relations

(1.18) $\quad |\, Re\, z\,|, \quad |\, Im\, z\,| \;\leq\; |\, z\,|$

For further convenience, we shall consider a *quantum configuration space* which is one dimensional.

Here however, there are two ways to proceed.

The simpler one is to model the one dimensional configuration space by the usual $\mathbb{R}$, in which case the *wave functions* will be given by

(1.19) $\quad \psi : \mathbb{R} \longrightarrow \mathbb{C}_{\mathcal{F}}$

This means that the only difference with the usual quantum mechanical setup is that, this time, the wave functions can take values in the reduced power algebra extension $\mathbb{C}_{\mathcal{F}}$ of $\mathbb{C}$.

Alternatively, one can be more consistent in the re-formulation of Physics in terms of reduced power algebras, and model not only the values of the wave functions, but also their one dimensional configuration space variables with the reduced power algebra $\mathbb{R}_{\mathcal{F}}$ which is an extension of $\mathbb{R}$. Thus in this second case, the wave functions would be

(1.20) $\quad \psi : \mathbb{R}_{\mathcal{U}} \longrightarrow \mathbb{C}_{\mathcal{F}}$

where $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$.

The first of these two alternatives will be developed in the next section. The second and yet more general alternative is treated elsewhere.

2. An Extension of the Heisenberg Uncertainty

In order to avoid unnecessary technical complications concerning the integrations related to wave functions ψ in (1.19), we shall only consider those of them which are of the following particular *step function* type

(2.1) $\quad \psi = \sum_{1 \leq h \leq m} \gamma_h H_h$

where $m \geq 1$, and $\gamma_h \in \mathbb{C}_{\mathcal{F}}$, while $H_h : \mathbb{R} \longrightarrow \{0, 1\}$ are step functions such that $H_h(x) = 1$, when $x \in I_h$, and $H_h(x) = 0$, when $x \notin I_h$. Here $I_h = [a_{h-1}, a_h) \subset \mathbb{R}$ are usual intervals, where $\infty \leq a_0 < a_1 < a_2 \leq \ldots < a_m \leq \infty$ are usual real numbers, with a_0 and a_m possibly minus and plus infinity, respectively.

The set of all such functions wave functions ψ in (2.1) is denoted by

$$(2.2) \quad \mathcal{S}_{\mathcal{F}}(\mathbb{R})$$

and this set replaces the usual Hilbert space $L^2(\mathbb{R})$ of complex valued square integrable wave functions ψ defined on the configuration space $\mathbb{R}$ of a one dimensional quantum system.

We note here one of the major *advantages* in the above use of reduced power algebras. Namely, each of the wave functions $\psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$ can easily be integrated on the whole of $\mathbb{R}$ regardless of the possible infinite length of some of the intervals I_h, or of the infinite value of some of the γ_h. Indeed, owing to the reduced power algebra structure of $\mathbb{C}_{\mathcal{F}}$, one simply obtains in $\mathbb{C}_{\mathcal{F}}$ the algebraically perfectly well defined value

$$(2.3) \quad \int_{\mathbb{R}} \psi(x) dx = \sum_{1 \leq h \leq m} (a_h - a_{h-1}) \gamma_h \in \mathbb{C}_{\mathcal{F}}$$

which is always a well defined element in $\mathbb{C}_{\mathcal{F}}$, and thus available in a correct and rigorous manner for all the algebraic operations in the algebra $\mathbb{C}_{\mathcal{F}}$, even if that value may turn out to be an infinitesimal, finite, or an infinitely large element in $\mathbb{C}_{\mathcal{F}}$. In this way there is no need to impose any integrability type conditions on the wave functions $\psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$, much unlike in the usual case, where the square integrability condition $\int_{\mathbb{R}} |\psi(x)|^2 dx < \infty$ is required, since within the usual Archimedean framework of $\mathbb{C}$, or $\mathbb{R}$, one cannot perform most of the usual algebraic operations with infinitely large quantities.

Now it follows easily that $\mathcal{S}_{\mathcal{F}}(\mathbb{R})$ is a vector space over $\mathbb{C}$, and in fact, it is a commutative algebra over $\mathbb{C}$. Furthermore, one can define on it the extension of the usual scalar product given by

$$(2.4) \quad < \psi, \chi > \; = \; \int_{\mathbb{R}} \overline{\psi(x)} \chi(x) dx \in \mathbb{C}_{\mathcal{F}}$$

for all $\psi, \chi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$. This extended scalar product has the following properties :

(2.5) It is linear over $\mathbb{C}_{\mathcal{F}}$, therefore also over $\mathbb{C}$, in the second argument.

$$(2.6) \quad < \chi, \psi > \; = \; \overline{< \psi, \chi >}, \quad \psi, \chi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$$

$$(2.7) \quad < \psi, \psi > \in \mathbb{R}^+_{\mathcal{F}}, \quad \psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$$

and for $\psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$, one has

$$(2.8) \quad < \psi, \psi > \; = \; 0 \quad \Longleftrightarrow \quad \psi = 0 \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$$

Also, we have the extension of the classical Schwartz inequality

$$(2.9) \quad | < \psi, \chi > | \; \leq \; < \psi, \psi >^{1/2} < \chi, \chi >^{1/2}, \quad \psi, \chi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$$

Lastly, we can consider the set

$$(2.10) \quad \mathcal{L}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$$

of all *linear operators* $A : \mathcal{S}_{\mathcal{F}}(\mathbb{R}) \longrightarrow \mathcal{S}_{\mathcal{F}}(\mathbb{R})$. Such an operator will be called *Hermitian*, if and only if

$$(2.11) \quad < A\psi, \chi > \; = \; < \psi, A\chi >, \quad \psi, \chi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$$

and we denote by

$$(2.12) \quad \mathcal{H}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$$

the set of all such Hermitian operators.

With these preparations, we can now proceed to obtain the Heisenberg uncertainty relation for arbitrary operators $A, B \in \mathcal{H}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$.

Given $A \in \mathcal{L}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$ and $\psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$, we denote

$$(2.13) \quad < A >_\psi \;=\; < \psi, A\psi > \in \mathbb{C}_{\mathcal{F}}$$

and call it the *expectation value* of A in the state ψ. Further, we denote

$$(2.14) \quad \Delta_\psi A = (< A^2 >_\psi \, - (< A >_\psi)^2\,)^{1/2}$$

and call it the *uncertainty* of A in the state ψ.

Theorem 2.1. (Extended Heisenberg Uncertainty Relation)

Given $A, B \in \mathcal{H}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$ and $\psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$ such that $< \psi, \psi >= 1$, then we have

$$(2.15) \quad \Delta_\psi A \; \Delta_\psi B \;\geq\; | < [A, B] >_\psi |/2$$

where $[A, B] = AB - BA$.

Proof.

We start with

Lemma 2.1.

Let $A \in \mathcal{H}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$, $\psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$, then

$$(2.16) \quad < A >_\psi \in \mathbb{R}_{\mathcal{F}}$$

Proof.

We have in view of (2.13), (2.11)

$$< A >_\psi \;=\; < \psi, A\psi > \;=\; < \psi, A\psi >$$

thus (2.4), (2.3) imply

$$< \psi, A\psi > \ = \ \overline{< \psi, A\psi >}$$

hence (1.13) completes the proof.

$\square$

Let us now denote

(2.17) $\quad A_1 = A - < A >_\psi I, \quad B_1 = B - < B >_\psi I$

where $I \in \mathcal{H}(\mathcal{S}_\mathcal{F}(\mathbb{R}))$ is the identity operator. Then

(2.18) $\quad A_1, B_1 \in \mathcal{H}(\mathcal{S}_\mathcal{F}(\mathbb{R}))$

Indeed, let $\eta, \chi \in \mathcal{S}_\mathcal{F}(\mathbb{R})$, then in view of (2.11) and Lemma 2.1., we have

$$\begin{aligned} < A_1\eta, \chi > \ &= \ < A\eta, \chi > - \overline{< A >_\psi} < \eta, \chi > \ = \\ &= \ < \eta, A\chi > - < A >_\psi < \eta, \chi > \ = \ < \eta, A\chi > - < \eta, < A >_\psi \chi > \ = \\ &= \ < \eta, A_1\chi > \end{aligned}$$

and similarly with B_1.

Next we prove

(2.19) $\quad [A_1, B_1] = [A, B]$

which is obtained as follows. We have from (2.17)

$$A_1 B_1 = AB - < A >_\psi B - < B >_\psi A + < A >_\psi < B >_\psi I$$

thus

$$B_1 A_1 = BA - < B >_\psi A - < A >_\psi B + < B >_\psi < A >_\psi I$$

hence (2.19).

Further, for $\psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$ with $< \psi, \psi >= 1$, we have

$$(2.20) \quad < A_1\psi, A_1\psi > \; = \; (\Delta_\psi A)^2$$

Indeed, in view of (2.18), we obtain

$$< A_1\psi, A_1\psi > \; = \; < \psi, (A_1)^2\psi > \; = \; < \psi, (A - < A >_\psi I)^2 \, \psi > \; =$$

$$= \; < \psi, (A^2 - 2 < A >_\psi A + (< A >_\psi)^2 I) \, \psi > \; =$$

$$= \; < \psi, A^2\psi > - 2 < A >_\psi < \psi, A\psi > + (< A >_\psi)^2 < \psi, \psi > \; =$$

$$= \; < A^2 >_\psi - 2(< A >_\psi)^2 + (< A >_\psi)^2 \; = \; < A^2 >_\psi -(< A >_\psi)^2$$

In view of the above, we have for $\psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$ with $< \psi, \psi >= 1$, the relations

$$< \psi, [A, B]\psi > \; = \; < \psi, [A_1, B_1]\psi > \; = \; < \psi, A_1 B_1 \psi > - < \psi, B_1 A_1 \psi > \; =$$

$$= \; < A_1\psi, B_1\psi > - < B_1\psi, A_1\psi > \; = \; < A_1\psi, B_1\psi > - \overline{< A_1\psi, B_1\psi >} \; =$$

$$= \; 2i \, Im < A_1\psi, B_1\psi >$$

Consequently

$$| < \psi, [A, B]\psi > | \; = \; 2 | \, Im < A_1\psi, B_1\psi > |$$

However, in view of (1.8), (1.9), (1.14), we have for $z \in \mathbb{C}_{\mathcal{F}}$ the relations

$$| \, Re \, z \, |, \quad | \, Im \, z \, | \; \leq \; | \, z \, |$$

3. The Extended Wintner Theorem on Unbounded Operators

In the context of the Heisenberg uncertainty relation, the non-commutativity of operators involved is crucial, since for commutative operators the respective inequality is obviously trivially satisfied, thus there is never any uncertainty, see (2.15).

Within the usual mathematical context of Quantum Mechanics given by complex Hilbert spaces H and linear operators on them, it turns out that certain simple non-commutativity relations for linear operators, such as (3.1) below, necessarily imply their unboundedness. Thus the need to consider unbounded, however, densely defined and closed operators on such Hilbert spaces, and the fundamental operators of position and momentum are well known to be among them.

Needless to say, this fact is a rather inconvenient one, since it complicates considerably the mathematical apparatus involved in Quantum Mechanics. And it was precisely the avoidance of such a complication which led von Neumann to his second mathematical model for Quantum Mechanics, namely, the one based on starting with algebras of observables, and then defining the states. Such an approach is obviosuly a reversal of the way in von Neumann's first model based on Hilbert spaces of states, where the observables are then defined as Hermitian operators.

The classical result regarding the inevitability of unbounded operators in von Neumann's first model of Quantum Mechanics is given in

Wintner's Theorem

Let H be a complex Hilbert space and A, B two bounded linear operators on it. Then there cannot be any nonzero constant $c \in \mathbb{C}$, such that the non-commutation relation holds

(3.1) $\quad [A, B] = AB - BA = cI$

where I is the identity operator on H.

□

The special relevance of this result is in the fact that the position and momentum operators do satisfy a non-commutation relation of type (3.1), this therefore being the reason they cannot be given by bounded operators.

Here, we give an extended version of Wintner's Theorem to the case of linear operators in $\mathcal{L}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$, see (2.2), (2.10), which are defined based on scalars given by reduced power algebras $\mathbb{R}_{\mathcal{F}}$ or $\mathbb{C}_{\mathcal{F}}$.

The following simple linear functional analytic notions, extended to the case of reduced power algebras, will be needed.

For any wave function $\psi = \sum_{1 \leq h \leq m} \gamma_h H_h \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$, see (2.1), (2.2), we define its *norm* by

(3.2) $\quad || \, \psi \, || = \sup_{1 \leq h \leq m} | \, \gamma_h \, | \in \mathbb{R}^+_{\mathcal{F}}$

We note that, with values in $\mathbb{R}^+_{\mathcal{F}}$, and not merely in $\mathbb{R}^+ = [0, \infty)$, as is the usual case with wave functions $\psi \in L^2(\mathbb{R})$, this norm (3.2) is always well defined, regardless of the γ_h being infinitesimal, finite or infinitely large elements in $\mathbb{C}_{\mathcal{F}}$.

Now, a linear operator $A \in \mathcal{L}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$ is called *bounded*, if and only if there exists $M \in \mathbb{R}^+_{\mathcal{F}}$, such that

(3.3) $\quad || \, A\psi \, || \; \leq \; M || \, \psi \, ||, \quad \psi \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$

and in this case we denote by

(3.4) $\quad \mathcal{M}_A \subseteq \mathbb{R}^+_{\mathcal{F}}$

the set of all such M.

We will also need the following partial order relation on $\mathbb{R}_{\mathcal{F}}$. Given $u = (u_n)_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}}, \; v = (v_n)_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}}$, we define

(3.5) $\quad u \; \leq \; v \quad \Longleftrightarrow \quad v - u \in \mathbb{R}^+_{\mathcal{F}}$

Clearly, with the partial order (3.5), $\mathcal{M}_A$ has the property

(3.6) $\quad M' \in \mathbb{R}^+_{\mathcal{F}}, \; M' \geq M \in \mathcal{M}_A \quad \Longrightarrow \quad M' \in \mathcal{M}_A$

The interest in dealing with $\mathcal{M}_A$ is that, in this way, we can avoid the

issue of considering the existence, and of the properties of inf $\mathcal{M}_A$ in the reduced power algebra $\mathbb{R}_{\mathcal{F}}$. Here we note that in the usual case of operators on Hilbert spaces, instead of (3.4) one obviously has $\mathcal{M}_A \subseteq \mathbb{R}^+ = [0, \infty)$, thus inf $\mathcal{M}_A \in \mathbb{R}^+$ always exists, and it is denoted by $||A||$, hence the above issue simply does not arise.
However, as the following two easy to prove Lemmas show it, we can to a good extent avoid that issue even in the general case of arbitrary reduced power algebras $\mathbb{R}_{\mathcal{F}}$.

Lemma 3.1.

Let $\psi, \ \psi' \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$ and $c \in \mathbb{C}_{\mathcal{F}}$, then

1) $|| \, \psi + \psi' \, || \leq || \, \psi \, || + || \, \psi' \, ||$

2) $|| \, c \, \psi \, || = | \, c \, | \; || \, \psi \, ||$

3) $|| \, \psi \, || = 0 \quad \Longleftrightarrow \quad \psi = 0 \in \mathcal{S}_{\mathcal{F}}(\mathbb{R})$

Lemma 3.2.

Let be any bounded operators $A, B \in \mathcal{L}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$ and $c \in \mathbb{C}_{\mathcal{F}}$. Then the following hold

1) $\forall \ K \in \mathcal{M}_A, \ L \in \mathcal{M}_B \ :$

$\exists \ M \in \mathcal{M}_{A+B} \ :$

$M \leq K + L$

2) $\forall \ K \in \mathcal{M}_A \ :$

$\exists \ M \in \mathcal{M}_{cA} \ :$

$M \leq |c| K$

3) $\forall \ K \in \mathcal{M}_A, \ L \in \mathcal{M}_B$:

$\exists \ M \in \mathcal{M}_{AB}$:

$M \leq KL$

The extension of the Wintner Theorem can now be formulated as follows

Theorem 3.1.

Let $A, B \in \mathcal{L}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$ two bounded linear operators. Then there cannot be any nonzero constant $c \in \mathbb{C}_{\mathcal{F}}$, such that the non-commutation relation holds

$$(3.7) \quad [A, B] = AB - BA = cI$$

where I is the identity operator on $\mathcal{S}_{\mathcal{F}}(\mathbb{R})$.

Proof.

Obviously, it suffices to consider the case $c = 1$. Let us then assume $A, B \in \mathcal{L}(\mathcal{S}_{\mathcal{F}}(\mathbb{R}))$ two bounded linear operators, such that

$$(3.8) \quad AB - BA = I$$

Then by induction, one obtains

$$(3.9) \quad nB^{n-1} = AB^n - B^n A, \quad n \geq 1$$

Indeed, for $n = 1$, the relation (3.9) reduces to (3.8). Assuming now that (3.9) holds for a certain $n \geq 1$, we have then

$$(3.10) \quad \begin{aligned} (n+1)B^n &= nB^{n-1}B + B^n I = \\ &= (AB^n - B^n A)B + B^n(AB - BA) + AB^{n+1} - B^{n+1}A \end{aligned}$$

Now in view of Lemma 3.2. applied to (3.9), one obtains

$$
\begin{aligned}
& \forall \;\; n \geq 1, \;\; K \in \mathcal{M}_A, \;\; L \in \mathcal{M}_B \;\; : \\
(3.11) \qquad & \exists \;\; M \in \mathcal{M}_{B^{n-1}} \;\; : \\
& \qquad nM \leq 2KLM
\end{aligned}
$$

consequently, we must have $M = 0$, for some $n \geq 1$. And then, (3.9) implies $B^{n-1} = 0$, and thus successively, $B = 0$, and finally $I = 0$, which of course is absurd.

4. Question on Two Fundamental Physical Constants

Re-formulating Physics in terms of scalars given by reduced power algebras leads naturally to the following two questions

- Is it possible that Planck's constant h is in fact an *infinitesimal* in some reduced power algebra $\mathbb{R}_\mathcal{F}$?
- Is it possible that the maximum speed of propagation of physical effects is not finite, but rather an *infinitely large* quantity in some reduced power algebra $\mathbb{R}_\mathcal{F}$?

The motivation for these two questions appears quite natural, as soon as one becomes more familiar with the non-Archimedean structure of reduced power algebras, [3-10]. Indeed, that non-Archimedean structure leads to the presence of three types of elements in such algebras, namely : infinitesimals, finite elements, and infinitely large elements.

The essential fact in this regard, however, is that the above classification in three types of elements is *relative*. Namely, it is implied by the fact that, when constructing reduced power algebras $\mathbb{R}_\mathcal{F}$, one starts by defining the usual real numbers in $\mathbb{R}$ as being the finite ones. Indeed, such reduced power algebras have a highly complex and rich *self-similar* structure. And it is easy to see that, due to that structure, one is in fact *not* obliged to choose the usual real numbers in $\mathbb{R}$ as being the finite ones. On the contrary, that self-similar structure renders the concept of "finite elements" highly relative, by allowing a wide range of other choices. In this way, elements which in a choice are finite, may become infinitesimal or infinitely large in other choices, and vice-versa.

Consequently, when using reduced power algebras in Physics, thus non-Archimedean scalar structures, one is *no longer* obliged to have both the Planck constant and that of the maximum speed of propagation of physical effects finite, and thus having the only possible difference between between them reduced to a large but finite factor.

5. Question on the Status of the Heisenberg Uncertainty in Quantum Mechanics

As shown in [2], it is possible to present most of Quantum Mechanics *without* recourse to the Heisenberg uncertainty principle. This fact seems particularly surprising to most of physicists, and so far, it seems not to have been given a proper consideration.

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Part III : No-Cloning in Reduced Power Algebras

1. Preliminaries

The No-Cloning property in Quantum Computation is known not to depend on the unitarity of the operators involved, but only on their linearity. Based on that fact, here it is shown that the No-Cloning property remains valid when Quantum Mechanics is re-formulated within far wider frameworks of *scalars*, namely, one or the other of the infinitely many *reduced power algebras* which can replace the usual real numbers $\mathbb{R}$, or complex numbers $\mathbb{C}$.

A remarkable feature of the so called No-Cloning property in Quantum Computation, [3,2], is that it is but a rather elementary and direct consequence of the *linearity* property of unitary operators on finite dimensional complex Hilbert spaces, and in fact, it does *not* require that the respective operators by unitary. The fact that unitary operators are involved in Quantum Computation is natural and unavoidable, since in Quantum Mechanics it is axiomatic that the evolution of a quantum systems which is not under measurement is given by such operators, being described by the Schrödinger equation.

Based on the above elementary fact underlying usual No-Cloning, here we shall show that the No-Cloning property remains valid when Quantum Mechanics is re-formulated within what appears to be a far more wide and appropriate framework of *scalars*, namely, any one of the infinitely many algebras which belong to the class of *reduced power algebras*, [4-12].

One of the essential features of scalars in algebras of reduced powers is that, in addition to being finite, just as the usual real or complex numbers, such scalars in algebras of reduced powers can also be *infinitesimal*, or on the contrary, *infinitely large*. Consequently, vast opportunities for algebraic operations are opened, and also, for appropriate physical interpretations. For instance, one may consider the possibility that the Planck constant h is a nonzero positive infinitesimal, and/or the speed of light c is positive and infinitely large, [12, section 4].

2. An Extension of the No-Cloning Property

We recall that the field $\mathbb{R}$ of usual real numbers can be extended into any of the infinitely many possible so called *reduced power algebras* $\mathbb{R}_{\mathcal{F}}$, where $\mathcal{F}$ suitable filters on the set $\mathbb{N}$ of natural numbers, see Appendix, and for further details [12, pp. 3-6], [4-11]. Similarly, the field $\mathbb{C}$ of usual complex numbers can be extended into any of the infinitely many possible *reduced power algebras* $\mathbb{C}_{\mathcal{F}}$. Furthermore, some of these algebras $\mathbb{R}_{\mathcal{F}}$ and $\mathbb{C}_{\mathcal{F}}$ are themselves fields, namely, when $\mathcal{F}$ are ultrafilters on the set $\mathbb{N}$ of natural numbers.

Let us now recall that in usual Quantum Computation, a quantum register of one qubit is represented as a vector in the complex Hilbert space $\mathbb{C}^2$. And in general, a quantum register of $n \geq 1$ qubits is represented by the n-fold tensor product

$$(2.1) \quad H_n = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \approx \mathbb{C}^{2^n}$$

Here, we shall replace such usual quantum registers of $n \geq 1$ qubits by the larger spaces

$$(2.2) \qquad H_{\mathcal{F},n} = (\mathbb{C}_{\mathcal{F}})^2 \otimes \ldots \otimes (\mathbb{C}_{\mathcal{F}})^2$$

with $n \geq 1$ factors, which again are vector spaces over $\mathbb{C}$, as can easily be seen in [12, pp. 3-6], [4-11]. Furthermore, they possess an extended scalar product (A.20) - (A.26) which gives them properties similar with the usual Hilbert spaces, properties sufficient in order to establish the extended version of the No-Cloning property.

What is important to note is that, since the No-Cloning property does *not* in fact require the unitarity of the operators involved, but only their *linearity*, we can proceed with the extension of the No-Cloning property to quantum registers given by the vector spaces $H_{\mathcal{F},n}$ in (2.2), without having to consider on them any usual Hilbert space structure, and instead, by only using the above mentioned extended Hilbert space structure of these spaces.

In order to make clear this argument, let us briefly recall the usual No-Cloning property, [3].

First, let us note that, scientists are on occasion giving names to new phenomena in ways which are not thoroughly well considered, and thus may lend themselves to misinterpretation. One such case is, unfortunately, with the term *No-Cloning* used in Quantum Computation. What is in fact going on here is that, quite surprisingly, quantum computers do *not* allow the copying of *arbitrary* qubits. And here by "copying" one means the precise reproduction any finite number of times of a given arbitrary qubit, a reproduction which does *not* destroy the original qubit which is being reproduced.
Thus a more proper term would be the somewhat longer one of *no arbitrary copying*.
Yet in spite of that, plenty of copying can be done by quantum computers, as will be seen in the sequel.

In order better to understand the issue, let us start by considering copying classical bits. For that purpose we can use the classical version of the quantum CNOT gate, [2,3], operating this time on bits $a,\ b \in \{\ 0, 1\ \}$, namely

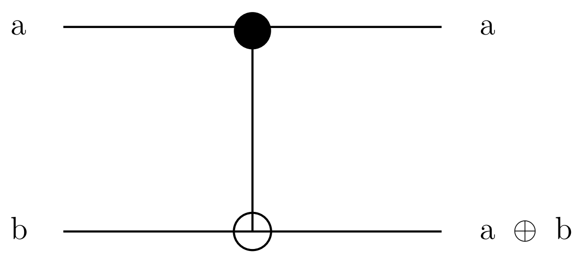

Fig. 2.1.

Now, if we fix $b = 0$, then for an arbitrary input bit $a \in \{\ 0, 1\ \}$, we shall obtain as output two copies of a.

Strangely enough, a similar copying of arbitrary quantum bits cannot be performed by quantum systems, as was discovered in 1982 by W K Wooters and W H Zurek, [2,3].
Of course, as well known, [2,3], each qubit contains a double infinity of classical information since it can be an arbitrary point on the Bloch sphere, which is much unlike the situation with one single bit. In this way, the ability to copy arbitrary qubits is considerably more demanding than copying arbitrary classical bits.

Let us now turn to this issue in some more detail. First we present a simple and somewhat intuitive argument. We assume that we have a quantum system S which allows one qubit at input and has one qubit at output. The output facility we shall use as a "blank sheet" on which we want to copy an arbitrary input qubit $\mid \psi > \ \in \mathbb{C}^2$. We can assume that the initial state of the "blank sheet" at the output is given by a fixed qubit $\mid \chi_0 > \ \in \mathbb{C}^2$. Thus we start with the setup

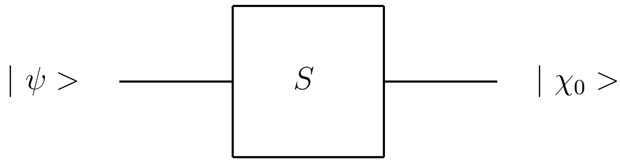

Fig. 2.2.

and would like to end up with the setup

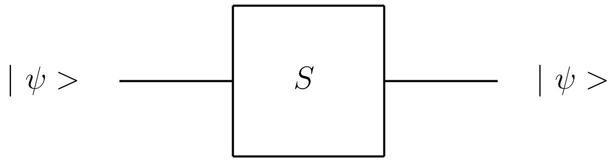

Fig. 2.3.

However, as quantum processes evolve through unitary operators when not subjected to measurement, it means that we are looking for such a unitary operator $U : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$, and one which would act according to

$$(2.3) \quad U(\, | \, \psi > \; \otimes \; | \, \chi_0 > \,) \; = \; | \, \psi > \; \otimes \; | \, \psi >, \quad | \, \psi > \; \in \mathbb{C}^2$$

Before going further, let us immediately remark here that a unitary operator U, which therefore is linear, is not likely to satisfy (2.3), in view of the fact that this is a *nonlinear* relation in $| \, \psi > \; \in \mathbb{C}^2$, and in particular, its left hand term is linear in $| \, \psi >$, while its right hand term is a quadratic in $| \, \psi >$.

And now, let us return to a more precise argument. Since $| \, \psi > \; \in \mathbb{C}^2$ is assumed to be arbitrary in (2.3), we can write that relation for any $| \, \psi_1 >, \; | \, \psi_2 > \; \in \mathbb{C}^2$. Thus we obtain

$$
(2.4) \quad \begin{aligned} U(\mid \psi_1 > \ \otimes \mid \chi_0 > \ ) \ &= \ \mid \psi_1 > \ \otimes \mid \psi_1 > \\ U(\mid \psi_2 > \ \otimes \mid \chi_0 > \ ) \ &= \ \mid \psi_2 > \ \otimes \mid \psi_2 > \end{aligned}
$$

Now if we take the inner product of these two relations and recall that U was supposed to be unitary, we obtain

$$
(2.5) \quad < \psi_1 \ \mid \psi_2 > \ = \ (\ < \psi_1 \mid \psi_2 > \ )^2
$$

which implies that either $< \psi_1 \ \mid \psi_2 > \ = \ 0$, or $< \psi_1 \ \mid \psi_2 > \ = \ 1$. This means that the two arbitrary quantum states $\mid \psi_1 >, \mid \psi_2 > \ \in \mathbb{C}^2$ are always either orthogonal, or identical from quantum point of view, which is clearly absurd.

The general and rigorous argument is as follows. We consider a quantum system whose state space is $\mathbb{C}^m$, for a certain integer $m \geq 2$. Further, we fix in this state space an arbitrary orthonormal basis $\mid \psi_1 >, \ . \ . \ . \ , \mid \psi_m > \ \in \mathbb{C}^m$. Finally, we assume that the state $\mid \psi_1 >$ will function as the "blank sheet" on which we want to copy arbitrary states $\mid \psi > \ \in \mathbb{C}^m$.

Then the desired copying machine of arbitrary states in $\mathbb{C}^m$ will be given by a unitary operator $U : \mathbb{C}^m \otimes \mathbb{C}^m \longrightarrow \mathbb{C}^m \otimes \mathbb{C}^m$, for which we have

$$
(2.6) \quad U(\mid \psi > \ \otimes \mid \psi_1 > \ ) \ = \ \mid \psi > \ \otimes \mid \psi >, \quad \mid \psi > \ \in \mathbb{C}^m
$$

And now we can prove that for $n \geq 2$, there does *not* exist such a copying machine U.

Indeed, if we assume that $n \geq 2$, then we do have at least the two orthonormal states $\mid \psi_1 >, \mid \psi_2 > \ \in \mathbb{C}^m$. Thus (2.6) gives

$$(2.7) \quad \begin{aligned} &U(\mid \psi_1 > \ \otimes \ \mid \psi_1 > \ ) \ = \ \mid \psi_1 > \ \otimes \ \mid \psi_1 > \\ &U(\mid \psi_2 > \ \otimes \ \mid \psi_1 > \ ) \ = \ \mid \psi_2 > \ \otimes \ \mid \psi_2 > \\ &U(\ (\mid \psi_1 > \ + \mid \psi_2 > \ ) \ \otimes \ \mid \psi_1 > \ ) \ = \\ &\qquad = \ (\mid \psi_1 > \ + \mid \psi_2 > \ ) \ \otimes \ (\mid \psi_1 > \ + \mid \psi_2 > \ ) \end{aligned}$$

Now the linearity of U gives together with the first two relations above

$$(2.8) \quad \begin{aligned} &U(\ (\mid \psi_1 > \ + \mid \psi_2 > \ ) \ \otimes \ \mid \psi_1 > \ ) \ = \\ &\quad = \ U(\mid \psi_1 > \ \otimes \ \mid \psi_1 > \ ) \ + \ U(\mid \psi_2 > \ \otimes \ \mid \psi_1 > \ ) \ = \\ &\quad = \ \mid \psi_1 > \ \otimes \mid \psi_1 > \ + \mid \psi_2 > \ \otimes \mid \psi_2 > \end{aligned}$$

Thus (2.8) with the last relation in (2.7) imply

$$(2.9) \quad \begin{aligned} &(\mid \psi_1 > \ + \mid \psi_2 > \ ) \ \otimes \ (\mid \psi_1 > \ + \mid \psi_2 > \ ) \ = \\ &\qquad\qquad = \ \mid \psi_1 > \ \otimes \mid \psi_1 > \ + \mid \psi_2 > \ \otimes \mid \psi_2 > \end{aligned}$$

or in other words

$$(2.10) \quad \mid \psi_1 > \ \otimes \ \mid \psi_2 > \ + \mid \psi_2 > \ \otimes \ \mid \psi_1 > \ = \ 0$$

which is obviously false.

Let us point out two facts with respect to the above no-cloning result.

First, in the more general second proof, we did *not* use the fact that U is unitary, and only made use of its linearity, when we obtained (2.8). In the first proof, on the other hand, the fact that U is unitary was essential in order to obtain (2.5).

Second, it is important to understand properly the meaning of the above limitation implied by No-Cloning. Indeed, while it clearly does not allow the copying of arbitrary qubits, it does nevertheless allow the copying of a *large range* of qubits.

For instance, in terms of the second proof, let $I = \{ 1, \ . \ . \ . \ , n \}$ be the set of indices of the respective orthonormal basis

$$| \psi_1 >, \ . \ . \ . \ , | \psi_n > \quad \in \mathbb{C}^m$$

Further, let us consider the partially defined function

$$c : I \times I \ \longrightarrow \ I \times I$$

given by $c\ (i, 1) = (i, i)$, with $1 \leq i \leq n$. Then clearly, c is injective on the domain on which it is defined. Therefore, c can be extended to the whole of $I \times I$, so as still to remain injective, and in fact, to become bijective as well. And obviously, there are many such extensions when $n \geq 2$.

Now we can define a mapping U by

$$U(\ | \psi_i > \ \otimes \ | \psi_j > \ ) \ = \ | \psi_k > \ \otimes \ | \psi_l >$$

where $1 \leq i, \ j \leq n$ and $c\ (i, j) = (k, l)$. Since c is bijective on $I \times I$, this mapping U will be a permutation of the respective basis in $\mathbb{C}^m \ \otimes \ \mathbb{C}^m$, therefore it extends in a unique manner to a linear and unitary mapping

$$U : \mathbb{C}^m \ \otimes \ \mathbb{C}^m \ \longrightarrow \ \mathbb{C}^m \ \otimes \ \mathbb{C}^m$$

And now it follows that

$$U(\ | \psi_i > \ \otimes \ | \psi_1 > \ ) \ = \ | \psi_i > \ \otimes \ | \psi_i >, \qquad 1 \leq i \leq n$$

thus indeed U is a copying machine with the "blank sheet" $| \psi_1 >$, and it can copy onto this "blank sheet" *all* the qubits in the given orthonormal basis $| \psi_1 >, \ . \ . \ . \ , | \psi_n >$ of $\mathbb{C}^m$. And in any such basis, with the exception of the fixed "blank sheet" $| \psi_1 >$, all the other qubits $| \psi_2 >, \ . \ . \ . \ , | \psi_n >$ are *arbitrary*, within the constraint that together they have to form an orthonormal basis.

Returning now to the extended situation in (2.2), we obtain the following No-Cloning property

Theorem 2.1. (Extended No-Cloning)

Given any extended quantum register $H_{\mathcal{F},n}$, and $\psi_1, \ldots, \psi_m \in (\mathbb{C}_{\mathcal{F}})^m$ orthonormal vectors, where $n, m \geq 2$. Then there does *not* exist any linear operator

$$(2.11) \quad U : (\mathbb{C}_{\mathcal{F}})^m \otimes (\mathbb{C}_{\mathcal{F}})^m \longrightarrow (\mathbb{C}_{\mathcal{F}})^m \otimes (\mathbb{C}_{\mathcal{F}})^m$$

such that

$$(2.12) \quad U(\psi \otimes \psi_1) = \psi \otimes \psi, \quad \psi \in \mathbb{C}^m$$

Proof.

We note that the relations (2.7) - (2.10) extend easily to (2.11), (2.12).

Appendix

For convenience, we shall recall in a particular case the construction, [4-11], as reviewed in [12, pp. 3-6], of *reduced power algebras*. Given any *filter* $\mathcal{F}$ on $\mathbb{N}$, we define

$$(\text{A.1}) \quad \mathcal{I}_{\mathcal{F}} = \{ \; v = (v_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \; \mid \; \{ \; n \in \mathbb{N} \mid v_n = 0 \; \} \in \mathcal{F} \; \}$$

which is a *proper ideal* in the algebra $\mathbb{R}^{\mathbb{N}}$. Thus we obtain the *reduced power algebra* associated to $\mathcal{F}$ as the quotient algebra

$$(\text{A.2}) \quad \mathbb{R}_{\mathcal{F}} = \mathbb{R}^{\mathbb{N}} / \mathcal{I}_{\mathcal{F}}$$

Furthermore, this algebra which is commutative, is also a strict extension of the field $\mathbb{R}$ of the usual real numbers, according to the embedding of algebras

$$(\text{A.3}) \quad \mathbb{R} \ni x \longmapsto (x, x, x, \ldots) + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} = \mathbb{R}^{\mathbb{N}} / \mathcal{I}_{\mathcal{F}}$$

In a similar manner one can obtain reduced power algebras extending the field $\mathbb{C}$ of the usual complex numbers. Namely, let us denote by

$$(A.4) \quad \mathcal{J}_{\mathcal{F}} = \{\ w = (w_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \ \mid\ \{\ n \in \mathbb{N} \mid w_n = 0\ \} \in \mathcal{F}\ \}$$

which is a *proper ideal* in the algebra $\mathbb{C}^{\mathbb{N}}$. Thus we obtain the *reduced power algebra* associated to $\mathcal{F}$ as the quotient algebra

$$(A.5) \quad \mathbb{C}_{\mathcal{F}} = \mathbb{C}^{\mathbb{N}} / \mathcal{J}_{\mathcal{F}}$$

Furthermore, this algebra which is commutative, is also a strict extension of the field $\mathbb{C}$ of the usual complex numbers, according to the embedding of algebras

$$(A.6) \quad \mathbb{C} \ni z \longmapsto (z, z, z, \ldots) + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}} = \mathbb{C}^{\mathbb{N}} / \mathcal{J}_{\mathcal{F}}$$

We now establish a natural connection between the algebras $\mathbb{R}_{\mathcal{F}}$ and $\mathbb{C}_{\mathcal{F}}$.

In this regard, we note the following connection between the ideals $\mathcal{I}_{\mathcal{F}}$ and $\mathcal{J}_{\mathcal{F}}$. Namely

$$(A.7) \quad \begin{array}{l} w = (w_n = u_n + iv_n)_{n \in \mathbb{N}} \in \mathcal{J}_{\mathcal{F}} \iff \\ \qquad \iff u = (u_n)_{n \in \mathbb{N}},\ v = (v_n)_{n \in \mathbb{N}} \in \mathcal{I}_{\mathcal{F}} \end{array}$$

where $u_n, v_n \in \mathbb{R}$. It follows that we have the algebra homomorphisms

$$(A.8) \quad \begin{array}{l} Re : \mathbb{C}_{\mathcal{F}} \ni w = (w_n = u_n + iv_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \longmapsto \\ \qquad \longmapsto u = (u_n)_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} \end{array}$$

$$(A.9) \quad \begin{array}{l} Im : \mathbb{C}_{\mathcal{F}} \ni w = (w_n = u_n + iv_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \longmapsto \\ \qquad \longmapsto v = (v_n)_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} \end{array}$$

as well as the algebra embeddings

$$(A.10) \quad \mathbb{R}_{\mathcal{F}} \ni u = (u_n)_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}} \longmapsto u = (u_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}}$$

$$\text{(A.11)} \quad \mathbb{R}_{\mathcal{F}} \ni v = (v_n)_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}} \longmapsto iv = (iv_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}}$$

Let us also define the surjective linear mapping

$$\text{(A.12)} \quad \begin{gathered} \mathbb{C}_{\mathcal{F}} \ni w = (w_n = u_n + iv_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \longmapsto \\ \longmapsto \overline{w} = (\overline{w_n} = u_n - iv_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}} \end{gathered}$$

As a consequence, we obtain

$$\text{(A.13)} \quad w = (w_n = u_n + iv_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \in \mathbb{C}_{\mathcal{F}}, \quad \overline{w} = w \quad \Longrightarrow \quad w \in \mathbb{R}_{\mathcal{F}}$$

Lastly, we can define the *absolute value* on $\mathbb{C}_{\mathcal{F}}$, by the mapping

$$\text{(A.14)} \quad \begin{gathered} \mathbb{C}_{\mathcal{F}} \ni z = (w_n = u_n + iv_n)_{n \in \mathbb{N}} + \mathcal{J}_{\mathcal{F}} \longmapsto \\ \longmapsto |z| = (|w_n| = \sqrt{(}u_n^2 + v_n^2))_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} \end{gathered}$$

Let us denote

$$\text{(A.15)} \quad \mathbb{R}_{\mathcal{F}}^{+} = \{ \ u = (u_n)_{n \in \mathbb{N}} + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}} \ \mid \ \{ \ n \in \mathbb{N} \mid u_n \geq 0 \ \} \in \mathcal{F} \ \}$$

then we obtain the surjective mapping

$$\text{(A.16)} \quad \mathbb{C}_{\mathcal{F}} \ni z \longmapsto |z| \in \mathbb{R}_{\mathcal{F}}^{+}$$

and for $z \in \mathbb{C}_{\mathcal{F}}$, we have

$$\text{(A.17)} \quad |z| = 0 \quad \Longleftrightarrow \quad z = 0$$

Now, in view of (A.8), (A.9), (A.14), we have for $z \in \mathbb{C}_{\mathcal{F}}$ the relations

$$\text{(A.18)} \quad |\, Re\, z\,|, \quad |\, Im\, z\,| \ \leq \ |\, z\,|$$

where the partial order $\leq$ is defined on $\mathbb{C}_{\mathcal{F}}$ by

$$\text{(A.19)} \quad u \leq v \quad \Longleftrightarrow \quad v - u \in \mathbb{R}_{\mathcal{F}}^{+}$$

Lastly, for $m \geq 1$, we define an *extended scalar product*

$$(A.20) \quad <,>: (\mathbb{C}_{\mathcal{F}})^m \times (\mathbb{C}_{\mathcal{F}})^m \longrightarrow \mathbb{C}_{\mathcal{F}}$$

by

$$(A.21) \quad <(z_1, \ldots, z_m), (w_1, \ldots, w_m)> \; = \; \overline{z_1}w_1 + \ldots + \overline{z_m}w_m \in \mathbb{C}_{\mathcal{F}}$$

for $\psi = (z_1, \ldots, z_m), \; \chi = (w_1, \ldots, w_m) \in (\mathbb{C}_{\mathcal{F}})^m$.

Then this extended scalar product has the properties

(A.22) It is linear over $\mathbb{C}_{\mathcal{F}}$, therefore also over $\mathbb{C}$, in the second argument.

$$(A.23) \quad <\chi, \psi> \; = \; \overline{<\psi, \chi>}, \quad \psi, \chi \in (\mathbb{C}_{\mathcal{F}})^m$$

$$(A.24) \quad <\psi, \psi> \in \mathbb{R}^+_{\mathcal{F}}, \quad \psi \in (\mathbb{C}_{\mathcal{F}})^m$$

and for $\psi \in (\mathbb{C}_{\mathcal{F}})^m$, one has

$$(A.25) \quad <\psi, \psi> \; = \; 0 \quad \Longleftrightarrow \quad \psi = 0 \in (\mathbb{C}_{\mathcal{F}})^m$$

Also, we have the extension of the classical Schwartz inequality

$$(A.26) \quad | <\psi, \chi> | \; \leq \; <\psi, \psi>^{1/2} <\chi, \chi>^{1/2}, \quad \psi, \chi \in (\mathbb{C}_{\mathcal{F}})^m$$

Two vectors $\psi, \chi \in (\mathbb{C}_{\mathcal{F}})^m$ are called *orthogonal*, if and only if $<\psi, \chi> \; = \; 0$.

Two orthogonal vectors $\psi, \chi \in (\mathbb{C}_{\mathcal{F}})^m$ are called *orthonormnal*, if and only if $<\psi, \psi> \; = \; <\chi, \chi> \; = \; 1$.

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